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Error calculus and regularity of Poisson functionals: 
the lent particle method.

Nicolas BOULEAU*

Abstract

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet 
forms to Poisson random measures.

Résumé

Calcul d’erreur et régularité des fonctionnelles de Poisson : la méthode de 
la particule prêtée. Nous proposons une nouvelle méthode pour appliquer le 
calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson.

1 Notation and basic formulae.

Let us consider a local Dirichlet structure with carré du champ \((X, \mathcal{X}, \nu, d, \gamma)\) where 
\((X, \mathcal{X}, \nu)\) is a \(\sigma\)-finite measured space called bottom-space. Singletons are in \(\mathcal{X}\) and \(\nu\) is 
diffuse, \(d\) is the domain of the Dirichlet form \(\epsilon[u] = 1/2 \int \gamma[u]d\nu\). We denote \((a, D(a))\) 
the generator in \(L^2(\nu)\) (cf. §).

A random Poisson measure associated to \((X, \mathcal{X}, \nu)\) is denoted \(N\). \(\Omega\) is the configuration 
space of countable sums of Dirac masses on \(X\) and \(\mathcal{A}\) is the \(\sigma\)-field generated by \(N\), of law 
\(P\) on \(\Omega\). The space \((\Omega, \mathcal{A}, P)\) is called the up-space. We write \(N(f)\) for \(\int f dN\). If \(p \in [1, \infty]\) 
the set \(\{e^{iN(f)} : f \text{ real, } f \in L^1 \cap L^2(\nu)\}\) is total in \(L^p_c(\Omega, \mathcal{A}, P)\). We put \(\tilde{N} = N - \nu\). The 
relation \(\mathbb{E}(\tilde{N}f)^2 = \int f^2 d\nu\) extends and gives sense to \(\tilde{N}(f), f \in L^2(\nu)\). The Laplace 
functional and the differential calculus with \(\gamma\) yield

\[\forall f \in d, \forall h \in D(a) \quad \mathbb{E}[e^{i\tilde{N}(f)}(\tilde{N}(g[h]) + \frac{i}{2} \tilde{N}(\gamma[f,h]))] = 0.\]

2 Product, particle by particle, of a Poisson random 
measure by a probability measure.

Given a probability space \((R, \mathcal{R}, \rho)\), let us consider a Poisson random measure \(N \odot \rho\) 
on \((X \times R, \mathcal{X} \times \mathcal{R})\) with intensity \(\nu \times \rho\) such that for \(f \in L^1(\nu)\) and \(g \in L^1(\rho)\) if

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$N(f) = \sum f(x_n)$ then $(N \otimes \rho)(fg) = \sum f(x_n)g(r_n)$ where the $r_n$’s are i.i.d. independent of $N$ with law $\rho$. Calling $(\Omega, \mathcal{A}, \mathbb{P})$ the product of all the factors $(R, \mathcal{R}, \rho)$ involved in the construction of $N \otimes \rho$, we obtain the following properties: For an $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$-measurable and positive function $F$, $E \int F(\omega, x, r)N \otimes \rho(dxdr) = \int F \ d\rho \ dN \ \mathbb{P}$-a.s.

Let us denote by $\mathbb{P}_N$ the measure $\mathbb{P}(d\omega)N_\omega(dx)$ on $(\Omega \times X, \mathcal{A} \times \mathcal{X})$. We have

Lemma 2.1 Let $F$ be $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$-measurable, $F \in L^2(\mathbb{P}_N \times \rho)$ and such that

\[ \int F(\omega, x, r) \rho(dr) = 0 \quad \mathbb{P}_N \text{-a.s., then } \int F \ d(N \otimes \rho) \text{ is well defined, belongs to } L^2(\mathbb{P} \times \mathbb{P}) \]

and

\[ \mathbb{E}(\int F \ d(N \otimes \rho))^2 = \int F^2 \ dN \ d\rho \quad \mathbb{P} \text{-a.s.} \]  

The argument consists in considering $F_n$ satisfying

$\mathbb{E} \int F_n^2 \ d\nu < +\infty$ and $\mathbb{E} \int (|F_n| \ d\nu)^2 \ d\rho < +\infty$ and then using the relation

\[ \mathbb{E}(\int F_n \ d(N \otimes \rho))^2 = (\int F_n \ d\rho \ d\nu)^2 - \int (\int F_n \ d\rho)^2 \ d\nu + \int F_n^2 \ d\nu \ d\rho \quad \mathbb{P} \text{-a.s.} \]

3 Construction by Friedrichs’ method and expression of the gradient.

a) We suppose the space by $d$ of the bottom structure is separable, then a gradient exists (cf. [2] Chap. V, p.225 et seq.). We denote it $\nabla$ and choose it with values in the space $L^2(R, \mathcal{R}, \rho)$. Thus, for $u \in d$ we have $u^b \in L^2(\nu \times \rho)$, $\gamma[u] = \int (u^b)^2 \ d\rho$ and $\nabla$ satisfies the chain rule. We suppose in addition, what is always possible, that $\nabla$ takes its values in the subspace orthogonal to the constant 1, i.e.

\[ \forall u \in d \quad \int u^b \ d\rho = 0 \quad \nu \text{-a.s.} \]

This hypothesis is important here as in many applications (cf. [2] Chap V §4.6). We suppose also, but this is not essential (cf. [2] p44) $1 \in d_{\text{loc}}$ $\gamma[1] = 0$ so that $1^b = 0$.

b) We define a pre-domain $\mathcal{D}_0$ dense in $L^2(\mathbb{P})$ by

\[ \mathcal{D}_0 = \{ \sum_{p=1}^m \lambda_p e^{m}(f_p) \in \mathbb{N}; m \in \mathbb{N}, \lambda_p \in \mathbb{C}, f_p \in \mathcal{D}(a) \cap L^1(\nu) \}. \]

c) We introduce the creation operator inspired from quantum mechanics (see [5], [6], [1], [2], [3], [4] and [10] among others) defined as follows

\[ \epsilon^+_x(\omega) \text{ equals } \omega \text{ if } x \in \text{supp}(\omega), \text{ and equals } \omega + \epsilon_x \text{ if } x \notin \text{supp}(\omega) \]

so that

\[ \epsilon^+_x(\omega) = \omega \quad \mathcal{N}_\omega \text{-a.e. } x \quad \text{and } \epsilon^+_x(\omega) = \omega + \epsilon_x \quad \nu \text{-a.e. } x \]

This map is measurable and the Laplace functional shows that for an $\mathcal{A} \times \mathcal{X}$-measurable $H \geq 0$,

\[ \mathbb{E} \int \epsilon^+ H \ d\nu = \mathbb{E} \int H \ d\nu. \]
Let us remark also that by (5), for \( F \in \mathcal{L}^2(\mathbb{P}_N \times \rho) \)

\[
(7) \quad \int \varepsilon^+ F \, d(N \circ \rho) = \int F \, d(N \circ \rho) \quad \mathbb{P} \times \hat{\mathbb{P}} \text{-a.s.}
\]

\( d \) We defined a gradient \( \xi \) for the up-structure on \( \mathcal{D}_0 \) by putting for \( F \in \mathcal{D}_0 \)

\[
(8) \quad F^\xi = \int (\varepsilon^+ F)^\gamma \, d(N \circ \rho)
\]

this definition being justified by the fact that for \( \mathbb{P}\text{-a.e.} \, \omega \) the map \( y \mapsto F(\varepsilon_y^+(\omega)) - F(\omega) \) is in \( d \), \( \varepsilon^+ F \) belongs to \( L^\infty(\mathbb{P}) \otimes d \) algebraic tensor product, and \( (\varepsilon^+ F - F)^\gamma = (\varepsilon^+ F)^\gamma \in L^2(\mathbb{P}_N \times \rho) \).

For \( F, G \in \mathcal{D}_0 \) of the form

\[
F = \sum_p \lambda_p e^{i\tilde{N}(f_p)} = \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)) \quad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \ldots, \tilde{N}(g_n))
\]

we compute using (2), (3) and (7) (in the spirit of prop. 1 of \([3]\) or lemma 1.2 of \([3]\))

\[
(9) \quad \hat{\mathbb{E}}[F^\xi G^\xi] = \sum_{p,q} \lambda_p \mu_q e^{i\tilde{N}(f_p) - i\tilde{N}(g_q)} \gamma[f_p, g_q]
\]

and we have

**Proposition 3.1** If we put \( A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(f_p)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])) \) it comes

\[
(10) \quad \mathbb{E}[A_0[F]G] = -\frac{1}{2} \mathbb{E} \sum_{p,q} \Phi_p \overline{\Phi_q}' N(\gamma[f_p, g_q]).
\]

In order to show that \( A_0[F] \) does not depend on the form of \( F \), by (10) it is enough to show that the expression \( \sum_{p,q} \Phi_p \overline{\Phi_q}' N(\gamma[f_p, g_q]) \) depends only on \( F \) and \( G \). But this comes from (9) since \( F^\xi \) and \( G^\xi \) depend only on \( F \) and \( G \).

By this proposition, \( A_0 \) is symmetric on \( \mathcal{D}_0 \), negative, and the argument of Friedrichs applies (cf. \([3]\) p4). \( A_0 \) extends uniquely to a selfadjoint operator \( (A, \mathcal{D}(A)) \) which defines a closed positive (hermitian) quadratic form \( \mathcal{E}[F] = -\mathbb{E}[A[F][F]] \). By (10) contractions operate and (cf. \([3]\)) \( \mathcal{E} \) is a Dirichlet form which is local with carré du champ denoted \( \Gamma \) and the up-structure obtained \( (\Omega, A, \mathbb{P}, \mathcal{D}, \Gamma) \) satisfies

\[
(11) \quad \forall f \in d, \quad \tilde{N}(f) \in \mathcal{D} \quad \text{and} \quad \Gamma[\tilde{N}(f)] = N(\gamma[f])
\]

The operator \( \xi \) extends to a gradient for \( \Gamma \) as a closed operator from \( L^2(\mathbb{P}) \) into \( L^2(\mathbb{P} \times \hat{\mathbb{P}}) \) with domain \( \mathcal{D} \) which satisfies the chain rule and may be computed on functionals \( \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)), \Phi \) Lipschitz and \( C^1 \) and their limits in \( \mathcal{D} \) (as done in \([3]\)).

Formula (8) for \( \xi \) can be extended from \( \mathcal{D}_0 \) to \( \mathcal{D} \). Let us introduce the space \( \mathcal{D} \) closure of \( \mathcal{D}_0 \otimes d \) for the norm

\[
\|H\|_\mathcal{D} = \left( \mathbb{E} \int \gamma[H(\omega, .)](x) \, N(dx) \right)^{1/2} + \mathbb{E} \int |\gamma[H(\omega, x)]\xi(x)| \, N(dx)
\]

where \( \xi > 0 \) is a fixed function such that \( N(\xi) \in L^2(\mathbb{P}) \).
Theorem 3.1  The formula $F^t = \int (\varepsilon^+ F)^b \ d(N \circ \rho)$ decomposes as follows

$$F \in \mathbb{D} \quad \xrightarrow{\varepsilon^+} \quad \varepsilon^+ F \in \mathbb{D} \quad \mapsto \quad (\varepsilon^+ F)^b \in L^2(\mathbb{P}_N \times \rho) \quad \xrightarrow{d(N \circ \rho)} \quad F^t \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one, $L^2_0(\mathbb{P}_N \times \rho)$ denoting the closed subspace of $L^2(\mathbb{P}_N \times \rho)$ of $\rho$-centered elements, and we have

$$\Gamma[F] = \hat{\mathbb{E}}|F^t|^2 = \int \gamma[\varepsilon^+ F] \ dN.$$  

4  The lent particle method.

Let us consider, for instance, a real process $Y_t$ with independent increments and Lévy measure $\sigma$ integrating $x^2$, $Y_t$ being supposed centered without Gaussian part. We assume that $\sigma$ has an $l.s.c.$ density so that a local Dirichlet structure may be constructed on $\mathbb{R} \setminus \{0\}$ with carré du champ $\gamma[f] = x^2 f''(x)$. If $N$ is the random Poisson measure with intensity $dt \times \sigma$ we have $\int_0^t h(s) \ dY_s = \int \int_{[0,t]} (s)h(s)xN(dsdx)$ and the choice done for $\gamma$ gives $\Gamma[\int_0^t h(s) \ dY_s] = \int_0^t h^2(s) d[Y,Y]_s$ for $h \in L^2_{loc}(dt)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-}) \ dY_s$ where $\varphi$ is Lipschitz and $C^1$, we have two ways:

a) We may represent the gradient $\gamma$ as $Y^2 = B_{[0,T]}$, where $B$ is a standard auxiliary independent Brownian motion. Then by the chain rule $V^2 = \int_0^T \varphi'(Y_{s-})(Y_{s-})^2 \ dY_s + \int_0^t \varphi(Y_{s-}) dB_{[Y_s]}$ now, using $(Y_{s-})^2 = (Y^2)_-$, a classical but rather tedious stochastic computation yields

$$\Gamma[V] = \hat{\mathbb{E}}[V^{2\gamma}] = \sum_{\alpha \leq t} \Delta Y_\alpha^2 (\int_0^t \varphi'(Y_{s-}) \ dY_s + \varphi(Y_{s-}))^2.$$  

Since $V$ has real values the energy image density property holds, and $V$ has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (13).

b) Another more direct way consists in applying the theorem. For this we define $b$ by choosing $\eta$ such that $\int_0^t \eta(r)dr = 0$ and $\int_0^t \eta^2(r)dr = 1$ and putting $f^b = x f'(x)\eta(r)$.

1°. First step. We add a particle $(\alpha, x)$ i.e. a jump to $Y$ at time $\alpha$ with size $x$ what gives

$$\varepsilon^+ V - V = \varphi(Y_{\alpha-}) x + \int_0^t \varphi'(Y_{s-}) x - \varphi(Y_{s-})) \ dY_s$$

2°. $V^b = 0$ since $V$ does not depend on $x$, and

$$(\varepsilon^+ V)^b = (\varphi(Y_{\alpha-}) x + \int_0^t \varphi'(Y_{s-}) x \ dY_s) \eta(r) \quad \text{because} \quad x^b = x \eta(r).$$

3°. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^2 dr = (\varphi(Y_{\alpha-}) x + \int_0^t \varphi'(Y_{s-}) x \ dY_s)^2$.

4°. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon^+ V] \ dN$ the integral in $N$ confuses $\varepsilon^+ \omega$ and $\omega$.

That gives $\int \gamma[\varepsilon^+ V] \ dN = \int (\varphi(Y_{\alpha-}) + \int_0^t \varphi'(Y_{s-}) \ dY_s)^2 x^2 N(d\alpha dx) \ dN$ and (13).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^b$ are non-local, but instead $F \mapsto \int (\varepsilon^+ F)^b \ d(N \circ \rho)$ and $F \mapsto \int \gamma[\varepsilon^+ F] \ dN$ are local : taking back the lent particle gives the locality.
References


