Delta Modulation for Multivariable Centralized Linear Networked Controlled Systems

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***Abstract*—This paper investigates the closed-loop properties of multivariable (MIMO[1]) linear systems where the sensed information is centralized and coded on the basis of a Δ-modulation algorithm often used for minimizing the numbers of transmitted bits. In particular we propose a new centralized vector coding algorithm that allows us to extend our previous results in [4] to any type of linear multivariable systems. In addition, we provide an estimation of the stability attraction domain, and we give some simulation results validating the proposed approach.**

***Index Terms*—Delta modulation, Networked controlled systems, NCS, quantized systems.**

I. INTRODUCTION

THis paper deals with the stabilization problem of a linear multivariable system through a communication network where information is transmitted via a particular coding algorithm. Coding algorithms seeking to transmit a minimum number of information bits are appealing in wireless networks since they allows a substantial channel bandwidth reduction. Many of such types of control architecture using that type of codes have been studied in the past. See [9], [5], [11], [12], [14], [10], [7], [1], [15] among others.

Delta modulation (Δ-M) is one alternative to minimize the numbers of bits to be coded. The reason is that innovation increments (with a granularity depending on a quantization factor Δ) are coded rather than the absolute value of the signal. Recent works in [4] have re-adapted the standard form of the delta modulation structure to their use in a feedback setup. One advantage of this type of strategy is that the coding algorithm can be built in a methodological and simple manner. A limitation is that re-synchronization may be needed, if the signal track is lost. Inspired by this approach several variants of [4] have been studied: asynchronous entropy coding [2], energy-aware coding [3], adaptive delta modulation [6], and gain scheduling multi-bit coding [8]. Except for the trivial case of diagonalizable multivariable system that can be reformulated as a set of n-scalar ones, all these works deal exclusively with scalar system.

In this paper, we present a generalization of the delta-modulation coding presented in [4], to MIMO systems. In particular we introduce a vector coding structure for *multivariable centralized* linear systems. The notion of centralization refers here to the fact that both the encoder-decoder and the control law use the full available information from all sensors. The idea is shown in Fig. 1, where we can see that all the sensed system outputs are collected in a *central* point, then transformed into a different coordinate-basis (using the transform matrix $\boldsymbol{T}_k$) before they are coded using a vector-coding algorithm. At the receiver side, it is similarly assumed that the transmitted information arrives to a central receiver, then decoded, and finally the control is computed using this centralized information. It is worth to notice that *decentralized* case is clearly much more constrained, even in absence of a coding process. A recent work [13] dealing with the case of decentralized multi-controller stability over communication channel illustrates well the fundamental difficulties, and provides an interesting preview on how to handle these problems when information is not centralized.

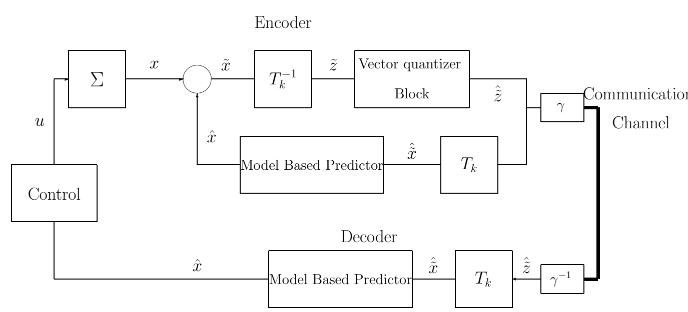

Fig. 1. NCS System with γ representing conversion from $\hat{\tilde{z}}$ to codeword.

The paper is organized as follows. After formalizing the problem in section II, we introduce in section III the general vector coding algorithm that can be adapted for all different forms of Jordan blocks resulting from the change of coordinate basis. Then, vector coding is performed in the transform domain. Vector coding here refers to the fact that a specific codeword is assigned for specific combinations between states. These are new information that are not present in standard scalar coding as they result from the combination of individual signals. Closed-loop stability properties resulting from this approach are also exposed here. Section V characterizes the attraction set associated to the previous local stability conditions. This allows a finer estimation of quantization values to

[1]MIMO : Multiple Input Multiple Output.

be used in the coding process. Finally simulation results are shown in section VI.

II. Problem formulation and assumptions

The problem considered here is the stabilization of a multivariable system in which sensor signals are centralized, and then transmitted through a digital communication link to the controller. At the controller side, the information is received in a unique point, and then decoding process provides the system n-dimensional estimated state, to be used for feedback. The coding design aims to achieve stability with a minimal information rate, thanks to a judicious coding strategy selection during the quantization step.

Let us assume the following:

- the coding process is centralized : a single encoder can be used to encode all the sensed states of the system,
- the encoded information is transmitted through a noiseless perfect transmission channel. Hence possible impairments (delay, errors) due to the transmission are not considered,
- information flow is unidirectional; the information is only transmitted from the encoder to the decoder,
- the encoder and decoder clocks are assumed to be synchronized, and samples are assumed to occur at each T_s.

The following notations will be used:

- n is the state dimension that corresponds to the number of sensors,
- m is the number of control inputs,
- $x_k = [x_k^1, \ldots, x_k^n]^T \in R^{(n\times 1)}$ is the n-dimensional sensed state vector at instant kT_s (each x_k^i corresponds to the $i-th$ sensor) ;
- $u_k = [u_k^1, \ldots, u_k^m]^T \in R^{(m\times 1)}$, is m-dimensional control input vector at instant kT_s.

The discretized system is described by:

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k \tag{1}$$

where $\boldsymbol{A} \in R^{(n\times n)}$, and $\boldsymbol{B} \in R^{(n\times m)}$. Moreover, the control law is given by

$$u_k = -\boldsymbol{K}\hat{x}_k \tag{2}$$

with $\boldsymbol{K} \in R^{(m\times n)}$ such that the eigenvalues of $\boldsymbol{A} - \boldsymbol{BK}$ are strictly lower in magnitude than 1. $\hat{x}_k$ is an estimation of x_k, and $\tilde{x}_k$ denotes the estimation error :

$$\tilde{x}_k = x_k - \hat{x}_k, \tag{3}$$

and, more generally, for a given signal s_k, $\hat{s}_k$ represent an estimated value of s_k and $\tilde{s}_k$ represent the error $s_k - \hat{s}_k$.

Without loss of generality, we suppose that system (1) is already expressed in its Jordan's form, such that $\boldsymbol{A}$ is of the form,

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{J}_{\lambda_1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{\lambda_l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_{\lambda_\gamma} \end{pmatrix} \tag{4}$$

where we assume that there are α Jordan's blocks, of dimension $\mu_l \times \mu_l$, with multi-valued real eigenvalue, and $\gamma - \alpha$ Jordan's blocks, of dimension $2\mu_l \times 2\mu_l$, with multi-valued complex conjugated eigenvalues.

For the multi-valued real eigenvalue case, the $\boldsymbol{J}_{\lambda_l}$, for $1 \leqslant l \leqslant \alpha$, are of the form,

$$\boldsymbol{J}_{\lambda_l} = \begin{pmatrix} \lambda_l & 1 & 0 & \\ 0 & \lambda_l & 1 & \\ & 0 & \lambda_l & 1 \\ & & 0 & \lambda_l \end{pmatrix} \tag{5}$$

and, for the multi-valued complex conjugated eigenvalues, the $\boldsymbol{J}_{\lambda_l}$, are, for all $\alpha + 1 \leqslant l \leqslant \gamma$, of the form,

$$\boldsymbol{J}_{\lambda_l} = \begin{pmatrix} |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 & 0 & \\ 0 & |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 & \\ & 0 & |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 \\ & & 0 & |\lambda_l|\boldsymbol{R}(\theta_l) \end{pmatrix} \tag{6}$$

where $\lambda_l = |\lambda_l|(\cos(\theta_l) + j\sin(\theta_l))$ describes the complex eigenvalues, with magnitude $|\lambda_l|$, and angle θ_l. $\boldsymbol{R}(\theta_l)$ is the rotation matrix associated to the polar form adopted above, i.e.

$$\boldsymbol{R}(\theta_l) = \begin{pmatrix} \cos(\theta_l) & \sin(\theta_l) \\ -\sin(\theta_l) & \cos(\theta_l) \end{pmatrix} \tag{7}$$

Remark 1: It is worthwhile underlining the fact that μ_l is not necessarily the multiplicity order of λ_l since the eigenvalues λ_l are not necessarily different from each other. Nevertheless, we have $\mu_1 + \ldots + \mu_\gamma = n$, n being the size of $\boldsymbol{A}$.

The case where $\boldsymbol{A}$ is diagonal ($\mu_l = 1$), with real-valued eigenvalues has been treated in [4]. In this paper we extend these results to the general case of multiple-valued, real and complex eigenvalues, with μ_l possibly different from 1.

III. Multivariable Δ-Modulation coding strategy

In this section, we present the multivariable coding strategy. This strategy is inspired from the Δ-modulation algorithm studied previously in [4] for the one-dimensional case. The n-dimensional case considered here does not result from the simple extension of the one-dimensional case, but requires a new vector coding strategy, and a particular change of coordinates (matrix T_k) for the multi-valued complex conjugated eigenvalue case. The role of the rotation matrix T_k is to align the direction of the eigenvector (signal oscillation) to the vector quantizer block.

A. Principle of multivariable coding and decoding process

Figure 1 shows the architecture of the proposed differential coding algorithm. It is composed of three main components:

- **The vector quantizer block** transforms the error $\tilde{z}_k$, into a finite codeword set, which is latter transformed into bits and sent through the communication channel,
- **The predictor**, that transforms back the codeword into a system state prediction $\hat{x}_k$
- **The rotation matrix** T_k transforms the estimation error $\tilde{x}_k$ between the signal x_k and its estimated (reconstructed) value $\hat{x}_k$ into a new set of coordinates $\tilde{z}_k$, i.e.

$$\tilde{z}_k = \boldsymbol{T}_k^{-1}\tilde{x}_k \tag{8}$$

As shown in the Figure 1, the encoding algorithm has the 3 components described above, while the decoding algorithm is just the predictor whose inputs are the received information codewords at the decoding side.

Each of these components are explained in detail next.

1) vector quantizer: it maps the transformed vector $\tilde{z}_k$ into the quantized vector $\hat{\tilde{z}}_k$. The multi-level quantizer is constructed as follows:

- we consider M_i (odd or even) subdivisions for each $\tilde{z}^i$ with respective quantization step Δ_i. The partition is centered at the origin,
- This partition generates an hypercube of dimension n with a total of $n_{\mathcal{C}} = \prod_{i=1}^{i=n} M_i$ quantized volumes (see example in Figure 2),
- To each quantized volume is associated a value for the quantized vector $\hat{\tilde{z}}_k$ (see example in the Table I).

The formula used to compute $\hat{\tilde{z}}_k$ is the following:

If M_i is odd, then $\hat{\tilde{z}}_k^i$ is given as:

$$\hat{\tilde{z}}_k^i = \begin{cases} (M_i-1)\Delta_i/2 & \text{if} \quad \mathcal{C}_1 \\ N\Delta_i & \text{if} \quad \mathcal{C}_2 \\ -(M_i-1)\Delta_i/2 & \text{if} \quad \mathcal{C}_3 \end{cases}$$

where the conditions $\mathcal{C}_i$ are:

$$\begin{array}{lll} \mathcal{C}_1 & : & \tilde{z}_k^i \geqslant (M_i-1)\Delta_i/2 \\ \mathcal{C}_2 & : & \tilde{z}_k^i \in [(N-1/2)\Delta_i, (N+1/2)\Delta_i], \\ & & (N \in \{-(M_i-1)/2, \dots, (M_i-1)/2\}) \\ \mathcal{C}_3 & : & \tilde{z}_k^i < -(M_i-1)\Delta_i/2 \end{array}$$

If M_i is even, then $\hat{\tilde{z}}_k^i$ is given as:

$$\hat{\tilde{z}}_k^i = \begin{cases} M_i/2\Delta_i & \text{if} \quad \mathcal{C}_1 \\ (N+1/2)\Delta_i & \text{if} \quad \mathcal{C}_2 \\ -M_i/2\Delta_i & \text{if} \quad \mathcal{C}_3 \end{cases}$$

where the conditions $\mathcal{C}_i$ are:

$$\begin{array}{lll} \mathcal{C}_1 & : & \tilde{z}_k^i \geqslant (M_i-1)/2\Delta_i \\ \mathcal{C}_2 & : & \tilde{z}_k^i \in [N\Delta_i, (N+1)\Delta_i], \\ & & (N \in \{-(M_i-1)/2, \dots, (M_i-1)/2\}) \\ \mathcal{C}_3 & : & \tilde{z}_k^i < -(M_i-1)\Delta_i/2 \end{array}$$

Remark 2: Before transmission, note that quantizer vector $\hat{\tilde{z}}_k^i$ is associated to a codeword of dimension $n_{\mathcal{C}}$ that can be coded directly into $R = \lceil \log_2(n_{\mathcal{C}}) \rceil$ bits, where $\lceil . \rceil$ denotes the ceil function.

2) Predictor: The estimation of the signal $\hat{x}_k$ is computed thanks to a model-based predictor:

$$\begin{aligned} \hat{x}_{k+1} &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\hat{x}_k + \boldsymbol{A}\hat{\tilde{x}}_k \\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\hat{x}_k + \boldsymbol{A}\boldsymbol{T}_k\hat{\tilde{z}}_k \end{aligned} \tag{9}$$

where the last expression results from the use of the inverse transformation matrix, i.e.

$$\tilde{x}_k = \boldsymbol{T}_k \tilde{z}_k \tag{10}$$

Due to the particular nature of this transformation (rotation matrix) its inverse always exists. Thus, using equations (8), (10) and (9), we get :

$$\tilde{z}_{k+1} = \boldsymbol{T}_{k+1}^{-1}\boldsymbol{A}\boldsymbol{T}_k(\tilde{z}_k - \hat{\tilde{z}}_k) \tag{11}$$

Note that, as this predictor is used at both the encoder and the decoder side, their respective initial conditions $\hat{x}_0$ and $\hat{\tilde{z}}_0$ are assumed to be the same.

3) Transformation matrix $\boldsymbol{T}_k$: The selection of this matrix for the general case is quite involved. In what follows we present two examples: one with a trivial choice of $\boldsymbol{T}_k = \boldsymbol{I}$, and an other where its choice depends on the eigenvalues position in the complex plane. The general case will be treated in detail in section IV.

B. Example 1: two-dimensional system with a real eigenvalue

Consider a system of the form (1), with

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

and some $\boldsymbol{B}$ such that $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Then, as the system does not contains oscillatory modes, we can take $\boldsymbol{T}_k = I_2$, where $\boldsymbol{I}_n$ denotes the n-entry identity matrix, which leads, with $\tilde{x}_k = \tilde{z}_k$, to

$$\tilde{z}_{k+1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\tilde{z}_k - \hat{\tilde{z}}_k)$$

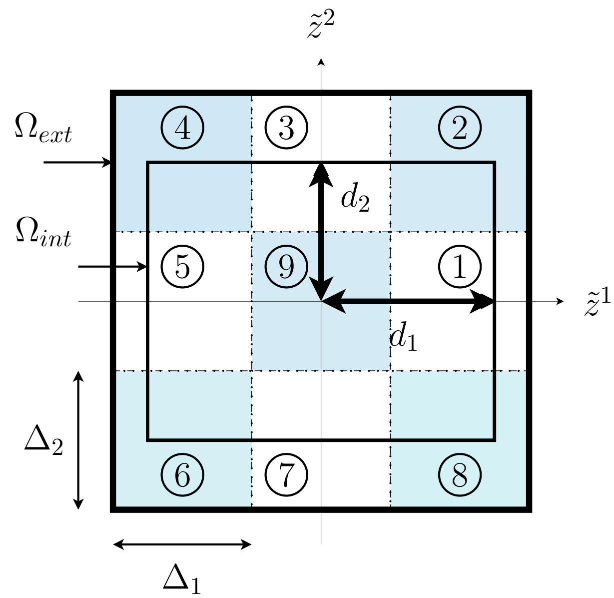

Fig. 2. Evolution of $\tilde{z}_k$ where $\tilde{z}_0$ begins in $\Omega^{\text{ext}} = \{[-1.5\Delta_1, 1.5\Delta_1] \times [-1.5\Delta_2, 1.5\Delta_2]\}$ and $\tilde{z}_k \in \Omega^{\text{int}} = \{[-d_1, d_1] \times [-d_2, d_2]\}$ and the dots delimit the nine subdivisions of the space.

Let us choose $M_i = 3$ subdivisions per signal, with a different step for each one; a quantization step of $\Delta_1 > 0$ for $\tilde{z}_k^1$, and $\Delta_2 > 0$ for $\tilde{z}_k^2$. This partition is shown in Figure 2, and the associated coding strategy in Table III-B

Now if we assume that $|\lambda| < 3$, and that the quantization steps are chosen such that

$$\Delta_2 < \Delta_1(3 - |\lambda|) \tag{12}$$

then it is easy to show that if the error signal $\tilde{z}_0$ is initiated inside the centered rectangle set Ω^{ext}, then the evolution of

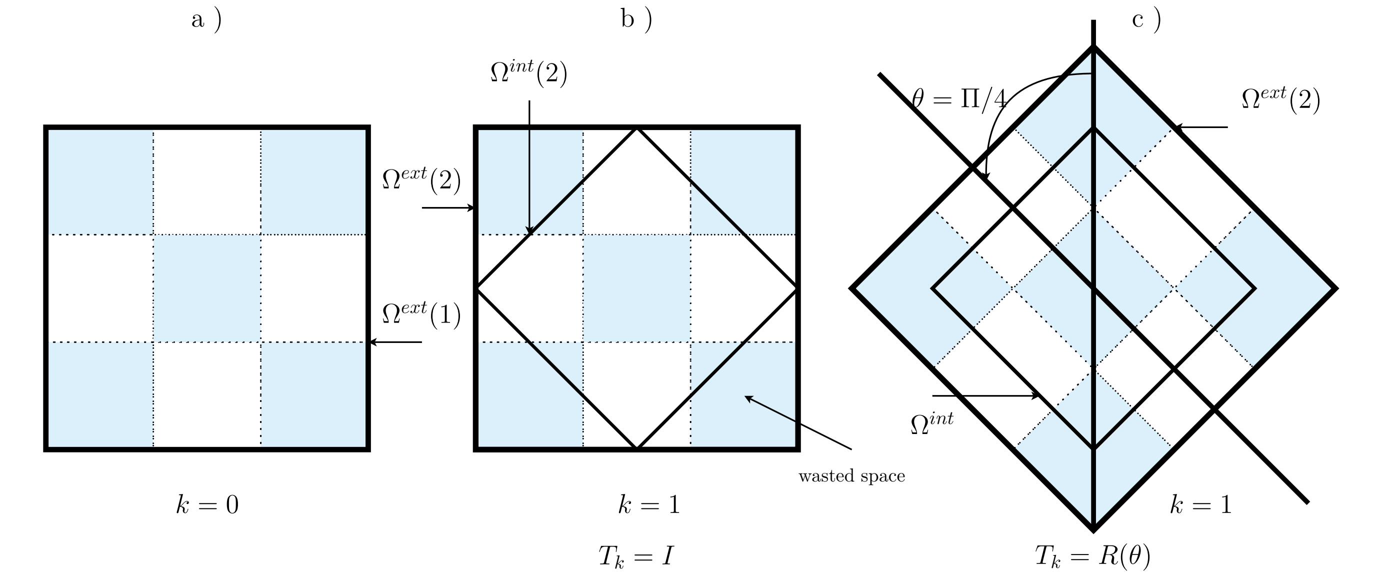

Fig. 3. Evolution of $\tilde{z}_k$, in the first figure, we choose that $\tilde{z}_k \in \Omega^{\text{ext}}(1)$ and in the second figure we see that $\tilde{z}_{k+1} \in \Omega^{\text{int}}$ and we see that if we code the signal $\tilde{z}_{k+1}$ we loose some space and, to ensure that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$, the maximal possible eigenvalue is $|\lambda| < 3/\sqrt{2}$. The third figure shows a forced rotation of the coder which permits to have better performances

TABLE I
CODING STRUCTURE RELATED TO FIGURE 2

Bits	Codeword	Value of $\hat{\tilde{z}}_k$
0000	1	$(\Delta_1, 0)$
0001	2	(Δ_1, Δ_2)
0010	3	$(0, \Delta_2)$
0011	4	$(-\Delta_1, \Delta_2)$
0100	5	$(-\Delta_1, 0)$
0101	6	$(-\Delta_1, -\Delta_2)$
0110	7	$(0, -\Delta_2)$
0111	8	$(\Delta_1, -\Delta_2)$
1000	9	$(0, 0)$

$\tilde{z}_k$ will enter (in one step) inside the set Ω^{int} as defined in Figure 2.

To see that, note that if $\tilde{z}_k \in \Omega^{\text{ext}}$, then we have $|\tilde{z}_k^i - \hat{\tilde{z}}_k^i| \leqslant \frac{\Delta_i}{2}, \forall i \in \{1, 2\}$. Now, from error equation in $\tilde{z}_k$, we have that $|\tilde{z}_{k+1}^1| < |\lambda|\frac{\Delta_1}{2} + \frac{\Delta_2}{2} = d_1$, and that $|\tilde{z}_{k+1}^2| < |\lambda|\frac{\Delta_2}{2} = d_2$. This defines the set Ω^{int}. From here it is obviously needed that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$, which lead to the condition (12).

C. Example 2: two-dimensional system with complex conjugate eigenvalues

Consider a system of the form (1), with

$$\boldsymbol{A} = |\lambda|\boldsymbol{R}(\pi/4)$$

with $\boldsymbol{R}(\pi/4)$ is defined in (7), and $\boldsymbol{B}$ such that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Suppose that we take $\boldsymbol{T}_k = \boldsymbol{I}_2$, which gives $\tilde{x}_k = \tilde{z}_k$ and from (3) we get

$$\tilde{z}_{k+1} = |\lambda|\boldsymbol{R}(\pi/4)(\tilde{z}_k - \hat{\tilde{z}}_k)$$

As in the former example, let us choose $M_i = 3$ subdivisions per signal, with a quantization step $\Delta_1 > 0$ for $\tilde{z}_k^1$, and $\Delta_2 > 0$ for $\tilde{z}_k^2$.

We suppose that the initial condition at $k = 0$ $\tilde{z}_0 \in \Omega^{\text{ext}}$ defined in the Figure 3 a), thus at $k = 1$ we obtain $\tilde{z}_1 \in \Omega^{\text{int}}$ (Figure 3 b)). It can be proved following similar steps as in Example 1 that Ω^{ext} is an invariant set if $|\lambda| < \frac{M_1}{\sqrt{2}}$ with $\Delta_1 = \Delta_2$. This condition is more conservative than the one obtained in Example III-B, where we only require that $|\lambda| < M_1$. It is also possible to retrieve the same result by redefining the transform matrix $\boldsymbol{T}_k$ as shown below

Let us choose $\boldsymbol{T}_k$ such that

$$\boldsymbol{T}_k = \boldsymbol{R}(k\pi/4)$$

Then $\tilde{z}_k = \boldsymbol{R}(-k\pi/4)\tilde{x}_k$ with $\boldsymbol{R}(\pi/4)^{-1} = \boldsymbol{R}(-\pi/4)$. Equation (3) becomes

$$\begin{aligned}\tilde{z}_{k+1} &= \boldsymbol{R}(-(k+1)\pi/4)|\lambda|\boldsymbol{R}(\pi/4)\boldsymbol{R}(-k\pi/4)^{-1}(\tilde{z}_k - \hat{\tilde{z}}_k) \\ &= |\lambda|\boldsymbol{R}(-(k+1)\pi/4)\boldsymbol{R}(\pi/4)\boldsymbol{R}(k\pi/4)(\tilde{z}_k - \hat{\tilde{z}}_k) \\ &= |\lambda|\boldsymbol{I}_2(\tilde{z}_k - \hat{\tilde{z}}_k)\end{aligned}$$

Hence, we obtain a fully decoupled system and it is straight forward to show that if $\tilde{z}_0$ begins in the set Ω^{ext}, it is necessary that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$ to ensure that Ω^{ext} is an invariant set, this condition leads to $|\lambda| < 3$ and a independent choice of Δ_1 and Δ_2. In this case, we see that we can find the same properties as in the real eigenvalues system. The generalization of this result needs an other transformation.

IV. CONSTRUCTION OF THE TRANSFORM MATRIX $\boldsymbol{T}_k$: GENERAL CASE

Consider a system of the form (1), with $\boldsymbol{A}$ defined in (4) and $\boldsymbol{B}$ such that $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. The error equation:

$$\tilde{x}_{k+1} = \boldsymbol{A}(\tilde{x}_k - \hat{\tilde{x}}_k)$$

As we have assumed that $\boldsymbol{A}$ is a block diagonal matrix, the associated stability properties can be analyzed separately for $\boldsymbol{J}_{\lambda_l}$. In the following paragraph, we will first deal with the case of real eigenvalues $1 \leqslant l \leqslant \alpha$ and latter we will focus on the complex conjugate case $\alpha+1 \leqslant l \leqslant \gamma$.

To simplify the notation, we only note $\tilde{x}_k$ instead of $\tilde{x}_k(l) \in R^{\mu_l}$, $\boldsymbol{J}_\lambda = \boldsymbol{J}_{\lambda_l}$ and $\mu = \mu_l$.

A. Case of multiple-valued real eigenvalues

Lemma 1: Case of multiple real eigenvalues. Assuming that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and suppose that

$$\tilde{z}_0 \in \Omega_{ext} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant M_i \frac{\Delta_i}{2}, 1 \leqslant i \leqslant \mu\}$$

and the quantization steps satisfy the equations

$$|\lambda| + \frac{\Delta_{i+1}}{\Delta_i} \leqslant M_i,\ 1 \leqslant i \leqslant \mu - 1 \tag{13}$$

Then

i) Ω^{ext} is an invariant set

ii) $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant 1$ where $\Omega^{\text{int}} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant |\lambda|\Delta_i/2 + \Delta_{i+1}/2$ $\forall i : 1 \leqslant i \leqslant \mu-1$ and $|\tilde{z}^\mu| \leqslant \lambda\Delta_\mu/2\}$

Proof: According to (5):

$$\tilde{z}^i_{k+1} = \lambda(\tilde{z}^i_k - \hat{\tilde{z}}^i_k) + (\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k) \tag{14}$$

$$\tilde{z}^\mu_{k+1} = \lambda(\tilde{z}^\mu_k - \hat{\tilde{z}}^\mu_k) \tag{15}$$

Given that $\hat{\tilde{z}}^{i+1}_k$ is quantized by the procedure given in section III-A1, we have $|\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k| \leqslant \frac{\Delta_{i+1}}{2}$. Then using (13), for $1 \leqslant l \leqslant \mu - 1$, we get

$$|\tilde{z}^i_{k+1}| \leqslant |\lambda||\tilde{z}^i_k - \hat{\tilde{z}}^i_k| + |\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k| \leqslant |\lambda|\frac{\Delta_i}{2} + \frac{\Delta_{i+1}}{2} \leqslant M_i\frac{\Delta_i}{2} \tag{16}$$

Finally, (13) implies that $|\lambda| < M_i$, so that

$$|\tilde{z}^\mu_{k+1}| \leqslant M_i \frac{\Delta_\mu}{2} \tag{17}$$

■

B. Case of complex conjugate eigenvalues.

We now consider the case where $\lambda \in \mathbb{C}$ for $\alpha+1 \leqslant l \leqslant \gamma$.

So, let us introduce the matrices $\boldsymbol{W}(\theta)$ and $\boldsymbol{Q}(\theta)$ defined by

$$\boldsymbol{W}(\theta) = \begin{pmatrix} \boldsymbol{R}(\theta) & 0 \\ 0 & \boldsymbol{R}(\theta) \end{pmatrix} \tag{18}$$

$$\boldsymbol{Q}(\theta) = \begin{pmatrix} \boldsymbol{R}(-\theta) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}(-\mu\theta) \end{pmatrix}. \tag{19}$$

It can be shown after a few calculations that

$$\boldsymbol{Q}^{-1}(\theta)\boldsymbol{W}^{-1}((k+1)\theta)\boldsymbol{J}_\lambda\boldsymbol{W}(k\theta)\boldsymbol{Q}(\theta) = \begin{pmatrix} |\lambda|I_2 & I_2 & 0 & 0 \\ 0 & |\lambda|I_2 & I_2 & 0 \\ \vdots & \ddots & |\lambda|I_2 & 0 \\ & \dots & 0 & |\lambda|I_2 \end{pmatrix} \widehat{=} \check{\boldsymbol{J}}_\lambda$$

Let us choose $\boldsymbol{T}_k = \boldsymbol{W}(k\theta)\boldsymbol{Q}(\theta)$. Then, as in the case of real-valued eigenvalues, we have

$$\tilde{z}_{k+1} = \check{\boldsymbol{J}}_\lambda(\tilde{z}_k - \hat{\tilde{z}}_k) \tag{20}$$

and $\check{\boldsymbol{J}}_\lambda$ is a block diagonal matrix, so that we can consider separately each block again.

Then, considering separately even indices and odd indices, we exactly recover the results of the case of real-valued eigenvalues. Indeed, if we denote $\tilde{z}^e_k = [\tilde{z}^2_k, \tilde{z}^4_k, \dots, \tilde{z}^{2\mu}_k]$ and $\tilde{z}^o_k = [\tilde{z}^1_k, \tilde{z}^3_k, \dots, \tilde{z}^{2\mu-1}_k]$, we have

$$\tilde{z}^e_{k+1} = \begin{pmatrix} |\lambda| & 1 & 0 & \\ 0 & |\lambda| & 1 & \\ & 0 & |\lambda| & 1 \\ & & 0 & |\lambda| \end{pmatrix} (\tilde{z}^e_k - \hat{\tilde{z}}^e_k) \tag{21}$$

$$\tilde{z}^o_{k+1} = \begin{pmatrix} |\lambda| & 1 & 0 & \\ 0 & |\lambda| & 1 & \\ & 0 & |\lambda| & 1 \\ & & 0 & |\lambda| \end{pmatrix} (\tilde{z}^o_k - \hat{\tilde{z}}^o_k) \tag{22}$$

Lemma 2: Case of multiple complex eigenvalues. Assuming that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and suppose that

$$\tilde{z}_0 \in \Omega^{\text{ext}} = \{\tilde{z} \in \mathbf{R}^{2\mu} : |\tilde{z}^i| \leqslant M_i \frac{\Delta_i}{2}\}$$

$$|\lambda| + \Delta_{i+2}/\Delta_i \leqslant M_i, \quad \forall i:\ 1 \leqslant i \leqslant 2\mu - 2 \tag{23}$$

Then we ensure that

i) Ω^{ext} is an invariant set

ii) $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant 1$ where $\Omega^{\text{int}} = \{\tilde{z} \in \mathbf{R}^{2\mu} : |\tilde{z}^i| \leqslant |\lambda|\Delta_i/2 + \Delta_{i+2}/2$ $1 \leqslant i \leqslant 2\mu-2$ and else $|\tilde{z}^i| \leqslant |\lambda|\Delta_i/2\}$

Proof: The proof is identical to the one derived for the demonstration of Lemma 1 in the case of real-valued eigenvalues. ■

C. General case: combined real and complex eigenvalues

Theorem 1: Suppose the system (2)

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k$$

with the pair $(\boldsymbol{A}, \boldsymbol{B})$ controllable.

And a channel rate R bounded by

$$\prod_{l=1, |\lambda_l|>1}^{n} \lceil |\lambda_l| \rceil < 2^R$$

Then, the coding structure that ensure that x_k is bounded, is realized with the Delta-modulation coding explained in section III where $\tilde{z}_k = \boldsymbol{T}_k^{-1}\tilde{x}_k$ with $\boldsymbol{T}_k$ defined as

$$\boldsymbol{T}_k = \begin{pmatrix} \boldsymbol{I}_{\mu_1} & 0 & 0 & 0... \\ 0 & \boldsymbol{I}_{\mu_\alpha} & 0 & 0... \\ 0 & 0... & \boldsymbol{W}_\iota(k\theta_\iota)\boldsymbol{Q}_\iota(k\theta_\iota) & 0 \\ 0 & 0... & 0 & \boldsymbol{W}_\gamma(k\theta_\gamma)\boldsymbol{Q}_\gamma(\theta_\gamma) \end{pmatrix} \tag{24}$$

with $\alpha + 1 = \iota$.
Then $\tilde{z}_{k+1} = \check{\boldsymbol{J}}(\tilde{z}_k - \hat{\tilde{z}}_k)$ and where $\boldsymbol{A} = \boldsymbol{T}_{k+1}\check{\boldsymbol{J}}\boldsymbol{T}_k^{-1}$

$$\check{\boldsymbol{J}} = \begin{pmatrix} \boldsymbol{J}_1 & 0 & \dots & 0 \\ \vdots & \boldsymbol{J}_\alpha & \dots & 0 \\ \vdots & 0 & \check{\boldsymbol{J}}_\iota & 0 \\ 0 & \dots & 0 & \check{\boldsymbol{J}}_\mu \end{pmatrix}$$

with the properties for M_i and Δ_i given in lemma 1 for real eigenvalues and lemma 2 for complex eigenvalues.

Proof: For each signal with instable open loop, one of the condition is $|\lambda_l| < M_i$, it is sufficient that $\lceil |\lambda_l| \rceil < M_i$ with $R = \log_2 \prod_{i=1}^{n} M_i$. If we multiply for all the coefficients, the result becomes

$$\prod_{l=1, |\lambda_l|>1}^{n} \lceil |\lambda_l| \rceil < 2^R$$

Using the previous lemmas, we ensure that $\tilde{x}$ is bounded.

$$x_{k+1} = (\boldsymbol{A} - \boldsymbol{BK})x_k + \boldsymbol{A}\tilde{x}_k$$

With the following system where $\boldsymbol{A} - \boldsymbol{BK}$ has its eigenvalues strictly inferior than 1, the authors of [4] have shown that the cascade system ensures that x_k is bounded. ■

V. Domain of attraction and new tuning policies for Δ_i

The aim of this section is twofold. First assuming the use of the tuning rule (13), we provide a less conservative method to estimate the attraction domain (named $\mathcal{B} \supset \Omega^{\text{ext}}$). Second, assuming the same attraction domain Ω^{ext}, we provide a new tuning rule for the Δ_i that,compared to previous rule given in (13), results in smaller values for Δ_i. As a consequence, the system precision can be improved. Specific simulation results concerning this last case, will be presented at the end of the paper.

A. Characterization of $\mathcal{B}$

Let assume that the Δ_i are tuned following the rule in (13), and denote $\mathcal{B}$ the new estimation of the attraction domain with $\Omega^{\text{ext}} \subset \mathcal{B} \subset \mathbf{R}^n$. Let $\mathcal{B}$ be defined as the compositions of the sub-sets $\mathcal{B}_{\lambda_l}$,

$$\mathcal{B} = \mathcal{B}_{\lambda_1} \times ... \times \mathcal{B}_{\lambda_w} \tag{25}$$

where the $\mathcal{B}_{\lambda_l}$ describes the attraction domain for the l-th Jordan's block, $\check{\boldsymbol{J}}_{\lambda_l}$, under consideration,

$$\tilde{z}_{k+1} = \underbrace{\begin{pmatrix} |\lambda_l| & 1 & 0 \\ 0 & |\lambda_l| & 1 \\ 0 & 0 & |\lambda_l| \end{pmatrix}}_{\check{\boldsymbol{J}}_{\lambda_l}} (\tilde{z}_k - \hat{\tilde{z}}_k)$$

This decomposition simplifies the analysis by looking at each block separately instead of considering the whole system together. Therefore, we only need to focus on a single block $\mathcal{B}_{\lambda_l}$, and repeat the same analysis for other block when needed.

Inspired by the Jordan block structure, assume in turn that $\mathcal{B}_{\lambda_l} = \mathcal{H}_{\lambda_l,1} \times ... \times \mathcal{H}_{\lambda_l,\mu_l}$ where each subset, $\mathcal{H}_{\lambda_l,1}$, correspond to a domain associated to each of the Jordan block components. For simplicity reasons, we omit the subindex λ_l in the sequel. Hence, we simply note $\mathcal{B} = \mathcal{H}_1 \times ... \times \mathcal{H}_{\mu_l}$.

Theorem 2: Assume that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and that Δ_i are tuned following the rule in (13), and suppose that

$$\tilde{z}_0 \in \mathcal{B} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant \gamma_i\}$$

with, for $1 \leqslant i \leqslant \mu - 1$,

$$\begin{aligned} \gamma_i &= \min\left((M-1)\Delta_i/2 + \varepsilon^i_{\text{max}}, (|\lambda||\hat{\tilde{z}}^i_k| - \varepsilon^{i+1}_{\text{max}})/(|\lambda| - 1)\right) \\ \varepsilon^{i+1}_{\text{max}} &\leqslant \min\left((M - |\lambda|)\Delta_i/2, (M - 1 - |\lambda|)\Delta_i/2 + \varepsilon^i_{\text{max}}\right) \end{aligned}$$

then:

i) $\mathcal{B}$ is an invariant set, i.e. $\tilde{z}_k \in \mathcal{B}\, \forall k \geqslant 0$.
ii) $\exists k_1 > 0$, such that, $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant k_1$. where Ω^{int} is the same set as defined in Lemma 1-*ii*).

Proof: Details of the proof are given in Appendix. ■

Note that this analysis allows us to obtain a bigger attraction domain than the one obtained in section IV. To see this, note that $\varepsilon^i_{\text{max}} \geqslant \Delta_i/2$, which implies that $\gamma_i \geqslant M\Delta_i/2$, and therefore we have that

$$\mathcal{B} \supset \Omega^{\text{ext}}$$

B. Tuning policies for Δ_i

Assume now that the attraction domain $\bar{\Omega}^{\text{ext}}$, is given by

$$\bar{\Omega}^{\text{ext}} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant \delta_i, 1 \leqslant i \leqslant \mu\}$$

where δ_i are arbitrary values specified by the user. Note that the specification above imposes, in the previous tuning method, that $M_i \frac{\Delta_i}{2} = \delta_i$, whereas theorem 3 below will show that the new values $\bar{\Delta}_i < \Delta_i = \frac{2\delta_i}{M_i}$ leading to a smaller convergence set $\bar{\Omega}^{\text{int}} \subset \Omega^{\text{int}}$, where Ω^{int} is the same set as defined in Lemma 1-*ii*).

Theorem 3: Suppose that $\tilde{z}_0 \in \bar{\Omega}^{\text{ext}}$, and let the following rule to be applied to select the coding levels,for $1 \leqslant i \leqslant \mu - 1$,

$$\begin{aligned} \bar{\Delta}_i &= 2\frac{|\lambda| - 1}{|\lambda|(M-1)}\delta_i + 2\frac{\delta_{i+1} - (M-1)\bar{\Delta}_{i+1}/2}{|\lambda|(M-1)} \\ \bar{\Delta}_\mu &= \delta_\mu(2(|\lambda| - 1))/|\lambda| \end{aligned}$$

Then:

i) $\bar{\Omega}^{\text{ext}}$ is an invariant set, and
ii) $\exists k_1 > 0$, such that, $\tilde{z}_k \in \bar{\Omega}^{\text{int}}$, $\forall k \geqslant k_1$, where $\bar{\Omega}^{\text{int}} \subset \Omega^{\text{int}}$ is given as:

$$\bar{\Omega}^{\text{int}} = \left\{ \tilde{z} \in \mathbf{R}^\mu : \left\{ \begin{array}{ll} |\tilde{z}^i| \leqslant |\lambda|\bar{\Delta}_i/2 + \bar{\Delta}_{i+1}/2 & 1 \leqslant i \leqslant \mu - 1 \\ |\tilde{z}^\mu| \leqslant |\lambda|\bar{\Delta}_\mu/2 & i = \mu \end{array} \right. \right\}$$

Proof: Property *i*) can be shown following the same proof as in part *i*) of Theorem 2, given in the Appendix. For the Property *ii*) the convergence of $\tilde{z}_k$ towards the set $\bar{\Omega}^{\text{int}}$ in finite time also follows the same lines as the proof of Theorem 2 and is omitted here.

Finally the fact that $\bar{\Omega}^{\text{int}} \subset \Omega^{\text{int}}$ follows by first observing that both sets $\bar{\Omega}^{\text{int}}$, and Ω^{int} have the same upper bound structure, and hence it is sufficient to prove that $\bar{\Delta}_i < \Delta_i$.

This last inequality follows from inspection comparing the definition of the $\bar{\Delta}_i$ given in the theorem with the ones resulting from the imposed constraints to the previous tuning method, i.e. $\Delta_i = \frac{2\delta_i}{M_i}$.

■

VI. Simulation results

The aim of this section, is to compare by simulations, the precision improvements that the second tuning method derived in previous section can provide. For this, we consider a second order system, already in it Jordan form:

$$\boldsymbol{A} = \begin{pmatrix} 1.1 & 1 \\ 0 & 1.1 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}$$

The controller is designed on the basis of a full static state feedback with the desired closed-loop eigenvalues located at $(0.5, 0.6)$. The control objective is to regulate the output states to a fix value; $x_1^{\text{ref}} = 1$, $x_2^{\text{ref}} = 1$. The desired attraction domain for the estimation error is specified as $(\delta_1, \delta_2) = (0.62, 0.52)$, and the initial error state are taken inside $\bar{\Omega}^{\text{ext}}$; $x_0 = (0.6, 0.5)$, and $\hat{x}_0 = (0, 0)$. We choose 2 word-code by signal namely 2 bits per unit of sampling time; $M_1 = M_2 = 2$.

Under this conditions on the quantization step Δ_i are computed according to the conditions given in theorem 1; $\Delta_1 = 0,62$ and $\Delta_2 = 0,52$. The $\bar{\Delta}_i$ are now computed following the procedure in Section V; $\bar{\Delta}_1 = 0,35$ and $\bar{\Delta}_2 = 0,057$.

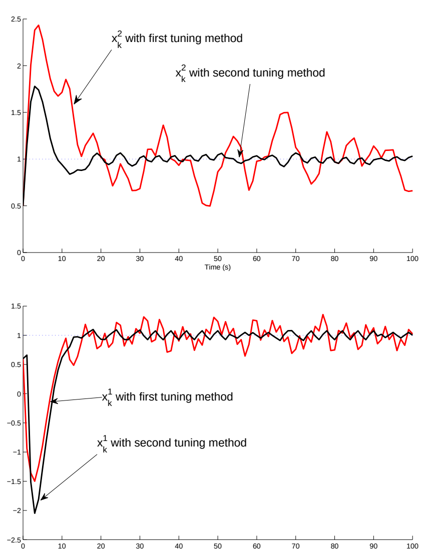

Fig. 4. Time-evolution of the closed-loop state x_k^2 (upper) and x_k^1(lower) using two different tuning methods discussed in this paper. The impact of quantization on the first state is less effective than on the second state.

Figure 4 shows the time-evolution of the resulting closed-loop signals (coding including). In both runs, the initial condition are the same, and as it was expected the second methods provides smaller values for the coding gains, resulting in a better signal reconstruction quality, and hence better regulation precision.

VII. Conclusion

In this paper, we have investigated the closed-loop properties of multivariable (MIMO) linear systems where the sensed information is centralized and coded on the basis of a Δ-modulation algorithm intended to be used for minimizing the number of transmitted bits.

In particular we had proposed a new centralized vector coding algorithm that allows us to extend our previous results in [4] to linear multivariable systems of arbitrarily dimension and arbitrarily structure (any canonical form with arbitrarily eigenvalues). The key feature allowing this results was based on the idea of performing the differential coding in a time-varying rotation coordinates associated to the well known canonical Jordan forms.

We have also shown that this fixed-gain simple and methodic coding strategy results in a ultimately uniformly (local) stability. We have also provided an estimation of the attraction domain, and a new method to tune the coding gains, resulting in closed-loop precision improvements. Simulation results have also been presented validating the proposed approach.

Future extensions of this work envision to devise adaptation rules for the coding gains, in order to generalize these results to global stability with an arbitrarily small convergence set precision.

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APPENDIX

The principle of this demonstration is based on a cascade argument. We assure that $\tilde{z}_k^1$ remains in $\mathcal{H}_1$ under conditions on $\tilde{z}_k^2$ and so on. In section IV, we have studied the comportment of $\tilde{z}_k$ in Ω^{ext}. Here we only interest us in the case where $\tilde{z}_k$ is outside Ω^{ext}, so we specify all the demonstration on the last quantization namely $|\hat{\tilde{z}}_k^i| = (M-1)\Delta_i/2$.

A. Determination of $\mathcal{H}_1 = \mathcal{H}_{\lambda,1}$

Here we determine conditions on $\tilde{z}_k^1$ and $\tilde{z}_k^2$ which permits to assure that the upper bound of $\mathcal{H}_1$ is bigger than (or equal to) $M\Delta_1/2$. We characterize the maximal value $\varepsilon_{\text{max}}^2 = \max_{k\in\mathbb{N}}(|\tilde{z}_k^2 - \hat{\tilde{z}}_k^2|)$, which yet permits that $\tilde{z}_k^1$ remains in $\mathcal{H}_1$. To this purpose, let us introduce the function $V_k^1 = |\tilde{z}_k^1|$ and its rate $\Delta V_k^1 = |\tilde{z}_{k+1}^1| - |\tilde{z}_k^1|$

$$\Delta V_k^1 \leqslant \quad |\lambda||\tilde{z}_k^1 - \hat{\tilde{z}}_k^1| - |\tilde{z}_k^1| + \varepsilon_{max}^2 \triangleq \psi_1(\tilde{z}_k^1, \varepsilon_{\text{max}}^2)$$

$$\Delta V_k^1 \leqslant \begin{cases} (|\lambda_l|-1)|\tilde{z}_k^1| - |\lambda|\hat{\tilde{z}}_k^1 + \varepsilon_{max}^2 & \text{if} \quad |\tilde{z}_k^1| > |\hat{\tilde{z}}_k^1| \\ -(|\lambda|+1)|\tilde{z}_k^1| + |\lambda||\hat{\tilde{z}}_k^1| + \varepsilon_{max}^2 & \text{if} \quad |\tilde{z}_k^1| \leqslant |\hat{\tilde{z}}_k^1| \end{cases}$$

where $|\hat{\tilde{z}}_k^1| = (M-1)\Delta_1/2$.

There exists a set where ψ_1 is negative if $\alpha_{1,1} < \alpha_{2,1}$ with $\alpha_{1,1} = (|\lambda||\hat{\tilde{z}}_k^1| + \varepsilon_{\text{max}}^2)/(|\lambda|+1)$ and $\alpha_{2,1} = (|\lambda||\hat{\tilde{z}}_k^1| - \varepsilon_{\text{max}}^2)/(|\lambda|-1)$. In the following analysis, we use a more restrictive condition to characterize $\mathcal{H}_1$. The function ψ_1 is positive for $(M-2)\Delta_1/2 \leqslant |\tilde{z}_k^1| \leqslant \alpha_{1,1}$, negative if $\alpha_{1,1} \leqslant |\tilde{z}_k^1| \leqslant \alpha_{2,1}$ and positive for $|\tilde{z}_k^1| > \alpha_{2,1}$. On this part, we interest ourselves in the zone where the rate of the function is positive and then negative. We search to find the maximal value of $\varphi(\tilde{z}_k^1, \varepsilon_{\text{max}}^2)$ with $|\tilde{z}_k^1| \in [(M-2)\Delta_1/2, \alpha_{1,1}]$. In this purpose, let us introduce φ_μ

$$|\tilde{z}_{k+1}^1| \leqslant \quad |\lambda||\tilde{z}_k^1 - \hat{\tilde{z}}_k^1| + \varepsilon_{\text{max}}^2 \triangleq \varphi_1(\tilde{z}_k^1, \varepsilon_{\text{max}})$$

$$|\tilde{z}_{k+1}^1| \leqslant \begin{cases} |\lambda|(|\tilde{z}_k^1| - |\hat{\tilde{z}}_k|) + \varepsilon_{\text{max}}^2 & \text{if} \quad |\tilde{z}_k^1| > |\hat{\tilde{z}}_k^1| \\ |\lambda|(|\hat{\tilde{z}}_k^1| - |\tilde{z}_k^1|) + \varepsilon_{\text{max}}^2 & \text{if} \quad |\tilde{z}_k^1| \leqslant |\hat{\tilde{z}}_k^1| \end{cases}$$

This function is piecewise described on each interval of step $\Delta_1/2$ and we easily obtain that the function φ_1 is totally increasing (decreasing) on each sector, hence the maximal values are at each extremity. So the value is $|\lambda|\Delta_1/2 + \varepsilon_{\text{max}}^2$ or α_1. Since $|\lambda| > 1$, we have $\alpha_1 < |\lambda|\Delta_1/2 + \varepsilon_{\text{max}}^2$, thus the maximal value is $|\lambda|\Delta_1/2 + \varepsilon_{\text{max}}^2$. To ensure that $\tilde{z}_k^1$ remains bounded, when the rate ΔV_k^1 is positive, the maximal value of $\tilde{z}^1$ takes at time $k+1$ has to be lower than α_2. So we obtain:

$$\varepsilon_{\text{max}}^2 \leqslant (M - |\lambda|)\Delta_1/2$$

Moreover, if we fix the worst case for the size of $\mathcal{H}_1$ namely $\varepsilon_{\text{max}}^2 = ((M-|\lambda|)\Delta_1/2$ we have $\gamma_1 = M\Delta_1/2$. This choice permits a bigger size for $\mathcal{H}_2$ due to the inequality between the two quantization steps in (13), we have $\varepsilon_{\text{max}}^2 \geqslant \Delta_2/2$, so we lose in conservatism and we obtained $|\tilde{z}_k^2 - \hat{\tilde{z}}_k^2| \leqslant \varepsilon_{\text{max}}^2$. Here we conclude that

$$\mathcal{H}_1 = \{\tilde{z}_k^1 : \quad |\tilde{z}_k^1| \leqslant \gamma_1\}$$

B. Determination of $\mathcal{H}_i = \mathcal{H}_{\lambda,i}$ $1 < i < \mu$

The analysis on $\mathcal{H}_i$ is almost the same as the analysis on $\mathcal{H}_1$ except that another constraint is imposed $\forall k \quad k > 0$ $|\tilde{z}_k^i| \leqslant |\hat{\tilde{z}}_k^i| + \varepsilon_{\text{max}}^i$ and we obtain:

$$\varepsilon_{\text{max}}^{i+1} \leqslant \min((M-|\lambda|)\Delta_i/2, (M-1-|\lambda|)\Delta_i/2 + \varepsilon_{\text{max}}^i)$$

With $\gamma_i = \min((M-1)\Delta_i/2 + \varepsilon_{\text{max}}^i, \alpha_{2,i})$, we can conclude that $\gamma_i \geqslant M\Delta_i/2$ and we have

$$\mathcal{H}_i = \{\tilde{z}_k^i : \quad |\tilde{z}_k^i| < \gamma_i\}$$

C. Determination of $\mathcal{H}_\mu = \mathcal{H}_{\lambda,\mu}$

We have determined conditions that $\tilde{z}_k^\mu$ needs to fill in order to characterize $\mathcal{H}_{\mu-1}$. Let us introduce the function $V_k^\mu = |\tilde{z}_k^\mu|$ and its rate $\nabla V_k^\mu = |\tilde{z}_{k+1}^\mu| - |\tilde{z}_k^\mu|$ to analyze the invariance set $\mathcal{H}_\mu$.

$$\Delta V_k^\mu \leqslant \quad |\lambda||\tilde{z}_k^\mu - \hat{\tilde{z}}_k^\mu| - |\tilde{z}_k^\mu| \triangleq \psi_\mu(\tilde{z}_k^\mu)$$

$$\Delta V_k^\mu \leqslant \begin{cases} (|\lambda_l|-1)|\tilde{z}_k^\mu| - |\lambda_l|\hat{\tilde{z}}_k^\mu & \text{if} \quad |\tilde{z}_k^\mu| > |\hat{\tilde{z}}_k^\mu| \\ -(|\lambda|+1)|\tilde{z}_k^\mu| + |\lambda||\hat{\tilde{z}}_k^\mu| & \text{if} \quad |\tilde{z}_k^\mu| \leqslant |\hat{\tilde{z}}_k^\mu| \end{cases}$$

where $|\hat{\tilde{z}}_k^\mu| = (M-1)/2\Delta_\mu$. There exists a set where ψ_μ is negative if $\alpha_{1,\mu} < \alpha_{2,\mu}$ with $\alpha_{1,\mu} = (|\lambda||\hat{\tilde{z}}_k^\mu|)/(|\lambda|+1)$ and $\alpha_{2,\mu} = (|\lambda||\hat{\tilde{z}}_k^\mu|)/(|\lambda|-1)$. In the following, we obtain a restrictive condition. If that zone exists, we must interest ourselves in the zone $|\tilde{z}_k^\mu| \in [(M-2)/2\Delta_1, \alpha_{1,\mu}]$ where the rate is positive. To understand its impact let us introduce φ_μ.

$$|\tilde{z}_{k+1}^\mu| \leqslant \quad |\lambda||\tilde{z}_k^\mu - \hat{\tilde{z}}_k^\mu| \triangleq \varphi_\mu(\tilde{z}_k^\mu)$$

$$|\tilde{z}_{k+1}^\mu| \leqslant \begin{cases} |\lambda|(|\tilde{z}_k^\mu| - |\hat{\tilde{z}}_k^\mu|) & \text{if} \quad |\tilde{z}_k^\mu| > |\hat{\tilde{z}}_k^\mu| \\ |\lambda|(|\hat{\tilde{z}}_k^\mu| - |\tilde{z}_k^\mu|) & \text{if} \quad |\tilde{z}_k^\mu| \leqslant |\hat{\tilde{z}}_k^\mu| \end{cases}$$

With the analysis of φ_μ we can show that on the zone where ΔV_k^μ is positive, we ensure that the worst case of $\tilde{z}_{k+1}^\mu$ is inferior than $\min(\varepsilon_{\text{max}}^\mu, \frac{|\lambda|(M-1)}{2(|\lambda|-1)})\Delta_\mu$. So we obtain that $\gamma_\mu = \min(\varepsilon_{\text{max}}^\mu, |\lambda|(M-1)\Delta_\mu/(2(|\lambda|-1)))$. We conclude that

$$\mathcal{H}_{\mu_l} = \{\tilde{z}^\mu : \quad |\tilde{z}_k^\mu| < \gamma_\mu\}$$

D. Convergence to Ω^{int}

Now, we obtain for $|\tilde{z}^\mu| < \gamma_\mu$ that its rate function is negative on $[M\Delta_\mu/2, \gamma_\mu]$, so we can ensure that there exists a k_μ such that $|\tilde{z}_{k_\mu}^\mu| < M\Delta_\mu/2$ and $|\tilde{z}_{k_\mu+1}^\mu| \leqslant |\lambda|\Delta_\mu/2$. For i from $\mu-1$ to 1, we can ensure that the rate function ΔV_k^i is negative in $[M\Delta_i/2, \gamma_i]$ after the time k_{i+1}. Hence we ensure that there exists a k_i such that $|\tilde{z}_{k_i}^i| < M\Delta_i/2$ that implies that $|\tilde{z}_{k_i+1}^i| \leqslant |\lambda|\Delta_i/2 + \Delta_{i+1}/2$. To conclude, we find a new attraction domain $\mathcal{B}$ less conservative than Ω^{ext}. Moreover $\exists k_1 : \forall k > k_1 + 1 \quad \tilde{z}_k \in \Omega^{\text{int}}$.