Simulations between triangular and hexagonal number-conserving cellular automata

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Abstract. A number-conserving cellular automaton is a cellular automaton whose states are integers and whose transition function keeps the sum of all cells constant throughout its evolution. It can be seen as a kind of modelization of the physical conservation laws of mass or energy. In this paper, we first propose a necessary condition for triangular and hexagonal cellular automata to be number-conserving. The local transition function is expressed by the sum of arity two functions which can be regarded as 'flows' of numbers. The sufficiency is obtained through general results on number-conserving cellular automata. Then, using the previous flow functions, we can construct effective number-conserving simulations between hexagonal cellular automata and triangular cellular automata.

Key words: Cellular automata; Number-conservation.

1 Introduction

A number-conserving cellular automaton (NCCA) is a cellular automaton (CA) such that all states of the cells are represented by integers and the sum of the numbers (states) of all cells of a global configuration is preserved throughout the computation. It can be thought as a kind of model of physical phenomena as, for example, fluid dynamics and highway traffic flow [7] and constitutes an alternative to differential equations.

There is a huge literature published in the domain which witnesses the great interest in number-conserving cellular automata which gathers together physicians, computer scientists and mathematicians. Actually, this particular model of CA applies to phenomena governed by conservation laws of mass or energy.

Boccara et al. [1] studied number conservation of one-dimensional CAs on circular configurations. Durand et al. [2,3] considered the two-dimensional case and the relations between several boundary conditions. These results are very useful for deciding whether a given CA is number-conserving but do not help much for the design of NCCAs with complex transition rules.

As for the rectangular von Neumann neighborhood case [5,8], several necessary and sufficient conditions to be number-conserving are shown. According

to these conditions, the local function of a rotation-symmetric NCCA is expressed by the sum of arity two functions as in [4]. Designing the functions, we constructed several NCCAs including a 14-state logically universal NCCA with rotation-symmetry [8], always with square neighborhoods.

In this paper, we show specific necessary conditions for triangular and hexagonal CAs to be number-conserving. Under some symmetry assuptions (rotation symmetry for the triangular case and permutation symmetry for the hexagonal one), we show that the local transition function can be decomposed into the sum of several *flow functions*, that is, functions only depending upon two variables. These flow functions are later used to design respective simulations between hexagonal cellular automata and triangular cellular automata if we assume both cellular automata to be permutation-symmetric.

This paper is organized as follows; section 2 recalls the classical definitions that will be used; section 3 exhibits the necessary conditions for triangular and hexagonal CAs to be number-conserving. And, finally, in section 4, we present number-conserving simulations between hexagonal cellular automata and triangular cellular automata under the permutation symmetry assumption.

2 Definitions

Definition 1. *A* deterministic two-dimensional radius one cellular automaton *is a 5-tuple defined by* $A = (\mathbb{Z}^2, n, Q, f, q)$*, where* $\mathbb{Z}$ *is the set of all integers,* $n \in \{3, 4, 6\}$ *is the number of neighbor cells (which implies a corresponding neighbor vector set, which is a finite and ordered set of distinct vectors from* $\mathbb{Z}^2$*:* $\{\overrightarrow{v_0}, \ldots, \overrightarrow{v_n}\}$*),* Q *is a non-empty finite set of internal states of each cell,* $f : Q^n \to Q$ *is a mapping called the* local transition function *and* $q \in Q$ *is a* quiescent state *that satisfies* $f(q, \cdots, q) = q$*.*

A *configuration* over Q is a mapping $\alpha : \mathbb{Z}^2 \to Q$. The set of all configurations over Q is denoted by $\mathrm{Conf}(Q)$, i.e., $\mathrm{Conf}(Q) = \{\alpha | \alpha : \mathbb{Z}^2 \to Q\}$. The function $F : \mathrm{Conf}(Q) \to \mathrm{Conf}(Q)$ is defined as follows and is called the *global function* of A induced by f: $\forall \alpha \in \mathrm{Conf}(Q), \forall \overrightarrow{v} \in \mathbb{Z}^2, F(\alpha)(\overrightarrow{v}) = f(\alpha(\overrightarrow{v}+\overrightarrow{v_0}), \ldots, \alpha(\overrightarrow{v}+\overrightarrow{v_n}))$. From now on, we will denote $\overrightarrow{v}$ by (x, y).

Let us denote by C_F the set of *finite configurations* i.e. which have a finite number of non-quiescent states. A cellular automaton A is *finite number-conserving* (FNC) when it satisfies

$$\forall \alpha \in C_F, \quad \sum_{(x,y)\in\mathbb{Z}^2} \{F(\alpha)(x, y) - \alpha(x, y)\} = 0.$$

And, according to [3], A is FNC if and only if it is number-conserving.

Next we define some symmetry conditions of common use (eg. see [5]).

Definition 2. *CA* A *is* rotation-symmetric *if its local function* f *satisfies:*

$$\forall g, s_i \in Q \ (1 \le i \le n), f(g, s_1, \cdots, s_n) = f(g, s_2, \cdots, s_n, s_1),$$

and A is permutation-symmetric *if its local function f satisfies:*

$$\forall g, s_i \in Q \ (i \leq 1 \leq n), \forall \pi \in S_n, f(g, s_1, \cdots, s_n) = f(g, s_{\pi(1)}, \cdots, s_{\pi(n)}),$$

where S_n denotes the permutation group with n elements.

2.1 Simulation

Below, we propose the definition of a step by step simulation between two CAs. It expresses that if a CA A simulates each step of CA B in τ units of time, there must exist effective applications between the corresponding configurations [6]:

Definition 3. *Let Conf_A and Conf_B be the two sets of CA configurations of (resp.) A and B. We say that A simulates each step of B in time τ (and we note $B \overset{\tau}{\prec} A$) if there exists a constant $\tau \in \mathbb{N}$ and two recursive functions $\kappa : \mathrm{Conf}_B \to \mathrm{Conf}_A$ and $\rho : \mathrm{Conf}_A \to \mathrm{Conf}_B$ such that $\kappa \circ \rho = Id$ and for all $c, c' \in \mathrm{Conf}_B$, there exists $c'' \in \mathrm{Conf}_A$ such that if $c' = F_B(c)$, $c'' = F_A^\tau(\kappa(c))$ with $\rho(c'') = c'$, where F_M denotes the global transition of CA M and F_M^t the t-th iterate of a global transition of CA M.*

Depending upon the value of τ, we say that the simulation is *elementary* if $\tau = 1$ and *simple* if $\tau = O(1)$.

3 Von Neumann neighborhood number-conserving CA

Durand et al. [2] proved a general necessary and sufficient condition for a NCCA with $n \times m$ neighbors to be number-conserving. With this condition, any local function can be decomposed into the summation of the local function in which several arguments are fixed to zero (which is a quiescent state). The drawback of this general statement is that it does not explicitely represent neither the movement of the numbers nor symmetries. In the sequel, we show novel necessary conditions for NCCA in different lattices structures, namely triangular and hexagonal. The case of the square grid was already considered in [8], where a necessary and sufficient condition for a von Neumann neighborhood CA to be number-conserving was shown.

3.1 Triangular number-conserving cellular automata

Theorem 1. *A deterministic two-dimensional rotation-symmetric triangular CA $A = (\mathbb{Z}^2, 3, Q, t, q)$ is number-conserving iff t satisfies*

$$\begin{aligned} &\exists \varphi : Q^2 \to \mathbb{Z}, \forall g, a, b, c \in Q, \\ &t(g, a, b, c) = g + \varphi(g, a) + \varphi(g, b) + \varphi(g, c) \\ &\varphi(g, a) = -\varphi(a, g). \end{aligned}$$

with $\varphi(g, a) = t(g, a, q, q) - t(g, q, q, q) - t(q, g, q, q) + q$.

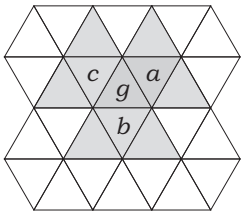


Fig. 1. A configuration in the triangular case.

Proof. Let $\delta(g,a,b,c) \equiv t(a,g,q,q) + t(b,g,q,q) + t(c,g,q,q) + t(g,a,b,c) + 2t(q,a,q,q) + 2t(q,b,q,q) + 2t(q,c,q,q) - a - b - c - g - 6q$. With respect to the configuration of Fig.1, only shadowed cells change their states in the next step. Then for any g,a,b,c in Q, $\delta(g,a,b,c) = 0$ is necessary to preserve number conservation. Let's consider the following equation.

$$\delta(g,a,b,c) - \delta(g,a,q,q) - \delta(g,q,b,q) - \delta(g,q,q,c) + 3\delta(g,q,q,q) = 0$$

To satisfy the number-conservation, it is also necessary. Finally, the following condition is necessary by expanding the equation.

$$t(g,a,b,c) = g+3q+t(g,a,q,q)+t(g,q,b,q)+t(g,q,q,c)-3t(g,q,q,q)-3t(q,g,q,q)$$

Let $\varphi(g,a) \equiv t(g,a,q,q) - t(g,q,q,q) - t(q,g,q,q) + q$, then $\varphi(g,a) + \varphi(a,g) = 2q - t(q,g,q,q) - t(q,a,q,q) - t(g,q,q,q) - t(a,q,q,q) + t(g,a,q,q) + t(a,g,q,q) = t(g,a,q,q) - t(g,q,q,q) - t(a,q,q,q) = 0$.

We use Durand et al. result [3] for proving the sufficiency. □

Remark 1. The condition also holds in the case of permutation-symmetry.

3.2 Hexagonal number-conserving cellular automata

Theorem 2. *A deterministic two-dimensional permutation-symmetric hexagonal CA, $\mathcal{H} = (\mathbb{Z}^2, 6, Q, \delta, q)$ is number-conserving iff its local transition function δ satisfies:*

$\exists \psi : Q^2 \to \mathbb{Z}, \quad \forall g,a,b,c,d,e,f,g \in Q,$
$\delta(g,a,b,c,d,e,f) = g + \psi(g,a) + \psi(g,b) + \psi(g,c) + \psi(g,d) + \psi(g,e) + \psi(g,f),$

with $\psi(g,x) = \delta(g,x,q,q,q,q,q) - \delta(g,q,q,q,q,q,q) - \delta(q,g,q,q,q,q,q) + q.$

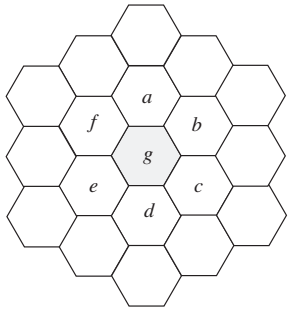


Fig. 2. A configuration in the hexagonal case.

Proof. We show that the condition is necessary. Let us assume that $\mathcal{H}$ is FNC; then according to Fig. 2,

$$\begin{array}{l} g+a+b+c+d+f+12q = \delta(g,a,b,c,d,e,f)+ \\ \delta(a,b,f,g,q,q,q)+\delta(b,a,c,g,q,q,q)+\delta(c,b,d,g,q,q,q)+ \\ \delta(d,c,e,g,q,q,q)+\delta(e,d,f,g,q,q,q)+\delta(f,a,e,g,q,q,q)+ \\ \delta(q,a,q,q,q,q,q)+\delta(q,b,q,q,q,q,q)+\delta(q,c,q,q,q,q,q)+ \\ \delta(q,d,q,q,q,q,q)+\delta(q,e,q,q,q,q,q)+\delta(q,a,q,q,q,q,q)+ \\ \delta(q,a,b,q,q,q,q)+\delta(q,b,c,q,q,q,q)+\delta(q,c,d,q,q,q,q)+ \\ \delta(q,d,e,q,q,q,q)+\delta(q,e,f,q,q,q,q)+\delta(q,f,a,q,q,q,q)+ \end{array} \quad (1)$$

The local function $\delta(g,a,b,c,d,e,f)$ which satisfies equation (1) only depends upon terms of four non-quiescent variables. The idea is to decrease this number of variables down to two, to finally get the binary ψ function. We first prove Lemma 1 which allows to decrease the number of variables in δ.

Lemma 1. *In the case of permutation-symmetry, the following equation holds for a hexagonal CA, $\mathcal{H} = (\mathbb{Z}^2, 6, Q, \delta, q)$: $\forall g,x,u,z \in Q$,*

$$\begin{array}{l} \delta(g,x,y,z,q,q,q) = g+x+y+z+12q-\delta(x,g,q,q,q,q,q)-\delta(q,x,y,g,q,q,q) \\ -\delta(y,g,q,q,q,q,q)-\delta(q,y,z,g,q,q,q)-\delta(z,g,q,q,q,q,q)-\delta(q,x,z,g,q,q,q) \\ -3\delta(q,x,q,q,q,q,q)-3\delta(q,y,q,q,q,q,q)-3\delta(q,z,q,q,q,q,q) \end{array}$$

Proof. Cancelling variables b, d and f by assigning them to the quiescent state q in equation (1) gives, because $\mathcal{H}$ is permutation-symmetric:

$$\begin{array}{l} g+a+c+e+12q = \delta(g,a,c,e,q,q,q)+\delta(a,g,q,q,q,q,q)+\delta(q,a,c,g,q,q,q)+ \\ \delta(c,g,q,q,q,q,q)+\delta(q,c,e,g,q,q,q)+\delta(e,g,q,q,q,q,q)+\delta(q,a,e,g,q,q,q)+ \\ 3\delta(q,a,q,q,q,q,q)+3\delta(q,c,q,q,q,q,q)+3\delta(q,e,q,q,q,q,q) \end{array}$$

□

Lemma 2. *In the case of permutation-symmetry, the following equation holds for a hexagonal CA, $\mathcal{H} = (\mathbb{Z}^2, 6, Q, \delta, q)$: $\forall x,y \in Q$,*

$$\begin{array}{rl} \delta(q,x,y,q,q,q,q) = & 11q+x+y \\ & -5\delta(q,x,q,q,q,q,q)-5\delta(q,y,q,q,q,q,q) \\ & -\delta(x,q,q,q,q,q,q)-\delta(y,q,q,q,q,q,q) \end{array}$$

Proof. Lemma 2 is proved by setting g, b, c, e and f to q in equation (1). □

Lemma 3. *In the case of permutation-symmetry, the following equation holds for a hexagonal CA, $\mathcal{H} = (\mathbb{Z}^2, 6, Q, \delta, q)$: $\forall x, y \in Q$,*

$$\begin{aligned} 8q + x + y = {} & 3\delta(q, x, q, q, q, q, q) + 3\delta(q, y, q, q, q, q, q) \\ & + \delta(q, x, y, q, q, q, q) + \delta(q, y, x, q, q, q, q) \\ & + \delta(x, y, q, q, q, q, q) + \delta(y, x, q, q, q, q, q). \end{aligned}$$

Proof. Lemma 3 is proved by replacing g, c, d and e by q in equation (1). □

Lemma 4. *In the case of permutation-symmetry, the following equation holds for a hexagonal CA, $\mathcal{H} = (\mathbb{Z}^2, 6, Q, \delta, q)$: $\forall x, y \in Q$,*

$$x = -6q + 6\delta(q, x, q, q, q, q, q) + \delta(x, q, q, q, q, q, q).$$

The proof of Lemma 4 is straightforward.

We now prove Theorem 2. We first make a repeated use of Lemma 1 by sustracting it from equation (1) with suitable variables substitutions and we obtain equation (2) which only depends upon terms in two non-quiescent variables.

$$\begin{aligned} & 5(a+b+c+d+e+f) + 7g + 174q + \\ & \delta(a,b,q,q,q,q,q) + \delta(a,f,q,q,q,q,q) + \delta(b,a,q,q,q,q,q) + \delta(b,c,q,q,q,q,q) + \\ & \delta(c,b,q,q,q,q,q) + \delta(c,d,q,q,q,q,q) + \delta(d,c,q,q,q,q,q) + \delta(d,e,q,q,q,q,q) + \\ & \delta(e,d,q,q,q,q,q) + \delta(e,f,q,q,q,q,q) + \delta(f,a,q,q,q,q,q) + \delta(f,e,q,q,q,q,q) + \\ & \delta(g,a,q,q,q,q,q) + \delta(g,b,q,q,q,q,q) + \delta(g,c,q,q,q,q,q) + \delta(g,d,q,q,q,q,q) + \\ & \delta(g,e,q,q,q,q,q) + \delta(g,f,q,q,q,q,q) = \delta(g,a,b,c,d,e,f) + 12\delta(g,q,q,q,q,q,q) + \\ & 16(\delta(q,a,q,q,q,q,q) + \delta(q,b,q,q,q,q,q) + \delta(q,c,q,q,q,q,q) + \delta(q,d,q,q,q,q,q) + \\ & \delta(q,e,q,q,q,q,q) + \delta(q,f,q,q,q,q,q)) + 18\delta(q,g,q,q,q,q,q) + 2\delta(q,a,f,q,q,q,q) + \\ & 2\delta(q,a,g,q,q,q,q) + 2\delta(q,b,a,q,q,q,q) + 2\delta(q,b,g,q,q,q,q) + 2\delta(q,c,b,q,q,q,q) + \\ & 2\delta(q,c,g,q,q,q,q) + 2\delta(q,d,c,q,q,q,q) + 2\delta(q,d,g,q,q,q,q) + 2\delta(q,e,d,q,q,q,q) + \\ & 2\delta(q,e,g,q,q,q,q) + 2\delta(q,f,e,q,q,q,q) + 2\delta(q,f,g,q,q,q,q) + 2\delta(q,g,a,q,q,q,q) + \\ & 2\delta(q,g,b,q,q,q,q) + 2\delta(q,g,c,q,q,q,q) + 2\delta(q,g,d,q,q,q,q) + 2\delta(q,g,e,q,q,q,q) + \\ & 2\delta(q,g,f,q,q,q,q) + 3(\delta(q,a,b,q,q,q,q) + \delta(q,b,c,q,q,q,q) + \delta(q,c,d,q,q,q,q) + \\ & \delta(q,d,e,q,q,q,q) + \delta(q,e,f,q,q,q,q) + \delta(q,f,a,q,q,q,q)) + 7(\delta(a,q,q,q,q,q,q) + \\ & \delta(b,q,q,q,q,q,q) + \delta(c,q,q,q,q,q,q) + \delta(d,q,q,q,q,q,q) + \delta(e,q,q,q,q,q,q) + \\ & \delta(f,q,q,q,q,q,q)) + \delta(q,a,c,q,q,q,q) + \delta(q,a,e,q,q,q,q) + \delta(q,b,d,q,q,q,q) + \\ & \delta(q,b,f,q,q,q,q) + \delta(q,c,e,q,q,q,q) + \delta(q,d,f,q,q,q,q). \end{aligned} \tag{2}$$

We observe that we have two kinds of terms in equation (2):

- $\delta(q, x, y, q, q, q, q)$;
- $\delta(x, y, q, q, q, q, q)$.

The former will be changed by the repeated use of Lemma 2 and the latter by the repeated use of Lemma 3. This yields to equation (3).

$$\begin{aligned}
&13a + 13b + 13c + 13d + 13e + 13f + 17g + 570q + \delta(g, a, b, c, d, e, f) = \\
&13\delta(a, q, q, q, q, q, q) + 13\delta(b, q, q, q, q, q, q) + 13\delta(c, q, q, q, q, q, q) + \\
&13\delta(d, q, q, q, q, q, q) + 13\delta(e, q, q, q, q, q, q) + 13\delta(f, q, q, q, q, q, q) + \\
&\delta(g, a, q, q, q, q, q) + \delta(g, b, q, q, q, q, q) + \delta(g, c, q, q, q, q, q) + \\
&\delta(g, d, q, q, q, q, q) + \delta(g, e, q, q, q, q, q) + \delta(g, f, q, q, q, q, q) + \\
&78\delta(q, c, q, q, q, q, q) + 78\delta(q, e, q, q, q, q, q) + 78\delta(q, b, q, q, q, q, q) + \\
&78\delta(q, f, q, q, q, q, q) + 78\delta(q, a, q, q, q, q, q) + 78\delta(q, d, q, q, q, q, q) + \\
&12\delta(g, q, q, q, q, q, q).
\end{aligned} \tag{3}$$

The sum of the neighbors $13(a + b + c + d + e + f)$ is removed by the repeated application of Lemma 4. Every time we use Lemma 4, we also cancel terms of the form $78\delta(q, x, q, q, q, q, q) + 13\delta(x, q, q, q, q, q, q)$ in the rhs of equation 3. All remaining terms in the rhs of equation 3 are like $\delta(g, x, q, q, q, q, q)$.

We use Durand et al. result [3] for proving the sufficiency. This proves Theorem 2. □

All above computations were made using computer algebra systems and the spreadsheets can be obtained from the authors.

4 Simulations between hexagonal and triangular NCCAs

In this section, we propose effective mutual simulations between triangular and hexagonal permutation-symmetric NCCAs.

4.1 Elementary simulation of a triangular NCCA by a hexagonal NCCA

Proposition 1. *For any triangular permutation-symmetric NCCA $\mathcal{T}$, there is a hexagonal permutation-symmetric NCCA $\mathcal{H}$ such that $\mathcal{T} \overset{1}{\prec} \mathcal{H}$ and whose transition function is surjective.*

Proof. Let $\mathcal{T} = (\mathbb{Z}^2, 3, Q_\mathcal{T}, t_\mathcal{T}, q)$ be a triangular permutation-symmetric NCCA with flow function φ.

We construct $\mathcal{H} = (\mathbb{Z}^2, 6, Q_\mathcal{H}, t_\mathcal{H}, q')$ a permutation-symmetric hexagonal NCCA by designing its flow function ψ.

Let $q' \notin Q_\mathcal{T}$ and $Q_\mathcal{H} = Q_\mathcal{T} \cup \{q'\}$; the flow function ψ corresponding to $t_\mathcal{H}$ contains the following values:

$$\begin{aligned}
&\text{For each } x, y \in Q_\mathcal{T}, \text{assign } \psi(x, y) = \varphi(x, y), \\
&\text{For each } x \in Q_\mathcal{T}, \text{assign }, \psi(x, q') = 0.
\end{aligned}$$

Given an initial configuration of $\mathcal{T}$ (see Fig. 3), let the initial configuration of $\mathcal{H}$ be as depicted in Fig. 5. From the process of $\mathcal{H}$, it is clear that the cells

with value q' don't have any effect and the other non-quiescent cells simulate $\mathcal{T}$. Applying the algorithm in [5], it is possible to add extra-states produced by $t_{\mathcal{H}}$ to $Q_{\mathcal{H}}$ and make the local function $t_{\mathcal{H}}$ to be surjective. □

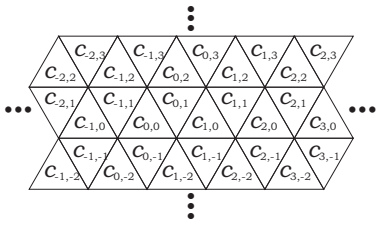


Fig. 3. An initial configuration of triangular CA $\mathcal{T}$.

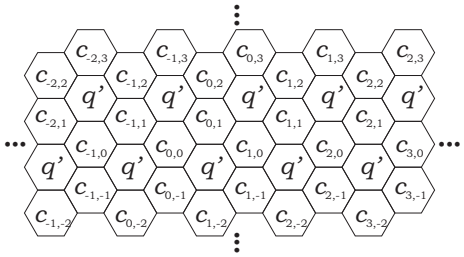


Fig. 4. An initial configuration of hexagonal CA $\mathcal{H}$.

4.2 Simple simulation of a hexagonal NCCA by a triangular NCCA

Proposition 2. *For any hexagonal permutation-symmetric NCCA $\mathcal{H}$, there is a triangular permutation-symmetric NCCA $\mathcal{T}$ such that $\mathcal{H} \overset{2}{\prec} \mathcal{T}$ and whose transition function is surjective.*

Proof. Let $\mathcal{H} = (\mathbb{Z}^2, 6, Q_{\mathcal{H}}, t_{\mathcal{H}}, q)$ be a hexagonal permutation-symmetric NCCA. We assume that $Q = \{s_0, s_1, \cdots, s_{m-1}\}$ and its flow function is ψ and $\mathcal{H}$ has an initial configuration as pictured in Fig. 5.

Without loss of generality, we can assume $0 < min(s_i) < max(s_j) < M$ for a constant M, because non-positive numbers in $Q_{\mathcal{H}}$ can be erased by adding a constant value to every state numbers and by changing the arguments of the rules.

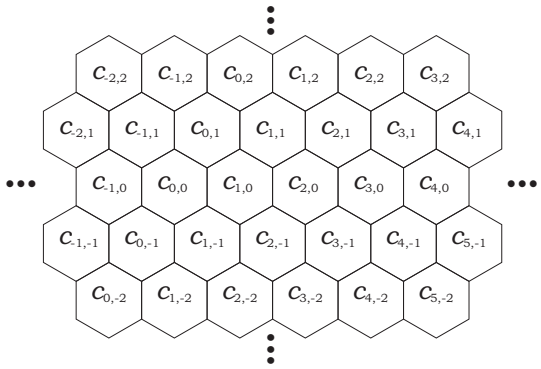


Fig. 5. An initial configuration of hexagonal CA $\mathcal{H}$.

We construct $\mathcal{T} = (\mathbb{Z}^2, 3, Q_{\mathcal{T}}, t_{\mathcal{T}}, 0)$ a permutation-symmetric triangular NCCA by designing its flow function φ.

First we assign numbers corresponding to each state in $Q_{\mathcal{H}}$. Let $p_i = 2^{\lceil \lg M \rceil + i} (i = 0, \cdots, m-1)$.

The flow function φ of $t_{\mathcal{T}}$ contains the following values:

For each $s_i \in Q_{\mathcal{H}}$, assign

$$\varphi(0, s_i) = p_i.$$

For each combination of $s_j, s_k \in Q_{\mathcal{H}}$,

$$\varphi(-3p_i, p_i + p_j + p_k) = p_i + \psi(s_i, s_j) + \psi(s_i, s_k).$$

All other values of φ are 0. The local function $t_{\mathcal{T}}$ can also be extended to be surjective.

The initial configuration of $\mathcal{T}$ is chosen as in Fig. 6.

We briefly explain how the rules work. In the first step, each nonzero state 'x' in Fig. 6 moves three values p_x to the neighboring zero cells; after this, the zero cells contain the value $p_x+p_y+p_z$ related to the three neighboring cells 'x','y','z'. In the second step, the cell on which we focus (was 'x') knows the values of two neighboring cells of $\mathcal{H}$ by the neighboring cell which value is $p_x + p_y + p_z$ and can move the values of $\psi(s_x, s_y) + \psi(s_x, s_z)$. □

5 Conclusion

In this paper, we have designed flow functions for number-conserving triangular and hexagonal cellular automata under the permutation-symmetry condition. This was also generalized to the rotation-symmetry for triangular NCCA. A simulation between triangular and hexagonal NCCA, and conversely, were also proposed.

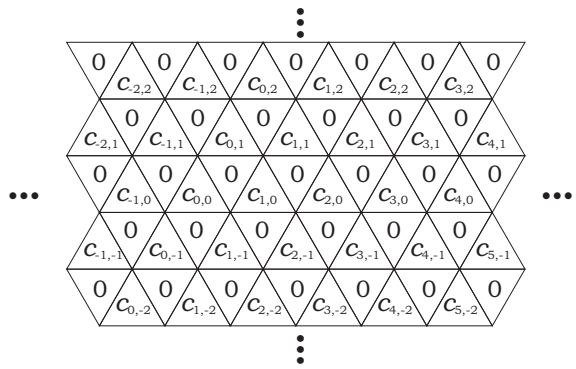


Fig. 6. An initial configuration of triangular CA $\mathcal{T}$.

This work can be extended in several ways. First, we'd like to know if the flow function we proposed in the permutation-symmetry hexagonal case also holds in the rotation-symmetric case. Then, we aim to go on in the number-conserving simulation of different neighborhoods like simulating a von Neumann neighborhood in a square lattice by a triangular NCCA.

Finally, it might be possible to generalize these results to the case of number-conserving cellular automata on Cayley graphs and to find flow functions in several cases and number-conserving simulations as well.

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