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Star Coloring of Graphs

Guillaume Fertin¹, André Raspaud², Bruce Reed³

¹LINA FRE 2729, Université de Nantes

2 rue de la Houssinière - BP 92208 - F44322 Nantes Cedex 3

²LaBRI UMR 5800, Université Bordeaux 1

351 Cours de la Libération - F33405 Talence Cedex

³Univ. Paris 6 - Equipe Combinatoire - Case 189 - 4 place Jussieu - F75005 Paris

fertin@irin.univ-nantes.fr, raspaud@labri.u-bordeaux.fr, reed@ecp6.jussieu.fr

Abstract

A *star coloring* of an undirected graph G is a proper vertex coloring of G (i.e., no two neighbors are assigned the same color) such that any path of length 3 in G is not bicolored. The *star chromatic number* of an undirected graph G , denoted by $\chi_s(G)$, is the smallest integer k for which G admits a star coloring with k colors. In this paper, we give the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs and 2-dimensional grids. We also study and give bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d -dimensional grids ($d \geq 3$), d -dimensional tori ($d \geq 2$), graphs with bounded treewidth and cubic graphs. We end this study by two asymptotic results, where we prove that, when d tends to infinity, (i) there exist graphs G of maximum degree d such that $\chi_s(G) = \Omega\left(\frac{d^{\frac{3}{2}}}{(\log d)^{\frac{1}{2}}}\right)$

and (ii) for any graph G of maximum degree d , $\chi_s(G) = O(d^{\frac{3}{2}})$.

Keywords : graphs, vertex coloring, proper coloring, star coloring.

1 Introduction

All graphs considered here are undirected. In the following definitions (and in the whole paper), the term *coloring* will be used to define *vertex coloring* of graphs. A *proper* coloring of a graph G is a labeling of the vertices of G such that no two neighbors in G are assigned the same label. Usually, the labeling (or coloring) of vertex x is denoted by $c(x)$. In the following, all the colorings that we will define and use are proper colorings.

Definition 1 (Star coloring) A star coloring of a graph G is a proper coloring of G such that no path of length 3 in G is bicolored.

We also introduce here the notion of *acyclic coloring*, that will be useful for our purpose.

Definition 2 (Acyclic coloring) An acyclic coloring of a graph G is a proper coloring of G such that no cycle in G is bicolored.

We define by *star chromatic number* (resp. *acyclic chromatic number*) of a graph G the minimum number of colors which are necessary to star color G (resp. acyclically color G). It is denoted $\chi_s(G)$ for *star* coloring, and $a(G)$ for *acyclic* coloring.

By extension, the star (resp. acyclic) chromatic number of a family \mathcal{F} of graphs is the minimum

number of colors that are necessary to star (resp. acyclically) color any graph belonging to \mathcal{F} . It is denoted $\chi_s(\mathcal{F})$ for star coloring and $a(\mathcal{F})$ for acyclic coloring.

The purpose of this paper is to determine and give properties on $\chi_s(\mathcal{F})$ for a large number of families of graphs. In Section 2, we present general properties for the star chromatic number of graphs. In the following sections (Sections 3 to 7), we determine precisely $\chi_s(\mathcal{F})$ for trees, cycles, complete bipartite graphs, outerplanar graphs and 2-dimensional grids and we give bounds on $\chi_s(\mathcal{F})$ for other families of graphs, such as planar graphs, hypercubes, d -dimensional grids, d -dimensional tori, graphs with bounded treewidth and cubic graphs. We end this paper by giving asymptotic bounds for the star chromatic number of graphs of order n and maximum degree d .

2 Generalities

We note that for any graph G , any star coloring of G is also an acyclic coloring of G : indeed, a cycle in G can be bicolored if and only if it is of even length, that is of length greater than or equal to 4. However, by definition of a star coloring, no path of length 3 in G can be bicolored. Hence, we get the following observation.

Observation 1 *For any graph G , $a(G) \leq \chi_s(G)$.*

Proposition 2 [FGR03] *For any graph G of order n and size m , $\chi_s(G) \geq \frac{2n+1-\sqrt{\gamma}}{2}$, where $\gamma = 4n(n-1) - 8m + 1$.*

Sketch of proof : Let us compute a lower bound for $a(G)$, the acyclic chromatic number of a graph G , with n vertices and m edges. Suppose we have acyclically colored G with k colors $1, 2, \dots, k$. Take any two of those colors, say i and j , and let V_i (resp. V_j) be the set of vertices of G that are assigned color i (resp. color j). $G[V_i \cup V_j]$, the graph induced by $V_i \cup V_j$, is by definition acyclic, i.e. it is a forest. Let e_{ij} be the number of edges of $G[V_i \cup V_j]$; thus $e_{ij} \leq |V_i| + |V_j| - 1$ (\mathbf{I}_{ij}). If we sum inequality (\mathbf{I}_{ij}) for all distinct pairs i, j with $1 \leq i \neq j \leq k$, we get that $\sum_{1 \leq i \neq j \leq k} e_{ij} \leq n(k-1) - \frac{k(k-1)}{2}$. Since $\sum_{1 \leq i \neq j \leq k} e_{ij} = m$, it now suffices to solve the inequality $k^2 - (2n+1)k + 2(m+n) \leq 0$. This gives $\frac{2n+1-\sqrt{\gamma}}{2} \leq k \leq \frac{2n+1+\sqrt{\gamma}}{2}$, where $\gamma = 4n(n-1) - 8m + 1$. However, the rightmost inequality does not give us any useful information, since for any graph with at least one edge we always have $\gamma \geq 1$ and thus $k \leq n+1$. Finally, we obtain that $a(G) \geq k \geq \frac{2n+1-\sqrt{\gamma}}{2}$. By Observation 1, we conclude that $\chi_s(G)$ also satisfies this inequality. \square

Actually, we can note that the star coloring is an acyclic coloring such that if we take two color classes then the induced subgraph is a forest composed only of stars. Star coloring was introduced in 1973 by Grünbaum [Grü73]. He linked star coloring to acyclic coloring by showing that any planar graph has an acyclic chromatic number less than or equal to 9, and by suggesting that this implies that any planar graph has a star chromatic number less than or equal to $9 \cdot 2^8 = 2304$.

However, this property can be generalized for any given graph G , as mentioned in [BKW99, BKRS00].

Theorem 3 (Relation acyclic/star coloring) [BKW99, BKRS00] *For any graph G , if the acyclic chromatic number of G satisfies $a(G) \leq k$, then the star chromatic number of G satisfies $\chi_s(G) \leq k \cdot 2^{k-1}$.*

As a corollary of this result, we can determine an upper bound for $\chi_s(\mathcal{P})$, where \mathcal{P} denotes the family of planar graphs. Indeed, Borodin [Bor79] showed that any planar graph has an acyclic coloring using at most 5 colors (we also note that there exists a graph of order 6 such that $a(G) = 5$). Thus we deduce that $\chi_s(\mathcal{P}) \leq 80$. However, a result from [NOdM01], applied to family \mathcal{P} , yields $\chi_s(\mathcal{P}) \leq 30$, and this result has later been improved in [ACK⁺02], where it is shown that $\chi_s(\mathcal{P}) \leq 20$. Concerning lower bounds, there exists a planar graph G_1 for which any

star coloring needs 6 colors (this graph is shown in Figure 1): indeed, we can first note that we need at least 4 colors to assign to vertices v_1, v_2, v_3 and v_4 , since the subgraph induced by those 4 vertices is isomorphic to K_4 . Let us then use colors 1,2,3 and 4 to color respectively v_1, v_2, v_3 and v_4 . However, four colors are not enough to star color G_1 , because if we are only allowed 4 colors then $c(v_5) = 3$, and in that case no color can be given to v_6 . Thus $\chi_s(G_1) \geq 5$.

Now suppose we are allowed 5 colors. If $c(v_5) = 3$, then $c(v_6) = 5$ and $c(v_7) = 5$; but in that case, it is impossible to assign a color to v_9 . Thus $c(v_5) = 5$. In that case, if $c(v_6) = 2$, then $c(v_7) = 5$ and v_8 cannot be assigned a color. Finally, if $c(v_6) = 5$, then either $c(v_7) = 1$ or $c(v_7) = 5$. In the first case, it is impossible to color v_8 , while in the second this implies $c(v_8) = 3$, and thus v_9 cannot be colored. Thus, 5 colors do not suffice to star color G_1 and consequently $\chi_s(G_1) \geq 6$ (we note that, on this example, it is possible to find a star coloring of G_1 with exactly 6 colors).

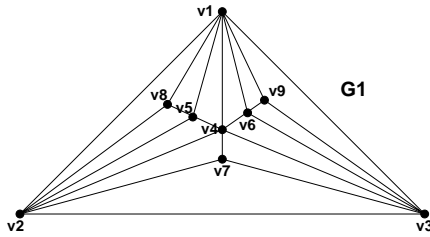


Figure 1: A planar graph G_1 for which $\chi_s(G_1) \geq 6$

The girth g of a graph G is the length of its shortest cycle. In [BKW99], it is proved that if G is planar with girth $g \geq 5$ (resp. $g \geq 7$), then $a(G) \leq 4$ (resp. $a(G) \leq 3$). Together with Theorem 3, we deduce:

Corollary 4 *If G is a planar graph with girth $g \geq 5$, then $\chi_s(G) \leq 32$. If G is a planar graph with girth $g \geq 7$, then $\chi_s(G) \leq 12$.*

However, a result from [NOdM01] implies that $\chi_s(G) \leq 14$ for any planar graph G of girth $g \geq 5$. When the girth g satisfies $g \geq 7$, then Theorem 1 in [NOdM01] implies that $\chi_s(G) \leq 14$, which does not improve the bound of Corollary 4.

Several graphs are cartesian product of graphs (Hypercubes, Grids, Tori), so it is interesting to have an upper bound for the star chromatic number of cartesian product of graphs. We recall that the cartesian product of two graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \square G'$, is the graph such that the set of vertices is $V \times V'$ and two vertices (x, x') and (y, y') are linked by an edge if and only if $x = y$ and $x'y'$ is an edge of G' or $x' = y'$ and xy is an edge of G .

Theorem 5 *For any two graphs G and H , $\chi_s(G \square H) \leq \chi_s(G) \cdot \chi_s(H)$*

Proof : Suppose that $\chi_s(G) = g$ and $\chi_s(H) = h$, and let \mathcal{C}_G (resp. \mathcal{C}_H) be a star coloring of G (resp. H) using g (resp. h) colors. In that case, we assign to any vertex (u, v) of $G \square H$ color $[\mathcal{C}_g(u), \mathcal{C}_h(v)]$. This coloring uses gh colors, and this defines a star coloring. Indeed, suppose that there exists a path P of length 3 that is bicolored in $G \square H$, with $V(P) = \{x, y, z, t\}$ and $E(P) = \{xy, yz, zt\}$. Depending on the composition of the ordered pairs corresponding to the vertices of the path, we have 8 possible paths. We will only consider 4 of them, because by permuting the first and second component of each ordered pair, we obtain the others. The 4 possible paths are:

- (1) $x = (u, v), y = (u, v_1), z = (u, v_2), t = (u, v_3)$
- (2) $x = (u, v), y = (u, v_1), z = (u, v_2), t = (u_4, v_2)$

$$(3) \ x = (u, v), y = (u, v_1), z = (u_2, v_1), t = (u_2, v_4)$$

$$(4) \ x = (u, v), y = (u, v_1), z = (u_2, v_1), t = (u_5, v_1)$$

Clearly, in the first case P cannot be bicolored, since the path v, v_1, v_2, v_3 is not bicolored in H . For the second case: y and t have different colors (v_1v_2 is an edge of H). For the third case: x and z have different colors (vv_1 is an edge of H). The same argument works for the last case. \square

Observation 6 For any graph G and for any $1 \leq \alpha \leq |V(G)|$, let G_1, \dots, G_p be the p connected components obtained by removing α vertices from G . In that case, $\chi_s(G) \leq \max_i \{\chi_s(G_i)\} + \alpha$.

Proof : Star color each G_i , and reconnect them by adding the α vertices previously deleted, using a new color for each of the α vertices. Any path of length 3 within a G_i will be star colored by construction, and if this path begins in G_i and ends in G_j with $i \neq j$, then it contains at least one of the α vertices, which has a unique color. Thus the path of length 3 cannot be bicolored, and we get a star coloring of G . \square

Remark 7 For any $\alpha \geq 1$, the above result is optimal for complete bipartite graphs $K_{n,m}$. Wlog, suppose $n \leq m$ and let $\alpha = n$. Remove the $\alpha = n$ vertices of partition V_n . We then get m isolated vertices, which can be independently colored with a single color. Then, give a unique color to the $\alpha = n$ vertices. We then get a star coloring with $n + 1$ colors ; this coloring can be shown to be optimal by Proposition 12.

We recall that the independence number of a graph G , $\alpha(G)$, is the cardinality of a largest independent set in G .

Observation 8 For any graph G , $\chi_s(G) \leq 1 + |V(G)| - \alpha(G)$, where $\alpha(G)$ is the independence number of G .

Proof : Let S be a maximum independent set of G . Color each vertex of S with color c , and give new pairwise distinct colors to all the other vertices. This coloring has the desired number of colors. It is clearly a proper coloring, and it is also a star coloring, because there is only one color which is used at least twice. \square

Remark 9 The above result is optimal for complete p -partite graphs K_{s_1, s_2, \dots, s_p} .

3 Trees, Cycles, Complete Bipartite Graphs, Hypercubes

Proposition 10 Let \mathcal{F}_r be the family of forests such that r is the maximum radius over all the trees contained in \mathcal{F}_r . In that case, $\chi_s(\mathcal{F}_r) = \min\{3, r + 1\}$.

Proof : Let F be a forest contained in \mathcal{F}_r . When $r = 0$, the result is trivial (F holds no edge). When $r = 1$, F is composed of isolated vertices and of stars. Hence, we color each isolated vertex, as well as the center of each star in F with color 1, and all the remaining vertices with color 2. This is obviously a proper coloring, and since in that case there is no path of length 3, it is consequently a star coloring as well. Now we assume $r \geq 2$. We then arbitrarily root each tree composing F , and we color each vertex v , of depth d_v in F , as follows: $c(v) = d_v \bmod 3$. Clearly, this is a proper coloring of F and it is easy to see that it is a star coloring. \square

Proposition 11 Let C_n be the cycle with $n \geq 3$ vertices.

$$\chi_s(C_n) = \begin{cases} 4 & \text{when } n = 5 \\ 3 & \text{otherwise} \end{cases}$$

Proof : It can be easily checked that $\chi_s(C_5) = 4$. Now let us assume $n \neq 5$. Clearly, 3 colors at least are needed to star color C_n . We now distinguish 3 cases: first, if $n = 3k$, we color alternatively the vertices around the cycle by colors 1,2 and 3. Thus, for any vertex u , its two neighbors are assigned distinct colors, and consequently this is a valid star coloring. Hence $\chi_s(C_{3k}) \leq 3$. Suppose now $n = 3k+1$. In that case, let us color $3k$ vertices of C_n consecutively, by repeating the sequence of colors 1,2 and 3. There remains 1 uncolored vertex, to which we assign color 2. One can check easily that this is also a valid star coloring, and thus $\chi_s(C_{3k+1}) \leq 3$. Finally, let $n = 3k+2$. Since the case $n = 5$ is excluded here, we can assume $k \geq 2$. Thus $n = 3(k-1) + 5$, with $k-1 \geq 1$. In that case, let us color $3(k-1)$ consecutive vertices along the cycle, alternating colors 1,2 and 3. For the 5 remaining vertices, we give the following coloring: 2, 1, 2, 3, 2. It can be checked that this is a valid star coloring, and thus $\chi_s(C_{3k+2}) \leq 3$ for any $k \geq 2$. Globally, we have $\chi_s(C_n) = 3$ for any $n \neq 5$, and the result is proved. \square

Proposition 12 *Let $K_{n,m}$ be the complete bipartite graph with $n+m$ vertices. Then $\chi_s(K_{n,m}) = \min\{m, n\} + 1$.*

Proof : Wlog, let $n \leq m$. The upper bound of $n+1$ immediately follows from Observation 6 (cf. Remark 7 for a detailed proof).

Now let us prove that $\chi_s(K_{n,m}) \geq n+1$: if each of the partite set contains at least two vertices with the same color, then there exists a bicolored 4-cycle, and the coloring we have is not a star coloring. If not, then the number of colors used is greater than or equal to $n+1$. Hence the result. \square

Theorem 13 (Star-Coloring of hypercube of dimension d , H_d) *For any d -dimensional hypercube H_d , $\lceil \frac{d+3}{2} \rceil \leq \chi_s(H_d) \leq d+1$.*

Proof : The lower bound is a direct application of Proposition 2, where $n = 2^d$ and $m = d \cdot 2^{d-1}$. Indeed, we have $\chi_s(H_d) \geq \frac{2n+1-\sqrt{\gamma}}{2}$, where $\gamma = 4n(n-1) - 8m + 1$. Let us prove now that $\chi_s(H_d) > \frac{d+2}{2}$. For this, let us show that $f(n, d) = \frac{2n+1-\sqrt{\gamma}}{2} - \frac{d+2}{2} > 0$. Note that $f(n, d) = \frac{(2n-1-d)-\sqrt{\gamma}}{2} = \frac{(2n-1-d)-\sqrt{\gamma}}{2} \cdot \frac{(2n-1-d)+\sqrt{\gamma}}{(2n-1-d)+\sqrt{\gamma}}$. That is, $f(n, d) = \frac{(2n-1-d)^2 - \gamma}{2((2n-1-d)+\sqrt{\gamma})}$. However, $D(n, d) = 2((2n-1-d) + \sqrt{\gamma})$ is positive for any $d \geq 1$, since $\sqrt{\gamma} \geq 1$ in all circumstances and $n = 2^d$. Hence, it suffices to show that $f'(n, d) = (2n-1-d)^2 - \gamma$ is positive in order to prove the lower bound. $f'(n, d) = (2n-1-d)^2 - (4n^2 - 4n - 8m + 1)$, thus $f'(n, d) = d^2 - 4nd + 2d - 8m$. Since $m = \frac{nd}{2}$, we conclude that $f'(n, d) = d(d+2) > 0$ for any $d \geq 1$. Hence we have $\chi_s(H_d) > \frac{d+2}{2}$, that is $\chi_s(H_d) \geq \lceil \frac{d+3}{2} \rceil$.

In order to prove the upper bound, we give the following coloring: suppose the vertices of H_d are labeled according to their binary representation ; that is, every vertex $u \in V(H_d)$ can be labeled as follows: $u = b_1b_2 \dots b_d$, with every $b_i \in \{0, 1\}$, $1 \leq i \leq d$. We then assign a color $c(u)$ to u according to the following equation: $c(u) = \sum_{i=1}^d i \cdot b_i \pmod{d+1}$. We know by [FGR03] that this coloring \mathcal{C} is acyclic. Moreover, in [FGR03] it has been shown that any bicolored path in H_d can only appear in a copy of a 2-dimensional cube H_2 . Since \mathcal{C} is acyclic, we conclude that no bicolored path of length strictly greater than 2 can appear, and thus \mathcal{C} is a star coloring. \square

4 d -dimensional Grids

In this section, we study the star chromatic number of grids. More precisely, we give the star chromatic number of 2 dimensional grids, and we extend this result in order to get bounds on the star chromatic number of grids of dimension d .

We recall that the 2-dimensional grid $G(n, m)$ is the cartesian product of two paths of length $n-1$ and $m-1$. Wlog, we will always consider in the following that $m \geq n$. A summary of the results is given in Table 1 ; those results are detailed below.

	$m = 2$	$m = 3$	$m \geq 4$
$n = 2$	3	4	4
$n = 3$	xxx	4	4
$n \geq 4$	xxx	xxx	5

Table 1: Star coloring of 2-dimensional grids $G(n, m)$ ($n \leq m$)

Proposition 14 $\chi_s(G(2, 2)) = 3$, and for any $m \geq 4$, $\chi_s(G(2, m)) = \chi_s(G(3, m)) = 4$.

Proof : The first result is trivial.

It is easily checked that $\chi_s(G(2, m)) \geq 4$ for any $m \geq 3$, since star coloring of $G(2, 3)$ requires at least 4 colors, and since $G(2, 3)$ is a subgraph of $G(2, m)$ for any $m \geq 3$. Moreover, as can be seen in Figure 2 (left), it is possible to find a 4 star coloring of $G(2, m)$. Thus $\chi_s(G(2, m)) \leq 4$, and altogether we have $\chi_s(G(2, m)) = 4$.

The fact that $\chi_s(G(3, m)) \geq 4$ for any $m \geq 3$ is trivial, since $G(2, 3)$ is a subgraph of $G(3, m)$. Moreover, Figure 2 (right) shows a 4 star coloring of $G(3, m)$ for any $m \geq 3$. \square

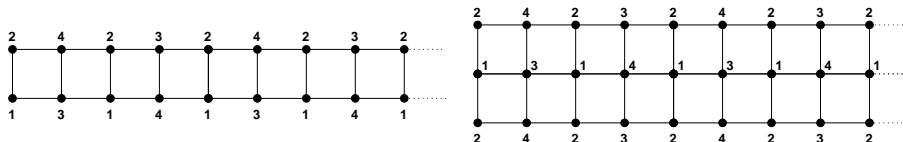


Figure 2: 4 star colorings of $G(2, m)$ (left) and $G(3, m)$ (right)

Theorem 15 (Star-Coloring of the 2-dimensional grid) For any n and m such that $\min\{n, m\} \geq 4$, $\chi_s(G(n, m)) = 5$.

Proof : By a rather tedious case by case analysis (confirmed by the computer), it is possible to show that 5 colors at least are needed to color $G(4, 4)$. Hence, for any n and m such that $\min\{n, m\} \geq 4$, $\chi_s(G(n, m)) \geq 5$.

The upper bound is a corollary of Theorem 16 below. \square

We know that d -dimensional grids G_d are isomorphic to the cartesian product of d paths. Hence, by Theorem 5 and Proposition 10, we get an upper bound for $\chi_s(G_d)$: $\chi_s(G_d) \leq 3^d$ for any $d \geq 1$. However, it is possible to do much better, and to prove that the star chromatic number of the d -dimensional grid is linear in d : this is the purpose of the following theorem.

Theorem 16 (Star-Coloring of the d -dimensional grid G_d) Let G_d be any d -dimensional grid, $d \geq 1$. Then $2 + \left\lfloor d - \sum_{i=1}^d \frac{1}{n_i} \right\rfloor \leq \chi_s(G_d) \leq 2d + 1$.

Proof : The lower bound is a direct application of this inequality on the lower bound of Proposition 2, and is similar to a proof of [FGR03] concerning acyclic coloring in G_d . Indeed, Proposition 2 yields that for any graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, we have $\chi_s(G) \geq \frac{2n+1-\sqrt{\gamma}}{2}$, with $\gamma = 4n(n-1) - 8m + 1$. This can also be written as follows: $\chi_s(G) \geq \frac{(2n+1)^2 - \gamma}{2(2n+1 + \sqrt{\gamma})}$, that is $\chi_s(G) \geq \frac{4(m+n)}{1+2n+\sqrt{\gamma}}$, or $a(G) \geq \frac{m+n}{n} \cdot \frac{4n}{1+2n+\sqrt{\gamma}}$.

However, $\gamma = (2n-1)^2 - 8m$, hence $\gamma < (2n-1)^2$, that is $1 + 2n + \sqrt{\gamma} < 4n$. Thus we have $\chi_s(G) > 1 + \frac{m}{n}$.

Now, if we replace respectively n and m by $|V(G_d)|$ and $|E(G_d)|$, we end up with the result. Indeed, it is well known that $|V(G_d)| = n_1 \times \dots \times n_d$ and $|E(G_d)| = n_1 \times \dots \times n_d \times (d - \sum_{i=1}^d \frac{1}{n_i})$.

We also note that this means that for “sufficiently large” grids (for instance, when each $n_i \geq d$), we have $\chi_s(G_d) \geq d + 1$, and that the “worst” case appears when each $n_i = 2$, $1 \leq i \leq d$; in that case G_d is isomorphic to the hypercube of dimension d , H_d , and the lower bound of Theorem 13 applies.

Concerning the upper bound, we use here a proof that is close to the one given in [FGR03] concerning acyclic coloring. Let us represent each vertex u of $G_d = G(n_1, n_2, \dots, n_d)$ by its coordinates in each dimension, that is $u = (x_1, x_2, \dots, x_d)$ where each x_i , $1 \leq i \leq d$ satisfies $0 \leq x_i \leq n_i - 1$. Now, we define a vertex coloring for G_d as follows: for each $u \in V(G_d)$ such that $u = (x_1, x_2, \dots, x_d)$, we let $c(u) = \sum_{i=1}^d i \cdot x_i \pmod{2d+1}$. First, we prove that this coloring is proper. Indeed, suppose that 2 vertices u and u' , differing on the j -th coordinate, $1 \leq j \leq d$, are assigned the same color $c(u) = c(u')$. Hence we have $u = (x_1, x_2, \dots, x_j, \dots, x_d)$ and $u' = (x_1, x_2, \dots, x_j \pm 1, \dots, x_d)$. Then, since $c(u) = c(u')$, we have $j \cdot x_j + \sum_{i=1, i \neq j}^d i \cdot x_i = j \cdot (x_j \pm 1) + \sum_{i=1, i \neq j}^d i \cdot x_i$, that is $\pm j = 0 \pmod{2d+1}$. Since $1 \leq j \leq d$, we conclude that this cannot happen.

Now let us show that this coloring is a star coloring. More precisely, we show that it is a 2 distance coloring, that is, no two vertices at distance less than or equal to 2 are assigned the same color. Consequently, no path of length 3 can be bicolored, and thus it is also a star coloring. We have seen that no two neighbors are assigned the same color. Now let us prove that for any two vertices u and u'' lying at distance 2, we cannot have $c(u) = c(u'')$. Indeed, u and u'' being at distance exactly 2 in G_d , their coordinates differ (i) either in two dimensions j_1 and j_2 (ii) or in a single dimension j . Case (ii) can be solved easily: we are in the case where $u = (x_1, x_2, \dots, x_j, \dots, x_d)$ and $u'' = (x_1, x_2, \dots, x_j \pm 2, \dots, x_d)$, and by the same computations as above, we end up with $\pm 2j = 0 \pmod{2d+1}$, a contradiction since $1 \leq j \leq d$. If we are in Case (i), the same argument applies, and by a similar computation, we end up with $\pm j_1 \pm j_2 = 0 \pmod{2d+1}$, a contradiction too since $1 \leq j_1 \neq j_2 \leq d$. Thus our coloring is a 2 distance coloring of G_d , and consequently a star coloring of G_d . We then conclude that $\chi_s(G_d) \leq 2d + 1$. \square

Remark 17 We note that for dimensions 1 and 2, the upper bound given by Theorem 16 for d -dimensional grids is tight (cf. Proposition 10 when $d = 1$ and Theorem 15 when $d = 2$).

5 d -dimensional Tori

In this section, we give bounds on the star chromatic number of d -dimensional tori for any $d \geq 2$.

In the following, for any $n_i \geq 3$, $1 \leq i \leq d$, we denote by $TG_d = TG(n_1, n_2, \dots, n_d)$ the toroidal d -dimensional grid having n_i vertices in dimension i . We recall that TG_d is the cartesian product of d cycles of length n_i , $1 \leq i \leq d$.

Theorem 18 (Star coloring of d -Dimensional Tori)

$$d + 2 \leq \chi_s(TG_d) \leq \begin{cases} 2d + 1 & \text{when } 2d + 1 \text{ divides each } n_i \\ 2d^2 + d + 1 & \text{otherwise} \end{cases}$$

Proof : The lower bound is obtained by Proposition 2, using arguments that are very similar to the ones developed for hypercubes in proof of Theorem 13. Indeed, we know that $\chi_s(TG_d) \geq \frac{2|V|+1-\sqrt{\gamma}}{2}$, where $\gamma = 4|V|(|V|-1) - 8|E| + 1$. Let $N = |V|$; we know that $|E| = dN$ where $N = \prod_{i=1}^d n_i$. Now, if we prove that $\frac{2|V|+1-\sqrt{\gamma}}{2} - (d+1) > 0$, this will imply that $\chi_s(TG_d) > d + 1$. Let $f(N, d) = \frac{2N+1-\sqrt{\gamma}}{2} - (d+1)$. Note that $f(N, d) = \frac{(2N-1-2d)-\sqrt{\gamma}}{2} = \frac{(2N-1-2d)-\sqrt{\gamma}}{2} \cdot \frac{(2N-1-2d)+\sqrt{\gamma}}{(2N-1-2d)+\sqrt{\gamma}}$. That is, $f(N, d) = \frac{(2N-1-2d)^2 - \gamma}{2((2N-1-2d)+\sqrt{\gamma})}$. However, $D(N, d) = 2((2N-1-2d) + \sqrt{\gamma})$ is strictly positive for any $d \geq 1$, since we always have $\sqrt{\gamma} \geq 1$ and $N \geq 3^d$ (in order to get a torus, the number of vertices n_i in each dimension i must be at least equal to 3). Hence, it suffices to show that $f'(N, d) = (2N-1-2d)^2 - \gamma$ is positive in order to prove the lower bound. $f'(N, d) =$

$(2N - 1 - 2d)^2 - (4N^2 - 4N - 8Nd + 1)$, thus $f'(N, d) = 4d^2 + 4d$. Hence, we conclude that $f'(N, d) > 0$ for any $d \geq 1$. Hence we have $\chi_s(TG_d) > d + 1$, that is $\chi_s(TG_d) \geq d + 2$.

The upper bound in the case where $2d + 1$ divides each n_i comes from the study of the non-toroidal grid G_d , and the coloring given in Theorem 16. It is easy to see that this coloring remains a star coloring of TG_d when $2d + 1$ divides each n_i , $1 \leq i \leq d$, and thus we have $\chi_s(TG_d) \leq 2d + 1$ in that case.

When $2d + 1$ does not divide each n_i , then consider the subgraph of TG_d that consists of a (non-toroidal) d -dimensional grid $G'_d = G(n_1 - 1, n_2 - 1, \dots, n_d - 1)$. We can star color G'_d with $2d + 1$ colors as shown in Theorem 16. Now, in order to avoid a bicolored path of length 3 due to the wrap-around in each dimension, it suffices to use new colors to color the ‘‘borders’’ of TG_d (that is, the vertices of TG_d that do not appear in G'_d). These vertices form a graph G_0 , where $V(G_0)$ can be partitioned in d classes V_1, \dots, V_d , such that the subgraph G_i induced by V_i , $1 \leq i \leq d$, is a $(d - 1)$ -dimensional non-toroidal grid. Hence, if for any $1 \leq i \leq d$, we use new colors to star color G_i (using the coloring described in proof of Theorem 16), then we get a star coloring of TG_d ; indeed, by construction any bicolored path that could occur would be either (i) between vertices of G'_d and vertices of G_i for some $1 \leq i \leq d$, or (ii) between vertices of G_i and vertices of G_j for some $1 \leq i \neq j \leq d$. However, any vertex u of G'_d has only one edge leading to a vertex of G_i , for any given $1 \leq i \leq d$. Hence u cannot be connected to two vertices of G_i , and thus cannot be connected to two vertices being assigned the same color c . We then conclude that case (i) cannot occur. The same argument holds for case (ii): a given vertex u of G_i has at most one connection leading to G_j , $1 \leq i \neq j \leq d$. Overall, the suggested coloring uses $2d + 1$ colors (to color G'_d), to which we must add d times $2(d - 1) + 1$ colors (to color each G_i). Thus, globally, this coloring needs $(2d + 1) + d \cdot (2(d - 1) + 1)$ colors, that is $2d^2 + d + 1$ colors. \square

6 Graphs with Bounded Treewidth

The notion of treewidth was introduced by Robertson and Seymour [RS83]. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i | i \in I\}, T = (I, F))$ where $\{X_i | i \in I\}$ is a family of subsets of V , one for each node of T , and T a tree such that:

- (1) $\bigcup_{i \in I} X_i = V$
- (2) For all edges $vw \in E$, there exists an $i \in I$ with $v \in X_i$ and $w \in X_i$
- (3) For all $i, j, k \in I$: if j is on the path from i to k in T , then $X_i \cap X_k \subseteq X_j$

The *width* of a tree decomposition $(\{X_i | i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The *treewidth* of a graph G is the minimum width over all possible tree decomposition of G .

We will prove the following theorem.

Theorem 19 *If a graph G is of treewidth at most k , then $\chi_s(G) \leq k(k + 3)/2 + 1$*

Actually we will prove Theorem 19 for k -trees, because it is well known that the treewidth of a graph G is at most k ($k > 0$) if and only if G is a partial k -tree [Bod98].

We recall the definition of a k -tree [BLS99]:

- (1) a clique with k -vertices is a k -tree
- (2) If $T = (V, E)$ is a k -tree and C is a clique of T with k vertices and $x \notin V$, then $T' = (V \cup \{x\}, E \cup \{cx : c \in C\})$ is a k -tree.

If a k -tree has exactly k vertices, then it is a clique by definition. If not, it contains at least a $(k + 1)$ -clique; moreover, it is easy to see that the greedy coloring with $k + 1$ colors of a k -tree gives an acyclic coloring. Hence we can deduce by Theorem 3 that for any k -tree T_k with at least $k + 1$ vertices, $k + 1 \leq \chi_s(T_k) \leq k \cdot 2^{k-1}$. However, a much better upper bound can be derived. This is the purpose of Theorem 20 below.

Theorem 20 *For any $k \geq 1$ and any k -tree T_k :*

- $\chi_s(T_k) = k$ if $|V(T_k)| = k$;
- $k + 1 \leq \chi_s(T_k) \leq k(k + 3)/2 + 1$ otherwise.

Proof : Consider a k -tree G . We recall that a k -tree G is an intersection graph [MM99] and can be represented by a tree T and a subtree S_v for each v in G s.t.:

- (1) $uv \in E(G) \iff S_u \cap S_v \neq \emptyset$
- (2) for any $t \in T$, $|\{v : t \in S_v\}| = k + 1$

We can see that by considering the tree decomposition of the k -tree. The tree T is the one of the tree decomposition and the subtree S_v for $v \in V(G)$ is exactly the subtree of T containing the nodes of T corresponding to the subsets of the tree decomposition containing v .

We root T at some node r and, for each vertex v of G , let $t(v)$ be the first node of S_v obtained when traversing T in preorder (i.e. $t(v)$ is the “highest” node of S_v). We choose some fixed preorder and order the nodes as v_1, v_2, \dots, v_n so that for any $i < j$, $t(v_i)$ is considered in the preorder before $t(v_j)$. We color the nodes in this order, using $k(k + 3)/2 + 1$ colors.

For each i we let:

$$X_{v_i} = \bigcup_{S_{v_j} \ni t(v_i)} \{v_l \neq v_i : S_{v_l} \ni t(v_j)\}$$

We now show that $|X_{v_i}| \leq k(k + 3)/2$. Indeed let $A = \{a_1, a_2, \dots, a_k, v_i\}$ be the subset of vertices corresponding to $t(v_i)$. We assume that $t(a_i)$ is before $t(a_{i+1})$ ($i \in \{1, \dots, k - 1\}$) in the preorder. We first give an upper bound for the number of subtrees S_{v_l} which contain $t(a_i)$ ($v_l \notin A$). The number of S_{v_l} ($v_l \notin A$) for $i \in \{1, \dots, k\}$ which contain $t(a_1)$ is at most k , because the corresponding subset can intersect A only in a_1 . The number of S_{v_l} ($v_l \notin A$) which contain $t(a_2)$ is at most $k - 1$, a_1 is in the subset corresponding to $t(a_2)$ and we do not count S_{a_1} . It is easy to see that the number of S_{v_l} ($v_l \notin A$) which contain $t(a_i)$ is at most $k + 1 - i$, because the subset corresponding to $t(a_i)$ contains $\{a_1, a_2, \dots, a_i\}$. In total we have at most $\sum_{i=1}^{i=k} i = k(k + 1)/2$ subtrees S_{v_l} with $v_l \notin A$ containing $t(a_i)$ $i \in \{1, \dots, k\}$. Now we have to add the number of S_{a_i} , this gives $|X_{v_i}| \leq k(k + 3)/2$, without counting v_i . We color v_i with any color not yet used on X_{v_i} . This clearly yields a proper coloring, indeed if xy is an edge of G then $S_x \cap S_y \neq \emptyset$ and either $t(x) \in S_y$ or $t(y) \in S_x$, hence by construction x and y have different colors. We claim it also yields a coloring with no bichromatic P_4 (a path of length 3): assume the contrary, and let $\{x, y, z, w\}$ be this P_4 labeled so that

- (1) $t(x)$ is the first of $t(x), t(y), t(z), t(w)$ considered in the preorder,
- (2) x and y have the same color,
- (3) xz and yz are in $E(G)$.

We have to notice that x, z, y are not in the same clique K_{k+1} of the graph G corresponding to a node of T , because by construction this would imply that the colors are different. Now $S_x \cap S_y = \emptyset$, $S_z \cap S_x \neq \emptyset$, $S_z \cap S_y \neq \emptyset$, so by (1) $t(y)$ is in S_z . Further since $zx \in E(G)$, by (1) we have $t(z)$ is in S_x . So $x \in X_y$, contradicting the fact that x and y get the same color. \square

We can notice that for 1-trees (i.e. the usual trees), the upper bound we obtain matches the one given by Proposition 10. For 2-trees, the upper bound is optimal because of graph of Figure 3, which has been shown by computer to have a star chromatic number equal to 6.

We also note for completeness that while this paper was submitted, it has been shown independently in [ACK⁺02] that for any graph G of treewidth k , $\chi_s(G) \leq k(k + 1)/2$, and that there exist graphs of treewidth k for which this bound is reached.

In the following, we will denote by \mathcal{O} the family of outerplanar graphs.

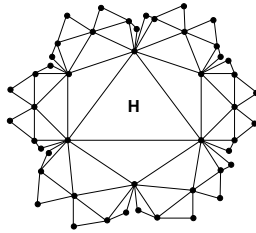


Figure 3: A 2-tree H that satisfies $\chi_s(H) = 6$

Corollary 21 $\chi_s(\mathcal{O}) = 6$.

Proof : It is well-known that any outerplanar graph is a partial 2-tree: thus, by Theorem 20, $\chi_s(\mathcal{O}) \leq 6$. Moreover the graph in Figure 3, which is also outerplanar, has a star chromatic number equal to 6. Thus, we conclude that $\chi_s(\mathcal{O}) \geq 6$, which proves the corollary. \square

7 Cubic Graphs

Observation 22 Let G be a graph of order n and G^2 be the square graph of G . In that case, $\chi_s(G) \leq \chi(G^2)$, where $\chi(G)$ denotes the (proper) chromatic number of G .

Proof : For any graph $G = (V, E)$, take its square graph $G^2 = (V, E \cup \mathcal{E})$, where \mathcal{E} is the set of edges joining any vertices at distance 2 in G . Any proper coloring \mathcal{C} of G^2 is a star coloring of G : indeed G^2 contains all the edges of G , thus \mathcal{C} is also proper in G . Now take any path of length 2 in G , say (u, v, w) . In G^2 , u , v and w are assigned 3 distinct colors by \mathcal{C} . Hence, no path of length 2 in G can be bicolored ; consequently, no path of length 3 is bicolored either, and \mathcal{C} is a star coloring of G . \square

It is a well known result that any graph of maximum degree d can be properly colored with at most $d + 1$ colors (cf. for instance [Viz64]). Since G^2 is of maximum degree d^2 when G is of maximum degree d , we deduce the following corollary.

Corollary 23 Let G be a graph of order n and of maximum degree d . Then $\chi_s(G) \leq d^2 + 1$.

Now we turn to the case where $d = 3$, that is cubic graphs. By Corollary 23 above, we deduce that for any cubic graph G , $\chi_s(G) \leq 10$. We also note that the result given in [NOdM01] yields the same upper bound. However, it is possible to slightly improve this bound to 9, by using the more general proposition below (which is itself an improvement of Corollary 23).

Proposition 24 For any graph of G maximum degree $d \geq 2$, $\chi_s(G) \leq d^2$.

Proof : Take G^2 , the square graph of G . Since G is of maximum degree d , G^2 is of maximum degree at most d^2 . If G^2 is not isomorphic to the complete graph K_{d^2+1} , then, by Brooks' theorem, we have directly $\chi(G^2) \leq d^2$. Hence, by Observation 22, we have $\chi_s(G) \leq d^2$.

Now, if G^2 is isomorphic to K_{d^2+1} , then the independence number of G , $\alpha(G)$, satisfies $\alpha(G) \geq 2$ (since $2 \leq d < d^2$). Then, by applying Observation 8, and since $|V(G)| = d^2 + 1$, we obtain $\chi_s(G) \leq 1 + (d^2 + 1) - 2$, that is $\chi_s(G) \leq d^2$. \square

Proposition 25 Let \mathcal{C} denote the family of cubic graphs. We have $6 \leq \chi_s(\mathcal{C}) \leq 9$.

Proof : The upper bound is a direct consequence of Proposition 24, applied to the case $d = 3$.

The lower bound is given by the cubic graph G_s given in Figure 4 (note that this graph turns out to be a snark, that is a nontrivial cubic graph whose edges cannot be properly colored by

three colors). We now show that its star chromatic number is equal to 6. Indeed, suppose that $\chi_s(G_s) \leq 5$. Then, take 5 vertices in G_s that induce a C_5 . There are now two options: either 4 or 5 colors are used to color those 5 vertices (we know by Proposition 11 that at least 4 colors are needed to color this C_5). Let us detail each of those two cases:

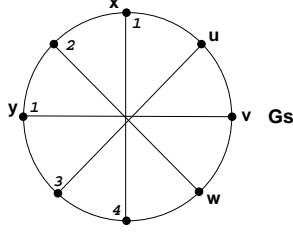


Figure 4: A cubic graph G_s such that $\chi_s(G_s) = 6$

- **Case 1:** only 4 colors are used. In that case, one of these colors (wlog, say color 1) must be used twice in this C_5 , and thus be assigned on two non-neighboring vertices x and y (hence, x and y are at distance 2) ; the three other colors are used exactly once for each of the three remaining vertices in C_5 . Wlog, let those vertices be colored as shown in Figure 4. In that case, u can be assigned either color 4 or 5. If $c(u) = 4$, then $c(v) = 5$ and no color can be given to w . If $c(u) = 5$, then v can be assigned either color 3 or 4. But in both cases, no color can be assigned to w . Hence this case cannot happen.
- **Case 2:** Since Case 1 cannot appear, this means that any C_5 in G_s must be colored with 5 different colors. By a somewhat tedious but easy case by case analysis, it can be seen that it is not possible to star color G_s in 5 colors satisfying this property.

We conclude that $\chi_s(G_s) \geq 6$, and equality holds by respectively assigning to the vertices of the outer cycle the colors 1,2,3,4,5,3,1 and 6. \square

8 Asymptotic Results

In this section, we give asymptotic results that apply to graphs of order n and maximum degree d . They rely heavily on Lovász's local Lemma, and are shown thanks to proofs that are similar to the ones given by Alon et al. [AMR91] about acyclic coloring.

We first recall Lovász's local Lemma below ([EL75], see also [Spe87]).

Lemma 26 (Lovász's local Lemma [EL75]) *Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. Let the graph $H = (V, E)$ on the nodes $\{1, 2, \dots, n\}$ be a dependency graph for the events A_i (that is, two events A_i and A_j will share an edge in H iff they are dependent). If there exist real numbers $0 \leq y_i < 1$ such that for all i we have*

$$Pr(A_i) \leq y_i \cdot \prod_{\{i,j\} \in E} (1 - y_j)$$

then

$$Pr(\cap_{i=1}^n \bar{A}_i) \geq \prod_{i=1}^n (1 - y_i) > 0$$

Theorem 27 *Let $G = (V, E)$ be a graph with maximum degree d . Then $\chi_s(G) \leq \lceil 20d^{\frac{3}{2}} \rceil$.*

Proof : Let $x = \lceil 20d^{\frac{3}{2}} \rceil$, and let us color $V(G)$ with x colors, where for each vertex v of $V(G)$, the color is independently chosen randomly according to a uniform distribution on $\{1, 2, \dots, x\}$. Let \mathcal{C} define this application. What we want to show here is that with non zero probability, \mathcal{C} is a star coloring of G .

For this, we define a family of events on which we will apply Lovász's local Lemma. This will imply that with non zero probability, none of these events occur. If our events are chosen so that if none of them happens, then our coloring is a star coloring of G , the theorem will be proved. Let us now describe the two types of events we have chosen.

- **Type I:** For each pair of adjacent vertices u and v in G , let $A_{u,v}$ be the event that $c(u) = c(v)$.
- **Type II:** For each path of length 3 uvw in G , let $B_{u,v,w}$ be the event that $c(u) = c(w)$ and $c(v) = c(x)$.

By definition of star coloring, it is straightforward that if none of these two events occur, then \mathcal{C} is a star coloring of G . Now, let us show that with strictly positive probability none of these two events occur. We will apply here the local Lemma : to this end, we construct a graph H whose nodes are all the events of the two types, and in which two nodes X_{S_1} and Y_{S_2} ($X, Y \in \{A, B\}$) are adjacent iff $S_1 \cap S_2 \neq \emptyset$. Since the occurrence of each event X_{S_1} depends only on the color of the vertices in S_1 , H is a dependency graph for these events, because even if the colors of all vertices of G but those in S_1 are known, the probability of X_{S_1} remains unchanged. Now, a vertex of H will be said to be of type $i \in \{I, II\}$ if it corresponds to an event of type i . We now want to estimate the degree of a vertex of type i in H . This is the purpose of the following observation. \square

Observation 28 For any vertex v in a graph G of maximum degree d , we have:

- (1) v belongs to at most d edges of G ;
- (2) v belongs to at most $2d^3$ paths of length 3 in G .

Proof : (1) Is straightforward since the maximum degree in G is d .

(2) Suppose first that v is an end vertex of such a path. Then v belongs to at most $d(d-1)^2 \leq d^3$ paths of length 3. Now suppose v is not an end vertex of such a path. Then one of its neighbors x must be an end vertex of this path. There are d ways to choose x , and the number of paths of length 3 with end vertex x going through v is at most $(d-1)^2$. Thus there are $d(d-1)^2 \leq d^3$ paths of length 3 for which v is an internal vertex. Globally, we have that v belongs to at most $2d^3$ paths of length 3 in G . \square

Lemma 29 For $i, j \in \{I, II\}$, the (i, j) entry matrix M given below is an upper bound on the number of vertices of type j which are adjacent to a vertex of type i in the dependency graph H .

	I	II
I	$2d$	$4d^3$
II	$4d$	$8d^3$

Proof : Take a vertex v_I of type I in H , and let us give an upper bound on the number of vertices of type I in H that are neighbors of v_I . v_I corresponds to an event $A_{u,v}$ that implies two vertices u and v in G . Thus, by definition of the event graph H , v_I is connected to all the vertices that correspond to events $A_{u,y}$ and $A_{z,v}$ for all vertices y that are neighbors of u in G and all the vertices z that are neighbors of v in G . Since by Observation 28(1), there are at most d vertices that are neighbor of u in G (resp. of v in G), the entry $M(I, I)$ is upperly bounded by $2d$. The entries $M(I, II)$, $M(II, I)$ and $M(II, II)$ are computed in a similar way, using the results of Observation 28(1) and (2). \square

Now, let us come back to our coloring \mathcal{C} . The following observation is straightforward.

Observation 30

(1) For each event A of type I, $Pr(A) = \frac{1}{x}$

(2) For each event B of type II, $Pr(B) = \frac{1}{x^2}$

Now, in order to apply Lovász's local Lemma, there remains to choose the $y_i, i \in \{1, 2\}$, where $0 \leq y_i < 1$. For this, we choose that $y_i = \frac{2}{x^i}, i \in \{1, 2\}$. In order to be able to apply Lovász's local Lemma, it is necessary to prove that:

$$\frac{1}{x} \leq \frac{2}{x} \left(1 - \frac{2}{x}\right)^{2d} \left(1 - \frac{2}{x^2}\right)^{4d^3}$$

and

$$\frac{1}{x^2} \leq \frac{2}{x^2} \left(1 - \frac{2}{x}\right)^{4d} \left(1 - \frac{2}{x^2}\right)^{8d^3}$$

Clearly, if the second inequality is satisfied, the first is satisfied too. In order to prove that it is satisfied, let $S = \left(1 - \frac{2}{x}\right)^{4d} \left(1 - \frac{2}{x^2}\right)^{8d^3}$. We have that $S \geq \left(1 - \frac{8d}{x}\right) \left(1 - \frac{16d^3}{x^2}\right)$, that is $S \geq \left(1 - \frac{2}{5\sqrt{d}}\right) \left(1 - \frac{1}{25}\right)$. It is easy to check that in that case $2S \geq 1$ for any $d \geq 1$. Hence Lovász's local Lemma applies, which means that \mathcal{C} is a star coloring of G with non zero probability. This proves Theorem 27.

We note that this result improves the one given by [NOdM01] or the easy Observation 22, that yields a star coloring with $O(d^2)$ colors, while Theorem 27 yields a star coloring with $O(d^{3/2})$ colors.

Theorem 31 *There exists a graph G of maximum degree d such that*

$$\chi_s(G) \geq \varepsilon \frac{d^{\frac{3}{2}}}{(\log d)^{\frac{1}{2}}}$$

where ε is an absolute constant.

Proof : First, we note that we make no attempt to maximize the constant here.

Let us show that, for a random graph G (with the "random" notion to be defined later) of order n , we have

$$Pr\{\chi_s(G) > \frac{n}{2}\} \rightarrow 1 \text{ as } n \rightarrow \infty. \text{ (P1)}$$

For this, we put

$$p = c \left(\frac{\log n}{n}\right)^{\frac{1}{3}}$$

where c is independent of n , to be chosen later. Now let $V = \{1, 2, \dots, n\}$ be a set of n labeled vertices ; for our purpose, we put n divisible by 4. Let $G = (V, E)$ be a random graph on V , where for each pair (u, v) in V^2 we choose to put an edge with probability p . If d denotes the maximum degree of G , then it is a well-known fact that $Pr\{d \leq 2np\} \rightarrow 1$ as $n \rightarrow \infty$ [Bol85]. With our choice of p , this gives here:

$$Pr\{d \leq 2cn^{\frac{2}{3}}(\log n)^{\frac{1}{3}}\} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Now, in order to prove (P1), we show the following lemma.

Lemma 32 *For any fixed partition of V in $k \leq \frac{n}{2}$ color classes, the probability that this partition*

is a star coloring of G is upperly bounded by $(1 - p^3)^{\binom{\frac{n}{4}}{2}}$.

Proof of Lemma 32 Let $V_1, V_2 \dots V_k$ be the parts of the partition of the set V of vertices of G . For any V_i of odd cardinality, we omit one vertex. Thus we end up with at least $n - k \geq \frac{n}{2}$ vertices altogether, that lie in k even disjoint parts. We partition each of those even parts in sets of 2 vertices $U_1, U_2 \dots U_r$, where $r \geq \frac{n}{4}$; in that case, the two vertices in each U_i are colored with the same color. Now take any pair (U_i, U_j) : if 3 edges connect this pair, then there exists a path of length 3 in G that is bicolored, and our coloring is not a star coloring. However, it is easy to see that the probability for which this case happens is equal to $4p^3 - 3p^4$, which is always greater than or equal to p^3 . Since there are at least $\binom{\frac{n}{4}}{2}$ pairs of the form (U_i, U_j) , the probability that our coloring is a star coloring does not exceed:

$$(1 - p^3)^{\binom{\frac{n}{4}}{2}}$$

□

Since there are less than n^n partitions of V , we see that the probability that there exists a star coloring of G with at most $\frac{n}{2}$ colors does not exceed

$$\begin{aligned} n^n (1 - p^3)^{\binom{\frac{n}{4}}{2}} &< n^n \cdot \exp\left\{-\binom{\frac{n}{4}}{2} p^3\right\} \\ &= \exp\left\{n \log n - \binom{\frac{n}{4}}{2} c^3 \log n/n\right\} \end{aligned}$$

With the right choice of c , this probability is in $o(1)$ as $n \rightarrow \infty$. Indeed, it suffices here that $c^3 > 32$ (for instance, $c = 4$).

Thus, we end up with the following two statements: $Pr\{d \leq 2cn^{\frac{2}{3}}(\log n)^{\frac{1}{3}}\} \rightarrow 1$ as $n \rightarrow \infty$, and $Pr\{\chi_s(G) > \frac{n}{2}\} \rightarrow 1$ as $n \rightarrow \infty$.

Now, it suffices to see that $\frac{x}{\log x}$ is an increasing function of x , for any $x > e$. Thus, for any a, b such that $e < a < b$, we have $\frac{a}{\log a} < \frac{b}{\log b}$. In other words, if we have $d \leq 2cn^{\frac{2}{3}}(\log n)^{\frac{1}{3}}$, then we have $d^3 \leq 16c^4 n^2 (\log n)$, or $\frac{d^3}{3 \log d} \leq \frac{(2c)^4 n^2 (\log n)}{4 \log 2c + 2 \log n + \log \log n}$. That is, $\frac{d^3}{3 \log d} \leq \frac{(2c)^4 n^2 \log n}{2 \log n}$. Thus $\frac{d^3}{3 \log d} \leq \frac{(2c)^4 n^2}{2}$ and we conclude that $n > \varepsilon \frac{d^{\frac{3}{2}}}{\log d^{\frac{1}{2}}}$, where ε is an absolute constant. This proves the result. □

9 Conclusion

In this paper, we have provided many new results concerning the star chromatic number of different families of graphs. In particular, we have provided exact results for trees, cycles, complete bipartite graphs, outerplanar graphs and 2-dimensional grids. We have also determined bounds for the chromatic number in several other families of graphs, such as planar graphs, hypercubes, d -dimensional grids ($d \geq 3$), d -dimensional tori ($d \geq 2$), graphs with bounded treewidth and cubic graphs. We have also determined several more general properties concerning the star chromatic number: notably, using the techniques of [AMR91], we have shown that the star chromatic number of a graph of maximum degree d is $O(d^{\frac{3}{2}})$ and that for every d , there exists a graph of maximum degree d whose star chromatic number exceeds $\varepsilon \frac{d^{\frac{3}{2}}}{\log d^{\frac{1}{2}}}$ for some positive absolute constant ε .

A large number of problems remain open here, such as getting optimal results for other families of graphs, or refining our non optimal bounds; getting one or several methods to provide good lower bounds for the star chromatic number is also another challenging problem.

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References

- [ACK⁺02] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, and R. Ramamurthi. Coloring with no 2-colored P_4 's. 2002. Submitted for publication.
- [AMR91] N. Alon, C. McDiarmid, and B. Reed. Acyclic coloring of graphs. *Random Structures and Algorithms*, 2(3):277–288, 1991.
- [BKRS00] O.V. Borodin, A.V. Kostochka, A. Raspaud, and E. Sopena. Acyclic k -strong coloring of maps on surfaces. *Mathematical Notes*, 67 (1):29–35, 2000.
- [BKW99] O.V. Borodin, A.V. Kostochka, and D.R. Woodall. Acyclic colourings of planar graphs with large girth. *J. London Math. Soc.*, 60 (2):344–352, 1999.
- [BLS99] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes A survey*. SIAM Monographs on D.M. and Applications, 1999.
- [Bod98] H.L. Bodlander. A partial k -arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209:1–45, 1998.
- [Bol85] B. Bollobás. *Random Graphs*. Academic Press, London, 1985.
- [Bor79] O.V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25:211–236, 1979.
- [EL75] P. Erdős and L. Lovász. Infinite and finite sets. *A. Hajnal et al. Eds, North-Holland, Amsterdam*, 1975.
- [FGR03] G. Fertin, E. Godard, and A. Raspaud. Acyclic and k -distance coloring of the grid. *Information Processing Letters*, 87(1):51–58, 2003.
- [Grü73] B. Grünbaum. Acyclic colorings of planar graphs. *Israel J. Math.*, 14(3):390–408, 1973.
- [MM99] T.A. McKee and F.R. McMorris. *Topics in intersection graph theory*. SIAM Monographs on D.M. and Applications, 1999.
- [NOdM01] J. Nešetřil and P. Ossona de Mendez. Colorings and homomorphisms of minor closed classes. *Technical report 476, Centre de Recerca Matemática*, 2001.
- [RS83] N. Robertson and P.D. Seymour. Graph minors. I. excluding a forest. *J. Combin. Theory Ser. B*, 35:39–61, 1983.
- [Spe87] J. Spencer. *Ten Lectures on the Probabilistic Method*. SIAM, Philadelphia, 1987.
- [Viz64] V.G. Vizing. On an estimate of the chromatic class of a p -graph. *Diskrete Analiz.*, 3:25–30, 1964. In Russian.