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Abstract

Clique family inequalities $a\sum_{v\in W} x_v + (a-1)\sum_{v\in W'} x_v \leq a\delta$ form an intriguing class of valid inequalities for the stable set polytopes of all graphs. We prove firstly that their Chvátal-rank is at most a, which provides an alternative proof for the validity of clique family inequalities, involving only standard rounding arguments. Secondly, we strengthen the upper bound further and discuss consequences regarding the Chvátal-rank of subclasses of claw-free graphs.

For any polyhedron P, let P^I denote the convex hull of all integer points in P. Chvátal [4] (and implicitly Gomory [9]) introduced a method to obtain approximations of P^I outgoing from P as follows. If $\sum a_i x_i \leq b$ is valid for P and has integer coefficients only, then $\sum a_i x_i \leq \lfloor b \rfloor$ is a Chvátal-Gomory cut for P. Define P' to be the set of points satisfying all Chvátal-Gomory cuts for P, and let $P^0 = P$ and $P^{t+1} = (P^t)'$ for non-negative integers t. Obviously $P^I \subseteq P^t \subseteq P$ for every t. An inequality $\sum a_i x_i \leq b$ is said to have Chvátal-rank at most t if it is a valid inequality for the polytope P^t. Chvátal showed in [4] that for each polyhedron P there exists a finite $t \geq 0$ with $P^t = P^I$; the smallest such t is the *Chvátal-rank* of P.

The fractional matching polytope is a famous example of a polytope with Chvátal-rank one [4]. In this note, we consider the Chvátal-rank of the fractional stable set polytope $P = \text{QSTAB}(G)$. In particular, P^I is the stable set polytope $\text{STAB}(G)$.

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The *stable set polytope* STAB(G) of a graph G is defined as the convex hull of the incidence vectors of all its stable sets (in a stable set all nodes are mutually nonadjacent). A canonical relaxation of STAB(G) is the *fractional stable set polytope* QSTAB(G) given by all "trivial" facets, the nonnegativity constraints $x_v \geq 0$ for all nodes v of G, and by the *clique constraints* $\sum_{v \in Q} x_v \leq 1$ for all cliques $Q \subseteq G$ (in a clique all nodes are mutually adjacent). Clearly, STAB(G) $\subseteq$ QSTAB(G) and STAB(G)=QSTAB$(G)^I$ holds for all graphs G. We say that a graph class $\mathcal{G}$ has Chvátal-rank t if t is the minimum value such that QSTAB$(G)^t$ = STAB(G) for all $G \in \mathcal{G}$. We have STAB(G) = QSTAB(G) if and only if G is perfect [5], that is perfect graphs form exactly the class of graphs with Chvátal-rank zero.

To describe the stable set polytopes of imperfect graphs, we consider two natural generalizations of clique constraints: 0/1-constraints associated with arbitrary induced subgraphs, and a/(a-1)-valued constraints associated with families of cliques. *Rank constraints* are 0/1-inequalities

$$\sum_{v \in G'} x_v \ \leq \ \alpha(G')$$

associated with induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G'. *Clique family inequalities* $(\mathcal{Q}, p)$

$$a \sum_{v \in V_p} x_v + (a-1) \sum_{v \in V_{p-1}} x_v \leq a\delta \tag{1}$$

rely on the intersection of cliques within a family $\mathcal{Q}$, where V_p (resp. V_{p-1}) contains all nodes belonging to at least p (resp. exactly $p-1$) cliques in $\mathcal{Q}$, and $a = p - r$ with $r = |\mathcal{Q}| \bmod p$ and $\delta = \lfloor \frac{|\mathcal{Q}|}{p} \rfloor$ holds.

Both types of inequalities are valid for the stable set polytopes of all graphs: rank constraints by the choice of the right hand side, and clique family inequalities by [1, 11].

It is known from [6] that the Chvátal-rank of rank constraints of a graph with n nodes is $\Omega((n/\log n)^{\frac{1}{2}})$ and from [7] that the split rank of clique family inequalities is one, that is, clique family inequalities are simple split cuts (split cuts were studied in [2]).

The aim of this note is to establish $\min\{r, p-r\}$ as upper bound of the Chvátal-rank for general clique family inequalities. We close with remarks regarding Chvátal-ranks of *quasi-line graphs* (where the neighbors of any node split into two cliques), as their stable set polytopes are completely described by nonnegativity, clique, and clique family inequalities [7].

The Chvátal-rank of clique family inequalities. The following observation will be crucial for the proofs: summing up the clique inequalities corresponding to the cliques in $\mathcal{Q}$ and possibly adding nonnegativity constraints $-x_v \leq 0$ for those nodes $v \in V_p$ which are contained in more than p cliques, we obtain that

$$p \sum_{v \in V_p} x_v + (p-1) \sum_{v \in V_{p-1}} x_v \quad \leq \quad p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \tag{2}$$

is valid for QSTAB(G).

Theorem 1 *Let $(\mathcal{Q}, p)$ be a clique family inequality and let $r = |\mathcal{Q}| \pmod p$. For every $1 \leq i \leq p - r$, the inequality $h(i)$*

$$i \sum_{v \in V_p} x_v + (i-1) \sum_{v \in V_{p-1}} x_v \leq i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i and, thus, $(\mathcal{Q}, p)$ has Chvátal-rank at most $p - r$.

Proof: For every $1 \leq i \leq p - r$, let $H(i)$ be the assertion : "The inequality $h(i)$ has Chvátal-rank at most i." The proof is performed by induction on i:

$H(1)$ is true: Inequality (2) implies that $\sum_{v \in V_p} x_v \leq \frac{|\mathcal{Q}|}{p}$ is valid for QSTAB(G), hence $\sum_{v \in V_p} x_v \leq \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$ has Chvátal-rank 1, as required.

Induction step: assume that $H(i)$ is true and $i < p - r$. To prove that $H(i+1)$ holds, we show that $h(i+1)$ is a Chvátal-Gomory cut from $h(i)$ and Inequality (2). Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned} \lambda i + \mu p &= i + 1 \\ \lambda(i-1) + \mu(p-1) &= i \\ \left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor &= (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \end{aligned}$$

Indeed, $\lambda = \frac{p-i-1}{p-i}$, $\mu = \frac{1}{p-i}$ are solutions, as $\left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor$ $= \left\lfloor (\lambda i + \mu p) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \frac{r}{p-i} \right\rfloor = (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$, since $0 \leq r/(p-i) < 1$. $\square$

Note that the proof of Theorem 1 yields an alternative proof for the validity of clique family inequalities for the stable set polytope of any graph, involving only standard rounding arguments.

Furthermore, we obtain that every *rank* clique family inequality has Chvátal-rank one. This is particularly nice, as neither general rank constraints nor general clique family inequalities have this property [6, 11], but the combination of both.

However, the upper bound established in Theorem 1 gets weaker if r gets smaller; we therefore improve the upper bound for $r < p/2$.

Theorem 2 *Every clique family inequality* $(\mathcal{Q}, p)$ *with* $r = |\mathcal{Q}| \pmod p$ *has Chvátal-rank at most* r *if* $0 \le r < p - r$.

Proof: For every $0 \le i \le r$, let $G(i)$ be the assertion : "The inequality $g(i)$: $(p-i)\sum_{v \in V_p} x_v + (p-i-1)\sum_{v \in V_{p-1}} x_v \le (p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i$ has Chvátal-rank at most i." The proof is performed by induction on i:

$G(0)$ is true due to Inequality (2).

Induction step: assume that $G(i)$ is true and $i < r$. To prove that $G(i+1)$ holds, we show that $g(i+1)$ is a Chvátal-Gomory cut from $g(i)$ and $h(i)$. Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned} \lambda(p-i) + \mu i &= p - i - 1 \\ \lambda(p-i-1) + \mu(i-1) &= p - i - 2 \\ \left\lfloor \lambda \left[(p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i \right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right\rfloor &= (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1 \end{aligned}$$

Indeed, $\lambda = \frac{p-2i-1}{p-2i}$, $\mu = \frac{1}{p-2i}$ are solutions as $\left\lfloor \lambda \left[(p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i \right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right\rfloor = \left\lfloor (\lambda(p-i) + \mu i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \lambda(r-i) \right\rfloor = \left\lfloor (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1 + \frac{p-i-r}{p-2i} \right\rfloor = (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1$ since $0 \le \frac{p-i-r}{p-2i} < 1$. □

Thus, Theorem 1 and Theorem 2 together imply:

Corollary 3 *Every clique family inequality* $(\mathcal{Q}, p)$ *has Chvátal-rank at most* $\min\{r, p-r\}$ *where* $r = |\mathcal{Q}| \pmod p$. *In particular, a clique family inequality* $(\mathcal{Q}, p)$ *has Chvátal-rank at most* $\frac{p}{2}$.

Consequences for quasi-line graphs. We now discuss consequences of the above results for quasi-line graphs, as all non-trivial, non-clique facets of their stable set polytopes are clique family inequalities according to [7].

Calling a graph G *rank-perfect* if $\mathrm{STAB}(G)$ has rank constraints as only non-trivial facets, Theorem 1 implies that rank-perfect subclasses of quasi-line graphs have Chvátal-rank 1. This verifies Edmond's conjecture that the

Chvátal-rank of claw-free graphs is one for the class of semi-line graphs, as they are rank-perfect [3].

A *semi-line graph* is a line graph or a quasi-line graph without a representation as fuzzy circular interval graph. A *line graph* $L(G)$ is obtained by turning adjacent edges of a root graph G into adjacent nodes of $L(G)$. Fuzzy circular interval graphs are defined as follows. Let $\mathcal{C}$ be a circle, $\mathcal{I}$ a collection of intervals in $\mathcal{C}$ without proper containments and common endpoints, and V a finite multiset of points in $\mathcal{C}$. The *fuzzy circular interval graph* $G(V, \mathcal{I})$ has node set V and two nodes are adjacent if both belong to one interval $I \in \mathcal{I}$, where edges between different endpoints of the same interval may be omitted.

As the only not rank-perfect quasi-line graphs are fuzzy circular interval, it suffices to restrict to this class in order to discuss the Chvátal-rank for quasi-line graphs. Giles and Trotter [8] exhibited a fuzzy circular interval graph with a clique family $\mathcal{Q}$ of size 37 such that $(\mathcal{Q}, 8)$ induces a facet. Oriolo noticed in [11] that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at least* 2. This example disproves Edmonds' conjecture for fuzzy circular interval graphs. On the other hand, Theorem 1 shows that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at most 3*, since $r = 5$ and so $p - r = 3$.

Furthermore, Giles and Trotter [8] introduced a sequence of fuzzy circular interval graphs G^k for $k \geq 1$ and showed that each of them admits a clique family facet $(\mathcal{Q}, k+2)$ with $|\mathcal{Q}| = 2k(k+2)+1$ and coefficients k and $k+1$; Theorem 2 ensures that these facets have Chvátal-rank 1 since $r = 1$ holds in all cases.

Webs W_n^k are special fuzzy circular interval graphs with nodes $0, \ldots, n-1$ and edges ij iff $\min\{|i-j|, n-|i-j|\} < k$. Liebling et al. [10] exhibited a sequence of webs $W_{(2a+3)^2}^{2(a+2)}$ for $a \geq 1$, each with a $(a+1)/a$-valued clique family facet $(\mathcal{Q}, a+2)$ with $|\mathcal{Q}| = (a+2)(2a+3)$. Since $(a+2)(2a+3) = 1$ (mod $a+2$), Theorem 2 shows that also these facets have Chvátal-rank 1.

The authors conjectured in [12] and Stauffer proved in [13] that all non-rank facets of webs W_n^k are clique family inequalities $(\mathcal{Q}, k'+1)$ associated with subwebs $W_{n'}^{k'} \subset W_n^k$ where the maximum cliques $\{i, \ldots, i+k\}$ of W_n^k starting in nodes i of the subweb $W_{n'}^{k'}$ yield the clique family $\mathcal{Q}$ of size n' where $(k'+1) \not| \; n'$ and $k' < k$. Thus, for any *fixed* k, the Chvátal-rank of all webs W_n^k is at most $\frac{k-1}{2}$. However, it is very likely that there exist sequences of webs inducing clique family facets $(\mathcal{Q}, p)$ with arbitrarily high p and $2p = |\mathcal{Q}|$ having Chvátal-rank $\frac{p}{2}$. Thus, also the Chvátal-rank of webs and, therefore, of quasi-line graphs could be arbitrarily large, as for general claw-free graphs [6].

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