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On uniqueness of large solutions of nonlinear parabolic equations in nonsmooth domains

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Abstract We study the existence and uniqueness of the positive solutions of the problem (P):
\[ \partial_t u - \Delta u + u^q = 0 \quad (q > 1) \] in \( \Omega \times (0, \infty) \), \( u = \infty \) on \( \partial \Omega \times (0, \infty) \) and \( u(., 0) \in L^1(\Omega) \), when \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). We construct a maximal solution, prove that this maximal solution is a large solution whenever \( q < N/(N-2) \) and it is unique if \( \partial \Omega = \partial \Omega^c \). If \( \partial \Omega \) has the local graph property, we prove that there exists at most one solution to problem (P).

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1 Introduction

Let \( q > 1 \) and let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega := \Gamma \). It has been proved by Keller [5] and Osserman [11] that there exists a maximal solution \( \overline{u} \) to the stationary equation
\[ -\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega. \] (1.1)

When \( 1 < q < N/(N-2) \) this maximal solution is a large solution in the sense that
\[ \lim_{\rho(x) \to 0} \overline{u}(x) = \infty \] (1.2)

where \( \rho(x) = \text{dist}(x, \partial \Omega) \). Furthermore Véron proves in [12] that \( \overline{u} \) is the unique large solution whenever \( \partial \Omega = \partial \Omega^c \). When \( q \geq N/(N-2) \) his proof of uniqueness does not apply. Marcus and Véron prove in [7] that, there exists at most one large solution, provided \( \partial \Omega \) is locally the graph of a continuous function. The aim of this article is to extend these questions to the parabolic equation
\[ \partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty). \] (1.3)

We are interested into positive solutions which satisfy
\[ \lim_{t \to 0} u(., t) = f \quad \text{in } L^1_{\text{loc}}(\Omega), \] (1.4)
where \( f \in L^1_{\text{loc}}(\Omega) \) and
\[
\lim_{(x,t) \to (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0,\infty). \tag{1.5}
\]

Notice that if the initial and boundary conditions are exchanged, i.e. \( u(.,t) \) blows-up when \( t \to 0 \) and coincides with a locally integrable function on \( \Gamma \times (0,\infty) \), this problem is associated with the study of the initial trace, and much work has been done by Marcus and Vérón [9] in the case of a smooth domain. In particular they obtain the existence and uniqueness when \( q \) is subcritical, i.e. \( 1 < q < 1 + 2/N \).

In this article we prove two series of results:

**Theorem A** Assume \( q > 1 \) and \( \Omega \) is a bounded domain. Then for any \( f \in L^1_{\text{loc}}(\Omega) \) there exists a maximal solution \( \overline{u}_f \) to problem (2.5) satisfying (1.4). If \( 1 < q < N/(N-2) \) and \( \partial \Omega = \partial \overline{\Omega} \), \( \overline{u}_f \) is the unique solution of the problem which satisfies (1.5).

The proof of uniqueness is based upon the construction of self-similar solutions of (2.5) in \( \mathbb{R}^N \setminus \{0\} \times (0,\infty) \), with a persistent strong singularity on the axis \( \{0\} \times (0,\infty) \) and a zero initial trace on \( \mathbb{R}^N \setminus \{0\} \). This solution, which is studied in Appendix, is reminiscent of the very singular solution of Brezis, Peletier and Terman [2], although the method of construction is far different. The uniqueness is a delicate adaptation to the parabolic framework of the proof by contradiction of [12].

**Theorem B** Assume \( q > 1 \), \( \Omega \) is a bounded domain and \( \partial \Omega \), is locally a continuous graph. Then for any \( f \in L^1_{\text{loc}}(\Omega) \) there exists at most one solution to problem (2.5) satisfying (1.4) and (1.5).

For proving this result, we adapt the idea which was introduced in [7] of constructing local super and subsolutions by small translations of the domain, but the non-uniformity of the boundary blow-up creates an extra-difficulty. In an appendix we study a self-similar equation which plays a key-role in our construction,
\[
\begin{aligned}
H'' + \left( \frac{N-1}{r} + \frac{r}{2} \right) H' + \frac{1}{q-1} H - |H|^{q-1} &= 0 \\
\lim_{r \to 0} H(r) &= \infty \\
\lim_{r \to \infty} r^{2/(q-1)} H(r) &= 0.
\end{aligned} \tag{1.6}
\]

We prove the existence and the uniqueness of the positive solution of (1.6) when \( 1 < q < N/(N-2) \) and we give precise asymptotics when \( r \to 0 \) and \( r \to \infty \).

This article is organised as follows: 1- Introduction. 2- The maximal solution 3- The case \( 1 < q < N/(N-2) \). 4- The local continuous graph property. 5- Appendix.

### 2 The maximal solution

In this section \( \Omega \) is an open domain of \( \mathbb{R}^N \), with a compact boundary \( \Gamma := \partial \Omega \). If \( G \) is any open subset of \( \mathbb{R}^N \) and \( 0 < T \leq \infty \), we denote \( Q_T^G := G \times (0,T) \). If \( f \in L^1_{\text{loc}}(\Omega) \), we
consider the problem

$$\begin{cases} 
\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q^\Omega_{\infty}, \\
\lim_{t \to 0} u(\cdot, t) = f(\cdot) \quad \text{in } L^1_{loc}(\Omega), \\
\lim_{(x,t) \to (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0, \infty). 
\end{cases}$$

(2.1)

By the next result, we reduce the lateral blow-up condition by a locally uniform one in which we set \( \rho(x) = \text{dist}(x, \Gamma) \).

**Lemma 2.1** The following two conditions are equivalent

$$\lim_{(x,t) \to (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0, \infty)$$

(2.2)

and

$$\lim_{\rho(x) \to 0} u(x,t) = \infty \quad \text{uniformly on } [\tau, T],$$

(2.3)

for any \( 0 < \tau < T < \infty \).

**Proof.** It is clear that (2.3) is equivalent to the fact that (2.2) holds uniformly on \( \Gamma \times [\tau, T] \). By contradiction, we assume that (2.2) does not hold uniformly for some \( T > \tau > 0 \). Then there exists \( \beta > 0 \) such that for any \( \delta > 0 \), there exist two couples \((y_\delta, s_\delta) \in \Gamma \times (0, \infty)\) and \((x_\delta, t_\delta) \in \Omega \times [\tau, T] \) such that

$$|x_\delta - y_\delta| + |t_\delta - s_\delta| \leq \delta \quad \text{and} \quad u(x_\delta, t_\delta) \leq \beta.$$  

(2.4)

Taking \( \delta = 1/n, \ n \in \mathbb{N}^* \), we can assume that \( \{\delta\} \) is discrete and that \( y_\delta \to y \in \Gamma \) and \( s_\delta \to s \in [\tau, T] \). Thus \( x_\delta \to y \) and \( t_\delta \to s \). Therefore (2.4) contradicts (2.2). \( \square \)

**Theorem 2.2** For any \( q > 1 \) and \( f \in L^1_{loc,+}(\Omega) \), there exists a maximal solution \( u := \mathbf{u}_f \) of

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q^\Omega_{\infty}$$

(2.5)

which satisfies

$$\lim_{t \to 0} u(\cdot, t) = f(\cdot) \quad \text{in } L^1_{loc}(\Omega).$$

(2.6)

**Proof.** Let \( \Omega_n \) be an increasing sequence of smooth bounded domains such that \( \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega \) and \( \cup \Omega_n = \Omega \). For each \( n \) let \( u_{n,f} \) be the increasing limit when \( k \to \infty \) of the \( u_{n,k,f} \) solution of

$$\begin{cases} 
\partial_t u_{n,k,f} - \Delta u_{n,k,f} + u_{n,k,f}^q = 0 \quad \text{in } Q^\Omega_{\infty}, \\
u_{n,k,f}(x,t) = k \quad \text{in } \partial \Omega_n \times (0, \infty), \\
u_{n,k,f}(x,0) = f \chi_{\Omega_n} \quad \text{in } \Omega_n.
\end{cases}$$

(2.7)

By the maximum principle and a standard approximation argument \( n \mapsto u_{n,k,f} \) is decreasing thus \( n \mapsto u_{n,f} \) too. The limit \( \mathbf{u}_f \) of the \( u_{n,f} \) satisfies (2.5) and (2.6). It is independent of the exhaustion \( \{\Omega_n\} \) of \( \Omega \). Let \( u \) be a positive solution of (2.5) in \( Q^\Omega_{\infty} \) which satisfies (2.6). Since the initial trace of \( u \) is a locally integrable function, \( u^q \in L^1_{loc}(\Omega \times [0, \infty)) \). By
Fubini we can assume that, for any $n$, $u \in L^1_{\text{loc}}(\partial \Omega_n \times [0, \infty))$. Because $(u - u_{n,k,f})_+ \leq u$ and tends to 0 when $k \to \infty$, it follows by Lebesgue’s theorem that
\[
\lim_{k \to \infty} \| (u - u_{n,k,f})_+ \|_{L^1(\partial \Omega_n \times (0,T))} = 0 \quad \forall T > 0.
\]
Applying the maximum principle in $\Omega_n \times (0, \infty)$ yields to
\[
u \leq \lim_{k \to \infty} u_{n,k,f} = u_{n,f} \Rightarrow u \leq \lim_{n \to \infty} u_{n,f} = u_f.
\]

**Theorem 2.3** For any $q > 1$ and $f \in L^1_{\text{loc}}(\Omega)$, there exists a minimal nonnegative solution $\bar{u}_f$ of (2.5) in $Q^\Omega$ which satisfies (2.6).

**Proof.** The scheme of the construction is similar to the one of $\pi_f$: with the same exhaustion $\{\Omega_n\}$ of $\Omega$, we consider the solution $u_{n,0,f}$ solution of
\[
\begin{cases}
\partial_t u_{n,0,f} - \Delta u_{n,0,f} + u_{n,0,f}^q = 0 & \text{in } Q^\Omega_n \\
u_{n,0,f}(x,t) = 0 & \text{in } \partial \Omega_n \times (0, \infty) \\
u_{n,0,f}(x,0) = f \chi_{\Omega_n} & \text{in } \Omega_n.
\end{cases}
\]
By the maximum principle, $n \mapsto u_{n,0,f}$ is increasing and dominated by $\bar{u}_f$. Therefore it converges to some solution $u_f$ of (2.5), which satisfies (2.6) as $u_{n,0,f}$ and $\pi_f$ do it. Using the same argument as in the proof of Theorem 2.2 there holds $u_{n,0,f} \leq u$ in $Q^\Omega_n$ for a suitable exhaustion. Thus $u_f \leq u$.

**Remark.** Because of the lack of regularity of $\partial \Omega$, there is no reason for $\bar{u}_f$ (resp $u_f$) to tend to infinity (resp. zero) on $\partial \Omega \times (0, \infty)$.

The next statement will be very useful for proving uniqueness results.

**Theorem 2.4** Assume $q > 1$, $f \in L^1_{\text{loc}}(\Omega)$ and $u_f$ is a nonnegative solution of (2.5) satisfying (2.6). Then there exists a nonnegative solution $u_0$ of (2.5) satisfying
\[
\lim_{t \to 0} u_0(\cdot, t) = 0 \quad \text{in } L^1_{\text{loc}}(\Omega),
\]

such that
\[
0 \leq u_f - \underline{u}_f \leq u_0 \leq u_f,
\]
and
\[
0 \leq \pi_f - u_f \leq \underline{u}_f - u_0.
\]

**Proof.** Step 1: construction of $u_0$. The function $w = u_f - \underline{u}_f$ is a nonnegative subsolution of (2.5) which satisfies
\[
\lim w(\cdot, t) = 0 \quad \text{in } L^1_{\text{loc}}(\Omega).
\]
Using the above considered exhaustion of $\Omega$, we denote by $v_n$ the solution of
\[
\begin{cases}
\partial_t v_n - \Delta v_n + v_n^q = 0 & \text{in } Q^\Omega_n \\
v_n(x,t) = u_f - \underline{u}_f & \text{in } \partial \Omega_n \times (0, \infty) \\
v_n(x,0) = 0 & \text{in } \Omega_n.
\end{cases}
\]
By the maximum principle
\[ u_f - u_f \leq v_n \leq u_f \quad \text{in } Q^\Omega_\infty. \]
Therefore \( v_{n+1} \geq v_n \) on \( \partial \Omega_n \times (0, \infty) \); this implies that the same inequality holds in \( Q^\Omega_\infty \).
If we denote by \( u_0 \) the limit of the \( \{ v_n \} \), it is a solution of (2.5) in \( Q^\Omega_\infty \).

**Step 2: proof of (2.11).** We follow a method introduced in [8] in a different context. For \( n \in \mathbb{N} \) and \( k > 0 \) fixed, we set
\[ Z_{f,n} = u_{f,n} - u_f \quad \text{and} \quad Z_{0,n} = u_{0,n} - u_0, \]
where we assume that the \( n \) are chosen such that \( u_f, u_0 \in L^1_{\text{loc}}(\partial \Omega_n \times [0, \infty)) \), and
\[ \phi(r, s) = \begin{cases} r^q - s^q & \text{if } r \neq s \\ r - s & \text{if } r = s. \end{cases} \]
By convexity,
\[ \left\{ \begin{array}{l} r_0 \geq s_0, \ r_1 \geq s_1 \\ r_1 \geq r_0, \ s_1 \geq s_0 \end{array} \right. \implies \phi(r_1, s_1) \geq \phi(r_0, s_0). \]
Therefore
\[ \phi(u_{f,n}, u_f) \geq \phi(u_{0,n}, u_0) \quad \text{in } Q^\Omega_\infty, \]
and
\[ 0 = \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + u_{f,n}^q - u_f^q - u_{0,n}^q + u_0^q \]
\[ = \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)Z_{f,n} - \phi(u_{0,n}, u_0)Z_{0,n}, \]
which implies
\[ \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{0,n}) \leq 0. \]
But \( Z_{f,n} - Z_{0,n} = 0 \) in \( \Omega_n \times \{ 0 \} \) and
\[ \int_0^\infty \int_{\partial \Omega_n} |Z_{f,n} - Z_{0,n}| \, dS \, dt = 0 \]
by approximations. By the maximum principle \( Z_{f,n,k} - Z_{0,n,k} \leq 0 \). Letting \( n \to \infty \) yields to
\[ u_f - u_f \leq u_0 - u_0, \]
which ends the proof. \( \square \)
3 The case $1 < q < N/(N - 2)$

In this section we assume that $\Omega$ is a domain of $\mathbb{R}^N$ with a compact boundary. We first prove that the maximal solution is a large solution.

**Theorem 3.1** Assume $1 < q < N/(N - 2)$ and $f \in L^1_{\text{loc}}(\Omega)$. Then the maximal solution $\overline{u}_f$ of (2.5) in $Q^f_1$ which satisfies (2.6) satisfies also (2.3).

**Proof.** In Appendix we construct the self-similar solution $V := V_N$ of (2.5) in $Q^\infty_{\Omega} \setminus \{0\}$ which has initial trace zero in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$
\lim_{|x| \to 0} V_N(x, t) = \infty,
$$

locally uniformly on $[\tau, \infty)$, for any $\tau > 0$. Furthermore $V_N(x, t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$. If $a \in \partial \Omega$, the restriction to $\Omega_n$ of the function $V_N(x - a, t)$ is bounded from above by $u_{a_n}$.

Letting $n \to \infty$ yields to

$$
V_N(x - a, t) \leq \overline{u}_f(x, t) \quad \forall (x, t) \in Q^\infty_{\Omega}.
$$

(3.1)

If we consider $x \in \Omega$ and denote by $a_x$ a projection of $x$ onto $\partial \Omega$, there holds

$$
t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) = V_N(x - a_x, t) \leq \overline{u}_f(x, t).
$$

(3.2)

Using (3.2), we derive that $\overline{u}_f$ satisfies (2.3). \hfill \square

**Theorem 3.2** Assume $1 < q < N/(N - 2)$, $f \in L^1_{\text{loc}}(\Omega)$ and $\partial \Omega = \partial \Omega^f$. Then $\overline{u}_f$ is the unique solution of (2.5) in $Q^f_1$ which satisfies (2.6) and (2.3).

**Proof.** Assume that $u_f$ is a solution of (2.5) in $Q^f_1$ such that (2.6) and (2.3) hold. By Theorem 2.4 there exists a positive solution $u_0$ with zero initial trace such that

$$
0 \leq u_f - u_0 \leq \overline{u}_f
$$

(3.3)

and (2.11) are satisfied. Since $\overline{u}_f(x, t) \leq ((q - 1)t)^{-1/(q-1)}$ (notice that this last expression is the maximal solution of (2.5) in $Q^\infty_{\Omega}$), the function $u_0$ satisfies also (2.3). Therefore, it is sufficient to prove that $\overline{u}_0 = u_0 := u$.

Step 1: bilateral estimates. Since $\partial \Omega = \partial \Omega^f$, for any $a \in \partial \Omega$, there exists a sequence $\{a_n\} \subset \Omega^f$ converging to $a$. If $u$ is any solution of (2.5) in $Q^f_1$ which satisfies (2.3) and (2.9), there holds

$$
V_N(x - a_n, t) \leq u(x, t) \implies V_N(x - a, t) \leq u(x, t).
$$

In particular, if $a = a_x$, we see that $u$ satisfies (3.2). In order to obtain an estimate from above we consider for $r < \rho(x)$ the solution $(y, t) \mapsto u_{x,r}(y, t)$ of

$$
\begin{cases}
\partial_t u_{x,r} - \Delta u_{x,r} + u_{x,r}^q = 0 & \text{in } Q^\infty_{B_r(x)} \\
\lim_{(y, t) \to (z, 0)} u_{x,r}(y, t) = 0 & \forall z \in B_r(x) \\
\lim_{|x| \to r} u_{x,r}(x, t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{ for any } \tau > 0
\end{cases}
$$

(3.4)
Then \( \mathfrak{m}_0(y, t) \leq u_{x,y}(y, t) \Rightarrow \mathfrak{m}_0(y, t) \leq u_{x,\rho(x)}(y, t) \) \( \forall (y, t) \in Q_{B^\infty}^{B^\infty(x)} \).

In particular, with \( u_{0,r} = u_r \),
\[
\mathfrak{m}_0(x, t) \leq u_{\rho(x)}(0, t) = (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2).
\]

Therefore
\[
t^{-1/(q-1)}H_N(\rho(x)/\sqrt{t}) \leq u(x, t) \leq \mathfrak{m}_0(x, t) \leq (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2). \tag{3.5}
\]

The function \( s \mapsto u_1(0, s) \) is increasing by the same argument as the one of Corollary \ref{5.2} and bounded from above by the unique solution \( P \) of
\[
\begin{cases}
-\Delta P + P^q = 0 & \text{in } B_1 \\
\lim_{|x| \to 1} P(x) = \infty.
\end{cases} \tag{3.6}
\]

Therefore it converges to \( P \) locally uniformly in \( B_1 \) and \( \lim_{s \to \infty} u_1(0, s) = P(0) \). Thus
\[
t/(\rho(x))^2 \to \infty \Rightarrow (\rho(x))^{-2/(q-1)}u_1(0, t/(\rho(x))^2) \approx P(0)(\rho(x))^{-2/(q-1)}. \tag{3.7}
\]

On the other hand, if \( t/(\rho(x))^2 \to \infty \), equivalently \( \rho(x)/\sqrt{t} \to 0 \),
\[
t^{-1/(q-1)}H_N(\rho(x)/\sqrt{t}) \approx \lambda_{N,q}t^{-1/(q-1)}(\rho(x)/\sqrt{t})^{-2/(q-1)} = \lambda_{N,q}(\rho(x))^{-2/(q-1)}, \tag{3.8}
\]

by \eqref{5.4}.

Next, in order to obtain an estimate from above of \( u_1(0, s) \) when \( s \to 0 \), we compare \( u_1 \) to a solution \( u_\Theta \) of \eqref{2.5} in \( Q_{B_1}^\infty \), where \( \Theta \) is a polyhedra inscribed in \( B_1 \); this polyhedra is a finite intersection of half spaces \( \Gamma_i \) containing \( \Pi \). In each of the half space \( \Gamma_i \), with boundary \( \gamma_i \), we can consider the solution \( W_i \) of \eqref{2.5} in \( Q_{B_1}^\infty \) which tends to infinity on \( \gamma_i \times (0, \infty) \) and has value 0 on \( \Gamma_i \times (0, \infty) \). This solution depends only on the distance to \( \gamma_i \) and \( t \). Thus it is expressed by the function \( V_i \) defined in Proposition \ref{5.1} when \( N = 1 \). Moreover, since a sum of solutions is a super solution,
\[
u_1 \leq u_\Theta \leq \sum_i W_i \Rightarrow u_1(0, s) \leq \sum_i H_i \left( \text{dist } (0, \gamma_i)/\sqrt{s} \right). \tag{3.9}
\]

We can choose the hyperplanes \( \gamma_i \) such that for any \( \delta \in (0, 1) \), there exists \( C_\delta \in \mathbb{N} \) such that
\[
u_1(0, s) \leq C_\delta H_i(1 - \delta)/\sqrt{s}). \tag{3.10}
\]

Using \eqref{5.3} we derive
\[
u(x, t) \geq c_{N,q}(\rho(x))^{2/(q-1)-N_i}t^{N/2-1/(q-1)}e^{-(\rho(x))^2/4t},
\]
when \( \rho(x)/\sqrt{t} \to \infty \), and
\[
\mathfrak{m}_0(x, t) \leq C H_i((1-\delta)\rho(x)/\sqrt{t}) \leq C(1-\delta)^2/(q-1)(\rho(x))^{2/(q-1)-1}t^{1/2-1/(q-1)}e^{-(1-\delta)\rho(x)^2/4t}.
\]

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Therefore, there exists $\theta > 1$ such that
\[
\overline{u}_0(x, t) \leq C(\rho(x))^{2/(q-1) - N} \lambda_{N,q}^{1/2 - 1/(q-1)} e^{-(\rho(x))^2/4\theta t} \leq C u(x, \theta t),
\] (3.11)
when $\rho(x)/\sqrt{t} \to \infty$. Finally, when $m^{-1} \leq \rho(x)/\sqrt{t} \leq m$ for some $m > 1$, (3.5) shows that
\[
(\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2) \text{ and } t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t})
\] are comparable. In conclusion, there exist constants $C > P(0)/\lambda_{N,q} > 1$ and $\theta > 1$ such that
\[
u(x, t) \leq \overline{u}_0(x, t) \leq C u(x, \theta t) \quad \forall (x, t) \in Q^\Omega_\infty.
\] (3.12)

**Step 2: End of the proof.** Let $\tau > 0$ and $C' > C$ be fixed. The function
\[
\quad t \mapsto u_\tau(x, t) := C' u(x, t + \theta \tau)
\]
is a supersolution of (2.5) in $\Omega \times (0, \infty)$ which satisfies $u_\tau(x, 0) = C' u(x, \theta \tau) > \overline{u}_0(x, \tau)$ by (3.12). Furthermore,
\[
C' u(x, t + \theta \tau) \geq C'(t + \theta \tau)^{-1/(q-1)} H_N(\rho(x)/\sqrt{t + \theta \tau}) \geq C' \lambda_{N,q} (1 + o(1))(\rho(x))^{-2/(q-1)},
\]
as $\rho(x) \to 0$, locally uniformly for $t \in [0, \infty)$. Similarly,
\[
\overline{u}_\tau(x, t + \tau) \leq (\rho(x))^{-2/(q-1)} u_1(0, t + \tau)/(\rho(x))^2 = P(0)(1 + o(1))(\rho(x))^{-2/(q-1)},
\]
as $\rho(x) \to 0$, and also locally uniformly for $t \in [0, \infty)$. Therefore $(\overline{u}_\tau(x, t) - u_\tau(x, t))_+$ vanishes in a neighborhood of $\partial \Omega \times [0, T]$ for any $T > 0$. By the maximum principle
\[
u(x, t) \geq \overline{u}_\tau(x, t) \quad \forall (x, t) \in \Omega \times (0, \infty).
\]
Letting $\tau \to 0$ and $C' \to C$ yields to
\[
u(x, t) \leq \overline{u}_0(x, t) \leq C u(x, t) \quad \forall (x, t) \in Q^\Omega_\infty.
\] (3.13)

The conclusion of the proof is contradiction, following an idea introduced in [8] and developed by [12] in the elliptic case. We assume $u \neq \overline{u}_0$, thus $u < \overline{u}_0$. By convexity the function
\[
\quad w = u - \frac{1}{2C}(\overline{u}_0 - u)
\]
is a supersolution and $w < u$. Moreover $w > w' := ((1 + C)/2C) u$ and $w'$ is a subsolution. Consequently, there exists a solution $u_1$ of (2.5) which satisfies
\[
u' < u_1 \leq w \implies \overline{u}_0 - u_1 \geq \left(1 + K^{-1}\right) (\overline{u}_0 - u) \quad \text{in } Q^\Omega_\infty.
\] (3.14)
Notice that $u_1$ satisfies (2.9) and (2.3), therefore it satisfies (3.13) as $u$ does it. Replacing $u$ by $u_1$ and introducing the supersolution
\[
u_1 = u_1 - \frac{1}{2C}(\overline{u}_0 - u_1)
\]
and the subsolution $w'_1 := ((1 + C)/2C) u_1$ we see that there exists a solution $u_2$ of (2.5) such that
\[
u'_1 \leq u_2 \implies \overline{u}_0 - u_2 \geq \left(1 + K^{-1}\right)^2 (\overline{u}_0 - u) \quad \text{in } Q^\Omega_\infty.
\] (3.15)
By induction, we construct a sequence of positive solutions $u_k$ of (2.5), subject to (2.9) and (2.3) such that
\[ u_0 - u_k \geq (1 + K^{-1})^k (u_0 - u) \quad \text{in} \quad Q_\infty^\Omega. \] (3.16)
This is clearly a contradiction since $(1 + K^{-1})^k \to \infty$ as $k \to \infty$ and $u_0$ is locally bounded in $Q_\infty^\Omega$.

\[ \square \]

4 The local continuous graph property

In this section, we assume that $\partial \Omega$ is compact and is locally the graph of a continuous function, which means that there exists a finite number of open sets $\Omega_j$ ($j = 1, \ldots, k$) such that $\Gamma \cap \Omega_j$ is the graph of a continuous function. Our main result is the following

**Theorem 4.1** Assume $q > 1$ and $f \in L_{\text{loc}}^1(\Omega)$. Then there exists at most one positive solution of (2.5) in $Q_\infty^\Omega$ satisfying (2.6) and (2.3).

Suppose $u_f$ satisfies (2.5) in $Q_\infty^\Omega$ satisfying (2.6) and (2.3), then clearly the maximal solution $\overline{u}$ endows the same properties. In order to prove that $u_f = \overline{u}$, we can assume that $f = 0$ by Theorem 2.4. We denote by $u$ this large solution with zero initial trace. We consider some $j \in \{1, \ldots, k\}$, perform a rotation, denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ the coordinates in $\mathbb{R}^N$ and represent $\Gamma \cap \Omega_j$ as the graph of a continuous positive function $\phi$ defined in $C = \{x' \in \mathbb{R}^{N-1} : |x'| \leq R\}$. We identify $C$ with $\{x = (x', 0) : |x'| \leq R\}$ and set
\[ \Gamma_1 = \{x = (x', \phi(x')) : x' \in C\}, \]
\[ \Gamma_2 = \{x = (x', x_N) : x' \in \partial C, 0 \leq x_N < \phi(x')\}, \]
and
\[ G_R = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi(x')\}. \]
We can assume that $\overline{G}_R \subset \Omega \cup \Gamma_1$.
\[ \inf\{\phi(x') : x' \in C\} = R_0 > 0 \quad \text{and} \quad \sup\{\phi(x') : x' \in C\} = R_1 > R_0. \]
For $\sigma > 0$, small enough, we consider $\phi_\sigma \in C^\infty(C)$ satisfying
\[ \phi(x') - \sigma/2 \leq \phi_\sigma(x') \leq \phi(x') + \sigma/2 \quad \forall x' \in C, \]
and set
\[ G_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') - \sigma\} \]
and
\[ G_{\sigma,R} = \{x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') + \sigma\}. \]
The upper boundaries of $G_\sigma$ and $G_\sigma^\prime$ are defined by
\[ \Gamma_{1,\sigma} = \{x = (x', \phi_\sigma(x') - \sigma) : x' \in C\}, \]
\[ \Gamma_{1,\sigma}^\prime = \{x = (x', \phi_\sigma(x') + \sigma) : x' \in C\}, \]
and the remaining boundaries are
\[ \Gamma_{2, \sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') - \sigma \}, \]
\[ \Gamma'_{2, \sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') + \sigma \}. \]

In order to have the monotonicity of the domains, we can also assume
\[ \phi_\sigma(x') - \sigma < \phi_{\sigma'}(x') - \sigma' < \phi_\sigma(x') + \sigma' < \phi_{\sigma'}(x') + \sigma \quad \forall 0 < \sigma' < \sigma \quad \forall x' \in C, \quad (4.1) \]

thus, under the condition \( 0 < \sigma' < \sigma \),
\[ G_{\sigma, R} \subset G'_{\sigma', R} \subset G_{R} \subset G'_{\sigma, R}. \quad (4.2) \]

The localization procedure is to consider the restriction of \( u \) to \( Q^{G_{\sigma,R}}_{\infty} := G_{R} \times (0, \infty) \), thus \( u \) is regular in \( G_{R} \cup \Gamma_{2} \times [0, \infty) \) and satisfies
\[ \lim_{x_N \to \phi(x')} u(x', x_N, t) = \infty, \quad (4.3) \]

uniformly with respect to \( (x', t) \in C \times [\tau, T] \), for any \( 0 < \tau < T \). We construct \( v_\sigma \) as solution of
\[ \partial_t v_\sigma - \Delta v_\sigma + v_\sigma^q = 0 \quad \text{in} \quad Q^{G_{\sigma,R}}_{\infty} := G_{\sigma,R} \times (0, \infty), \quad (4.4) \]

subject to the initial condition
\[ \lim_{t \to 0} v_\sigma(x, t) = 0 \quad \text{locally uniformly in} \quad G_{\sigma,R}, \quad (4.5) \]

and the boundary conditions
\[ \lim_{x_N \to \phi(x') - \sigma} v_\sigma(x', x_N, t) = \infty \quad \forall (x', t) \in C \times (0, \infty], \quad (4.6) \]

uniformly on any set \( K \times [\tau, T] \), where \( T > \tau > 0 \) and \( K \) is a compact subset of \( C \) and
\[ v_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma_{2, \sigma} \times [0, \infty). \quad (4.7) \]

We also construct \( w_\sigma \) as solution of
\[ \partial_t w_\sigma - \Delta w_\sigma + w_\sigma^q = 0 \quad \text{in} \quad Q^{G'_{\sigma,R}}_{T} := G'_{\sigma,R} \times (0, \infty), \quad (4.8) \]

subject to the initial condition
\[ \lim_{t \to 0} w_\sigma(x, t) = 0 \quad \text{locally uniformly in} \quad G'_{\sigma,R}, \quad (4.9) \]

and the boundary conditions
\[ \left\{ \begin{array}{l} (i) \quad w_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma'_{1, \sigma} \times [0, T], \vspace{1em} \\
(i') \quad \lim_{(x,s) \to (y,t)} w_\sigma(x, t) = \infty \quad \forall (y, s) \in \Gamma'_{2, \sigma} \times [0, T]. \end{array} \right. \quad (4.10) \]

The functions \( v_\sigma \) and \( w_\sigma \) inherit the following properties in which the local graph property plays a fundamental role, allowing translations of the truncated domains in the \( x_N \)-direction.
Lemma 4.2 For \( \sigma > \sigma' > 0 \) there holds
\[
v_{\sigma'} \leq v_{\sigma} \quad \text{in } Q_{\infty}^{G_{\sigma,R}}, \tag{4.11}
\]
\[
w_{\sigma'} \leq w_{\sigma} \quad \text{in } Q_{\infty}^{G_{\sigma,R}}, \tag{4.12}
\]
(i) \( v_{\sigma}(x', x_N - 2\sigma, t) \leq u(x', x_N, t) \quad \text{in } Q_{\infty}^{G_{\sigma,R}} \)
(ii) \( u(x', x_N, t) \leq v_{\sigma}(x, t) + w_{\sigma}(x, t) \quad \text{in } Q_{\infty}^{G_{\sigma,R}}. \tag{4.13} \)

Proof. The inequalities (4.11) and (4.12) are the direct consequence of the fact that the domains \( G_{\sigma,R} \) and \( G_{\sigma',R} \) are Lipschitz and the functions \( v_{\sigma} \) and \( w_{\sigma} \) are constructed by approximations of solutions of (2.5) with bounded boundary data. For proving (4.13)-(i), we compare, for \( \tau > 0 \), \( u(x, t - \tau) \) and \( v_{\sigma}(x', x_N - 2\sigma, t) \) in \( Q_{\infty}^{G_{\sigma,R}} \). Because \( u \) satisfies (2.3), and \( v_{\sigma}(x', x_N - 2\sigma, 0) = 0 \) in \( G_{R} \), (4.13)-(i) follows by the maximum principle. The proof of (4.13)-(ii) needs no translation, but the fact that the sum of two solutions is a supersolution.

Corollary 4.3 There exist \( v_0 = \lim_{\sigma \to 0} v_{\sigma} \) and \( w_0 = \lim_{\sigma \to 0} w_{\sigma} \) and there holds
\[
v_0 \leq u \leq v_0 + w_0 \quad \text{in } Q_{\infty}^{G_{\sigma,R}}. \tag{4.14}
\]

Moreover, the functions \( t \mapsto v_0(x, t) \) and \( t \mapsto w_0(x, t) \) are increasing on \((0, \infty), \forall \sigma \in G_{R}\).

Proof. The first assertion follows from (4.11)-(4.12), and (4.14) from (4.13). Since \( v_0 \) is the limit, when \( \sigma \to 0 \) of \( v_{\sigma} \) which satisfy equation (4.1) in \( Q_{T}^{G_{\sigma,R}} \), initial condition (4.15) and boundary conditions (4.6), (4.7), it is sufficient to prove the monotonicity of \( t \mapsto v_{\sigma}(\cdot, t) \), Moreover \( v_0 \) is the limit, when \( k \) tends to infinity of the \( v_{k,\sigma} \) solutions of (2.5) in \( Q_{T}^{G_{\sigma,R}} \), which satisfy the same boundary conditions as \( v_{\sigma} \) on \( \Gamma_{2,\sigma} \times [0, T] \), the same zero initial condition and
\[
\lim_{x_N \to \phi(x') - \sigma} v_{k,\sigma}(x', x_N, t) = k.
\]

For \( \tau > 0 \), we define \( V_{\tau} \) by \( V_{\tau}(x, t) = (v_{k,\sigma}(x, t) - v_{k,\sigma}(x, t + \tau)) \). Because \( \partial G_{\sigma,R} \) is Lipschitz and \( V_{\tau} \) is a subsolution of (2.5) which vanishes on \( \partial G_{\sigma,R} \times [0, T] \) and at \( t = 0 \), it is identically zero. This implies \( v_{k,\sigma}(x, t) \leq v_{k,\sigma}(x, t + \tau) \), and the monotonicity property of \( v_0 \), by strict maximum principle and letting \( \sigma \to 0 \). The proof of the monotonicity of \( w_0 \) is similar.

The key step of the proof is the following result.

Proposition 4.4 Let \( \epsilon, \tau > 0 \). Then there exists \( \delta_{\epsilon} > 0 \) such that, if we denote \( G_{\delta,R} = \{ x = (x', x_N) : |x'| < R' \text{ and } \phi(x') - \delta \leq x_N < \phi(x') \} \), there holds, for \( R' < R/\sqrt{N - 1} \),
\[
w_0(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q_{\infty}^{G_{\delta,R}}. \tag{4.15}
\]
Proof. Using the result in Appendix, we recall that $V := V_1$ is the unique positive and self-similar solution of the problem

$$
\begin{align*}
\partial_t V - \partial_{zz} V + V^q &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\
\lim_{t \to 0} V(z, t) &= 0 \quad \forall z > 0 \\
\lim_{z \to 0} V(z, t) &= \infty \quad \forall t > 0,
\end{align*}
$$

and it is expressed by $V_1(z, t) = t^{-(q-1)H_1(x/\sqrt{t})}$, where $H_1$ satisfies $(5.2)$-$(5.3)$ with $N = 1$. We set $R_N = R/\sqrt{N} - 1$ so that $C_\infty := \{x' = (x_1, \ldots, x_{N-1}) : \sup_{j \leq N-1} |x_j| < R_N \} \subset C' = \{x' : |x'| \leq R\}$ and we define

$$
\tilde{w}(x, t) = W(x_N, t) + \sum_{j=1}^{N-1} (W(x_j - R, t) + W(R - x_j, t)).
$$

The function $\tilde{w}$ a super solution in $\Theta \times \mathbb{R}_+$ where $\Theta := \{x', x_N) : x' \in C_\infty, x_N > 0\}$ which blows up on

$$
\{x : x_N = 0, \sup_{j \leq N-1} |x_j| \leq R\} \cup \{x : x_N \geq 0, x_j = \pm R\}.
$$

Therefore $v_0 \leq \tilde{w}$ in $Q^{GR}_\infty$. Moreover $\tilde{w}(x, t) \to 0$ when $t \to 0$, uniformly on

$$
G_{\alpha, R} := \{x = (x_1, x_2) : |x_1| \leq R', \alpha \leq x_2 \leq \phi(x_1)\},
$$

for any $\alpha \in (0, R_0]$ and $R' \in (0, R_N)$. Since for any $\tau > 0$, $v_0(x, t + \tau) \to \infty$ when $\rho(x) \to 0$, locally uniformly on $[0, \infty)$, and $\tilde{w}(x, t)$ remains uniformly bounded on $Q^{G_{\alpha, R}'}_{\infty}$, for any $\delta > R_0$, it follows that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$
w_0(x, t) \leq \tilde{w}(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q^{G_{\alpha, R}'}_{\infty}.
$$

□

Proof of Theorem 4.1. Assume $u$ is a solution of $(2.5)$ satisfying $(2.6)$ and $(2.3)$. Then there holds in $Q^{G_{\alpha, R}'}_{\infty}$,

$$
v_0(., t) \leq u(., t) \leq v_0(., t) + \epsilon v_0(., t + \tau). \quad (4.17)
$$

Therefore

$$
v_0(., t + \tau) \leq u(., t + \tau) \leq v_0(., t + \tau) + \epsilon v_0(., t + 2\tau),
$$

from which follows

$$(1 + \epsilon)u(., t + \tau) \geq (1 + \epsilon)v_0(., t + \tau) \geq v_0(., t) + \epsilon v_0(., t + \tau)$$
since \( t \mapsto v_0(\cdot,t) \) is increasing by Corollary 4.3. The maximal solution \( \overline{u}_0 \) satisfies (1.17) too; consequently the following inequality is verified in \( Q^{G_{\Omega},w} \),

\[
(1 + \epsilon)u(\cdot,t + \tau) \geq \overline{u}_0(\cdot,t). \tag{4.18}
\]

Since \( \partial \Omega \) is compact, there exists \( \delta^* > 0 \) such that (4.18) holds whenever \( t \in [0,T] \) (\( T > 0 \) arbitrary) and \( \rho(x) \leq \delta^* \). Furthermore

\[
\lim_{t \to 0} \max\{ (\overline{u}_0(x,t) - (1 + \epsilon)u(x,t + \tau))_+ : \rho(x) \geq \delta^* \} = 0
\]

because of (2.6). Since \( (\overline{u}_0(x,t) - (1 + \epsilon)u(x,t + \tau))_+ \) is a subsolution, which vanishes at \( t = 0 \) and near \( \partial \Omega \times [0,T] \), it follows that (4.18) holds in \( Q^0_T \). Letting \( \epsilon \to 0 \) and \( \tau \to 0 \) yields to \( u \geq \overline{u}_0 \). \( \square \)

**Remark.** The existence of large solutions when \( q \geq N/(N-2) \) is a difficult problem as it is already in the elliptic case. We conjecture that the necessary and sufficient conditions, obtained by Dhersin-Le Gall when \( q = 2 \) \([3]\) and Labutin \([6]\) in the general case \( q > 1 \), and expressed by mean of a Wiener type criterion involving the \( C^{\alpha}_\Omega \)-Bessel capacity, are still valid. As in \([3]\), it is clear that if \( \partial \Omega \) satisfies the exterior segment property and \( 1 < q < (N-1)/(N-3) \), then \( \overline{u}_0 \) is a large solution.

### 5 Appendix

The proof of this result is based upon the existence of solution of (2.5) in \( Q^{\mathbb{R}^N \setminus \{0\}}_{\infty} \) with a persistent singularity on \( \{0\} \times [0,\infty) \).

**Proposition 5.1** For any \( q > 1 \), there exists a unique positive function \( V := V_N \) defined in \( \mathbb{R}_+ \times \mathbb{R}_+ \) satisfying, for any \( \tau > 0 \)

\[
\begin{aligned}
\partial_t V - \Delta V + V^q &= 0 & \text{in } \mathbb{R}^N_0 \\
\lim_{(x,t) \to (y,0)} V(x,t) &= 0 & \forall y \in \mathbb{R}^N \setminus \{0\} \\
\lim_{|x| \to 0} V(x,t) &= \infty & \text{locally uniformly on } [\tau,\infty), \text{ for any } \tau > 0
\end{aligned} \tag{5.1}
\]

Then \( V_N(x,t) = t^{1/(q-1)}H_N(|x|/\sqrt{t}) \), where \( H := H_N \) is the unique positive function satisfying

\[
\begin{aligned}
H'' + \left( \frac{N-1}{r} + \frac{r}{2} \right) H' + \frac{1}{q-1} H - H^q &= 0 & \text{in } \mathbb{R}_+ \\
\lim_{r \to 0} H(r) &= \infty \\
\lim_{r \to \infty} r^{2/(q-1)} H(r) &= 0
\end{aligned} \tag{5.2}
\]

Furthermore there holds

\[
H_N(r) = c_{N,q} r^{2/(q-1)-N} e^{-r^2/4} (1 + O(r^{-2})) \quad \text{as } r \to \infty, \tag{5.3}
\]

and

\[
H_N(r) = \lambda_{N,q} r^{-2/(q-1)} (1 + O(r)) \quad \text{as } r \to 0, \tag{5.4}
\]

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Proof. If we assume \(1 < q < N/(N-2)\), the \(C_{2.1,q'}\) parabolic capacity of the axis \(\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1}\) is positive, therefore there exists a unique solution \(u := u_{\mu}\) to the problem

\[
\partial_t u - \Delta u + |u|^{q-1}u = \mu \quad \in \mathbb{R}^N \times \mathbb{R},
\]

(see [1]) where \(\mu\) is the uniform measure on \(\{0\} \times \mathbb{R}_+\) defined by

\[
\int \zeta d\mu = \int_0^\infty \zeta(0,t) dt \quad \forall \zeta \in C^\infty_c(\mathbb{R}^{N+1}).
\]

If we denote \(T_\ell[u](x,t) = \ell^{2/(q-1)}u(\ell x, \ell^2 t)\) for \(\ell > 0\), then \(T_\ell\) leaves the equation \((2.5)\) invariant, and \(T_\ell[u_\mu] = u_{\ell^{2/(q-1)} - N\mu}\). If we replace \(\mu\) by \(k\mu\) \((k > 0)\), we obtain

\[
T_\ell[u_{k\mu}] = u_{\ell^{2/(q-1)} - N\mu}.
\]

Moreover, any solution of \((2.5)\) in \(\mathbb{R}^N \setminus \{0\} \times \mathbb{R}_+\) which vanishes on \(\mathbb{R}^N \setminus \{0\} \times \{0\}\) is bounded from above by the maximum solution \(u := U\) of

\[
-\Delta u + u^q = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

This is obtained by considering the solution \(U_\epsilon\) of

\[
\begin{cases}
-\Delta u + u^q = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\
\lim_{|x| \to \epsilon} u(x) = \infty.
\end{cases}
\]

Actually,

\[
U(x) := \lim_{\epsilon \to 0} U_\epsilon(x) = \lambda_{N,q}\lambda^{-2/(q-1)}_N \quad \text{with } \lambda_{N,q} := \left[\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - N\right)\right]^{1/(q-1)},
\]

an expression which exists since \(1 < q < N/(N-2)\). If we let \(k \to \infty\) in \((5.6)\), using the monotonicity of \(\mu \mapsto u_{\mu}\), we obtain that \(u_{k\mu} \to u_{\infty\mu}\), \(u_{\infty\mu} \leq U\) and

\[
T_\ell[u_{\infty\mu}] = u_{\ell^{2/(q-1)} - N\mu} = u_{\infty\mu} \quad \forall \ell > 0.
\]

This implies that \(u_{\infty\mu}\) is self-similar, that is

\[
u_{\infty\mu}(x, t) = t^{-1/(q-1)}h(x/\sqrt{t}).
\]

Furthermore, \(h(.)\) is positive and radial as \(x \mapsto u_{\mu}(x, t)\) is, and it solves

\[
h'' + \left(\frac{N-1}{r} + \frac{r}{2}\right)h' + \frac{1}{q-1}h - h^q = 0 \quad \text{in } \mathbb{R}_+.
\]

Since \(u_{\mu}(x,0) = 0\) for \(x \neq 0\), the a priori bounds \(u_{k\mu} \leq U\), the equicontinuity of the \(\{u_{k\mu}\}_{k>0}\) implies that \(u_{\infty\mu}(x,0) = 0\) for \(x \neq 0\); therefore

\[
\lim_{r \to \infty} r^{2/(q-1)}h(r) = 0.
\]
The same argument as the one used in the proof of Corollary 4.3 implies that $t \mapsto u_\mu(x,t)$ is increasing, therefore $\lim_{x \to 0} u_\mu(x,t) = \infty$ for $t > 0$. This implies $\lim_{r \to 0} h(r) = \infty$. Then the proof of (5.3) follows from [10, Appendix]. When $r \to 0$, $h$ could have two possible behaviours [3]:

(i) either

$$h(r) = \lambda_N q^r^{-2/(q-1)}(1 + O(r)),$$

(ii) or there exists $c \geq 0$ such that

$$h(r) = cm_N(r)(1 + O(r)),$$

where $m_N(r)$ is the Newtonian kernel if $N \geq 2$ and $m_1(r) = 1 + o(1)$.

If (ii) were true with $c > 0$ (the case $c = 0$ implying that $h = 0$ because of the behavior at $\infty$ and maximum principle), it would lead to

$$u_\infty(x) = c|x|^{2-N-t^{N-2-1/(q-1)}(1 + o(1))} \text{ as } x \to 0,$$

for all $t > 0$. Therefore

$$\int^T_\varepsilon \int_{B_1} u^q_{k\mu} \, dx \, dt < C(\varepsilon),$$

for any $\varepsilon > 0$ and $k \in (0, \infty)$. We write (5.5) under the form

$$\partial_t u_{k\mu} - \Delta u_{k\mu} = g_k + k\mu$$

where $g_k = -u^q_{k\mu}$, then $u_{k\mu} = u'_{k\mu} + u''_{k\mu}$, where

$$\partial_t u'_{k\mu} - \Delta u'_{k\mu} = k\mu$$

and

$$\partial_t u''_{k\mu} - \Delta u''_{k\mu} = g_k.$$ 

By linearity $u'_{k\mu} = k u'_{\mu}$. Because of (5.16) $u'_{k\mu}$ remains uniformly bounded in $L^1(B_1 \times (\varepsilon, T)$. This clearly contradicts $\lim_{k \to \infty} u'_{k\mu} = \infty$. Thus (5.4) holds. The proof of uniqueness is an easy adaptation of [3, Lemma 1.1]: the fact that the domain is not bounded being compensated by the strong decay estimate (5.3). This unique solution is denoted by $V_N$ and $h = H_N$. □

References


