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FUNCTIONAL INEQUALITIES FOR HEAVY TAILS DISTRIBUTIONS AND APPLICATION TO ISOPERIMETRY

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A . This paper is devoted to the study of probability measures with heavy tails. Using the Lyapunov function approach we prove that such measures satisfy different kind of functional inequalities such as weak Poincaré and weak Cheeger, weighted Poincaré and weighted Cheeger inequalities and their dual forms. Proofs are short and we cover very large situations. For product measures on $\mathbb{R}^n$ we obtain the optimal dimension dependence using the mass transportation method. Then we derive (optimal) isoperimetric inequalities. Finally we deal with spherically symmetric measures. We recover and improve many previous results.

Key words : weighted Poincaré inequalities, weighted Cheeger inequalities, Lyapunov function, weak inequalities, isoperimetric profile

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1. I , .

The subject of functional inequalities knows an amazing growth due to the numerous fields of application: differential geometry, analysis of p.d.e., concentration of measure phenomenon, isoperimetry, trends to equilibrium in deterministic and stochastic evolutions... Let us mention Poincaré, weak Poincaré or super Poincaré inequalities, Sobolev like inequalities, F-Sobolev inequalities (in particular the logarithmic Sobolev inequality), modified log-Sobolev inequalities and so on. Each type of inequality appears to be very well adapted to the study of one (or more) of the applications listed above. We refer to [36], [2], [30], [1], [40], [56], [37], [50], [10], [29] for an introduction.

If a lot of results are known for log-concave probability measures, not so much has been proved for measures with heavy tails (let us mention [48, 9, 22, 4, 18, 24]). In this paper the focus is on such measures with heavy tails and our aim is to prove functional and isoperimetric inequalities.

Informally measures with heavy tails are measures with tails larger than exponential. Particularly interesting classes of examples are either κ-concave probability measures, or sub-exponential like laws (or tensor products of any of them) defined as follows.

We say that a probability measure μ is κ-concave with $\kappa = -1/\alpha$ if

$$d\mu(x) = V(x)^{-(n+\alpha)}dx \tag{1.1}$$

with $V : \mathbb{R}^n \to (0, \infty)$ convex and $\alpha > 0$. Such measures have been introduced by Borell [26] in more general setting. See [18] for a comprehensive introduction and the more general definition of

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κ-concave probability measures. Prototypes of κ-concave probability measures are the generalized Cauchy distributions

$$d\mu(x) = \frac{1}{Z}\left((1+|x|^2)^{1/2}\right)^{-(n+\alpha)} \tag{1.2}$$

for $\alpha > 0$, which corresponds to the previous description since $x \mapsto (1+|x|^2)^{1/2}$ is convex. In some situations we shall also consider $d\mu(x) = (1/Z)\left((1+|x|)\right)^{-(n+\alpha)}$. Note that these measures are Barenblatt solutions in porous medium equations and appears naturally in weighted porous medium equations, giving the decay rate of this nonlinear semigroup towards the equilibrium measure, see [54, 32].

We may replace the power by an exponential yielding the notion of sub-exponential law, i.e. given any convex function $V : \mathbb{R}^n \to (0,\infty)$ and $p > 0$, we shall say that

$$d\mu(x) = e^{-V(x)^p}\,dx$$

is a sub-exponential like law. A typical example is $V(x) = |x|$.

Heavy tails measures are now particularly important since they appear in various areas: fluid mechanics, mathematical physics, statistical mechanics, mathematical finance ... Since previous results in the literature are not optimal, our main goal is to study the isoperimetric problem for heavy tails measures. This will lead us to consider various functional inequalities (weak Cheeger, weighted Cheeger, converse weighted Cheeger). Let us explain why.

Recall the isoperimetric problem.

Denote by d the Euclidean distance on $\mathbb{R}^n$. For $h \geq 0$ the closed h-enlargement of a set $A \subset \mathbb{R}^n$ is $A_h := \{x \in M;\ d(x,A) \leq h\}$ where $d(x,A) := \inf\{d(x,a);\ a \in A\}$ is $+\infty$ by convention for $A = \emptyset$. We may define the boundary measure, in the sense of μ, of a Borel set $A \subset \mathbb{R}^n$ by

$$\mu_s(\partial A) := \liminf_{h\to 0^+} \frac{\mu(A_h \setminus A)}{h}.$$

An isoperimetric inequality is of the form

$$\mu_s(\partial A) \geq F(\mu(A)) \qquad \forall A \subset \mathbb{R}^n \tag{1.3}$$

for some function F. Their study is an important topic in geometry, see e.g. [49, 8]. The first question of interest is to find the optimal F. Then one can try to find the optimal sets for which (1.3) is an equality. In general this is very difficult and the only hope is to estimate the isoperimetric profile defined by

$$I_\mu(a) := \inf\{\mu_s(\partial A);\ \mu(A) = a\}, \quad a \in [0,1].$$

Note that the isoperimetric inequality (1.3) is closely related to concentration of measure phenomenon, see [20, 41]. For a large class of distributions μ on the line with exponential or faster decay, it is possible to prove [25, 51, 15, 19, 5, 10, 11, 46] that the isoperimetric profile I_{μ^n} of the n-tensor product μ^n is (up a to universal, hence dimension free constants) equal to I_μ.

For measures with heavy tails, this is no more true. Indeed, if μ is a probability measure on $\mathbb{R}$ such that there exist $h > 0$ and $\varepsilon > 0$ such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$, one has

$$\mu^n(A + [-h,h]^n) \geq \frac{1}{2} + \varepsilon, \tag{1.4}$$

then μ has exponential tails, that is there exist positive constants C_1, C_2 such that $\mu([x,+\infty)) \leq C_1 e^{-C_2 x}$, $x \in \mathbb{R}$, see [52].

Therefore, for measures with heavy tails, the isoperimetric profile as well as the concentration of measure for product measure should heavily depend on n. Some bounds on I_{μ^n}, not optimal in n, are obtained in [9] using weak Poincaré inequality. The non optimality is mainly due to the fact that $\mathbb{L}_2$ inequalities (namely weak Poincaré inequalities) are used. We shall obtain optimal bounds, thus completing the pictures for the isoperimetric profile of tensor product of very general form of probability measures, using $\mathbb{L}_1$ inequalities called weak Cheeger inequalities we introduce now.

As noted by Bobkov [18], for measures with heavy tails, isoperimetric inequalities are equivalent to weak Cheeger inequalities. A probability measure is said to satisfy a weak Cheeger inequality if there exists some non-increasing function $\beta : (0,\infty) \to [0,\infty)$ such that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$, it holds

$$\int |f - m|\, d\mu \;\le\; \beta(s) \int |\nabla f|\, d\mu \;+\; s\,\mathrm{Osc}_\mu(f) \qquad \forall s > 0, \tag{1.5}$$

where m is a median of f for μ and $\mathrm{Osc}_\mu(f) = \operatorname{ess\,sup}(f) - \operatorname{ess\,inf}(f)$. The relationship between β in (1.5) and F in (1.3) is explained in Lemma 3.1 below. Since $\int |f - m|\, d\mu \le \frac{1}{2}\mathrm{Osc}_\mu(f)$, only the values $s \in (0, 1/2]$ are relevant.

Recall that similar weak Poincaré inequalities were introduced in [48], replacing the median by the mean and introducing squares.

Of course if $\beta(0) < +\infty$ we recover the usual Cheeger or Poincaré inequalities.

In order to get isoperimetric results, we thus investigate such inequalities. We use two main strategies. One is based on the Lyapunov function approach [4, 29, 3], the other is based on mass transportation method [34, 35] (see also [14, 53, 19, 21]). In the first case proofs are very short. The price to pay is a rather poor control on the constants, in particular in terms of the dimension. But we cover very general situations (not at all limited to κ-concave like measures). The second strategy gives very explicit controls on the constants, but results are limited to tensor products of measures on the line or spherically symmetric measures (but only for the $\mathbb{L}_2$ case).

This is not surprising in view of the analogue results known for log-concave measures for instance. Indeed recall that the famous conjecture of Kannan-Lovasz-Simonovits ([39]) telling that the Poincaré constant of log-concave probability measures only depends on their variance is still a conjecture. In this situation universal equivalence between Cheeger's inequality and Poincaré inequality is known ([42, 46]), and some particular cases (for instance spherically symmetric measures) have been studied ([17]). In our situation the equivalence between weak Poincaré and weak Cheeger inequalities does not seem to be true in general, so our results are in a sense the natural extension of the state of the art to the heavy tails situation.

The Lyapunov function approach appears to be a very powerful tool not only when dealing with the $\mathbb{L}_1$ form (1.5) but also with $\mathbb{L}_2$ inequalities.

This approach is well known for dynamical systems for example. It has been introduced by Khasminski and developed by Meyn and Tweedie ([43, 44, 45]) in the context of Monte Carlo algorithm (Markov chains). This dynamical approach is in some sense natural: consider the process whose generator is symmetric with respect to the studied measure (see next section for more precise definitions), Lyapunov conditions express that there is some drift (whose strength varies depending on the measure studied) which pushes the process to some natural, say compact, region of the state space. Once in the compact the process behaves nicely and pushed forward to it as soon as it escapes. It is then natural that it gives nice qualitative (but not so quantitative) proofs of total variation convergence of the associated semigroup towards its invariant measure and find applications in the study of the decay to equilibrium of dynamical systems, see e.g. [33, 38, 55, 4, 28]. It is also widely studied in statistics, see e.g. [43] and the references therein. In [4], connections are given between Lyapunov functions and functional inequalities of weak Poincaré type, improving some existing criteria discussed in [48, 9]. In this paper we give new types of Lyapunov functions (in the spirit of [3]) leading to quantitative improvements and in some sense optimal results. Actually we obtain four types of functional inequalities: weighted Cheeger (and weighted Poincaré inequalities)

$$\int |f-m|\,d\mu \;\leq\; C \int |\nabla f|\,\omega\,d\mu \tag{1.6}$$

and their dual forms called converse Cheeger (and converse Poincaré inequalities)

$$\inf_c \int |f-c|\,\omega\,d\mu \;\leq\; C \int |\nabla f|\,d\mu \tag{1.7}$$

where ω are suitable "weights" (see Section 2 for precise and more general definitions definitions).

Weighted Cheeger and weighted Poincaré inequalities were very recently studied by Bobkov and Ledoux [22], using functional inequalities of Brascamp-Lieb type. Their results apply to κ-concave probability measures. We recover their results with slightly worst constants but our approach also applies to much general type laws (sub-exponential for example).

Note that converse Poincaré inequalities appear in the spectral theory of Schrödinger operators, see [31]. We will not pursue this direction here.

Our approach might be summarized by the following diagram:

$$\begin{array}{ccccccc}
 & & & & & & \text{Transport} \\
 & & & & & & \Downarrow \\
 & & \text{Weighted Cheeger} & & \Rightarrow & & \text{Weighted Poincaré} \\
 & & & \Nwarrow & & \Nearrow & \\
 & & & & \text{Lyapunov} & & \Downarrow \\
 & & & \Swarrow & & \Searrow & \\
 & & \text{Converse Cheeger} & & & & \text{Converse Poincaré} \\
 & & \Downarrow & & & & \Downarrow \\
\text{Transport} & \Rightarrow & \text{Weak Cheeger} & & \Rightarrow & & \text{Weak Poincaré} \\
 & & \Updownarrow & & & & \Downarrow \\
 & & \text{Isoperimetry} & & \Rightarrow & & \text{Concentration}
\end{array}$$

Some points have to be underlined. As the diagram indicates, converse inequalities are suitable for obtaining isoperimetric (or concentration like) results, while (direct) weighted inequalities, though more natural, are not. Indeed, the tensorization property of the variance immediately shows that if μ satisfies a weighted Poincaré inequality with constant C and weight ω, then the tensor product μ^n satisfies the same inequality. Since we know that the concentration property for heavy tails measures is not dimension free, this implies that contrary to the ordinary or the weak Poincaré inequality, the weighted Poincaré inequality cannot capture the concentration property of μ. The other point is that the mass transportation method can also be used to obtain some weighted Poincaré inequalities, and weighted Poincaré inequalities via a change of function lead to converse Poincaré inequality (see [22]). The final point is that on most examples we obtain sharp weights (but non necessarily sharp constants), showing that (up to constants) our results are optimal.

The paper is organized as follows.

In Section 2 we prove that the existence of a Lyapunov function implies weighted Cheeger and weighted Poincaré inequalities and their converse.

Section 3 is devoted to the study of weak Cheeger inequalities and to their application to the isoperimetric problem. The Lyapunov function approach and the transport technique are used. Explicit examples are given.

Then, weighted Poincaré inequalities are proved in Section 4 for some spherically symmetric probability measures with heavy tails. We use there the transport technique.

We show in Section 5 how to obtain weak Poincaré inequalities from weak Cheeger and converse Poincaré inequalities.

Finally, the appendix is devoted to the proof of some technical results used in Section 3.

2. F ϕ-L

The purpose of this section is to derive weighted inequalities of Poincaré and Cheeger types, and their converse forms, from the existence of a ϕ Lyapunov function for the underlying diffusion operator. To properly define this notion let us describe the general framework we shall deal with.

Let E be some Polish state space equipped with a probability measure μ and a μ-symmetric operator L. The main assumption on L is that there exists some algebra $\mathcal{A}$ of bounded functions, containing constant functions, which is everywhere dense (in the $\mathbb{L}_2(\mu)$ norm) in the domain of L. This ensures the existence of a "carré du champ" Γ, *i.e.* for $f, g \in \mathcal{A}$, $L(fg) = fLg + gLf + 2\Gamma(f, g)$. We also assume that Γ is a derivation (in each component), *i.e.* $\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$. This is the standard "diffusion" case in [2] and we refer to the introduction of [27] for more details. For simplicity we set $\Gamma(f) = \Gamma(f, f)$. Note that, since Γ is a non-negative bilinear form (see [1, Proposition 2.5.2]), the Cauchy-Schwarz inequality holds: $\Gamma(f, g) \leq \sqrt{\Gamma(f)}\sqrt{\Gamma(g)}$. Furthermore, by symmetry,

$$\int \Gamma(f, g)d\mu = -\int f\, Lg\, d\mu\,. \tag{2.1}$$

Also, since L is a diffusion, the following chain rule formula $\Gamma(\Psi(f), \Phi(g)) = \Psi'(f)\Phi'(g)\Gamma(f, g)$ holds.

In particular if $E = \mathbb{R}^n$, $\mu(dx) = p(x)dx$ and $L = \Delta + \nabla \log p.\nabla$, we may consider the C^∞ functions with compact support (plus the constant functions) as the interesting subalgebra $\mathcal{A}$, and then $\Gamma(f, g) = \nabla f \cdot \nabla g$.

Now we define the notion of Φ-Lyapunov function.

Definition 2.2. *Let $W \geq 1$ be a smooth enough function on E and ϕ be a C^1 positive increasing function defined on $\mathbb{R}^+$. We say that W is a ϕ-Lyapunov function if there exist some set $K \subset E$ and some $b \geq 0$ such that*

$$LW \;\leq\; -\phi(W) \;+\; b\,\mathbb{1}_K\,.$$

This latter condition is sometimes called a "drift condition".

Note that, for simplicity of the previous definition, we did not (and we shall not) specify the underlying operator L.

Remark 2.3. One may ask about the meaning of LW in this definition. In the $\mathbb{R}^n$ case, we shall choose C^2 functions W, so that LW is defined in the usual sense. On more general state spaces of course, the easiest way is to assume that W belongs to the ($\mathbb{L}_2$) domain of L, in particular $LW \in \mathbb{L}_2$. But in some situations one can also relax the latter, provided all calculations can be justified. $\diamond$

2.1. **Weighted Poincaré inequality and weighted Cheeger inequality.** In this section we derive weighted Poincaré and weighted Cheeger inequalities from the existence of a ϕ-Lyapunov function.

Definition 2.4. *We say that μ satisfies a weighted Cheeger (resp. Poincaré) inequality with weight ω (resp. η) if for some $C, D > 0$ and all $g \in \mathcal{A}$ with μ-median equal to* 0*,*

$$\int |g|\, d\mu \;\leq\; C \int \sqrt{\Gamma(g)}\, \omega\, d\mu\,, \tag{2.5}$$

respectively, for all $g \in \mathcal{A}$,

$$Var_\mu(g) \;\leq\; D \int \Gamma(g)\, \eta\, d\mu\,. \tag{2.6}$$

The standard method shows that if (2.5) holds, then (2.6) also holds with $D = 4C^2$ and $\eta = \omega^2$ (see Corollary 2.15).
In order to deal with the "local" part $b\mathbb{1}_K$ in the definition of a ϕ-Lyapunov function, we shall use the notion of local Poincaré inequality we introduce now.

Definition 2.7. *Let $U \subset E$. We shall say that μ satisfies a local Poincaré inequality on U if there exists some constant κ_U such that for all $f \in \mathcal{A}$*

$$\int_U f^2\, d\mu \;\leq\; \kappa_U \int_E \Gamma(f) d\mu \;+\; (1/\mu(U)) \left(\int_U f\, d\mu\right)^2\,.$$

Notice that in the right hand side the energy is taken over the whole space E (unlike the usual definition). Moreover, $\int_U f^2\, d\mu - (1/\mu(U)) \left(\int_U f\, d\mu\right)^2 = \mu(U) \mathrm{Var}_{\mu_U}(f)$ with $\frac{d\mu_U}{d\mu} := \frac{\mathbb{1}_U}{\mu(U)}$. This justifies the name "local Poincaré inequality".
Now we state our first general result.

Theorem 2.8 (Weighted Poincaré inequality)**.** *Assume that there exists some ϕ-Lyapunov function $W \in \mathcal{A}$ (see Definition 2.2) and that μ satisfies a local Poincaré inequality on some subset $U \supseteq K$. Then for all $g \in \mathcal{A}$, it holds*

$$Var_\mu(g) \;\leq\; \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int \left(1 + \frac{1}{\phi'(W)}\right) \Gamma(g)\, d\mu\,. \tag{2.9}$$

Proof. Let $g \in \mathcal{A}$, choose c such that $\int_U (g-c) d\mu = 0$ and set $f = g - c$. Since $\mathrm{Var}_\mu(g) = \inf_a \int (g - a)^2\, d\mu$, we have

$$\mathrm{Var}_\mu(g) \leq \int f^2 d\mu \leq \int \frac{-LW}{\phi(W)}\, f^2\, d\mu + \int f^2\, \frac{b}{\phi(W)}\, \mathbb{1}_K\, d\mu\,.$$

To manage the second term, we first use that $\Phi(W) \geq \Phi(1)$. Then, the definition of c and the local Poincaré inequality ensures that

$$\begin{aligned} \int_K f^2\, d\mu \;&\leq\; \int_U f^2\, d\mu \\ &\leq\; \kappa_U \int_E \Gamma(f) d\mu \;+\; (1/\mu(U)) \left(\int_U f\, d\mu\right)^2 \\ &=\; \kappa_U \int_E \Gamma(g) d\mu\,. \end{aligned}$$

For the first term, we use Lemma 2.10 below (with $\psi = \phi$ and $h = W$). This ends the proof. □

Lemma 2.10. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 increasing function. Then, for any $f, h \in \mathcal{A}$,*

$$\int \frac{-Lh}{\psi(h)} f^2 \, d\mu \leq \int \frac{\Gamma(f)}{\psi'(h)} d\mu$$

Proof. By (2.1), the fact that Γ is a derivation and the chain rule formula, we have

$$\int \frac{-Lh}{\psi(h)} f^2 \, d\mu \;=\; \int \Gamma\left(h, \frac{f^2}{\psi(h)}\right) d\mu = \int \left(\frac{2\, f\, \Gamma(f,h)}{\psi(h)} - \frac{f^2 \psi'(h)\Gamma(h)}{\psi^2(h)}\right) d\mu\,.$$

Since ψ is increasing and according to Cauchy-Schwarz inequality we get

$$\begin{aligned}\frac{f\,\Gamma(f,h)}{\psi(h)} &\leq \frac{f\,\sqrt{\Gamma(f)\Gamma(h)}}{\psi(h)} = \frac{\sqrt{\Gamma(f)}}{\sqrt{\psi'(h)}} \cdot \frac{f\,\sqrt{\psi'(h)\Gamma(h)}}{\psi(h)} \\ &\leq \frac{1}{2}\frac{\Gamma(f)}{\psi'(h)} + \frac{1}{2}\,\frac{f^2\psi'(h)\,\Gamma(h)}{\psi^2(h)}.\end{aligned}$$

The result follows. □

Remark 2.11. To be rigorous one has to check some integrability conditions in the previous proof. If W belongs to the domain of L, the previous derivation is completely rigorous since we are first dealing with bounded functions g. If we do not have a priori controls on the integrability of LW (and $\Gamma(f, W)$) one has to be more careful.

In the $\mathbb{R}^n$ case there is no real difficulty provided K is compact and U is for instance a ball $B(0, R)$. To overcome all difficulties in this case, we may proceed as follows : we first assume that g is compactly supported and $f = (g - c)\chi$, where χ is a non-negative compactly supported smooth function, such that $\mathbb{1}_U \leq \chi \leq 1$. All the calculation above are thus allowed. In the end we choose some sequence χ_k satisfying $\mathbb{1}_{kU} \leq \chi_k \leq 1$, and such that $|\nabla\chi_k| \leq 1$, and we go to the limit. ◊

Remark 2.12. Very recently, two of the authors and various coauthors have pushed forward the links between Lyapunov functionals (and local inequalities) and usual functional inequalities. for example if ϕ (in the Lyapunov condition) is assumed to be linear, then we recover the results in [3], namely a Poincaré inequality (and a short proof of Bobkov's result on logconcave probability measure satisfying spectral gap inequality). If ϕ is superlinear, then the authors of [29] have obtained super-Poincaré inequalities, including nice alternative proofs of Bakry-Emery or Kusuocka-Stroock criterion for logarithmic Sobolev inequality. ◊

The same ideas can be used to derive $\mathbb{L}_1$ weighted Poincaré (or weighted Cheeger) inequalities.

Consider f an arbitrary smooth function with median w.r.t. μ equal to 0. Assume that W is a ϕ-Lyapunov function. Then if $f = g - c$,

$$\begin{aligned}\int |f| d\mu &\leq \int |f| \frac{-LW}{\phi(W)} d\mu + b \int_K \frac{|f|}{\phi(W)}\, d\mu \\ &\leq \int \Gamma\left(\frac{|f|}{\phi(W)}, W\right) + \frac{b}{\phi(1)} \int_K |f|\, d\mu \\ &\leq \int \frac{\Gamma(|f|, W)}{\phi(W)} d\mu - \int \frac{|f|\Gamma(W)\phi'(W)}{\phi^2(W)} d\mu + \frac{b}{\phi(1)} \int_K |f| d\mu\,.\end{aligned}$$

Now we use Cauchy-Schwarz for the first term (i.e. $\Gamma(u,v) \leq \sqrt{\Gamma(u)}\,\sqrt{\Gamma(v)}$) in the right hand side, we remark that the second term is negative since ϕ' is positive, and we can control the last one as before if we assume a local Cheeger inequality, instead of a local Poincaré inequality. We have thus obtained

Theorem 2.13. *Assume that there exists a ϕ-Lyapunov function W and μ satisfies some local Cheeger inequality*

$$\int_U |f|\, d\mu \;\leq\; \kappa_U \int_E \sqrt{\Gamma(f)} d\mu\,,$$

for some $U \supseteq K$ and all f with median w.r.t. $\mathbb{1}_U\, \mu/\mu(U)$ equal to 0. Then for all $g \in \mathcal{A}$ with median w.r.t. μ equal to 0, it holds

$$\int |g|\, d\mu \;\leq\; \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int \left(1 + \frac{\sqrt{\Gamma(W)}}{\phi(W)}\right) \sqrt{\Gamma(g)}\, d\mu\,. \tag{2.14}$$

Again one has to be a little more careful in the previous proof, with integrability conditions, but difficulties can be overcome as before.

It is well known that Cheeger inequality implies Poincaré inequality. This is also true for weighted inequalities:

Corollary 2.15. *Under the assumptions of Theorem 2.13, for all $g \in \mathcal{A}$, it holds*

$$Var_\mu(g) \;\leq\; 8\, \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right)^2 \int \left(1 + \frac{\Gamma(W)}{\phi^2(W)}\right) \Gamma(g)\, d\mu\,.$$

Proof. As suggested in the proof of Theorem 5.1 in [22], if g has a μ median equal to 0, $g_+ = \max(g,0)$ and $g_- = \max(-g,0)$ too. We may thus apply Theorem 2.13 to both g_+^2 and g_-^2, yielding

$$\int g_+^2\, d\mu \;\leq\; 2\max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int g_+\, \sqrt{\Gamma(g_+)}\, \left(1 + \frac{\sqrt{\Gamma(W)}}{\phi(W)}\right) d\mu$$

and similarly for g_-. Applying Cauchy-Schwarz inequality, and using the elementary $(a+b)^2 \leq 2a^2 + 2b^2$ we get that

$$\int g_+^2\, d\mu \leq\; 8\, \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right)^2 \int \left(1 + \frac{\Gamma(W)}{\phi^2(W)}\right) \Gamma(g_+)\, d\mu$$

and similarly for g_-. To conclude the proof, it remains to sum-up the positive and the negative parts and to notice that $\mathrm{Var}_\mu(g) \leq \int g^2\, d\mu$. □

Note that the forms of weight obtained respectively in Theorem 2.8 and last corollary are different. But, up to constant, they are of the same order in all examples we shall treat in the following section.

2.2. **Examples in $\mathbb{R}^n$.** We consider here the $\mathbb{R}^n$ situation with $d\mu(x) = p(x)dx$ and $L = \Delta + \nabla \log p.\nabla$, p being smooth enough. We can thus use the argument explained in remark 2.11 so that as soon as W is C^2 one may apply Theorem 2.8 and Theorem 2.13.

Recall the following elementary lemma whose proof can be found in [3].

Lemma 2.16. *If V is convex and $\int e^{-V(x)}\, dx < +\infty$, then*

1. *for all x, $x.\nabla V(x) \geq V(x) - V(0)$,*
2. *there exist $\delta > 0$ and $R > 0$ such that for $|x| \geq R$, $V(x) - V(0) \geq \delta\, |x|$.*

We shall use this Lemma in the following examples. Our first example corresponds to the convex case discussed by Bobkov and Ledoux [22].

Proposition 2.17 (Cauchy type law)**.** *Let $d\mu(x) = (V(x))^{-(n+\alpha)}\,dx$ for some positive convex function V and $\alpha > 0$. Then there exists $C > 0$ such that for all g*

$$Var_\mu(g) \ \le\ C \int |\nabla g(x)|^2\,(1+|x|^2)\,d\mu(x),$$

$$\int |g-m|\,d\mu \ \le\ C \int |\nabla g(x)|\,(1+|x|)\,d\mu(x),$$

where m stands for a median of g under μ.

Remark 2.18. The restriction $\alpha > 0$ is the same as in [22].

Proof. By Lemma 2.19 below, there exists a ϕ-Lyapunov function W satisfying $(1/\phi'(W))(x) = \frac{k}{c(k-2)}|x|^2$ for x large. Hence, in order to apply Theorem 2.8 it remains to recall that since $d\mu/dx$ is bounded from below and from above on any ball $B(0,R)$, μ satisfies a Poincaré inequality and a Cheeger inequality on such subset, hence a local Poincaré (and Cheeger) inequality in the sense of definition 2.7 (or Theorem 2.13). This ends the proof. □

Lemma 2.19. *Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ with V and α as in Proposition 2.17. Then, there exists $k > 2$, $b, R > 0$ and $W \ge 1$ such that*

$$LW \le -\phi(W) + b\mathbb{1}_{B(0,R)}$$

with $\phi(u) = cu^{(k-2)/k}$ for some constant $c > 0$. Furthermore, one can choose $W(x) = |x|^k$ for x large.

Proof. Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ and choose $W \ge 1$ smooth and satisfying $W(x) = |x|^k$ for $|x|$ large enough and $k > 2$ that will be chosen later. For $|x|$ large enough we have

$$LW(x) = k\,(W(x))^{\frac{k-2}{k}}\left(n+k-2-\frac{(n+\alpha)\,x.\nabla V(x)}{V(x)}\right).$$

Using (1) in Lemma 2.16 (since $V^{-(n+\alpha)}$ is integrable e^{-V} is also integrable) we have

$$n+k-2-\frac{(n+\alpha)\,x.\nabla V(x)}{V(x)} \le k-2-\alpha+(n+\alpha)\frac{V(0)}{V(x)}.$$

Using (2) in Lemma 2.16 we see that we can choose $|x|$ large enough for $\frac{V(0)}{V(x)}$ to be less than ε, say $|x| > R_\varepsilon$. It remains to choose $k > 2$ and $\varepsilon > 0$ such that

$$k + n\varepsilon - 2 - \alpha(1-\varepsilon) \le -\gamma$$

for some $\gamma > 0$. We have shown that, for $|x| > R_\varepsilon$,

$$LW \le -k\gamma\phi(W),$$

with $\phi(u) = u^{\frac{k-2}{k}}$ (which is increasing since $k > 2$). A compacity argument achieves the proof. □

Remark 2.20. The previous proof gives a non explicit constant C in terms of α and n. This is mainly due to the fact that we are not able to control properly the local Poincaré and Cheeger inequalities on balls for the general measures $d\mu = (V(x))^{-(n+\alpha)}\,dx$. More could be done on specific laws.

Our next example deals with sub-exponential distributions.

Proposition 2.21 (Sub exponential like law)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}$ for some positive convex function V and $p > 0$. Then there exists $C > 0$ such that for all g*

$$Var_\mu(g) \;\leq\; C \int |\nabla g(x)|^2 \left(1 + (1 + |x|)^{2(1-p)}\right) d\mu(x)\,,$$
$$\int |g - m|\, d\mu \;\leq\; C \int |\nabla g(x)| \left(1 + (1 + |x|)^{(1-p)}\right) d\mu(x)\,,$$

where m stands for a median of g under μ.

Remark 2.22. For $p < 1$ we get some weighted inequalities, while for $p \geq 1$ we see that (changing C into $2C$) we obtain the usual Poincaré and Cheeger inequalities. For $p = 1$, one recovers the well known fact (see [39, 16]) that Log-concave distributions enjoy Poincaré and Cheeger inequalities. Moreover, if we consider the particular case $d\mu(x) = (1/Z_p)\, e^{-|x|^p}$ with $0 < p < 1$, and choose $g(x) = e^{|x|^p/2}\, \mathbb{1}_{[0,R]}(x)$ for $x \geq 0$ and $g(-x) = -g(x)$, we see that the weight is optimal in Proposition 2.21.

Proof. The proof follows the same line as the proof of Proposition 2.17, using Lemma 2.23 below. $\square$

Lemma 2.23. *Let $L = \Delta - pV^{p-1}\nabla V \nabla$ for some positive convex function V and $p > 0$. Then, there exists $b, c, R > 0$ and $W \geq 1$ such that*

$$LW \leq -\phi(W) + b\mathbb{1}_{B(0,R)}$$

with $\phi(u) = u\, \log^{2(p-1)/p}(c + u)$ increasing. Furthermore, one can choose $W(x) = e^{\gamma |x|^p}$ for x large.

Proof. We omit the details since we can mimic the proof of Lemma 2.19. $\square$

Remark 2.24. Changing the values of b and R, only the values of $\Phi(u)$ in the large are relevant. In other words, one could take Φ to be an everywhere increasing function which coincides with $u\, \log^{2(p-1)/p}(u)$ for the large u's, choosing the constants b and R large enough.

2.3. **Example on the real line.** In this section we give examples on the real line where other techniques can also be done.

Note that in both previous examples we used a Lyapunov function $W = p^{-\gamma}$ for some well chosen $\gamma > 0$. In the next result we give a general statement using such a Lyapunov function in dimension 1.

Proposition 2.25. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$ for a smooth potential V. We assume for simplicity that V is symmetric. Furthermore, we assume that V is concave on $(R, +\infty)$ for some $R > 0$ and that $\left(V''/|V'|^2\right)(x) \;\to\; r > -1/2$ as $x \to \infty$. Then for some $S > R$ and some $C > 0$, it holds*

$$Var_\mu(g) \;\leq\; C \int |g'(x)|^2 \left(1 + \frac{\mathbb{1}_{|x|>S}}{|V'|^2(x)}\right) d\mu(x)\,,$$
$$\int |g - m|\, d\mu \;\leq\; C \int |g'(x)| \left(1 + \frac{\mathbb{1}_{|x|>S}}{|V'|(x)}\right) d\mu(x)$$

where m is a median of g under μ.

Proof. Since V' is non-increasing on $(R, +\infty)$ it has a limit l at $+\infty$. If $l < 0$, V goes to $-\infty$ at $+\infty$ with a linear rate, contradicting $\int e^{-V}dx < +\infty$. Hence $l \geq 0$, V is increasing and goes to $+\infty$ at $+\infty$.

Now choose $W = e^{\gamma V}$ (for large $|x|$). We have

$$LW = \left(\gamma V'' - (\gamma-\gamma^2)|V'|^2\right) W$$

so that for $0<\gamma<1$ we have $LW \leq -(\gamma-\gamma^2)|V'|^2 W$ at infinity. We may thus choose $\phi(W) = (\gamma-\gamma^2)|V'|^2 W$. The corresponding ϕ can be built on $(W^{-1}(R), +\infty)$ where W is one to one. On the other hand,

$$\phi'(W)\, W' = (\gamma-\gamma^2) V' W \left(2V'' + \gamma|V'|^2\right),$$

so that, since $W'>0$, $V'>0$ and $V''/|V'|^2 > -1/2$ asymptotically, ϕ is non-decreasing at infinity for a well chosen γ. Then, it is possible to build ϕ on a compact interval $[0,a]$ in order to get a smooth increasing function on the whole $\mathbb{R}_+$.
Since $d\mu/dx$ is bounded from above and below on any compact interval, a local Poincaré inequality and a local Cheeger inequality hold on such interval. hence, it remains to apply Theorem 2.8 and Theorem 2.13, since at infinity $\phi'(W)$ behaves like $|V'|^2$. □

Remark 2.26. The example of Proposition 2.21 enters the framework of this proposition, and the general Cauchy distribution $V(x) = c\log(1+|x|^2)$ does if $c>1$, since $V''/|V'|^2$ behaves asymptotically as $-1/2c$. Note that the weight we obtain is of good order, applying the inequality with approximations of $e^{V/2}$.
It is possible to extend the previous proposition to the multi-dimensional setting, but the result is quite intricate. Assume that $V(x)\to+\infty$ as $|x|\to+\infty$, and that V is concave (at infinity). The same $W=e^{\gamma V}$ furnishes $LW/W = \gamma\Delta V - (\gamma-\gamma^2)|\nabla V|^2$. Hence we may define

$$\phi(u) = (\gamma-\gamma^2)\, u \inf_{A(u)} |\nabla V|^2 \quad \text{with} \quad A(u) = \{x; V(x) = \log(u)/\gamma\}$$

at least for large u's. The main difficulty is to check that ϕ is increasing. This could probably be done on specific examples.

It is known that Hardy-type inequalities are useful tool to deal with functional inequalities of Poincaré type in dimension 1 (see [13, 12] for recent contributions on the topic). We shall use now Hardy-type inequalities to relax the hypothesis on V and to obtain the weighted Poincaré inequality of Proposition 2.25. However no similar method (as far as we know) can be used for the weighted Cheeger inequality, making the ϕ-Lyapunov approach very efficient.

Proposition 2.27. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$ for a smooth potential V that we suppose for simplicity to be even. Let $\varepsilon\in(0,1)$. Assume that there exists $x_0\geq 0$ such that V is twice differentiable on $[x_0,\infty)$ and*

$$V'(x)\neq 0, \qquad \frac{|V''(x)|}{V'(x)^2} \leq 1-\varepsilon, \qquad \forall x\geq x_0.$$

Then, for some $C>0$, it holds

$$Var_\mu(g) \leq C \int |g'(x)|^2 \left(1 + \frac{\mathbb{1}_{|x|>x_0}}{|V'|^2(x)}\right)\mu(dx).$$

Proof. Given η and using a result of Muckenhoupt [47], one has for any G

$$\int_0^{+\infty} (G(x)-G(0))^2\,\mu(dx) \leq 4B \int_0^{+\infty} G'(x)^2\,(1+\eta^2(x))\,\mu(dx),$$

with $B = \sup_{y>0} \left(\int_y^{+\infty} e^{-V(x)} dx\right) \left(\int_0^y \frac{e^{V(x)}}{1+\eta^2(x)}\, dx\right)$. Hence, since V is even and $\mathrm{Var}_\mu(g) \le \int_{-\infty}^0 (G(x) - G(0))^2 d\mu + \int_0^{+\infty} (G(x) - G(0))^2 d\mu$, the previous bound applied twice leads to

$$\mathrm{Var}_\mu(g) \le 4B \int |g'(x)|^2 \left(1 + \eta(x)^2\right) d\mu.$$

In particular, one has to prove that

$$B = \sup_{y>0} \left(\int_y^{+\infty} e^{-V(x)} dx\right) \left(\int_0^y \frac{e^{V(x)}}{1 + \frac{\mathbb{1}_{|x|>x_0}}{|V'|^2(x)}}\, dx\right) < \infty$$

Consider $y \ge x_0$. Then, (note that $V' > 0$ since it cannot change sign and e^{-V} is integrable),

$$\begin{aligned}
\int_{x_0}^y \frac{e^{V(x)}}{1 + \frac{\mathbb{1}_{|x|>x_0}}{|V'|^2(x)}}\, dx &= \int_{x_0}^y \frac{V'(x) e^{V(x)}}{V'(x) + \frac{1}{V'(x)}}\, dx = \left[\frac{e^V}{V' + \frac{1}{V'}}\right]_{x_0}^y + \int_{x_0}^y e^V \frac{V''((V')^2 - 1)}{((V')^2+1)^2} \\
&\le \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}} + (1-\varepsilon) \int_{x_0}^y e^V \frac{(V')^2 |(V')^2 - 1|}{((V')^2+1)^2} \\
&\le \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}} + (1-\varepsilon) \int_{x_0}^y \frac{e^V}{1 + \frac{1}{(V')^2}}
\end{aligned}$$

where in the last line we used that $x^2|x^2 - 1|/(x^2+1)^2 \le 1/(1 + \frac{1}{x^2})$ for $x = V' > 0$. This leads to

$$\int_{x_0}^y \frac{e^{V(x)}}{1 + \frac{\mathbb{1}_{|x|>x_0}}{|V'|^2(x)}}\, dx \le \frac{1}{\varepsilon} \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}}.$$

Similar calculations give (we omit the proof)

$$\int_y^{+\infty} e^{-V(x)} dx \le \frac{1}{\varepsilon} \frac{e^{-V(y)}}{V'(y)} \qquad \forall y \ge x_0.$$

Combining these bounds and using a compactness argument on $[0, x_0]$, it is not hard to show that B is finite. □

We end this section with distributions in dimension 1 that do not enter the framework of the two previous propositions. Moreover, the laws we have considered so far are κ concave for $\kappa > -\infty$. The last examples shall satisfy $\kappa = -\infty$.

Example 2.28. Let $q > 1$ and define

$$d\mu(x) = (1/Z_q)\ ((2 + |x|) \log^q(2 + |x|))^{-1}\ dx \ = \ V_q^{-1}(x) dx \qquad x \in \mathbb{R}.$$

The function V_q is convex but V_q^γ is no more convex for $\gamma < 1$ (hence $\kappa = -\infty$). We may choose $W(x) = (2 + |x|)^2 \log^a(2 + |x|)$ (at least far from 0), which is a ϕ-Lyapunov function for $\phi(u) = \log^{a-1}(2 + |u|)$ provided $q > a > 1$ (details are left to the reader). We thus get a weighted inequality

$$\mathrm{Var}_\mu(g) \ \le \ C \int |\nabla g(x)|^2 \left(1 + x^2 \log^2(2 + |x|)\right) d\mu(x)\,. \tag{2.29}$$

Unfortunately we do not know whether the weight is correct in this situation. The usual choice g behaving like $\sqrt{(2 + |x|) \log^q(2 + |x|)}$ on $(-R, R)$ furnishes a variance behaving like R but the right hand side behaves like $R \log^2 R$.

We may even find a Lyapunov functional in the case $V(x) = x \log x \log^q(\log x)$ for large x and $q > 1$, *i.e* choose $W(x) = 1 + |x|^2 \log(2 + |x|) \log^c \log(2e + |x|)$ with $1 < c < q$ for which $\phi(x)$ is merely $\log^{c-1} \log(2e+|x|)$ so that the weight in the Poincaré inequality is $1+|x|^2 \log^2(2+|x|) \log^2 \log(2e+|x|)$. ◊

2.4. **Converse inequalities.** This section is dedicated to the study of converse inequalities from ϕ-Lyapunov function. We start with converse Poincaré inequalities and then we study converse Cheeger inequalities.

Definition 2.30. *We say that μ satisfies a converse weighted Cheeger (resp. Poincaré) inequality with weight ω if for some $C > 0$ and all $g \in \mathcal{A}$*

$$\inf_c \int |g-c|\, \omega \, d\mu \ \le\ C \int \sqrt{\Gamma(g)}\, d\mu \,, \tag{2.31}$$

respectively, for all $g \in \mathcal{A}$,

$$\inf_c \int |g-c|^2\, \omega \, d\mu \ \le\ C \int \Gamma(g)\, d\mu \,. \tag{2.32}$$

2.4.1. *Converse Poincaré inequalities.* In [22, Proposition 3.3], the authors perform a change of function in the weighted Poincaré inequality to get

$$\inf_c \int (f-c)^2\, \omega \, d\mu \le \int |\nabla f|^2 d\mu.$$

This method requires that the constant D in the weighted Poincaré inequality (2.6) (with weight $\eta(x) = (1+|x|)^2$) is not too big. The same can be done in the general situation, provided the derivative of the weight is bounded and the constant is not too big.

But instead we can also use a direct approach from ϕ-Lyapunov functions.

Theorem 2.33 (Converse Poincaré inequality)**.** *Under the assumptions of Theorem 2.8, for any $g \in \mathcal{A}$, it holds*

$$\inf_c \int (g-c)^2\, \frac{\phi(W)}{W}\, d\mu \le (1 + b\kappa_U) \int \Gamma(g)\, d\mu \,. \tag{2.34}$$

Proof. Rewrite the drift condition as

$$w := \frac{\phi(W)}{W} \le -\frac{LW}{W} + b\, \mathbb{1}_K \,,$$

recalling that $W \ge 1$. Set $f = g - c$ with $\int_U (g-c) d\mu = 0$. Then,

$$\inf_c \int (g-c)^2\, \frac{\phi(W)}{W}\, d\mu \le \int f^2\, w\, d\mu \le \int -\frac{LW}{W}\, f^2\, d\mu \ +\ b \int_K f^2\, d\mu \,.$$

The second term in the right hand side of the latter can be handle using the local Poincaré inequality, as in the proof of Theorem 2.8 (we omit the details). We get $\int_K f^2\, d\mu \le \kappa_U \int \Gamma(g) d\mu$. For the first term we use Lemma 2.10 with $\psi(x) = x$. This achieves the proof. □

Remark 2.35. In the proof the previous theorem, we used the inequality

$$\int \frac{-LW}{W}\, f^2\, d\mu \ \le\ \int \Gamma(f)\, d\mu.$$

By [29, Lemma 2.12], it turns out that the latter can be obtained without assuming that Γ is a derivation. In particular the previous Theorem extends to any situation where L is the generator of a μ-symmetric Markov process (including jump processes) in the form

$$\inf_c \int (g-c)^2 \, \frac{\phi(W)}{W} \, d\mu \leq (1+b\kappa_U) \int -g\, Lg\, d\mu \, .$$

$\diamond$

Now we give two examples to illustrate our result.

Proposition 2.36 (Cauchy type law)**.** *Let $d\mu(x) = (V(x))^{-(n+\alpha)}\, dx$ for some positive convex function V and $\alpha > 0$. Then there exists $C > 0$ such that for all g*

$$\inf_c \int (g(x)-c)^2 \, \frac{1}{1+|x|^2} \, d\mu(x) \ \leq \ C \int |\nabla g|^2 \, d\mu \, .$$

Proof. It is a direct consequence of Theorem 2.33 and Lemma 2.19. □

Proposition 2.37 (Sub exponential like law)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}$ for some positive convex function V and $p \in (0,1)$. Then there exists $C > 0$ such that for all g*

$$\inf_c \int (g(x)-c)^2 \, \frac{1}{1+|x|^{2(1-p)}} \, d\mu(x) \ \leq \ C \int |\nabla g|^2 \, d\mu \, .$$

Proof. Again it is a direct consequence of Theorem 2.33 and Lemma 2.23. □

2.4.2. *Converse Cheeger inequalities.* Here we study the harder converse Cheeger inequalities. The approach by ϕ-Lyapunov functions works but some additional assumptions have to be done.

Theorem 2.38 (Converse Cheeger inequality)**.** *Under the hypotheses of Theorem 2.13, assume that K is compact and that either*

(1) $|\Gamma(W, \Gamma(W))| \leq 2\delta\phi(W)\,(1+\Gamma(W))$ *outside K, for some $\delta \in (0,1)$*

or

(2) $\Gamma(W, \Gamma(W)) \geq 0$ *outside K.*

Then, there exists a constant $C > 0$ such that for any $g \in \mathcal{A}$, it holds

$$\inf_c \int |g-c| \, \frac{\phi(W)}{\sqrt{1+\Gamma(W)}} \, d\mu \leq C \int \sqrt{\Gamma(g)} d\mu \, .$$

Remark 2.39. Note that using Cauchy-Schwarz inequality, Assumption (1) is implied by $\Gamma(\Gamma(W)) \leq 4\delta^2\phi(W)^2(1+\Gamma(W))$ outside K.
On the other hand, in dimension 1 for usual diffusions, we have $\Gamma(W, \Gamma(W)) = 2\,|W'|^2\, W''$. Hence this term is non negative as soon as W is convex outside K.

Proof. Let $g \in \mathcal{A}$ and set $f = g - c$ with c satisfying $\int_U (g-c) d\mu = 0$. Recall that $LW \leq -\phi(W) + b\mathbb{1}_K$. Hence,

$$\frac{\phi(W)}{\sqrt{1+\Gamma(W)}} \leq -\frac{LW}{\sqrt{1+\Gamma(W)}} + \frac{b\mathbb{1}_K}{\sqrt{1+\Gamma(W)}} \leq -\frac{LW}{\sqrt{1+\Gamma(W)}} + b\mathbb{1}_K .$$

In turn,

$$\int |f| \, \frac{\phi(W)}{\sqrt{1+\Gamma(W)}} \, d\mu \quad \leq \quad - \int \frac{|f|}{\sqrt{1+\Gamma(W)}} \, LW \, d\mu \ + \ b \int_K |f| \, d\mu \, .$$

To control the first term we use (2.1), the fact that Γ is a derivation and Cauchy-Schwarz inequality to get that

$$\begin{aligned} -\int \frac{|f|}{\sqrt{1+\Gamma(W)}} LW\,d\mu &= \int \Gamma\left(\frac{|f|}{\sqrt{1+\Gamma(W)}}, W\right) d\mu \\ &= \int \frac{\Gamma(|f|, W)}{\sqrt{1+\Gamma(W)}}\,d\mu + \int |f|\Gamma\left(\frac{1}{\sqrt{1+\Gamma(W)}}, W\right) d\mu \\ &\leq \int \sqrt{\Gamma(f)}\,d\mu - \int |f|\,\frac{\Gamma(W,\Gamma(W))}{2(1+\Gamma(W))^{\frac{3}{2}}}\,d\mu\,. \end{aligned}$$

Now, we divide the second term of the latter in sum of the integral over K and the integral outside K. Set $M := \sup_K \frac{|\Gamma(W,\Gamma(W))|}{2(1+\Gamma(W))^{\frac{3}{2}}}$. Under Assumption (2), the integral outside K is non-positive, thus we end up with

$$\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \leq \int \sqrt{\Gamma(f)}\,d\mu + (M+b)\int_K |f|\,d\mu$$

while under Assumption (1), we get

$$\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \leq \int \sqrt{\Gamma(f)}\,d\mu + (M+b)\int_K |f|\,d\mu + \delta\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu$$

In any case the term $\int_K |f|\,d\mu$ can be handle using the local Cheeger inequality (we omit the details): we get $\int_K |f|d\mu \leq \kappa_U \int \sqrt{\Gamma(g)}d\mu$. This ends the proof, since $\Gamma(f) = \Gamma(g)$. ◻

We apply our result to Cauchy type laws.

Proposition 2.40 (Cauchy type laws)**.** *Let $d\mu(x) = (V(x))^{-(n+\alpha)}\,dx$ for some positive convex function V and some $\alpha > 0$. Then, there exists $C > 0$ such that for any g, it holds*

$$\inf_c \int |g(x) - c|\,\frac{1}{1+|x|}\,\mu(dx) \leq C\int |\nabla g| d\mu\,.$$

Proof. By Lemma 2.19, we know that $W(x) = |x|^k$ (for x large) is a ϕ-Lyapunov function for $\phi(u) = c|u|^{(k-2)/k}$. Note that $\Gamma(W,\Gamma(W))(x) = (2k-2)k^2|x|^{3k-4}$ at infinity. Hence Assumption (2) of the previous theorem holds and the theorem applies. This leads to the expected result. ◻

The same argument works for sub exponential distributions (we omit the proof).

Proposition 2.41 (Sub exponential type laws)**.** *Let $d\mu = (1/Z_p)\,e^{-V^p}$ for some positive convex function V and $p \in (0,1)$. Then there exists $C > 0$ such that for all g*

$$\inf_c \int |g(x) - c|\,\frac{1}{1+|x|^{1-p}}\,d\mu(x) \leq C\int |\nabla g| d\mu\,.$$

We end this section with an example in dimension 1. Consider $d\mu = e^{-V}$ on $\mathbb{R}$, and $W = e^{\gamma V}$ for some $\gamma < 1$. The function W is convex in the large as soon as $\limsup(|V''|/|V'|^2) < \gamma$ at infinity. Hence we can use remark 2.39 and the previous theorem to get that, under the hypothesis of Proposition 2.25, for some $S > 0$ and $C > 0$

$$\inf_c \int |g-c|\,(\mathbb{1}_{(-S,S)} + |V'|)\,d\mu \leq C\int |g'|\,d\mu\,.$$

(we used also that W is a Lyapunov function with ϕ satisfying $\phi(W) = (\gamma-\gamma^2)|V'|^2 W$ (which leads to $\phi(W)/\sqrt{1+\Gamma(W)}$ of the order of $|V'|$ in the large), see the proof of Proposition 2.25 for more details).

2.5. **Additional comments.** *Stability.* As it is easily seen, the weighted Cheeger and Poincaré inequalities (and their converse) are stable under log-bounded transformations of the measure. The Lyapunov approach encompasses a similar property with compactly supported (regular) perturbations. In fact the Lyapunov aproach is even more robust, let us illustrate it in the following example: suppose that the measure $\mu = e^{-V}dx$ satisfies a ϕ-Lyapunov condition with test function W and suppose that for large x, $\nabla V.\nabla W \geq \nabla V \nabla U$ for some regular (but possibly unbounded) U, then there exists $\beta > 0$ such that $d\nu = e^{-V+\beta U}dx$ satisfies a ϕ-Lyapunov condition with the same test function W and then the same weighted Poincaré or Cheeger inequality.

Manifold case. In fact, many of the results presented here can be extended to the manifold case, as soon as we can suppose that $V(x) \to \infty$ as soon as the geodesic distance (to some fixed points) grows to infinity and of course that a local Poincaré inequality or a local Cheeger inequality is valid. We refer to [29] for a more detailed discussion.

3. W .

In this section we recall first a result of Bobkov that shows the equivalence between the isoperimetric inequality and what we have called a weak Cheeger inequality (see 1.5).

Lemma 3.1 (Bobkov [18])**.** *Let μ be a probability measure on $\mathbb{R}^n$. There is an equivalence between the following two statements (where I is symmetric around $1/2$)*

(1) *for all $s > 0$ and all smooth f with μ median equal to 0,*

$$\int |f| \, d\mu \;\leq\; \beta(s) \int |\nabla f| \, d\mu \;+\; s \, \mathrm{Osc}_\mu(f) \, ,$$

(2) *for all Borel set A with $0 < \mu(A) < 1$,*

$$\mu_s(\partial A) \;\geq\; I(\mu(A)),$$

where β and I are related by the duality relation

$$\beta(s) \;=\; \sup_{s \leq t \leq \frac{1}{2}} \frac{t-s}{I(t)} \, , \qquad I(t) \;=\; \sup_{0 < s \leq t} \frac{t-s}{\beta(s)} \;\; \textit{for } t \leq \frac{1}{2} \, .$$

Here as usual $\mathrm{Osc}_\mu(f) = \mathrm{ess\,sup} f - \mathrm{ess\,inf} f$ *and* $\mu_s(\partial A) = \liminf_{h \to 0} \frac{\mu(0<d(x,A)<h)}{h}$.

Recall that in the weak Cheeger inequality, only the values $s \in (0, 1/2)$ are relevant since $\int |f| d\mu \leq \frac{1}{2}\mathrm{Osc}_\mu(f)$. Moreover this Lemma and its proof extend to the general case we are dealing with as soon as the general coarea formula is satisfied and provided one can approximate indicators by $\sqrt{\Gamma}$ of Lipschitz functions.

Thanks to the previous lemma, we see that isoperimetric results can be derived from weak Cheeger inequalities. We now give two different way to prove such inequalities. The first one is based on the ϕ-Lyapunov approach using the converse Cheeger inequalities proved in the previous section. The second one uses instead a transportation of mass technique.

3.1. **From converse Cheeger to weak Cheeger inequalities.** Here we shall first relate converse inequalities to weak inequalities, and then deduce some isoperimetric results on concrete examples.

Theorem 3.2. *Let μ be a probability measure and ω be a non-negative function satisfying $\bar{\omega} = \int \omega\, d\mu < +\infty$. Assume that there exists $C > 0$ such that*

$$\inf_c \int |g-c|\,\omega\, d\mu \le C \int \sqrt{\Gamma(g)}\, d\mu \qquad \forall g \in \mathcal{A}.$$

Define $F(u) = \mu(\omega < u)$ and $G(s) = F^{-1}(s) := \inf\{u; \mu(\omega \le u) > s\}$. Then, for all $s > 0$ and all $g \in \mathcal{A}$, it holds

$$\inf_c \int |g-c|\, d\mu \ \le\ \frac{C}{G(s)} \int \sqrt{\Gamma(g)}\, d\mu \ +\ s\,\mathrm{Osc}_\mu(f)\,.$$

Proof. Let $g \in \mathcal{A}$. Define $m_\omega \in \mathbb{R}$ to be a median of g under $\omega d\mu/\bar{\omega}$. We have

$$\begin{aligned} \inf_c \int |g-c|\, d\mu &\le \int |g-m_\omega| d\mu \\ &\le \int_{\omega \ge u} |g-m_\omega|\, \frac{\omega}{u}\, d\mu \ +\ \int_{\omega<u} |g-m_\omega|\, d\mu \\ &\le \frac{1}{u} \int |g-m_\omega|\,\omega\, d\mu \ +\ \mathrm{Osc}_\mu(g)\, F(u) \\ &= \frac{1}{u} \inf_c \int |g-c|\,\omega\, d\mu \ +\ \mathrm{Osc}_\mu(g)\, F(u)\,. \end{aligned}$$

It remains to apply the converse weighted Cheeger inequality and the definition of G. Note that if $F(u) = 0$ for $u \le u_0$ then $G(s) \ge u_0$. □

We illustrate this result on two examples.

Proposition 3.3 (Cauchy type laws)**.** *Let $d\mu(x) = V^{-(n+\alpha)}(x)dx$ with V convex and $\alpha > 0$. Recall that $\kappa = -1/\alpha$. Then, there exists a constant $C > 0$ such that for any f with μ-median 0,*

$$\int |f|\, d\mu \ \le\ C s^\kappa \int |\nabla f|\, d\mu \ +\ s\mathrm{Osc}_\mu(f) \qquad \forall s > 0.$$

Equivalently there exists $C' > 0$ such that for any $A \subset \mathbb{R}^n$,

$$\mu_s(\partial A) \ge C' \min\left(\mu(A), 1-\mu(A)\right)^{1-\kappa}.$$

Proof. By Proposition 2.40, μ satisfies a converse weighted Cheeger inequality with weight $\omega(x) = \frac{1}{1+|x|}$. So $F(u) = \mu(\omega < u) = \mu(u^{-1} - 1 < |x|)$. Since V is convex, $V(x) \ge \rho|x|$ for large $|x|$ (recall Lemma 2.16), hence using polar coordinates we have

$$\mu(|x| > R) = \int_{|x|>R} V^{-\beta}(x)\, dx \ \le\ \int_{|x|>R} \rho^{-\beta}|x|^{-\beta} dx \le\ cR^{n-\beta}\,,$$

for some $c = c(n, \alpha, \rho)$. The result follows by Theorem 3.2. The isoperimetric inequality follows at once by Lemma 3.1. □

Remark 3.4. The previous result recover Corollary 8.4 in [18] (up to the constants). Of course we do not attain the beautiful Theorem 1.2 in [18], where S. Bobkov shows that the constant C' only depends on κ and the median of $|x|$.

Proposition 3.5 (Sub exponential type laws)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}$ for some positive convex function V and $p \in (0,1)$. Then there exists $C > 1$ such that for all f with μ-median 0,*

$$\int |f|\, d\mu \;\leq\; C \log^{\frac{1}{p}-1}(C/s) \int |\nabla f|\, d\mu \;+\; s\mathrm{Osc}_\mu(f) \qquad \forall s \in (0,1).$$

Equivalently there exists $C' > 0$ such that for any $A \subset \mathbb{R}^n$,

$$\mu_s(\partial A) \geq C' \min(\mu(A), 1-\mu(A)) \log\left(\frac{1}{\min(\mu(A), 1-\mu(A))}\right)^{1-\frac{1}{p}}.$$

Proof. According to Proposition 2.41, μ verifies the converse Cheeger inequality with the weight function ω defined by $\omega(x) = 1/(1+|x|^{1-p})$ for all $x \in \mathbb{R}^n$. Moreover, since V is convex, it follows from Lemma 2.16 that there is some $\rho > 0$ such that $\int e^{\rho|x|^p}\, d\mu(x) < \infty$. Hence, applying Markov's inequality gives $\mu(|x| > R) \leq K e^{-\rho R^p}$, for some $K \geq 1$. Elementary calculations gives the result. □

3.2. **Weak Cheeger inequality via mass transport.** The aim of this section is to study how the isoperimetric inequality, or equivalently the weak Cheeger inequality, behave under tensor products. More precisely, we shall start with a probability measure μ on the real line $\mathbb{R}$ and derive weak Cheeger inequalities for μ^n with explicit constants.

We need some notations. For any probability measure μ (on $\mathbb{R}$) we denote by F_μ the cumulative distribution function of μ which is defined by

$$F_\mu(x) = \mu(-\infty, x], \quad \forall x \in \mathbb{R}.$$

It will be also convenient to consider the tail distribution function $\overline{F}_\mu$ defined by

$$\overline{F}_\mu(x) = 1 - F_\mu(x) = \mu(x, +\infty), \quad \forall x \in \mathbb{R}.$$

The isoperimetric function of μ is defined by

$$J_\mu = F'_\mu \circ F_\mu^{-1}. \tag{3.6}$$

In all the sequel, the two sided exponential measure $d\nu(x) = \frac{1}{2}e^{-|x|}\, dx$, $x \in \mathbb{R}$ will play the role of a reference probability measure. We will set $F_\nu = F$ and $J_\nu = J$ for simplicity. Note that the isoperimetric function J can be explicitly computed: $J(t) = \min(t, 1-t)$, $t \in [0,1]$.

3.2.1. *A general result.* We are going to derive a weak Cheeger inequality starting from a well known Cheeger inequality for ν^n obtained in [19] and using a transportation idea developed in [34]. Our result will be available for a special class of probability measures on $\mathbb{R}$ which is described in the following lemma.

Lemma 3.7. *Let μ be a symmetric probability measure on $\mathbb{R}$; the following propositions are equivalent*

1. *The function $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$,*
2. *The function J/J_μ is non increasing on $(0,1/2]$ and non decreasing on $[1/2,1)$.*

Furthermore, if $d\mu(x) = e^{-\Phi(|x|)}\, dx$ with $\Phi : \mathbb{R}^+ \to \mathbb{R}$ concave, then $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$.

Proof. The equivalence between (1) and (2) is easy to check. Now suppose that μ is of the form $d\mu(x) = e^{-\Phi(|x|)}\,dx$ with a concave Φ. Then for $r \in \mathbb{R}^+$,

$$(\log \overline{F}_\mu)''(r) = \frac{e^{-\Phi(r)}}{\left(\int_r^\infty e^{-\Phi(s)}\,ds\right)^2}\left(\Phi'(r)\int_r^\infty e^{-\Phi(s)}\,ds - e^{-\Phi(r)}\right)$$

where Φ' is the right derivative. Since Φ is concave, Φ' is non-increasing. It follows that

$$\Phi'(r)\int_r^\infty e^{-\Phi(s)}\,ds \geq \int_r^\infty \Phi'(s)e^{-\Phi(s)}\,ds = e^{-\Phi(r)}.$$

The result follows. □

Recall that distributions satisfying (1) in the previous lemma are known as “Decreasing Hazard Rate” distributions. We refer to [6] for some very interesting properties of these distributions (unfortunately less powerful than the Increasing Hazard Rate situation).

Using a mass transportation technique, we are now able to derive a weak Cheeger inequality for product measures on $\mathbb{R}^n$. Dimension dependence is explicit, as well as the constants.

Theorem 3.8. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$.*
Then, for any n, any bounded smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\int |f - m|\,d\mu^n \leq \kappa_1 \frac{s}{J_\mu(s)}\int |\nabla f|\,d\mu^n + \kappa_2 ns\mathrm{Osc}(f), \quad \forall s \in (0, 1/2), \tag{3.9}$$

where m is a median of f under μ^n, $\kappa_1 = 2\sqrt{6}$ and $\kappa_2 = 2(1 + 2\sqrt{6})$.

Remark 3.10. Note that $\int |f - m| d\mu^n \leq \mathrm{Osc}(f)$. Hence only the values $s \leq (\kappa_2 n)^{-1}$ are of interest in (3.9). ◊

Proof. Recall that ν is the two sided exponential distribution. Fix the dimension n and $r > 0$. By [19, Inequality (6.9)], any locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$ with $\int |h|\,d\nu^n < \infty$ satisfies

$$\int |h - m_{\nu^n}(h)|\,d\nu^n \leq \kappa_1 \int |\nabla h|\,d\nu^n \tag{3.11}$$

where $m_{\nu^n}(h)$ is a median of h for ν^n and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.
Consider the map $T^n : \mathbb{R}^n \to \mathbb{R}^n$, that pushes forward ν^n onto μ^n, defined by $(x_1, \ldots, x_n) \mapsto (T(x_1), \ldots, T(x_n))$ with $T = F_\mu^{-1} \circ F$. By construction, any $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $\int f(T^n)\,d\nu^n = \int f\,d\mu^n$.
Next, for $t \geq 0$ let $B(t) = \{x = (x_1, \ldots, x_n) : \max_i |x_i| \leq t\}$. Fix $a > 0$ that will be chosen later and consider $g : \mathbb{R} \to [0, 1]$ defined by $g(x) = \left(1 - \frac{1}{a}(x - r)_+\right)_+$ with $X_+ = \max(X, 0)$. Set $\varphi(x) = g(\max_i(|x_i|))$, $x \in \mathbb{R}^n$. The function φ is locally Lipschitz.
Finally let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and bounded. We assume first that 0 is a μ^n-median of f. Furthermore, by homogeneity of (3.9) we may assume that $\mathrm{Osc}(f) = 1$ in such a way that $\|f\|_\infty \leq 1$.

It follows from the definition of the median that

$$\begin{aligned}\int |f|\, d\mu^n &\le \int |f - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n \\ &\le \int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n + \int |f(1-\varphi)|\, d\mu^n \\ &\le \int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n + \mu^n\left(B(r)^c\right).\end{aligned}$$

Note that the assumption on $\log \overline{F}_\mu$ guarantees that $T' \circ T^{-1}$ is non-decreasing on $\mathbb{R}^+$. Hence, using (3.11), the triangle inequality in $\ell^2(\mathbb{R}^n)$, the fact that $0 \le \varphi \le 1$ on $\mathbb{R}^n$ and $\varphi = \partial_i\varphi = 0$ on $B(r+a)^c$ imply that

$$\begin{aligned}&\int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n = \int |(f\varphi)(T^n) - m_{\nu^n}((f\varphi)(T^n))|\, d\nu^n \\ &\le \kappa_1 \int \sqrt{\sum_{i=1}^n T'(x_i)^2\left((\varphi\partial_i f)(T^n) + (f\partial_i\varphi)(T^n)\right)^2}\, d\nu^n \\ &= \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(\varphi\partial_i f + f\partial_i\varphi\right)^2}\, d\mu^n \\ &\le \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(\varphi\partial_i f\right)^2}\, d\mu^n + \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(f\partial_i\varphi\right)^2}\, d\mu^n \\ &\le \kappa_1 T' \circ T^{-1}(r+a)\left(\int |\nabla f|\, d\mu^n + \int |\nabla\varphi|\, d\mu^n\right).\end{aligned}$$

Note that $|\nabla\varphi| \le 1/h$ on $B(r+a) \setminus B(r)$ and $|\nabla\varphi| = 0$ elsewhere μ^n-almost surely. Hence,

$$\int |f|\, d\mu^n \le \kappa_1 T' \circ T^{-1}(r+a)\left(\int |\nabla f| d\mu^n + \frac{1}{a}\mu^n\left(B(r+a)\setminus B(r)\right)\right) + \mu^n\left(B(r)^c\right). \tag{3.12}$$

Since μ is symmetric, we have

$$G(t) := \mu^n\left(B(t)\right) = \left(1 - 2\overline{F}_\mu(t)\right)^n.$$

Hence,

$$\begin{aligned}\lim_{a\to 0} \frac{1}{a}\mu^n\left(B(r+a)\setminus B(r)\right) = G'(r) &= 2nF'_\mu(r)\left(1 - 2\overline{F}_\mu(t)\right)^{n-1} \\ &\le 2nF'_\mu(r).\end{aligned}$$

On the other hand, since the function $x \mapsto 1-(1-2x)^n$ is concave on $[0,1/2]$, one has: $1-(1-2x)^n \le 2nx$ for all $x \in [0,1/2]$. As a consequence,

$$\mu^n(B(r)^c) = 1 - G(r) = 1 - (1 - 2\overline{F}_\mu(r))^n \le 2n\overline{F}_\mu(r),$$

for all $r \ge 0$.

Letting a go to 0 in (3.12) leads to

$$\int |f|\, d\mu^n \le \kappa_1 T' \circ T^{-1}(r) \int |\nabla f|\, d\mu^n + 2n\kappa_1 T' \circ T^{-1}(r) F'_\mu(r) + 2n\overline{F}_\mu(r).$$

Note that $T' \circ T^{-1} = J \circ F_\mu / F'_\mu = \min(F_\mu, 1 - F_\mu)/F'_\mu$. Hence, for $r \geq 0$,

$$T' \circ T^{-1}(r) F'_\mu(r) = \frac{1 - F_\mu(r)}{F'_\mu(r)} F'_\mu(r) = \overline{F}_\mu(r)\,.$$

It follows that

$$\int |f|\, d\mu^n \leq \kappa_1 \frac{\overline{F}_\mu(r)}{F'_\mu(r)} \int |\nabla f|\, d\mu^n + n\kappa_2 \overline{F}_\mu(r),$$

for all $r \geq 0$. Using the symmetry of μ it is easy to see that $F'_\mu \circ \overline{F}_\mu^{-1}(t) = J_\mu(t)$ for all $t \in (0, 1/2)$. Consequently, one has

$$\int |f|\, d\mu^n \leq \kappa_1 \frac{s}{J_\mu(s)} \int |\nabla f|\, d\mu^n + \kappa_2 n s,$$

for all $s \in (0, 1/2)$. For general $f : \mathbb{R}^n \to \mathbb{R}$ with μ^n-median m, we apply the result to $f - m$. This ends the proof. □

Combining this theorem with Bobkov's Lemma 3.1 we immediately deduce

Corollary 3.13. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$. Then, for any n, any Borel set $A \subset \mathbb{R}^n$ satisfies*

$$(\mu^n)_s(\partial A) \geq \frac{n\kappa_2}{\kappa_1} J_\mu \left(\frac{\min(\mu^n(A), 1 - \mu^n(A))}{2n\kappa_2} \right). \tag{3.14}$$

Proof. According to Lemma 3.1, if $\mu(A) \leq 1/2$ (the other case is symmetric), $(\mu^n)_s(\partial A) \geq I(\mu^n(A))$ with $I(t) = \sup_{0<s\leq t} \frac{t-s}{\beta(s)}$, for $t \leq 1/2$, where according to the previous theorem

$$\beta(s) = \frac{\kappa_1}{n\,\kappa_2} \, \frac{s}{J_\mu(s/n\kappa_2)}\,,$$

for $s \leq n\kappa_2/2$ hence for $s \leq 1/2$. This yields

$$I(t) = \sup_{0<s\leq t} \frac{t-s}{\kappa_1} \, \frac{J_\mu(s/n\kappa_2)}{(s/n\kappa_2)}\,.$$

In order to estimate I we use the following: first a lower bound is obtained for $s = t/2$ yielding the statement of the corollary. But next according to Lemma 3.7, the slope function $J_\mu(v)/v$ is non-decreasing, so that

$$I(t) \;\leq\; \sup_{0<s\leq t} \frac{t-s}{\kappa_1} \, \frac{J_\mu(t/n\kappa_2)}{(t/n\kappa_2)} \;\leq\; \frac{n\kappa_2}{\kappa_1} \, J_\mu(t/n\kappa_2)\,.$$

Remark that we have shown that for $t \leq 1/2$

$$\frac{n\kappa_2}{\kappa_1} \, J_\mu(t/2n\kappa_2) \;\leq\; I(t) \;\leq\; \frac{n\kappa_2}{\kappa_1} \, J_\mu(t/n\kappa_2)\,, \tag{3.15}$$

so that up to a factor 2 our estimate is of good order. □

3.2.2. *Application: Isoperimetric profile for product measures with heavy tails.* Here we apply the previous results to product of the measures

$$\mu(dx) = \mu_\Phi(dx) = Z_\Phi^{-1} \exp\{-\Phi(|x|)\} dx\,, \tag{3.16}$$

$x \in \mathbb{R}$, with Φ concave.

For even measures on $\mathbb{R}$ with positive density on a segment, Bobkov and Houdré [20, Corollary 13.10] proved that solutions to the isoperimetric problem can be found among half-lines, symmetric segments and their complements. More precisely, one has for $t \in (0,1)$

$$I_\mu(t) = \min\left(J_\mu(t), 2J_\mu\Big(\frac{\min(t,1-t)}{2}\Big)\right). \tag{3.17}$$

Under few assumptions on Φ, I_μ compares to the function

$$L_\Phi(t) = \min(t,1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{1}{\min(t,1-t)}\right),$$

where Φ' denotes the right derivative. More precisely,

Proposition 3.18. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let μ_Φ be defined in (3.16). Define F_μ and J_μ as in (3.6).*
Then,

$$\lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} = 1.$$

Consequently, if $\Phi(0) < \log 2$, L_Φ is defined on $[0,1]$ and there exist constants $k_1, k_2 > 0$ such that for all $t \in [0,1]$,

$$k_1 L_\Phi(t) \le J_\mu(t) \le k_2 L_\Phi(t).$$

Remark 3.19. This result appears in [7, 23] in the particular case $\Phi(x) = |x|^p$ and in [11] for Φ convex and $\sqrt{\Phi}$ concave. ◊

The previous results together with Corollary 3.13 lead to the following (dimensional) isoperimetric inequality.

Corollary 3.20. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$ and $\Phi(0) < \log 2$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$ be a probability measure on $\mathbb{R}$. Then,*

$$I_{\mu^n}(t) \ge c \min(t,1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{n}{\min(t,1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ independent on n.

Remark 3.21. Note that there is a gain of a square root with respect to the results in [9]. ◊

For the clarity of the exposition, the rather technical proofs of Proposition 3.18 and Corollary 3.20 are postponed to the Appendix.

We end this section with two examples.

Proposition 3.22 (Sub-exponential law)**.** *Consider the probability measure μ on $\mathbb{R}$, with density $Z_p^{-1}e^{-|x|^p}$, $p \in (0,1]$. There is a constant c depending only on p such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$,*

$$\mu_s^n(\partial A) \geq c \min(\mu^n(A), 1-\mu^n(A)) \log\left(\frac{n}{\min(\mu^n(A), 1-\mu^n(A))}\right)^{1-\frac{1}{\beta}}.$$

Proof. The proof follows immediately from Corollary 3.20. □

Remark 3.23. Let $I_{\mu^n}(t)$ be the isoperimetric profile of μ^n. The preceding bound combined with the upper bound of [9, Inequality (4.10)] gives

$$c(p)\,t\left(\log\left(\frac{n}{t}\right)\right)^{1-1/p} \leq I_{\mu^n}(t) \leq c'(p)t\log(1/t)\left(\log\left(\frac{n}{\log(1/t)}\right)\right)^{1-1/p}$$

for any $n \geq \log(1/t)/\log 2$ and $t \in (0,1/2)$. Hence, we obtain the right logarithmic behavior of the isoperimetric profile in term of the dimension n. This result extends the corresponding one obtained in section 3 for this class of examples. ◊

More generally consider the probability measure $\mu = Z^{-1}e^{-|x|^p \log(\gamma+|x|)^\alpha}$, $p \in (0,1]$, $\alpha \in \mathbb{R}$ and $\gamma = \exp\{2|\alpha|/(p(1-p))\}$ chosen in such a way that $\Phi(x) = |x|^p \log(\gamma+|x|)^\alpha$ is concave on $\mathbb{R}^+$. The assumptions of Corollary 3.20 are satisfied. Hence, we get that

$$I_{\mu^n}(t) \geq c(p,\alpha)t\left(\log\left(\frac{n}{t}\right)\right)^{1-1/p}\left(\log\log\left(e+\frac{n}{t}\right)\right)^{\frac{\alpha}{p}}, \qquad t \in (0,1/2).$$

Cauchy laws do not enter the framework of Corollary 3.20. Nevertheless, explicit computations can be done.

Proposition 3.24 (Cauchy distributions)**.** *Consider $d\mu(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}\,dx$ on $\mathbb{R}$, with $\alpha > 0$. There is $c > 0$ depending only on α such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$,*

$$\mu_s^n(\partial A) \geq c\frac{\min(\mu^n(A), 1-\mu^n(A))^{1+\frac{1}{\alpha}}}{n^{\frac{1}{\alpha}}}.$$

Proof. Since $1 - F_\mu(r) = \frac{1}{2(1+r)^\alpha}$ for $r \in \mathbb{R}^+$, $\log(1-F_\mu)$ is convex on $\mathbb{R}^+$. Moreover $J_\mu(t) = \alpha 2^{1/\alpha}\min(t,1-t)^{1+1/\alpha}$, and so the result follows by Corollary 3.13. □

Remark 3.25. Note that, since $J_\mu(t) = \alpha 2^{1/\alpha}\min(t,1-t)^{1+1/\alpha}$, one has

$$I_\mu(t) = \alpha t^{1+1/\alpha}, \qquad \forall t \in (0,1/2).$$

Hence, our results reads as

$$I_{\mu^n}(t) \geq c\,\frac{t}{n^{1/\alpha}}t^{1/\alpha}$$

for some constant c depending only on α. Together with [9, Inequality (4.9)] (for the upper bound) our results gives for any $n \geq \log(1/t)/\log 2$ and $t \in (0,1/2)$

$$c\,\frac{t}{n^{1/\alpha}}t^{1/\alpha} \leq I_{\mu^n}(t) \leq c'\,\frac{t}{n^{1/\alpha}}\log(1/t)^{1+1/\alpha}.$$

Again, we get the correct polynomial behavior in the dimension n. ◊

4. W P ´

In this section we deal with spherically symmetric probability measures $d\mu(x) = h(|x|)dx$ on $\mathbb{R}^n$ with $|\cdot|$ the Euclidean distance. In polar coordinates, the measure μ with density h can be viewed as the distribution of $\xi\theta$, where θ is a random vector uniformly distributed on the unit sphere S^{n-1}, and ξ (the radial part) is a random variable independent of θ with distribution function

$$\mu\,\{|x| \leq r\}) = n\omega_n \int_0^r s^{n-1} h(s)ds\,, \tag{4.1}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. We shall denote by $\rho_\mu(r) = n\omega_n r^{n-1}h(r)$ the density of the distribution of ξ, defined on $\mathbb{R}_+$.

Our aim is to obtain weighted Poincaré inequalities with explicit constants for μ on $\mathbb{R}^n$ of the forms $d\mu(x) = \frac{1}{Z}\frac{1}{(1+|x|)^{(n+\alpha)}}\,dx$ with $\alpha > 0$ or $d\mu(x) = \frac{1}{Z}e^{-|x|^p}\,dx$, with $p \in (0,1)$. To do so we will apply a general radial transportation technique which is explained in the following result.

Given an application $T : \mathbb{R}^n \to \mathbb{R}^n$, the image of μ under T is by definition the unique probability measure ν such that

$$\int f\,d\nu = \int f \circ T\,d\mu, \qquad \forall f.$$

In the sequel, we shall use the notation $T\sharp\mu$ to denote this probability measure.

Theorem 4.2 (Transportation method)**.** *Let μ and ν be two spherically symmetric probability measures on $\mathbb{R}^n$ and suppose that $\mu = T\sharp\nu$ with T a radial transformation of the form: $T(x) = \varphi(|x|)\frac{x}{|x|}$, with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing function with $\varphi(0) = 0$.*

If ν satisfies Poincaré inequality with constant C, then μ verifies the following weighted Poincaré inequality

$$Var_\mu(f) \leq C \int \omega(|x|)^2 |\nabla f|^2\,d\mu(x), \qquad \forall f,$$

with the weight ω defined by

$$\omega(r) = \max\left(\varphi' \circ \varphi^{-1}(r), \frac{r}{\varphi(r)}\right).$$

If one suppose that ν verifies Cheeger inequality with constant C, then μ verifies the following weighted Cheeger inequality

$$\int |f - m|\,d\mu \leq C \int \omega(|x|)|\nabla f|(x)\,d\mu(x), \qquad \forall f,$$

with the same weight ω as above and m being a median of f.

Finally, if the function φ is convex, then $\omega(r) = \varphi' \circ \varphi^{-1}(r)$.

Remark 4.3. In [57], Wang has used a similar technique to get weighted logarithmic Sobolev inequalities.

Proof. Consider a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$; it follows from the minimizing property of the variance and the Poincaré inequality verified by ν that

$$\mathrm{Var}_\mu(f) \leq \int \left(f - \int f\,d\nu\right)^2\,d\mu = \int \left(f(T) - \int f d\nu\right)^2\,d\nu \leq C \int |\nabla(f \circ T)|^2\,d\nu.$$

In polar coordinates we have

$$\begin{aligned}|\nabla(f\circ T)|^2 &= \left[\frac{\partial}{\partial r}(f\circ T)\right]^2+\frac{1}{r^2}\,|\nabla_\theta(f\circ T)|^2=\left(\frac{\partial f}{\partial r}\right)^2\circ T\times\varphi'^2+\frac{1}{r^2}\,|\nabla_\theta f|^2\\ &= \left(\frac{\partial f}{\partial r}\right)^2\circ T\times\left(\varphi'\circ\varphi^{-1}\circ\varphi\right)^2+\frac{1}{(\varphi^{-1}\circ\varphi)^2}\,|\nabla_\theta f|^2\circ T.\end{aligned}$$

Moreover, denoting by $d\theta$ the normalized Lebesgue measure on S^{n-1}, and using the notations introduced in the beginning of the section, the previous inequality reads

$$\begin{aligned}\mathrm{Var}_\mu(f) &\leq C\iint\left(\left(\frac{\partial f}{\partial r}\right)^2\circ T\times\left(\varphi'\circ\varphi^{-1}\circ\varphi\right)^2+\frac{1}{(\varphi^{-1}\circ\varphi)^2}\,|\nabla_\theta f|^2\circ T\right)\rho_\nu(r)drd\theta\\ &= C\iint\left(\left(\frac{\partial f}{\partial r}\right)^2\times\left(\varphi'\circ\varphi^{-1}\right)^2+\frac{1}{(\varphi^{-1})^2}\,|\nabla_\theta f|^2\right)\rho_\mu(r)drd\theta\\ &\leq C\iint\omega^2(r)\left(\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\,|\nabla_\theta f|^2\right)\rho_\mu(r)drd\theta\\ &= C\int\omega^2(|x|)|\nabla f|^2d\mu\end{aligned}$$

where we used the fact that the map φ transports $\rho_\nu\,dr$ onto $\rho_\mu\,dr$. The proof of the Cheeger case follows exactly in the same way.

Now, let us suppose that φ is convex. Since φ is convex and $\varphi(0)=0$, one has $\frac{\varphi(r)}{r}\leq\varphi'(r)$. This implies at once that $\omega(r)=\varphi'\circ\varphi^{-1}$ and achieves the proof. □

To apply Theorem 4.2, one needs a criterion for Poincaré inequality. The following theorem is a slight adaptation of a result by Bobkov [17, Theorem 1].

Theorem 4.4. *Let $d\nu(x)=h(|x|)\,dx$ be a spherically symmetric probability measure on $\mathbb{R}^n$. Define as before ρ_ν as the density of the law of $|X|$ where X is distributed according to ν and suppose that ρ_ν is a log-concave function. Then ν verifies the following Poincaré inequality*

$$Var_\nu(f)\leq C_\nu\int|\nabla f|^2\,d\nu,\qquad\forall f$$

with $C_\nu=12\left(\int r^2\rho_\nu(r)\,dr-\left(\int r\rho_\nu(r)\,dr\right)^2\right)+\frac{1}{n}\int r^2\rho_\nu(r)\,dr.$

Proof. We refer to [17]. □

Proposition 4.5 (Generalized Cauchy distributions)**.** *The probability measure $d\mu(x)=\frac{1}{Z}\frac{1}{(1+|x|)^{(n+\alpha)}}$ on $\mathbb{R}^n$ with $\alpha>0$ verifies the weighted Poincaré inequality*

$$Var_\mu(f)\leq C_{opt}\int(1+|x|)^2\,|\nabla f|^2\,d\mu(x),\qquad\forall f.$$

where the optimal constant C_{opt} is such that

$$\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}\leq C_{opt}\leq 14\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}.$$

Remark 4.6. Note that, comparing to integrals, we have

$$\frac{1}{\alpha^2} + \frac{n-1}{(\alpha+1)(\alpha+n)} \le \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2} \le \frac{1}{\alpha^2} + \frac{n-1}{\alpha(\alpha+n-1)}.$$

Since $\alpha^2 \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2} \to n$ when $\alpha \to \infty$, applying the previous weighted Poincaré inequality to $g(\alpha x)$, making a change of variables, and letting α tend to infinity lead to

$$\mathrm{Var}_\nu(f) \le 14n \int |\nabla f|^2 \, d\nu$$

with $d\nu(x) = (1/Z)e^{-|x|}dx$. Moreover, the optimal constant in the latter is certainly greater than n. This recover (with 14 instead of 13) one particular result of Bobkov [17].

Proof. Define $\psi(r) = \ln(1+r)$, $r > 0$ and let ν be the image of μ under the radial map $S(x) = \psi(|x|)\frac{x}{|x|}$. Conversely, one has evidently that μ is the image of ν under the radial map $T(x) = \varphi(|x|)\frac{x}{|x|}$, with $\varphi(r) = \psi^{-1}(r) = e^r - 1$ (which is convex). To apply Theorem 4.2, one has to check that ν verifies Poincaré inequality.

Elementary computations yield

$$\frac{d\nu}{dx}(x) = \frac{1}{Z}\left(\frac{e^{|x|}-1}{|x|}\right)^{n-1} e^{(1-n-\alpha)|x|} \qquad \text{and} \qquad \rho_\nu(r) = \frac{n\omega_n}{Z}\,(1-e^{-r})^{n-1}\,e^{-\alpha r}$$

It is clear that $\log \rho_\nu$ is concave. So we may apply Theorem 4.4 and conclude that ν verifies Poincaré inequality with the constant C_ν defined above.

Define

$$H(\alpha) = \int_0^{+\infty} e^{-\alpha r}(1-e^{-r})^{n-1}\,dr = \int_0^1 u^{\alpha-1}(1-u)^{n-1}\,du.$$

Then $\int r\rho_\nu(r)\,dr = -\frac{H'(\alpha)}{H(\alpha)}$ and $\int r^2\rho_\nu(r)\,dr = \frac{H''(\alpha)}{H(\alpha)}$. Integrations by parts yield

$$H(\alpha) = \frac{(n-1)!}{(\alpha+n-1)(\alpha+n-2)\cdots(\alpha)}.$$

So,

$$H'(\alpha) = -H(\alpha)\sum_{k=0}^{n-1}\frac{1}{\alpha+k} \quad \text{and} \quad H''(\alpha) = H(\alpha)\left[\left(\sum_{k=0}^{n-1}\frac{1}{\alpha+k}\right)^2 + \sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}\right].$$

This gives, using Cauchy-Schwarz inequality

$$C_\nu = 13\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2} + \frac{1}{n}\left(\sum_{k=0}^{n-1}\frac{1}{\alpha+k}\right)^2 \le 14\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}.$$

Now, suppose that there is some constant C such that the inequality $\mathrm{Var}_\mu(f) \le C\int(1+|x|)^2|\nabla f|^2\,d\mu$ holds for all f. We want to prove that $C \ge \sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}$. To do so let us test this inequality on the functions $f_a(x) = \frac{1}{(1+|x|)^a}$, $a > 0$. Defining $F(r) = \int \frac{1}{(1+|x|)^{n+r}}\,dr$, for all $r > 0$, one obtains immediately

$$C \ge \frac{1}{a^2}\frac{F(2a+\alpha)F(\alpha) - F(a+\alpha)^2}{F(\alpha)F(2a+\alpha)}.$$

But a Taylor expansion easily shows that the right hand side goes to $K = \frac{F''(\alpha)}{F(\alpha)} - \left(\frac{F'(\alpha)}{F(\alpha)}\right)^2$, so $C \geq K$. Easy computations give that $F(\alpha) = n\omega_n H(\alpha)$ and so $K = \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2}$. □

Proposition 4.7 (Sub-exponential laws)**.** *The probability measure $d\mu(x) = \frac{1}{Z}e^{-|x|^p}\,dx$ on $\mathbb{R}^n$ with $p \in (0,1)$ verifies the weighted Poincaré inequality*

$$Var_\mu(f) \leq C_{opt} \int |\nabla f|^2 |x|^{2(1-p)}\, d\mu(x),$$

where the optimal constant C_{opt} is such that

$$\frac{n}{p^3} \leq C_{opt} \leq 12\frac{n}{p^3} + \frac{n+p}{p^4}.$$

Remark 4.8. As for the Cauchy law, letting p go to 1 leads to

$$\mathrm{Var}_\nu(f) \leq (13n+1) \int |\nabla f|^2\, d\nu$$

with $d\nu(x) = (1/Z)e^{-|x|}dx$. Again this recover (with $13n+1$ instead of $13n$) one particular result of Bobkov [17].

Proof. We mimic the proof of the preceding example. Let $\psi(r) = \frac{1}{p}r^p$, $r \geq 0$ and define ν as the image of μ under the radial map $S(x) = \psi(|x|)\frac{x}{|x|}$. Easy calculations give that the radial part of ν has density ρ_ν defined by

$$\rho_\nu(r) = \frac{n\omega_n}{Z}\,(\beta u)^{\frac{n-p}{p}}\, e^{-pu}.$$

It is clearly a log-concave function on $[0,+\infty)$. Let us compute the constant C_ν appearing in Theorem 4.4. One has

$$\int r\rho_\nu(r)\,dr = \frac{1}{p}\frac{\Gamma(\frac{n}{p}+1)}{\Gamma(\frac{n}{p})} = \frac{n}{p^2},$$

and

$$\int r^2\rho_\nu(r)\,dr = \frac{1}{p^2}\frac{\Gamma(\frac{n}{p}+2)}{\Gamma(\frac{n}{p})} = \frac{n(n+p)}{p^4}.$$

Consequently,

$$C_\nu = 12\frac{n}{p^3} + \frac{n+p}{p^4}.$$

Now suppose that there is some C such that $\mathrm{Var}_\mu(f) \leq C\int |\nabla f|^2|x|^{2(1-p)}\,d\mu(x)$ holds for all f. To prove that $C \geq \frac{n}{p^3}$, we will test this inequality on the functions $f_a(x) = e^{-a|x|^p}$, $a > 0$. Letting $G(t) = \int e^{-t|x|^p}\,d\mu(x)$, we arrive at the relation

$$C \geq \frac{1}{\beta^2 a^2}\frac{G(1)G(2a+1) - G(a+1)^2}{G(1)G(2a+1)}, \qquad \forall a > 0.$$

Letting $a \to 0$, one obtains $C \geq \frac{1}{p^2}\left[\frac{G''(1)}{G(1)} - \left(\frac{G'(1)}{G(1)}\right)^2\right]$. The change of variable formula immediately yields $G(t) = t^{-\frac{n}{p}}G(1)$, and so $C \geq \frac{1}{p^2}\left[\frac{n(n+p)}{p^2} - \left(\frac{n}{p}\right)^2\right]$, which achieves the proof. □

5. L P ´ .

In this section we deal with weak Poincaré inequalities and work under the general setting of Section 2. One says that a probability measure μ verifies the weak Poincaré inequality if for all $f \in \mathcal{A}$,

$$\mathrm{Var}_\mu(f) \le \beta(s) \int \Gamma(f)\, d\mu + s\, \mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4),$$

where $\beta : (0, 1/4) \to \mathbb{R}^+$ is a non-increasing function. Note that the limitation $s \in (0, 1/4)$ comes from the bound $\mathrm{Var}_\mu(f) \le \mathrm{Osc}_\mu(f)^2/4$.

Weak Poincaré inequalities were introduced by Röckner and Wang in [48]. In the symmetric case, they describe the decay of the semi-group P_t associated to L (see [48, 4]). Namely for all bounded centered function f, there exists $\psi(t)$ tending to zero at infinity such that $\|P_t f\|_{L_2(\mu)} \le \psi(t)\|f\|_\infty$.

They found another application in concentration of measure phenomenon for sub-exponential laws in [9, Thm 5.1]. The approach proposed in [9] to derive weak Poincaré inequalities was based on capacity-measure arguments (following [13]). In this section, we give alternative arguments. One is based on converse Poincaré inequalities. This implies that weak Poincaré inequalities can be derived directly from the ϕ-Lyapunov function strategy, using Theorem 2.33. The second approach is based on a direct implication of weak Poincaré inequalities from weak Cheeger inequalities. In turn, one can use either (the mass-transport technique of) Theorem 3.8 in order to get precise bounds for measures on $\mathbb{R}^n$ which are tensor product of a measure on $\mathbb{R}$, or (via ϕ-Lyapunov functions) Theorem 3.2.

Converse Poincaré inequalities imply weak Poincaré inequalities as shown in the following Theorem.

Theorem 5.1. *Assume that μ satisfies a converse Poincaré inequality*

$$\inf_c \int (g-c)^2\, \omega\, d\mu \le C \int \Gamma(g)\, d\mu$$

for some non-negative weight ω, such that $\bar{\omega} = \int \omega d\mu < +\infty$. Define $F(u) = \mu(\omega < u)$ and $G(s) = F^{-1}(s) := \inf\{u; \mu(\omega \le u) > s\}$ for $s < 1$.
Then, for all $f \in \mathcal{A}$,

$$Var_\mu(f) \le \frac{C}{G(s)} \int \Gamma(f)\, d\mu + s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof follows the same line of reasoning as the one of Theorem 3.2. □

Weak Poincaré inequalities are also implied by weak Cheeger inequalities as stated in the following Lemma. The proof of the Lemma is a little bit more tricky than the usual one from Cheeger to Poincaré. We give it for completeness.

Lemma 5.2. *Let μ be a probability measure and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$. Assume that for any $f \in \mathcal{A}$ it holds*

$$\int |f - m|\, d\mu \le \beta(s) \int \sqrt{\Gamma(f)}\, d\mu + s\mathrm{Osc}(f) \qquad \forall s \in (0, 1)$$

where m is a median of f under μ. Then, any $f \in \mathcal{A}$ satisfies

$$\mathrm{Var}_\mu(f) \le 4\beta\left(\frac{s}{2}\right)^2 \int \Gamma(f) d\mu + s\mathrm{Osc}(f)^2 \qquad \forall s \in (0, 1/4). \tag{5.3}$$

Proof. let $f \in \mathcal{A}$. Assume that 0 is a median of f and by homogeneity of (5.3) that $\mathrm{Osc}(f) = 1$ (which implies in turn that $\|f\|_\infty \le 1$). Let m be a median of f^2. Applying the weak Cheeger inequality to f^2, using the definition of the median and the chain rule formula, we obtain

$$\int f^2\, d\mu \le \int |f^2 - m|\, d\mu \le 2\beta(s) \int |f| \sqrt{\Gamma(f)}\, d\mu + s\mathrm{Osc}(f^2) \qquad \forall s \in (0,1).$$

Since $\|f\|_\infty \le 1$ and $\mathrm{Osc}(f) = 1$, one has $\mathrm{Osc}(f^2) \le 2$. Hence, by the Cauchy-Schwarz inequality, we have

$$\int f^2\, d\mu \le 2\beta(s) \left(\int \Gamma(f)\, d\mu\right)^{\frac{1}{2}} \left(\int |f|^2\, d\mu\right)^{\frac{1}{2}} + 2s \qquad \forall s \in (0,1).$$

Hence,

$$\left(\int f^2\, d\mu\right)^{\frac{1}{2}} \le \beta(s) \left(\int \Gamma(f)\, d\mu\right)^{\frac{1}{2}} + \left(\beta(s)^2 \int \Gamma(f)\, d\mu + s\right)^{\frac{1}{2}}.$$

Since $\mathrm{Var}_\mu(f) \le \int f^2\, d\mu$, we finally get

$$\mathrm{Var}_\mu(f) \le 4\beta(s)^2 \int \Gamma(f)\, d\mu + 2s \qquad \forall s \in (0,1)$$

which is the expected result. □

Two examples follow.

Proposition 5.4 (Cauchy type laws)**.** *Let $d\mu(x) = V^{-(n+\alpha)}(x)dx$ with V convex on $\mathbb{R}^n$ and $\alpha > 0$. Recall that $\kappa = -1/\alpha$. Then there exists a constant $C > 0$ such that for all smooth enough $f : \mathbb{R}^n \to \mathbb{R}$,*

$$Var_\mu(f) \;\le\; C s^{2\kappa} \int |\nabla f|^2\, d\mu \;+\; s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof is a direct consequence of Proposition 3.3 together with Lemma 5.2 above. □

Remark 5.5. For the generalized Cauchy distribution $d\mu(x) = c_\beta\,(1 + |x|)^{-(n+\alpha)}$, this result is optimal for $n = 1$ and was shown in [48] (see also [9, Example 2.5]). For $n \ge 2$ the result obtained in [48] is no more optimal. In [4], a weak Poincaré inequality is proved in any dimension with rate function $\beta(s) \le c(p)\, s^{2p}$ for any $p < \kappa$. Here we finally get the optimal rate. Note however that the constant C may depend on n. ◊

Proposition 5.6 (Sub exponential type laws)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}$ for some positive convex function V on $\mathbb{R}^n$ and $p \in (0,1)$. Then there exists $C > 0$ such that for all f*

$$Var_\mu(f) \;\le\; C \left(\log\left(\frac{1}{s}\right)\right)^{2(\frac{1}{p}-1)} \int |\nabla f|^2\, d\mu \;+\; s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof is a direct consequence of Proposition 3.5 together with Lemma 5.2 above. □

By Lemma 5.2 above, we see that weak Poincaré inequalities can be derived from mass-transport arguments using Theorem 3.8. This is stated in the next Corollary.

Corollary 5.7. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$. Then, for any n, every function $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough satisfies*

$$Var_{\mu^n}(f) \le \kappa_1^2 \frac{s^2}{J_\mu(s/2)^2} \int |\nabla f|\, d\mu^n + 2\kappa_2 n s\mathrm{Osc}(f)^2, \qquad \forall s > 0. \tag{5.8}$$

with $\kappa_1 = 2\sqrt{6}$ *and* $\kappa_2 = 2(1 + 2\sqrt{6})$.

Proof. Applying Lemma 5.2 to μ^n together with Theorem 3.8 immediately yields the result. □

We illustrate this Corollary on two examples.

Proposition 5.9 (Cauchy distributions). *Consider* $d\mu(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}\, dx$ *on* $\mathbb{R}$, *with* $\alpha > 0$. *Then, there is a constant C depending only on* α *such that for all* $n \geq 1$

$$Var_{\mu^n}(f) \leq C\left(\frac{n}{s}\right)^{\frac{2}{\alpha}} \int |\nabla f|^2\, d\mu^n + s\mathrm{Osc}_{\mu^n}(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. Since $J_{m_\alpha}(t) = \alpha 2^{1/\alpha} t^{1+1/\alpha}$ for $t \in (0, 1/2)$, by Corollary 5.7, on $\mathbb{R}^n$, μ^n satisfies a weak Poincaré inequality with rate function $\beta(s) = C\left(\frac{n}{s}\right)^{\frac{2}{\alpha}}$, $s \in (0, \frac{1}{4})$. □

Proposition 5.10 (Sub-exponential law). *Consider the probability measure* μ *on* $\mathbb{R}$, *with density* $Z^{-1}e^{-|x|^p}$, $p \in (0, 1]$. *Then, there is a constant C depending only on p such that for all* $n \geq 1$

$$Var_{\mu^n}(f) \leq C\left(\log\left(\frac{n}{s}\right)\right)^{2(\frac{1}{p}-1)} \int |\nabla f|^2\, d\mu^n + s\mathrm{Osc}_{\mu^n}(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. By Corollary 3.20, $J_\mu(t)$ is, up to a constant, greater than or equal to $t\left(\log(1/t)\right)^{1-\frac{1}{p}}$ for $t \in [0, 1/2]$. Hence, by Corollary 5.7, μ^n satisfies a weak Poincaré inequality on $\mathbb{R}^n$, with the rate function $\beta(s) = C\left(\log\left(\frac{n}{s}\right)\right)^{2(\frac{1}{p}-1)}$, $s \in (0, \frac{1}{4})$. □

Remark 5.11. The two previous results recover the results of [9]. Note the difference between the results of Proposition 5.6 (applied to $V(x) = |x|$) and Proposition 5.10. This is mainly due to the fact that Proposition 5.6 holds in great generality, while Proposition 5.10 deals with a very specific distribution. The same remark applies to Propositions 5.4 and 5.9 since in the setting of Proposition 5.9, $2/\alpha = -2\kappa$.
However, it is possible to recover the results of Proposition 5.10 (resp. Propositions 5.9) applying Proposition 5.6 (resp. Propositions 5.4) to the sub-exponential (resp. Cauchy) measure on $\mathbb{R}$ and then to use the tensorization property [9, Theorem 3.1]. ◊

Remark 5.12. According to an argument of Talagrand (recalled in the introduction), if for all k, μ^k satisfies the same concentration property as μ, then the tail distribution of μ is at most exponential. So no heavy tails measure can satisfy a dimension-free concentration property.The concentration properties of heavy tailed measure are thus particularly intersting to study, and in particular the dimension dependence of the result. The first results in this direction using weak Poincaré inequalities were done in [9]. As converse Poincaré inequalities plus control of the tail of the weight lead to weak Poincaré inequality, and thus concentration, it is interesting to remark that in Theorem 4.1 and Corollary 4.2 in [22], Bobkov and Ledoux proved that if a weighted Poincaré inequality holds, any 1-Lipschitz function with zero mean satisfies

$$\| f \|_p \leq \frac{Dp}{\sqrt{2}} \, \| \sqrt{1+\eta^2} \|_p$$

for all $p \geq 2$. It follows that for all t large enough ($t > Dpe \, \| \sqrt{1+\eta^2} \|_p$),

$$\mu(|f| > t) \leq 2\left(\frac{D\,p\, \| \sqrt{1+\eta^2} \|_p}{t}\right)^p .$$

Hence the concentration function is controlled by some moment of the weight. Dimension dependence is hidden in this moment control. However if one is only interested in concentration properties, one could use directly weighted Poincaré inequalities. ◊

6. A

This appendix is devoted to the proofs of Proposition 3.18 and Corollary 3.20. Let us recall the first of these statements.

Proposition. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let μ_Φ be defined in (3.16). Define F_μ and J_μ as in (3.6).*
Then,

$$\lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} = 1.$$

Proof of Proposition 3.18. The proof follows the line of [11, Proposition 13]. By Point (*iii*) of Lemma 6.2 below, Φ' never vanishes. Under our assumptions on Φ we have $F_\mu(y) = \int_{-\infty}^y Z_\Phi^{-1} e^{-\Phi(|x|)} dx \sim Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ when y tends to $-\infty$. Thus using the change of variable $y = F_\mu^{-1}(t)$, we get

$$\begin{aligned} \lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} &= \lim_{y\to -\infty} \frac{e^{-\Phi(|y|)}}{Z_\Phi F_\mu(y) \Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})} \\ &= \lim_{y\to -\infty} \frac{\Phi'(|y|)}{\Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})}. \end{aligned}$$

By concavity of Φ we have $F_\mu(y) \geq Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ for all $y \leq 0$. Hence, since $\lim_\infty \Phi' = 0$, we have $\log \frac{1}{F_\mu(y)} \leq \Phi(|y|)$ when $y \ll -1$.
Then, a Taylor expansion of $\Phi' \circ \Phi^{-1}$ between $\log \frac{1}{F_\mu(y)}$ and $\Phi(|y|)$ gives

$$\frac{\Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})}{\Phi'(|y|)} = 1 + \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{F_\mu(y)} - \Phi(|y|) \right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}$$

for some $c_y \in [\log \frac{1}{F_\mu(y)}, \infty)$.
For $y \ll -1$, we have

$$\frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)} \leq F_\mu(y) \leq 2 \frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)}. \tag{6.1}$$

Hence, using Point (*iii*) of Lemma 6.2 below,

$$\begin{aligned} \left| \log \frac{1}{F_\mu(y)} - \Phi(|y|) \right| &= \Phi(|y|) - \log \frac{1}{F_\mu(y)} \\ &\leq \log \frac{2}{Z_\Phi} + \log \left(\frac{1}{\Phi'(|y|)} \right) \\ &\leq \log \frac{2}{Z_\Phi} + c \log(|y|) \end{aligned}$$

for some constant c and all $y \ll -1$.
On the other hand, when Φ^θ is convex and C^2, $(\Phi^\theta)''$ is non negative. This, together with Point (i) of Lemma 6.2, lead to

$$\left|\frac{\Phi''(x)}{\Phi'(x)}\right| = -\frac{\Phi''(x)}{\Phi'(x)} \leq (\theta-1)\frac{\Phi'(x)}{\Phi(x)} \leq \frac{c'}{x}$$

for some constant c' and $x \gg 1$. It follows that

$$\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \leq \frac{c'}{\Phi^{-1}\left(\log \frac{1}{F_\mu(y)}\right)}.$$

Now, by (6.1) and Point (iii) and (ii) of Lemma 6.2, we note that

$$\begin{aligned}
\log \frac{1}{F_\mu(y)} &\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) + \log(\Phi'(|y|)) \\
&\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) - c_3 \log(|y|)) \\
&\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) - \frac{c_3}{c_2} \log(\Phi(|y|))) \\
&\geq \frac{1}{2}\Phi(|y|)
\end{aligned}$$

provided $y \ll -1$. In turn, by Point (iv) of Lemma 6.2,

$$\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \leq \frac{c''}{|y|}$$

for some constant c''.
All these computations together give

$$\left|\frac{1}{\Phi'(|y|)}\left(\log\frac{1}{H(y)} - \Phi(|y|)\right)\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}\right| \leq c'' \frac{\log \frac{2}{Z_\Phi} + c\log(|y|)}{|y|\Phi'(|y|)}$$

which goes to 0 as y goes to $-\infty$ by Point (i) and (ii) of Lemma 6.2. This ends the proof. □

Lemma 6.2. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be an increasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Assume that $\int e^{-\Phi(|x|)}dx < \infty$. Then, there exist constants $c_1, c_3 > 1$, $c_2, c_4 \in (0,1)$ such that for x large enough,*
(i) $c_1^{-1} x\Phi'(x) \leq \Phi(x) \leq c_1 x\Phi'(x)$;
(ii) $\Phi(x) \geq x^{c_2}$;
(iii) $\Phi'(x) \geq x^{-c_3}$;
(iv) $\frac{1}{2}\Phi(x) \geq \Phi(c_4 x)$.

Proof. Let $\widetilde{\Phi} = \Phi - \Phi(0)$. Then, in the large, $\widetilde{\Phi}$ is concave and $(\widetilde{\Phi})^\theta$ is convex. Hence, the slope functions $\widetilde{\Phi}(x)/x$ and $(\widetilde{\Phi})^\theta/x$ are non-increasing and non-decreasing respectively. In turn, for x large enough,

$$x\Phi'(x) = x\widetilde{\Phi}'(x) \leq \widetilde{\Phi}(x) \leq \theta x \widetilde{\Phi}'(x) = \theta x\Phi'(x).$$

This bound implies in particular that $x\Phi'(x) \to \infty$ as x tends to infinity. Point (i) follows.

The second inequality in (i) implies that for x large enough,

$$\frac{\Phi'(x)}{\Phi(x)} \geq \frac{1}{c_1 x}. \tag{6.3}$$

Hence, for some x_0 large enough, integrating, we get

$$\log\Phi(x) \geq \log\Phi(x_0) + \frac{1}{c_1}(\log(x) - \log(x_0)) \geq \frac{1}{2c_1}\log(x) \quad \forall x \gg x_0.$$

Point (ii) follows.

Point (iii) follows from the latter and Inequality (6.3).

Take $c = \exp\{1/c_1\}$. By Point (i), we have for x large enough

$$\begin{aligned}
\Phi(cx) &= \Phi(x) + \int_x^{cx} \Phi'(t)dt \\
&\geq \Phi(x) + \int_x^{cx} \frac{\Phi(t)}{c_1 t}dt \\
&\geq \Phi(x)\left(1 + \int_x^{cx} \frac{1}{c_1 t}\right)dt \\
&= \Phi(x)\left(1 + \frac{\log c}{c_1}\right) = 2\Phi(x).
\end{aligned}$$

Point (iv) follows. □

Now let us recall the statement of Corollary 3.20.

Corollary. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$ and $\Phi(0) < \log 2$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$ be a probability measure on $\mathbb{R}$. Then,*

$$I_{\mu^n}(t) \geq c\min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log\frac{n}{\min(t, 1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ independent on n.

Proof of Corollary 3.20. Since Φ is concave, $\log(1 - F_\mu)$ is convex on $\mathbb{R}^+$. Applying Corollary 3.13 together with Proposition 3.18 lead to

$$I_{\mu^n}(t) \geq c\min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log\frac{n}{c'\min(t, 1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ and $c' > 1$ independent on n. It remains to prove that for all $t \in [0, 1/2]$,

$$t\Phi' \circ \Phi^{-1}\left(\log\frac{n}{c't}\right) \geq c'' t\Phi' \circ \Phi^{-1}\left(\log\frac{n}{t}\right)$$

for some constant $c'' > 0$. For $t \leq 1/2$ we have $1/(c't) \leq (1/t)^C$ for some $C > 1$. Hence, since $\Phi' \circ \Phi^{-1}$ is non-increasing,

$$\Phi' \circ \Phi^{-1}(\log\frac{n}{c't}) \geq \Phi' \circ \Phi^{-1}(C\log\frac{n}{t}).$$

Now note that Point (*iv*) of Lemma 6.2 is equivalent to say $\Phi^{-1}(2x) \leq \frac{1}{c_4}\Phi^{-1}(x)$ for x large enough. Hence $\Phi^{-1}(Cx) \leq \left(\frac{1}{c_4}\right)^{\lfloor \log_2 C \rfloor + 1} \Phi^{-1}(x)$. It follows that

$$\Phi' \circ \Phi^{-1}(\log \frac{n}{c't}) \geq \Phi' \left(\left(\frac{1}{c_4} \right)^{\lfloor \log_2 C \rfloor + 1} \Phi^{-1}(\log \frac{n}{t}) \right)$$

for t small enough. Finally, Point (*i*) and (*iv*) of Lemma 6.2 ensure that

$$\Phi' \left(\frac{1}{c_4} x \right) \geq \frac{c_4}{c_1} \frac{\Phi\left(\frac{x}{c_4}\right)}{x} \geq \frac{2c_4}{c_1} \frac{\Phi(x)}{x} \geq \frac{2c_4}{c_1^2} \Phi'(x).$$

Hence

$$t\Phi' \circ \Phi^{-1} \left(\log \frac{n}{c't} \right) \geq c'' t\Phi' \circ \Phi^{-1} \left(\log \frac{n}{t} \right)$$

for some constant $c'' > 0$ and t small enough, say for $t \leq t_0$. The expected result follows by continuity of $t \mapsto t\Phi' \circ \Phi^{-1}(\log \frac{n}{t})/t\Phi' \circ \Phi^{-1}(\log \frac{n}{c't})$ (on $[t_0, 1/2]$). $\square$

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