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Abstract : The purpose of this work is to solve exterior problems in the half-space for the Laplace operator. We give existence and unicity results in weighted L^p's theory with $1 < p < \infty$. This paper extends the studies done in [5] with Dirichlet and Neumann conditions.
Key words : Weighted Sobolev spaces; Laplacian; Dirichlet and Neumann boundary conditions; Exterior problems; Half-space.
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1 Introduction and preliminaries

Many problems in fluid dynamics, such as flows past obstacles, around corners or through pipes or apertures, are first conceptualized by Stokes or Navier-Stokes equations in unbounded domains. Our aim is to solve such systems in a particular unbounded domain for which any result is known. This domain, that we call exterior domain in the half-space, is the complement in the upper half-space of a compact region ω_0. We can see this geometry as an extension of the "classical" exterior domain, *i.e* the complement of ω_0 in the whole space. In a forthcoming paper, we study a Stokes system on such a domain but prior to that, it can be interesting to give results for the Laplace's equation. Thus, in this work, we want to solve the exterior Laplace's problem in the half-space.

First, let us recall some elements for the Laplace's equation in a classical exterior domain, domain which is the basis of ours. Several families of spaces are used for this operator, like the completion of $\mathcal{D}({}^c\omega_0)$ for the norm of the gradient in $L^p({}^c\omega_0)$ (where ${}^c\omega_0$ is the complement of ω_0 in $\mathbb{R}^n$), which has the inconvenient that, when $p \geq n$, some very treacherous Cauchy sequences exist in $\mathcal{D}(\overline{{}^c\omega_0})$ that do not converge to distributions, a behaviour carefully described in 1954 by Deny and Lions (*cf.* [9]). An other family of spaces is the subspace in $L^p_{loc}({}^c\omega_0)$ of functions whose gradients belong to $L^p({}^c\omega_0)$, subspace which have

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an imprecision at infinity inherent to the L^p_{loc} norm.

An other approach is to set problems in weighted Sobolev spaces where the growth or decay of functions at infinity are expressed by means of weights. These spaces have several advantages : they satisfy an optimal weighted Poincaré-type inequality ; they allow us to describe the behaviour of functions and not just of their gradient, which is vital from the mathematical and the numerical point of view.

Without being exhaustive, we can recall works of several authors who have contributed to the solution of Laplace's equation in a classical exterior domain by means of weighted Sobolev spaces : see Cantor [8], Giroire [10], Giroire and Nedelec [11], Nedelec [18], Nedelec and Planchard [19], Hsiao and Wendland [13], Leroux [14] and [15], McOwen [16] and Amrouche, Girault and Giroire [5].

In this paper, we choose to set our problems in weighted Sobolev spaces and we remind that here, our originality, with respect to results previously quoted, is to extend the resolution of the exterior Laplace's problem in the whole space to the exterior problem in the half-space. From this extension, comes an additional difficulty due to the nature of the boundary. Indeed, as it contains $\mathbb{R}^{n-1}$, it is not bounded anymore. So, we have to introduce weights even in the spaces of traces. We can cite Hanouzet [12] who has given the first results for such spaces in 1971 and Amrouche, Nečasovà [6] who have extended these results in 2001 to weighted Sobolev spaces which possess logarithmic weights (we just remind that logarithmic weights allow us to have a Poincaré-type inequality even in the "critical" cases ; see below for more details). Nevertheless, the half-space has a useful symmetric property what we use many times in this work.

Moreover, we deal with problems which have Dirichlet or Neumann conditions on the bounded boundary but also on the unbounded boundary, that is to say on $\mathbb{R}^{n-1}$. So, we want to solve the four following problems :

$$(\mathcal{P}_D) \ -\Delta u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma_0, \quad u = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_N) \ -\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_1}) \ -\Delta u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_2}) \ -\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ on } \Gamma_0, \quad u = g_1 \text{ on } \mathbb{R}^{n-1},$$

with Ω the complement of ω_0 in $\mathbb{R}^n_+$, where ω_0 is a compact and non-empty subset of $\mathbb{R}^n_+$, $n \geq 2$ and Γ_0 the boundary of ω_0. We supposed that Γ_0 is connected so that Ω is connected too. We suppose that Ω is of class $C^{1,1}$, even if, for some values of the exponent p, Ω can be less regular.

Each section of this paper is devoted to the study of one of the four problems. We will call $(\mathcal{P}_{M_1})$ and $(\mathcal{P}_{M_2})$ the first and the second mixed problem because of the presence of both a Dirichlet condition and a Neumann one. The main results of this work are Theorems 2.2, 3.3, 4.3 and 5.4.

We complete this introduction with a short review of the weighted Sobolev spaces and their trace spaces. For any integer q we denote by $\mathcal{P}_q$ the space of polynomials in n variables, of degree less than or equal to q, with the convention

that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative.
For any real number $p \in]1, +\infty[$, we denote by p' the dual exponent of p :

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a typical point of $\mathbb{R}^n$, $\boldsymbol{x'} = (x_1, \dots, x_{n-1})$ and let $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote its distance to the origin. We shall use two basic weights :

$$\rho = (1 + r^2)^{1/2} \quad \text{and} \quad lg\ \rho = ln(2 + r^2).$$

As usual, $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support, $\mathcal{D}'(\Omega)$ its dual space, called the space of distributions and $\mathcal{D}(\overline{\Omega})$ the space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$.

Then, we define the two following spaces :

$$W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{\omega_1} \in L^p(\Omega),\ \nabla u \in \boldsymbol{L}^p(\Omega)\}$$

and

$$W_1^{2,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{\omega_1} \in L^p(\Omega),\ \nabla u \in \boldsymbol{L}^p(\Omega),\ \rho\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega),\ i, j = 1, \dots, n\},$$

where ω_1 is defined by :

$$\omega_1 = \begin{cases} \rho & \text{if } n \neq p, \\ \rho\ lg\ \rho & \text{if } n = p. \end{cases}$$

They are reflexive Banach spaces equipped, respectively, with natural norms :

$$\|u\|_{W_0^{1,p}(\Omega)} = (\|\omega_1^{-1}u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\boldsymbol{L}^p(\Omega)})^{1/p}$$

and

$$\|u\|_{W_1^{2,p}(\Omega)} = (\|\omega_1^{-1}u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\boldsymbol{L}^p(\Omega)} + \sum_{1 \leq i,j \leq n} \|\rho\frac{\partial^2 u}{\partial x_i \partial x_j}\|^p_{L^p(\Omega)})^{1/p}.$$

We also define semi-norms :

$$|u|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$|u|_{W_1^{2,p}(\Omega)} = (\sum_{1 \leq i,j \leq n} \|\rho\frac{\partial^2 u}{\partial x_i \partial x_j}\|^p_{L^p(\Omega)})^{1/p}.$$

We set the following spaces :

$$\overset{\circ}{W}{}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_0^{1,p}(\Omega)}} \quad \text{and} \quad \overset{\circ}{W}{}_1^{2,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_1^{2,p}(\Omega)}},$$

and we easily check that

$$\overset{\circ}{W}{}_0^{1,p}(\Omega) = \{u \in W_0^{1,p}(\Omega), u = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}$$

and that

$$\overset{\circ}{W}{}_1^{2,p}(\Omega) = \{u \in W_1^{2,p}(\Omega), u = \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}$$

where the sense of traces of these functions are given below. The weights defined previously are chosen so that the space $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$ and in $W_1^{2,p}(\Omega)$ and so that the following Poincaré-type inequalities hold :

$$\forall u \in W_0^{1,p}(\Omega), \inf_{k \in \mathcal{P}_{[1-n/p]}} \|u + k\|_{W_0^{1,p}(\Omega)} \leq C \ |u|_{W_0^{1,p}(\Omega)}$$

noticing that $\mathcal{P}_{[1-n/p]} = \{0\}$ si $p < n$, and

$$\forall u \in \overset{\circ}{W}{}_0^{1,p}(\Omega), \|u\|_{W_0^{1,p}(\Omega)} \leq C \ |u|_{W_0^{1,p}(\Omega)}.$$

We have similar inequalities for the space $W_1^{2,p}(\Omega)$. These results are proved, in the general case, by Amrouche, Girault and Giroire [5] in an exterior domain and by Amrouche and Nečasovà [6] in the half-space. They are extended to this domain by an adequate partition of unity. We denote by $W_0^{-1,p}(\Omega)$ (respectively $W_{-1}^{-2,p}(\Omega)$) the dual space of $\overset{\circ}{W}{}_0^{1,p'}(\Omega)$ (respectively of $\overset{\circ}{W}{}_1^{2,p'}(\Omega)$). It is spaces of distributions.

Then, we define too, for $\ell \in \mathbb{R}$, the space $W_\ell^{0,p}(\Omega)$ by :

$$W_\ell^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \rho^\ell u \in L^p(\Omega)\}.$$

We have, if $n \neq p$, the continuous injections

$$W_0^{1,p}(\Omega) \subset W_{-1}^{0,p}(\Omega) \ \text{ and } \ W_1^{0,p'}(\Omega) \subset W_0^{-1,p'}(\Omega).$$

Now, we want to define the traces of functions of $W_0^{1,p}(\Omega)$ and $W_1^{2,p}(\Omega)$. These traces have a component on Γ_0 and an other component on $\mathbb{R}^{n-1}$. For the traces on Γ_0, we return to Adams [1] or Nečas [17] for the definition of the two spaces $W^{1-\frac{1}{p},p}(\Gamma_0)$ and $W^{2-\frac{1}{p},p}(\Gamma_0)$ and for the usual trace theorems. For the traces of functions on $\mathbb{R}^{n-1}$, we send back to Amrouche and Nečasovà [6] for general definitions and here, we define the three following spaces :

$$W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) = \{u \in \mathcal{D}'(\mathbb{R}^{n-1}), \omega_2^{-1+\frac{1}{p}} u \in L^p(\mathbb{R}^{n-1}),$$
$$\int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+p-2}} \ d\boldsymbol{x} d\boldsymbol{y} < \infty\},$$

where

$$\omega_2 = \begin{cases} \rho' & \text{if } n \neq p, \\ \rho'(lg \ \rho')^{p'} & \text{if } n = p, \end{cases}$$

with $\rho' = (1 + |x'|^2)^{1/2}$ and $lg \ \rho' = ln(2 + |x'|^2)$. It is a reflexive Banach space equipped with its natural norm

$$(\|\omega_2^{-1+\frac{1}{p}} u\|_{L^p(\mathbb{R}^{n-1})}^p + \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+p-2}} \ d\boldsymbol{x} d\boldsymbol{y})^{1/p}.$$

We show that the mapping

$$\gamma_0 : W_0^{1,p}(\mathbb{R}^n_+) \to W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$$
$$u \mapsto u_{|\mathbb{R}^{n-1}}$$

is continuous, onto and such that

$$\text{Ker } \gamma_0 = \overset{\circ}{W}{}_0^{1,p}(\mathbb{R}^n_+) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_0^{1,p}(\mathbb{R}^n_+)}}.$$

Then, we define

$$W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) = \{u \in W_{\frac{1}{p}}^{0,p}(\mathbb{R}^{n-1}), \rho' u \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})\},$$

and

$$W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1}) = \{u \in W_{\frac{1}{p}}^{1,p}(\mathbb{R}^{n-1}), \rho' \nabla u \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})\},$$

where

$$W_{\frac{1}{p}}^{1,p}(\mathbb{R}^{n-1}) = \{u \in \mathcal{D}'(\mathbb{R}^{n-1}), \omega_2^{-1+\frac{1}{p}} u \in L^p(\mathbb{R}^{n-1}), (\rho')^{\frac{1}{p}} \nabla u \in \boldsymbol{L}^p(\mathbb{R}^{n-1})\}.$$

Here again, we equip these spaces with their natural norm. As in [12], we can prove that :

$$\gamma : W_1^{2,p}(\mathbb{R}^n_+) \to W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1}) \times W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$$
$$u \mapsto (u_{|\mathbb{R}^{n-1}}, \frac{\partial u}{\partial \boldsymbol{n}}_{|\mathbb{R}^{n-1}})$$

is a mapping continuous, onto and such that

$$\text{Ker } \gamma = \overset{\circ}{W}{}_1^{2,p}(\mathbb{R}^n_+) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_1^{2,p}(\mathbb{R}^n_+)}}.$$

We define ω_0' the symmetric region of ω_0 with respect to $\mathbb{R}^{n-1}$, Γ_0' the boundary of ω_0', Ω' the symmetric region of Ω, $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \Gamma_0 \cup \Gamma_0'$.

We define too the following functions ℓ^* and ℓ_*. For $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$ and ℓ any function, we set :

$$\ell^*(\boldsymbol{x'}, x_n) = \begin{cases} \ell(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ -\ell(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$\ell_*(\boldsymbol{x'}, x_n) = \begin{cases} \ell(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ \ell(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Finally, we denote, for $p \in]1, \infty[$, $< .,. >_\Gamma$ the duality pairing $W^{-\frac{1}{p},p}(\Gamma)$, $W^{1-\frac{1}{p'},p'}(\Gamma)$, with $\Gamma = \Gamma_0$ or $\widetilde{\Gamma}_0$ and $< .,. >_{\mathbb{R}^{n-1}}$, the pairing $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, $W_0^{1-\frac{1}{p'},p'}(\mathbb{R}^{n-1})$.

We remind that in all this article, we suppose that Ω is of class $C^{1,1}$.

We will denote by C a positive and real constant which may vary from line to line.

2 The problem of Dirichlet

In this section, we want to solve the following problem of Dirichlet :

$$(\mathcal{P}_D)\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ u = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

First, we characterize the following kernel :

$$\mathcal{D}_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_0,\ z = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Proposition 2.1. *For any $p > 1$, $\mathcal{D}_0^p(\Omega) = \{0\}$.*

Proof- Let z be in $\mathcal{D}_0^p(\Omega)$, we define, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z^* \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega})$. For any $\varphi \in \mathcal{D}(\widetilde{\Omega})$, we have :

$$\begin{aligned} < \Delta z^*, \varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} &= < z^*, \Delta\varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= \int_\Omega z(\boldsymbol{x'}, x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} - \int_{\Omega'} z(\boldsymbol{x'}, -x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x}. \end{aligned}$$

Moreover

$$\int_\Omega z(\boldsymbol{x'}, x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} = - < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\mathbb{R}^{n-1}} .$$

Setting $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, we have $\psi \in \mathcal{D}(\widetilde{\Omega})$ and

$$\begin{aligned} \int_{\Omega'} z(\boldsymbol{x'}, -x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} &= \int_\Omega z(\boldsymbol{x'}, x_n)\Delta\psi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} \\ &= - < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\mathbb{R}^{n-1}} . \end{aligned}$$

Thus, we deduce that $< \Delta z^*, \varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} = 0$, *i.e* $\Delta z^* = 0$ in $\widetilde{\Omega}$. So, the function z^* is in the space $\mathcal{A}_0^p(\widetilde{\Omega})$ defined by :

$$\mathcal{A}_0^p(\widetilde{\Omega}) = \{v \in W_0^{1,p}(\widetilde{\Omega}),\ \Delta v = 0 \text{ in } \widetilde{\Omega},\ v = 0 \text{ on } \widetilde{\Gamma}_0\}.$$

Now, we use the characterization of $\mathcal{A}_0^p(\widetilde{\Omega})$ (see [5]). For this, we set μ_0 the function defined by :

$$\mu_0 = U * (\frac{1}{|\widetilde{\Gamma}_0|}\delta_{\widetilde{\Gamma}_0})$$

where $U = \dfrac{1}{2\pi}\ln(r)$ is the fundamental solution of the Laplace's equation in $\mathbb{R}^2$ and $\delta_{\widetilde{\Gamma}_0}$ is defined by :

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2),\ < \delta_{\widetilde{\Gamma}_0}, \varphi > = \int_{\widetilde{\Gamma}_0} \varphi\ d\sigma.$$

i) If $p < n$ or $p = n = 2$, then $\mathcal{A}_0^p(\widetilde{\Omega}) = \{0\}$ and $z^* = 0$ in $\widetilde{\Omega}$, *i.e* $z = 0$ in Ω and $\mathcal{D}_0^p(\Omega) = \{0\}$.

ii) If $p \geq n \geq 3$, then we have $z^* = c(\lambda - 1)$, where c is a real constant and λ is the unique solution in $W_0^{1,p}(\widetilde{\Omega}) \cap W_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta\lambda = 0 \text{ in } \widetilde{\Omega}, \quad \lambda = 1 \text{ on } \widetilde{\Gamma}_0.$$

Thus, on $\mathbb{R}^{n-1}$, $z^* = z = c(\lambda - 1) = 0$. This implies that $c = 0$ because otherwise, λ will be equal to 1 on $\mathbb{R}^{n-1}$, that is not possible because $1 \notin W_0^{\frac{1}{2},2}(\mathbb{R}^{n-1})$. Finally, we deduce that $z = 0$, *i.e* $\mathcal{D}_0^p(\Omega) = \{0\}$.

iii) If $p > n = 2$, then we have $z^* = c(\mu - \mu_0)$, where c is a real constant and the function μ is the unique solution in $W_0^{1,p}(\widetilde{\Omega}) \cap W_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta\mu = 0 \text{ in } \widetilde{\Omega}, \quad \mu = \mu_0 \text{ on } \widetilde{\Gamma}_0.$$

Thus, on $\mathbb{R}$, $z = c(\mu - \mu_0) = 0$. This implies again that $c = 0$ because otherwise μ will be equal to μ_0 on $\mathbb{R}$, that is not possible because $\mu_0 \notin W_0^{\frac{1}{2},2}(\mathbb{R})$. Indeed, let $\boldsymbol{x} = (\boldsymbol{x'}, 0)$ be in $\mathbb{R}$, like

$$\mu_0(\boldsymbol{x}) = \frac{1}{2\pi|\widetilde{\Gamma}_0|} \int_{\widetilde{\Gamma}_0} ln(|\boldsymbol{y} - \boldsymbol{x}|) \, d\sigma_{\boldsymbol{y}},$$

then $\mu_0(\boldsymbol{x'}) \geq C \; ln|\boldsymbol{x'}|$ if $|\boldsymbol{x'}| > \alpha$ with α enough big and

$$\int_{|\boldsymbol{x'}|>\alpha} \frac{|\mu_0(\boldsymbol{x'}, 0)|^2}{|\boldsymbol{x'}| \; log^2(2 + |\boldsymbol{x'}|)} \, d\boldsymbol{x'} \; \geq C \int_{|\boldsymbol{x'}|>\alpha} \frac{d\boldsymbol{x'}}{|\boldsymbol{x'}|} = +\infty$$

that is contradictory with $\mu_0 \in W_0^{\frac{1}{2},2}(\mathbb{R})$. Thus $c = 0$ and we deduce that $z = 0$, *i.e* $\mathcal{D}_0^p(\Omega) = \{0\}$. $\square$

Theorem 2.2. *For each $p > 1$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_0^{-1,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_D)$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \; C(\; \|f\|_{W_0^{-1,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{1}$$

.

Proof- i) We begin to show that solving $(\mathcal{P}_D)$ amounts to solve a problem with homogeneous boundary conditions. We know there exists $u_{g_1} \in W_0^{1,p}(\mathbb{R}^n_+)$ such that $u_{g_1} = g_1$ on $\mathbb{R}^{n-1}$ and

$$\|u_{g_1}\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \; \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})} \tag{2}$$

We set $u_1 = u_{g_1|\Omega}$. Then $u_1 \in W_0^{1,p}(\Omega)$ and the trace η of u_1 on Γ_0 is in $W^{1-\frac{1}{p},p}(\Gamma_0)$. Setting $z = u - u_1$, the problem $(\mathcal{P})$ is equivalent to the problem :

$$(\mathcal{P}_1) \quad -\Delta z = f + \Delta u_1 \text{ in } \Omega, \quad z = g_0 - \eta \text{ on } \Gamma_0, \quad z = 0 \text{ on } \mathbb{R}^{n-1}.$$

We set $g = g_0 - \eta$, and let $R > 0$ be such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$. The function h_0 defined by :

$$h_0 = g \text{ on } \Gamma_0, \quad h_0 = 0 \text{ on } \partial B_R,$$

is in $W^{1-\frac{1}{p},p}(\Gamma_0 \cup \partial B_R)$. We know there exists $u_{h_0} \in W^{1,p}(\Omega_R)$, where $\Omega_R = \Omega \cap B_R$, such that $u_{h_0} = h_0$ on $\Gamma_0 \cup \partial B_R$ and checking the estimate :

$$\|u_{h_0}\|_{W^{1,p}(\Omega_R)} \leq C \, \|h_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0 \cup \partial B_R)}.$$

We set

$$u_0 = u_{h_0} \text{ in } \Omega_R, \quad u_0 = 0 \text{ in } \Omega \setminus \Omega_R.$$

We have $u_0 \in W^{1,p}(\Omega)$, $u_0 = g$ on Γ_0, $u_0 = 0$ on $\mathbb{R}^{n-1}$ and u_0 checks

$$\|u_0\|_{W_0^{1,p}(\Omega)} \leq \; C(\, \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{3}$$

Finally, setting $v = z - u_0$, the problem $(\mathcal{P}_1)$ is equivalent to the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1},$$

where $h = f + \Delta u_1 + \Delta u_0 \in W_0^{-1,p}(\Omega)$.

ii) Now, we want to come back to a problem setted in the open region $\widetilde{\Omega}$, problem that we know solving. Let φ be in $\overset{\circ}{W}{}_0^{1,p'}(\widetilde{\Omega})$, we set for almost any $(\boldsymbol{x'}, x_n) \in \Omega$,

$$\pi\varphi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, x_n) - \varphi(\boldsymbol{x'}, -x_n).$$

It is obvious that $\pi\varphi \in \overset{\circ}{W}{}_0^{1,p'}(\Omega)$ and, for any $\varphi \in \overset{\circ}{W}{}_0^{1,p'}(\widetilde{\Omega})$, we define the operator h_π by

$$< h_\pi, \varphi > := < h, \pi\varphi >_{W_0^{-1,p}(\Omega) \times \overset{\circ}{W}{}_0^{1,p'}(\Omega)} .$$

We notice that h_π is in $W_0^{-1,p}(\widetilde{\Omega})$ and satisfies

$$\|h_\pi\|_{W_0^{-1,p}(\widetilde{\Omega})} \leq 2\|h\|_{W_0^{-1,p}(\Omega)}. \tag{4}$$

Now, we suppose that $p \geq 2$. Thanks to [5], we know there exists $w \in W_0^{1,p}(\widetilde{\Omega})$ solution of

$$-\Delta w = h_\pi \text{ in } \widetilde{\Omega}, \quad w = 0 \text{ on } \widetilde{\Gamma}_0,$$

checking the estimate

$$\|w\|_{W_0^{1,p}(\widetilde{\Omega})} \leq \; C \, \|h_\pi\|_{W_0^{-1,p}(\widetilde{\Omega})}. \tag{5}$$

The function $v = \dfrac{1}{2}\pi w$ belongs to $\overset{\circ}{W}{}_0^{1,p}(\Omega)$ and we have :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \; 2\|w\|_{W_0^{1,p}(\widetilde{\Omega})}. \tag{6}$$

Now, let us show that $-\Delta v = h$ in Ω *i.e* v solution of $(\mathcal{P}')$. Let φ be in $\mathcal{D}(\Omega)$, then :

$$\begin{aligned} 2 < \Delta v, \varphi >_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} &= 2 < v, \Delta\varphi >_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \\ &= \int_\Omega [w(\boldsymbol{x'}, x_n) - w(\boldsymbol{x'}, -x_n)] \Delta\varphi \, d\boldsymbol{x}. \end{aligned}$$

Moreover, setting $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, then $\psi \in \mathcal{D}(\Omega')$ and we have the relations

$$\int_\Omega w(\boldsymbol{x'}, x_n)\Delta\varphi \, d\boldsymbol{x} = < \Delta w, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$$

and

$$\int_\Omega w(\boldsymbol{x'}, -x_n)\Delta\varphi \, d\boldsymbol{x} = \int_{\Omega'} w(\boldsymbol{x'}, x_n)\Delta\psi \, d\boldsymbol{x} = < \Delta w, \psi >_{\mathcal{D}'(\Omega'),\mathcal{D}(\Omega')} .$$

Setting $\widetilde{\varphi}$ and $\widetilde{\psi}$ the extensions by 0 in $\widetilde{\Omega}$ of φ and ψ respectively, we deduce that :

$$\begin{aligned} 2 < \Delta v, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= < \Delta w, \widetilde{\varphi} - \widetilde{\psi} >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= - < h_\pi, \widetilde{\varphi} - \widetilde{\psi} >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= - < h, \pi\widetilde{\varphi} - \pi\widetilde{\psi} >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= -2 < h, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \end{aligned}$$

$i.e$ $-\Delta v = h$ in Ω. So, we have checked that, if $p \geq 2$, the operator

$$\Delta : \overset{\circ}{W}{}_0^{1,p}(\Omega) \mapsto W_0^{-1,p}(\Omega)$$

is a isomorphism, and, by duality, the operator

$$\Delta : \overset{\circ}{W}{}_0^{1,p'}(\Omega) \mapsto W_0^{-1,p'}(\Omega)$$

is an isomorphism too. So, if $p < 2$, the problem $(\mathcal{P}')$ has also a unique solution $v \in W_0^{1,p}(\Omega)$. Thus, the problem $(\mathcal{P}_D)$ has a unique solution for $1 < p < \infty$. Finally, thanks to (2), (3), (4), (5) and (6), we have the estimate (1). $\square$

3 The problem of Neumann

We remind that in this section and in the following ones, Ω is supposed to be of class $C^{1,1}$. In this section, we want to solve the following problem :

$$(\mathcal{P}_N) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Here, we suppose that $\dfrac{n}{p'} \neq 1$.

First, we characterize the following kernel :

$$\mathcal{N}_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Proposition 3.1. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, $\mathcal{N}_0^p(\Omega) = \mathcal{P}_{[1-n/p]}$.*

Proof- First, we notice that $\mathcal{P}_{[1-n/p]} \subset \mathcal{N}_0^p(\Omega)$. Let us show the other inclusion. Let z be in $\mathcal{N}_0^p(\Omega)$ and its associated function z_* which is in $W_0^{1,p}(\widetilde{\Omega})$. Like $\dfrac{\partial z}{\partial \boldsymbol{n}} = 0$ on Γ_0, we have $\dfrac{\partial z_*}{\partial \boldsymbol{n}} = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 2.1, that $\Delta z_* = 0$ in $\widetilde{\Omega}$. So, the function z_* belongs to the space $\{v \in W_0^{1,p}(\widetilde{\Omega}),\ \Delta v = 0 \text{ in } \widetilde{\Omega},\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0\}$ which is equal to $\mathcal{P}_{[1-n/p]}$ (see [5]). Thus, if $p < n$, $z_* = 0$ in $\widetilde{\Omega}$ and $z = 0$ in Ω. If $p \geq n$, z_* is constant in $\widetilde{\Omega}$, so z is constant in Ω. In other words, we have $\mathcal{N}_0^p(\Omega) = \mathcal{P}_{[1-n/p]}$. $\square$

The following theorem allows us to obtain strong solutions of the problem $(\mathcal{P}_N)$.

Theorem 3.2. *For each $p > 1$ checking $\dfrac{n}{p'} \neq 1$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying, if $p < \dfrac{n}{n-1}$, the following compatibility condition :*

$$\int_\Omega f\ d\boldsymbol{x} + \int_{\Gamma_0} g_0\ d\sigma + \int_{\mathbb{R}^{n-1}} g_1\ d\boldsymbol{x'} = 0, \tag{7}$$

the problem $(\mathcal{P}_N)$ has a unique solution $u \in W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}$. Moreover, u satisfies

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{8}$$

Proof- First, we notice that, thanks to the hypothesis on the data, any integral of (7) has a meaning when $p < \dfrac{n}{n-1}$, the last one being finished because $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \subset W_{\frac{1}{p}}^{0,p}(\mathbb{R}^{n-1}) \subset L^1(\mathbb{R}^{n-1})$. Moreover, like $\dfrac{n}{p'} \neq 1$, we have the injection $W_{\frac{1}{p}}^{0,p}(\mathbb{R}^{n-1}) \subset W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}_+^n)$ such that $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = g_1$ and $u_{g_1} = 0$ on $\mathbb{R}^{n-1}$ checking :

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}_+^n)} \leq\ C\ \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{9}$$

We set u_1 the restriction of u_{g_1} to Ω and η the normal derivative of u_1 on Γ_0. Finally, we set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Then, setting $v = u - u_1 \in W_1^{2,p}(\Omega)$, the problem $(\mathcal{P}_N)$ is equivalent to the following problem $(\mathcal{P}')$:

$$(\mathcal{P}')\begin{cases} -\Delta v = h & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

We build the two functions $h_* \in W_1^{0,p}(\widetilde{\Omega})$ and $g_* \in W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ which check, if $p < \dfrac{n}{n-1}$ and thanks to (7), the equality $\displaystyle\int_{\widetilde{\Omega}} h_*\ d\boldsymbol{x} + \int_{\widetilde{\Gamma}_0} g_*\ d\sigma = 0$. Thanks to

[5], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$, solution of

$$-\Delta w = h_* \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g_* \text{ on } \widetilde{\Gamma}_0,$$

checking :

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq C(\|h\|_{W_1^{0,p}(\widetilde{\Omega})} + \|g\|_{W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)}).$$

Now, let $w_0 \in W_1^{2,p}(\widetilde{\Omega})$ be a solution of the above problem. For almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n)$. As h_* is even with respect to x_n, we easily check that we have $-\Delta v_0 = h_*$ in $\widetilde{\Omega}$. Moreover, by the definition of the normal derivative on $\widetilde{\Gamma}_0$, we notice that we have, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$:

$$\frac{\partial v_0}{\partial \boldsymbol{n}}(\boldsymbol{x'}, x_n) = \frac{\partial w_0}{\partial \boldsymbol{n}}(\boldsymbol{x'}, -x_n).$$

As g_* is even with respect to x_n, we again easily check that we have $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_*$ on $\widetilde{\Gamma}_0$. So $v_0 \in W_1^{2,p}(\widetilde{\Omega})$ is solution of the same problem that w_0 satisfies. Thus, the difference $v_0 - w_0$ is equal to a constant c which is necessary nil. So $w_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n)$ and thus $\dfrac{\partial w_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. The restriction v of w_0 to Ω being in $W_1^{2,p}(\Omega)$, is solution of $(\mathcal{P}')$ and checks :

$$\|v\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq C(\|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}).$$

Finally, from this inequality and (9), comes the estimate (8). □

Now, we search weak solutions of the problem $(\mathcal{P}_N)$:

Theorem 3.3. *For each $p > 1$ checking $\dfrac{n}{p'} \neq 1$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying, if $p < \dfrac{n}{n-1}$, the following condition of compatibility :*

$$\int_\Omega f \; d\boldsymbol{x} + < g_0, 1 >_{\Gamma_0} + < g_1, 1 >_{\mathbb{R}^{n-1}} = 0, \tag{10}$$

the problem $(\mathcal{P}_N)$ has a unique solution $u \in W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}$. Moreover, u satisfies

$$\|u\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{11}$$

Proof- i) First, we suppose $\dfrac{n}{p'} > 1$.

Theorem 3.2 assures the existence of a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and checking

$$\|s\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \|s\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \; C\; \|f\|_{W_1^{0,p}(\Omega)}. \qquad (12)$$

Then, thanks to [2], there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+ \;\; \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

checking the estimate

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq \; C\; \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \qquad (13)$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$ and we want to solve the following problem :

$$(\mathcal{P}') \;\; \Delta v = 0 \text{ in } \Omega, \;\; \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \;\; \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. For almost any $(\boldsymbol{x'}, x_n) \in \Gamma_0$, we set

$$\pi\mu(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, x_n) + \mu(\boldsymbol{x'}, -x_n).$$

We notice that $\pi\mu \in W^{1-\frac{1}{p'},p'}(\Gamma_0)$ and we define

$$< g_\pi, \mu >:= < g, \pi\mu >_{\Gamma_0} .$$

It is obvious that $g_\pi \in W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and that g is the restriction of g_π to Γ_0. Moreover, we easily check that g_π is even with respect to x_n, *i.e*

$$< g_\pi, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \mu >_{\widetilde{\Gamma}_0},$$

where $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$ with $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$. Thanks to [5], there exists a function $w \in W_0^{1,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$ solution of the following problem :

$$\Delta w = 0 \text{ in } \widetilde{\Omega}, \;\; \frac{\partial w}{\partial \boldsymbol{n}} = g_\pi \text{ on } \widetilde{\Gamma}_0,$$

and checking :

$$\|w\|_{W_0^{1,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq \; C\; \|g_\pi\|_{W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)} \leq \; C\; \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}.$$

Let w_0 be a solution of the problem and we set for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$:

$$v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n).$$

The function v_0 is in $W_0^{1,p}(\widetilde{\Omega})$ and like $\Delta w_0 = 0$ on $\widetilde{\Omega}$, we easily check that Δv_0 is nil too. Thus, $\dfrac{\partial v_0}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$. Now, we want to show that

$\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$. Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. We know there exists $\varphi \in W_0^{1,p'}(\widetilde{\Omega})$ such that $\varphi = \mu$ on $\widetilde{\Gamma}_0$ and $\|\varphi\|_{W_0^{1,p'}(\widetilde{\Omega})} \leq C \; \|\mu\|_{W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)}$. We have :

$$< \frac{\partial v_0}{\partial \boldsymbol{n}}, \mu >_{\widetilde{\Gamma}_0} = \int_{\widetilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \; d\boldsymbol{x}.$$

For almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$. The function ψ is in $W_0^{1,p'}(\widetilde{\Omega})$ and we set $\xi \in W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$ the trace of ψ on $\widetilde{\Gamma}_0$. We notice that $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$. Moreover, we show that

$$\int_{\widetilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \; d\boldsymbol{x} = \int_{\widetilde{\Omega}} \nabla w_0 \cdot \nabla \psi \; d\boldsymbol{x}.$$

Thus,

$$< \frac{\partial v_0}{\partial \boldsymbol{n}}, \mu >_{\widetilde{\Gamma}_0} = < \frac{\partial w_0}{\partial \boldsymbol{n}}, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \mu >_{\widetilde{\Gamma}_0} .$$

So $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$ and v_0 is solution of the same problem that w_0 satisfies, which implies that $v_0 - w_0$ is a constant, constant which is necessary nil. The restriction of w_0 to Ω, that we note v, being in $W_0^{1,p}(\Omega)$, is solution of the problem $(\mathcal{P}')$ and we have the estimate

$$\|v\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \; C \; \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}. \tag{14}$$

Finally, the function $u = z + s + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and thanks to (12), (13) and (14), we have (11).

ii) Now, we suppose that $\dfrac{n}{p'} < 1$.

Let α be in $W_1^{0,p}(\Omega)$, β in $W^{1-\frac{1}{p},p}(\Gamma_0)$ and γ in $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ such that :

$$\int_\Omega \alpha \; d\boldsymbol{x} = \int_{\Gamma_0} \beta \; d\sigma = \int_{\mathbb{R}^{n-1}} \gamma \; d\boldsymbol{x'} = 1.$$

Here, we notice that we have $W^{1-\frac{1}{p},p}(\Gamma_0) \subset W^{-\frac{1}{p},p}(\Gamma_0)$ and $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \subset W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We set

$$F = \; (\int_\Omega f \; d\boldsymbol{x}) \; \alpha, \quad G_0 = < g_0, 1 >_{\Gamma_0} \beta \;\; \text{and } G_1 = < g_1, 1 >_{\mathbb{R}^{n-1}} \gamma.$$

Thanks to Theorem 3.2, we know there exists $r \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta r = f - F \text{ in } \Omega, \quad \frac{\partial r}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad \frac{\partial r}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

checking :

$$\|r\|_{W_0^{1,p}(\Omega)} \leq \|r\|_{W_1^{2,p}(\Omega)} \leq \; C \; \|f - F\|_{W_1^{0,p}(\Omega)} \leq \; C \; \|f\|_{W_1^{0,p}(\Omega)}. \tag{15}$$

Thanks to [2], like $< G_1 - g_1, 1 >_{\mathbb{R}^{n-1}} = 0$, there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 - G_1 \text{ on } \mathbb{R}^{n-1},$$

checking :

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \, \|g_1 - G_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})} \leq C \, \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{16}$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$ and satisfy the following equality :

$$< \eta, 1 >_{\Gamma_0} = 0.$$

We set $g = g_0 - G_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$, and we do the same reasonning as in the point **i)** to show there exists $v \in W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

checking :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \, \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)} \leq C(\, \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{17}$$

We notice that the compatibility condition on g_π is satisfied because $< g_\pi, 1 >_{\widetilde{\Gamma}_0} = 2 < g, 1 >_{\Gamma_0} = 0$. Finally, noticing that $F \in W_1^{0,p}(\Omega)$, $G_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$, $G_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and that the condition (10) is satisfied, thanks to Theorem 3.2, there exists a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta s = F \text{ in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = G_0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = G_1 \text{ on } \mathbb{R}^{n-1},$$

and checking the following estimate :

$$\|s\|_{W_0^{1,p}(\Omega)} \leq C(\, \|F\|_{W_1^{0,p}(\Omega)} + \|G_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|G_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{18}$$

Finally, the function $u = r + z + v + s \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and the estimate (11) is given by (15), (16) (17) and (18). $\square$

Remark : We notice that, when the data are more regular, the weak solution is also more regular ; in fact, it is the solution of Theorem 3.2.

4 The first mixed problem

In this section, we want to solve the following problem :

$$(\mathcal{P}_{M_1}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Here, we suppose too that $\frac{n}{p'} \neq 1$ and first, we characterize the following kernel :

$$\mathcal{E}_0^p(\Omega) = \{z \in \overset{\circ}{W}{}_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_0,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

We have the following result (we send back to the proof of Proposition 2.1 for the definition of μ_0) :

Proposition 4.1. *i) If $p < n$ or $p = n = 2$, then $\mathcal{E}_0^p(\Omega) = \{0\}$.*
ii) If $p \geq n \geq 3$, then $\mathcal{E}_0^p(\Omega) = \{c(\lambda - 1),\ c \in \mathbb{R}\}$ where λ is the unique solution in $\overset{\circ}{W}{}_0^{1,2}(\Omega) \cap \overset{\circ}{W}{}_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1)\quad \Delta\lambda = 0 \text{ in } \Omega,\quad \lambda = 1 \text{ on } \Gamma_0,\quad \frac{\partial \lambda}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

iii) If $p > n = 2$, then $\mathcal{E}_0^p(\Omega) = \{c(\mu - \mu_0),\ c \in \mathbb{R}\}$ where μ is the unique solution in $\overset{\circ}{W}{}_0^{1,2}(\Omega) \cap \overset{\circ}{W}{}_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_2)$:

$$(\mathcal{P}_2)\quad \Delta\mu = 0 \text{ in } \Omega,\quad \mu = \mu_0 \text{ on } \Gamma_0,\quad \frac{\partial \mu}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}.$$

Proof- Let z be in $\mathcal{E}_0^p(\Omega)$. We define, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z_* \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega})$, $z_* = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 2.1 that $\Delta z_* = 0$ in $\widetilde{\Omega}$. So the function z_* is in the space

$$\mathcal{A}_0^p(\widetilde{\Omega}) = \{z \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}),\ \Delta z = 0 \text{ in } \widetilde{\Omega},\ z = 0 \text{ on } \widetilde{\Gamma}_0\}$$

Now, we use the characterization of $\mathcal{A}_0^p(\widetilde{\Omega})$ (see [5]).

i) If $p < n$ or if $p = n = 2$, then $\mathcal{A}_0^p(\widetilde{\Omega}) = \{0\}$ which implies that $z_* = 0$ in $\widetilde{\Omega}$ and so $z = 0$ in Ω, *i.e* $\mathcal{E}_0^p(\Omega) = \{0\}$.

ii) If $p \geq n \geq 3$, then $z_* = c(\widetilde{\lambda} - 1)$, where c is a real constant and $\widetilde{\lambda}$ is the unique solution in $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta\widetilde{\lambda} = 0 \text{ in } \widetilde{\Omega},\quad \widetilde{\lambda} = 1 \text{ on } \widetilde{\Gamma}_0.$$

Now, we set, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, $\beta(\boldsymbol{x'}, x_n) = \widetilde{\lambda}(\boldsymbol{x'}, -x_n)$. We easily check that β, belonging to $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$, is solution of the same problem that $\widetilde{\lambda}$ satisfies, but this solution is unique, so we deduce that $\beta = \widetilde{\lambda}$ and so on $\mathbb{R}^{n-1}$, $\frac{\partial \widetilde{\lambda}}{\partial \boldsymbol{n}} = 0$. Thus, setting λ the restriction of $\widetilde{\lambda}$ to Ω, $\lambda \in \overset{\circ}{W}{}_0^{1,p}(\Omega) \cap \overset{\circ}{W}{}_0^{1,2}(\Omega)$ is solution of the problem $(\mathcal{P}_1)$. Moreover, this solution is unique. Indeed, if θ is an other solution, θ_* is solution of the same problem that $\widetilde{\lambda}$ satisfies in $\widetilde{\Omega}$, so $\theta_* = \widetilde{\lambda}$ in $\widetilde{\Omega}$ and $\theta = \lambda$ in Ω.

iii) If $p > n = 2$, so, we have $z_* = c(\widetilde{\mu} - \mu_0)$, where c is a real constant and $\widetilde{\mu}$ the unique solution in $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta\widetilde{\mu} = 0 \text{ in } \widetilde{\Omega},\quad \widetilde{\mu} = \mu_0 \text{ on } \widetilde{\Gamma}_0.$$

But, we notice that μ_0 can also be written

$$\mu_0(\boldsymbol{x}) = \frac{1}{2\pi|\widetilde{\Gamma}_0|}\int_{\widetilde{\Gamma}_0} ln(|\boldsymbol{y}-\boldsymbol{x}|)\; d\sigma_{\boldsymbol{y}}.$$

As $\widetilde{\Gamma}_0$ is symmetric with respect to $\mathbb{R}^{n-1}$, we deduce that μ_0 is symmetric too, and so $\dfrac{\partial \mu_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. Now, for $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $\xi(\boldsymbol{x'}, x_n) = \widetilde{\mu}(\boldsymbol{x'}, -x_n)$. We checks that ξ, belonging to $W_0^{1,p}(\widetilde{\Omega}) \cap W_0^{1,2}(\widetilde{\Omega})$, is solution of the same problem that $\widetilde{\mu}$ satisfies, but this solution being unique, we deduce that $\xi = \widetilde{\mu}$ and so, on $\mathbb{R}^{n-1}$, $\dfrac{\partial \widetilde{\mu}}{\partial \boldsymbol{n}} = 0$. Thus, setting μ the restriction of $\widetilde{\mu}$ to Ω, $\mu \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is solution of the problem $(\mathcal{P}_2)$ and we show that this solution is unique like in the point **ii)**. Noticing that we have also $\Delta\mu_0 = 0$ in Ω, the other inclusion becomes obvious. □

Let f be in $W_1^{0,p}(\Omega)$, g_0 in $W^{1-\frac{1}{p},p}(\Gamma_0)$ and g_1 in $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We remind that we search $u \in W_0^{1,p}(\Omega)$ solution of the problem $(\mathcal{P}_{M_1})$. We suppose that such a solution $u \in W_0^{1,p}(\Omega)$ exists. Then, for any $v \in W_0^{1,p'}(\Omega)$, we have :

$$\int_\Omega -v\Delta u \; d\boldsymbol{x} \;= \int_\Omega \nabla u\cdot\nabla v \; d\boldsymbol{x} - <\frac{\partial u}{\partial \boldsymbol{n}}, v>_{\Gamma_0\cup\mathbb{R}^{n-1}} .$$

In particular, for any $\varphi \in \mathcal{E}_0^{p'}(\Omega)$:

$$\int_\Omega f\varphi \; d\boldsymbol{x} \;= \int_\Omega \nabla u\cdot\nabla \varphi \; d\boldsymbol{x} - < g_1, \varphi>_{\mathbb{R}^{n-1}} .$$

We have too :

$$0 = \int_\Omega -u\Delta \varphi \; d\boldsymbol{x} = \int_\Omega \nabla \varphi\cdot\nabla u \; d\boldsymbol{x} - <\frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0>_{\Gamma_0} .$$

We deduce from this that if $u \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_{M_1})$, the data must check the following compatibility condition :

$$\forall\varphi \in \mathcal{E}_0^{p'}(\Omega), \quad \int_\Omega f\varphi \; d\boldsymbol{x} = <\frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0>_{\Gamma_0} - < g_1, \varphi>_{\mathbb{R}^{n-1}} . \tag{19}$$

Now, we are going to search strong solutions for the problem $(\mathcal{P}_{M_1})$.

Theorem 4.2. *For each $p > \dfrac{n}{n-1}$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_1^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)$ solution of $(\mathcal{P}_{M_1})$. Moreover, u satisfies*

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)} \le \; C(\; \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{20}$$

Proof- We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}^n_+)$ such that $u_{g_1} = 0$ and $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = g_1$ on $\mathbb{R}^{n-1}$ checking the estimate :

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}^n_+)} \le \; C\; \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{21}$$

We set u_1 the restriction of u_{g_1} to Ω and η the trace of u_1 on Γ_0. Then, we set $g = g_0 - \eta \in W^{2-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Now, we must find $v \in W_1^{2,p}(\Omega)$ solution of the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad v = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

For this, we define the functions $h_* \in W_1^{0,p}(\widetilde{\Omega})$ and $g_* \in W^{2-\frac{1}{p},p}(\widetilde{\Gamma}_0)$. Thanks to [5], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{A}_0^p(\widetilde{\Omega})$, solution of

$$-\Delta w = h_* \text{ in } \widetilde{\Omega}, \quad w = g_* \text{ on } \widetilde{\Gamma}_0,$$

and checking the estimate :

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{A}_0^p(\widetilde{\Omega})} \leq C(\|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}).$$

Let w_0 be a solution of this problem and for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set :

$$v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n).$$

Thanks to the symmetry of h_*, g_*, $\widetilde{\Omega}$ and $\widetilde{\Gamma}_0$ with respect to $\mathbb{R}^{n-1}$, we easily show that v_0 is solution of the same problem that w_0. Thus $v_0 = w_0 + k$ where $k \in \mathcal{A}_0^p(\widetilde{\Omega})$. Moreover, we show that $\dfrac{\partial k}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$ and we deduce that $\dfrac{\partial w_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$, so, the function v, restriction of w_0 to Ω, is in $W_1^{2,p}(\Omega)$, is solution of $(\mathcal{P}')$ and checks :

$$\|v\|_{W_1^{2,p}(\Omega)} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{22}$$

Finally, $u = v + u_1 \in W_1^{2,p}(\Omega)$ is solution of $(\mathcal{P}_{M_1})$ and (20) comes from (21) and (22). $\square$

Now we search weak solutions of the problem $(\mathcal{P}_{M_1})$. For this, in the following theorem, we shall introduce a lemma between points **i)** and **ii)**. This lemma, whose the proof shall use the result of the point **i)**, allows us to obtain an “inf-sup” condition, fundamental condition for the resolution of the point **ii)**.

Theorem 4.3. *For each $p > 1$ checking $\dfrac{n}{p'} \neq 1$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, checking, if $p < \dfrac{n}{n-1}$, the compatibility condition (19), there exists a unique $u \in W_0^{1,p}(\Omega)/\mathcal{E}_0^p(\Omega)$ solution of $(\mathcal{P}_{M_1})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)/\mathcal{E}_0^p(\Omega)} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{23}$$

Proof- i) First, we suppose $\dfrac{n}{p'} > 1$, *i.e* $p > \dfrac{n}{n-1}$.

Thanks to the previous theorem, we begin to show that there exists a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \quad s = 0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

and checking the estimate :

$$\|s\|_{W_0^{1,p}(\Omega)} \leq \|s\|_{W_1^{2,p}(\Omega)} \leq \; C \; \|f\|_{W_1^{0,p}(\Omega)}. \tag{24}$$

Moreover, thanks to [2], there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

and checking :

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq \; C \; \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{25}$$

We denote again by z the restriction of z to Ω, so $z \in W_0^{1,p}(\Omega)$ and $\Delta z = 0$ in Ω. Now, let η be the trace of z on Γ_0. The function η is in $W^{1-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$. Like done in the proof of Theorem 4.2 and thanks to [5], we show there exists $v \in W_0^{1,p}(\Omega)$ solution of :

$$\Delta v = 0 \text{ in } \Omega, \quad v = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and ckecking the estimate :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \; C \; \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \tag{26}$$

Finally, the function $u = s + z + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_{M_1})$ and the estimate (23) comes from (24), (25) and (26).

Now, we set

$$V_p = \{v \in W_0^{1,p}(\Omega), \; v = 0 \text{ on } \Gamma_0\},$$

and we introduce the following lemma to solve the point **ii)** of the theorem :

Lemma 4.4. *Let p be such that $p > \dfrac{n}{n-1}$. There exists a real constant $\beta > 0$ such that*

$$\inf_{\substack{w \in V_{p'} \\ w \neq 0}} \sup_{\substack{v \in V_p \\ v \neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \;\geq\; \beta,$$

and the operators B from $V_p / Ker\ B$ to $(V_{p'})'$ and B' from $V_{p'}$ to $(V_p)' \perp Ker\ B$ defined by :

$$\forall v \in V_p, \; \forall w \in V_{p'}, \; < Bv, w > = < v, B'w > = \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}$$

are isomorphisms.

Proof- We must firstly show an equivalent proposition to Proposition 3.2 of [3], $i.e$ for any $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, there exists $\boldsymbol{z} \in \overset{\circ}{H}_p(\Omega)$ and $\varphi \in V_p$, such that :

$$\begin{aligned} &\boldsymbol{g} = \nabla\varphi + \boldsymbol{z}, \\ &\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)} \leq \ C\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} \end{aligned}$$

where $C > 0$ is a real constant which depends only on Ω and p and

$$\overset{\circ}{H}_p(\Omega) = \{\boldsymbol{z} \in \boldsymbol{L}^p(\Omega),\ \text{div}\ \boldsymbol{z} = 0 \text{ in } \Omega,\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

The proof takes one's inspiration from the proof of [3], using the fact that thanks to [4], setting

$$\widetilde{\boldsymbol{g}} = \boldsymbol{g} \text{ in } \Omega,\ \widetilde{\boldsymbol{g}} = 0 \text{ in } \omega_0,\ \widetilde{\boldsymbol{g}} = 0 \text{ in } \mathbb{R}^n_-,$$

there exists $v \in W^{1,p}_0(\mathbb{R}^n)$ solution of

$$\Delta v = \text{div}\ \widetilde{\boldsymbol{g}} \text{ in } \mathbb{R}^n,$$

such that $\|\nabla v\|_{\boldsymbol{L}^p(\mathbb{R}^n)} \leq \ C\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$. We denote again by v the restriction of v to Ω. We notice that, thanks to [7], $(\boldsymbol{g} - \nabla v)\cdot\boldsymbol{n} \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ because div $(\boldsymbol{g} - \nabla v) = 0 \in W^{0,p}_1(\Omega)$. Moreover, thanks to the point **i)**, there exists a unique $w \in W^{1,p}_0(\Omega)$ solution of :

$$\Delta w = 0 \text{ in } \Omega,\quad w = -v \text{ on } \Gamma_0,\quad \frac{\partial w}{\partial \boldsymbol{n}} = (\boldsymbol{g} - \nabla v)\cdot\boldsymbol{n} \text{ on } \mathbb{R}^{n-1},$$

such that $\|\nabla w\|_{\boldsymbol{L}^p(\Omega)} \leq \ C\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$. Then, setting $\varphi = v + w$ and $\boldsymbol{z} = \boldsymbol{g} - \nabla\varphi$, we have the searched result, and, like done in [3], the "inf-sup" condition. The second part of the lemma comes from the Babuška-Brezzi's theorem (see [3] for example). $\square$

ii) We suppose $\dfrac{n}{p'} < 1$, $i.e$ $p < \dfrac{n}{n-1}$.

Thanks to Section 2, we know there exists a unique $z \in W^{1,p}_0(\Omega)$ solution of the problem

$$\Delta z = 0 \text{ in } \Omega,\quad z = g_0 \text{ on } \Gamma_0,\quad z = 0 \text{ on } \mathbb{R}^{n-1},$$

and checking the estimate :

$$\|z\|_{W^{1,p}_0(\Omega)} \leq \ C\ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \tag{27}$$

Like $\Delta z = 0 \in W^{0,p}_1(\Omega)$, $\eta = \dfrac{\partial z}{\partial \boldsymbol{n}}$ has a meaning in $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We set $g = g_1 - \eta \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and we want to solve the following problem $(\mathcal{P}')$:

$$(\mathcal{P}')\quad -\Delta v = f \text{ in } \Omega,\quad v = 0 \text{ on } \Gamma_0,\quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \mathbb{R}^{n-1}.$$

For this, for any $w \in V_{p'}$ we define the operator :

$$Tw = \int_\Omega f w\ d\boldsymbol{x} + < g, w >_{\mathbb{R}^{n-1}}\ .$$

We easily check that $T \in (V_{p'})'$. We define the following problem $(\mathcal{FV})$: find $v \in V_p$ such that for any $w \in V_{p'}$, we have :

$$\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x} = Tw.$$

We notice that if $v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}')$, it is also solution of $(\mathcal{FV})$. Conversely, let $v \in V_p$ be a solution of $(\mathcal{FV})$ and let φ be in $\mathcal{D}(\Omega) \subset V_{p'}$. So

$$< \Delta v, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = -\int_\Omega \nabla v \cdot \nabla \varphi \; d\boldsymbol{x} = -T\varphi = -< f, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},$$

i.e $-\Delta v = f$ in Ω. The function $\Delta v \in W_1^{0,p}(\Omega)$, so $\dfrac{\partial v}{\partial \boldsymbol{n}}$ has a meaning in $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Now, we want to show that we have $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on $\mathbb{R}^{n-1}$. We know that, for any $\mu \in W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})$, there exists $u_1 \in W_1^{2,p'}(\mathbb{R}^n_+)$ such that

$$u_1 = \mu \;\text{ and }\; \frac{\partial u_1}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

with $\|u_1\|_{W_1^{2,p'}(\mathbb{R}^n_+)} \leq C \; \|\mu\|_{W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})}$. We denote again by $u_1 \in W_1^{2,p'}(\Omega)$ the restriction of u_1 to Ω and $\xi \in W^{2-\frac{1}{p'},p'}(\Gamma_0)$ the trace of u_1 on Γ_0. There exists $u_0 \in W^{2,p'}(\Omega_R)$, where $R > 0$ is such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$ and $\Omega_R = \Omega \cap B_R$, checking

$$u_0 = \xi \;\text{ and }\; \frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad u_0 = \frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ on } \partial B_R$$

and

$$\|u_0\|_{W^{2,p'}(\Omega_R)} \leq \; C \; \|\xi\|_{W^{2-\frac{1}{p'},p'}(\Gamma_0)}$$

We set $\widetilde{u}_0$ the extension of u_0 by 0 outside B_R. We have $\widetilde{u}_0 \in W_1^{2,p'}(\Omega)$ and

$$\widetilde{u}_0 = \xi \;\text{ and }\; \frac{\partial \widetilde{u}_0}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad \widetilde{u}_0 = \frac{\partial \widetilde{u}_0}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

with $\|\widetilde{u}_0\|_{W_1^{2,p'}(\Omega)} \leq C \; \|u_1\|_{W_1^{2,p'}(\Omega)}$. We set $u = u_1 - \widetilde{u}_0 \in W_1^{2,p'}(\Omega)$, then u checks

$$u = 0 \text{ on } \Gamma_0, \quad u = \mu \;\text{ and }\; \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}$$

and

$$\|u\|_{W_1^{2,p'}(\Omega)} \leq \; C \; \|\mu\|_{W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})}$$

Thus, noticing that $u \in V_{p'}$ and $\mu \in W_0^{1-\frac{1}{p'},p'}(\mathbb{R}^{n-1})$ because, for any value of n and p', $W_1^{2,p'}(\Omega) \subset W_0^{1,p'}(\Omega)$, we have

$$\begin{aligned} < \frac{\partial v}{\partial \boldsymbol{n}}, \mu >_{\mathbb{R}^{n-1}} = < \frac{\partial v}{\partial \boldsymbol{n}}, u >_{\Gamma_0 \cup \mathbb{R}^{n-1}} &= \int_\Omega u \Delta v \; d\boldsymbol{x} + \int_\Omega \nabla v \cdot \nabla u \; d\boldsymbol{x} \\ &= -\int_\Omega f u \; d\boldsymbol{x} + Tu \\ &= < g, \mu >_{\mathbb{R}^{n-1}}, \end{aligned}$$

$i.e$ $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on $\mathbb{R}^{n-1}$. So, problems $(\mathcal{P}')$ and $(\mathcal{FV})$ are equivalents. Moreover, like $p < \dfrac{n}{n-1}$, $p' > \dfrac{n}{n-1}$ and we apply the previous lemma noticing that we have Ker $B = \mathcal{E}_0^{p'}(\Omega)$. We deduce that

$$B' \text{ is an isomorphism from } V_p \text{ to } (V_{p'})' \perp \mathcal{E}_0^{p'}(\Omega). \tag{28}$$

Moreover $T \in (V_{p'})' \perp \mathcal{E}_0^{p'}(\Omega)$. Indeed, for any $\varphi \in \mathcal{E}_0^{p'}(\Omega)$, we have

$$T\varphi = \int_\Omega f\varphi \, d\boldsymbol{x} + < g_1, \varphi >_{\mathbb{R}^{n-1}} - < \eta, \varphi >_{\mathbb{R}^{n-1}}$$

and

$$< \eta, \varphi >_{\mathbb{R}^{n-1}} = < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0 \cup \mathbb{R}^{n-1}} = \int_\Omega \nabla z \cdot \nabla \varphi \, d\boldsymbol{x} = < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0},$$

which implies, thanks to the condition (19), that $T\varphi = 0$. This allows us to deduce, thanks to (28), that there exists a unique $v \in V_p$ such that $B'v = T$, $i.e$ solution of $(\mathcal{FV})$ and consequently of $(\mathcal{P}')$ and we have the following estimate :

$$\|v\|_{W_0^{1,p}(\Omega)} \le C \, (\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{29}$$

Finally, $u = z + v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}_{M_1})$ and we have the estimate (23) thanks to (27) and (29). $\square$

Remark : We notice that when $p > \dfrac{n}{n-1}$ and when the data are more regular, the weak solution is more regular too ; it is in fact the solution of Theorem 4.2.

5 The second mixed problem

In this section, we want to solve the following problem :

$$(\mathcal{P}_{M_2}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_0 & \text{on } \Gamma_0, \\ u = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Here, we still suppose $\dfrac{n}{p'} \neq 1$ and, first, we characterize the following kernel :

$$\mathcal{F}_0^p(\Omega) = \{ z \in W_0^{1,p}(\Omega), \; \Delta z = 0 \text{ in } \Omega, \; \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \; z = 0 \text{ on } \mathbb{R}^{n-1} \}.$$

Proposition 5.1. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, $\mathcal{F}_0^p(\Omega) = \{0\}$.*

Proof- Let z be in $\mathcal{F}_0^p(\Omega)$. We define, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z^* \in W_0^{1,p}(\widetilde{\Omega})$. Then $\dfrac{\partial z^*}{\partial \boldsymbol{n}} = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 2.1 that $\Delta z^* = 0$ in $\widetilde{\Omega}$. The function z^* is in the space $\{z \in W_0^{1,p}(\widetilde{\Omega}),\ \Delta z = 0 \text{ in } \widetilde{\Omega},\ \dfrac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0\}$ which is equal to $\mathcal{P}_{[1-n/p]}$ (see [5]). Thus, if $p < n$, $z^* = 0$ in $\widetilde{\Omega}$ and $z = 0$ in Ω and if $p \geq n$, z^* is a constant in $\widetilde{\Omega}$ so z is constant in Ω, but $z = 0$ on $\mathbb{R}^{n-1}$, so $z = 0$ in Ω and $\mathcal{F}_0^p(\Omega) = \{0\}$. □

The following theorem allows us to obtain strong solutions of the problem $(\mathcal{P}_{M_2})$.

Theorem 5.2. *For each $p > \dfrac{n}{n-1}$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_1^{2,p}(\Omega)$ solution of $(\mathcal{P}_{M_2})$. Moreover, u satisfies*

$$\|u\|_{W_1^{2,p}(\Omega)} \leq C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \qquad (30)$$

Proof- We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}_+^n)$ such that $u_{g_1} = g_1$ and $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$, checking the estimate :

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}_+^n)} \leq C\ \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \qquad (31)$$

We set u_1 the restriction of u_{g_1} to Ω and η the normal derivative of u_1 on Γ_0. Then, we set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Now, we want to find $v \in W_1^{2,p}(\Omega)$ solution of the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1}.$$

We define the functions $h^* \in W_1^{0,p}(\widetilde{\Omega})$ and $g^* \in W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and, thanks to [5], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$, solution of

$$-\Delta w = h^* \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g^* \text{ on } \widetilde{\Gamma}_0,$$

and checking the estimate :

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq C(\ \|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}).$$

Let w_0 be a solution of this problem and, for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set :

$$v_0(\boldsymbol{x'}, x_n) = -w_0(\boldsymbol{x'}, -x_n).$$

We easily check that v_0 is solution of the same problem that w_0 satisfies. Thus $v_0 - w_0 \in \mathcal{P}_{[1-n/p]}$.

i) We suppose that $\dfrac{n}{p} > 1$. In this case, $v_0 = w_0$ in $\widetilde{\Omega}$ and we deduce that $w_0 = 0$ on $\mathbb{R}^{n-1}$. So, the function $v \in W_1^{2,p}(\Omega)$, restriction of w_0 to Ω is a solution of $(\mathcal{P}')$.

ii) We suppose that $\dfrac{n}{p} \leq 1$. In this case, $v_0 = w_0 + \alpha$ in $\widetilde{\Omega}$, where α is a real constant, and, setting $c = -\dfrac{1}{2}\alpha$, we deduce that $w_0 = c$ on $\mathbb{R}^{n-1}$. The function $v = w_{0|\Omega} - c$ is an element of $W_1^{2,p}(\Omega)$ and v is solution of $(\mathcal{P}')$.

Moreover, v, solution of $(\mathcal{P}')$, checks the estimate :

$$\|v\|_{W_1^{2,p}(\Omega)} \leq \; C(\; \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \qquad (32)$$

Finally, the function $u = v + u_1 \in W_1^{2,p}(\Omega)$ is solution of $(\mathcal{P}_{M_2})$ and the estimate (30) comes from (31) and (32). $\square$

Now, we search weak solutions of the problem $(\mathcal{P}_{M_2})$. We set

$$W_p = \{v \in W_0^{1,p}(\Omega), \; v = 0 \text{ on } \mathbb{R}^{n-1}\},$$

and we firstly give the following lemma that we demonstrate like to Lemma 4.4 reversing only Γ_0 and $\mathbb{R}^{n-1}$ (and so, using in its proof the result of the point **i)** of the following theorem) :

Lemma 5.3. *Let p be such that $p > \dfrac{n}{n-1}$. There exists a real constant $\beta > 0$ such that*

$$\inf_{\substack{w \in W_{p'} \\ w \neq 0}} \; \sup_{\substack{v \in W_p \\ v \neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \; \geq \; \beta,$$

and the operators B from $W_p/Ker \; B$ to$(W_{p'})'$ and B' from $W_{p'}$ to $(W_p)' \perp Ker \; B$ defined by :

$$\forall v \in W_p, \; \forall w \in W_{p'}, \; < Bv, w > \, = \, < v, B'w > \, = \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}$$

are isomorphisms.

Theorem 5.4. *For each $p > 1$ checking $\dfrac{n}{p'} \neq 1$, there exists $C = C(\omega_0, p) > 0$ depending only on ω_0 and p such that the following holds. For any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_{M_2})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \; C(\; \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \qquad (33)$$

Proof- i) We suppose $\dfrac{n}{p'} > 1$, *i.e* $p > \dfrac{n}{n-1}$.

First, we apply Theorem 5.2 to have the existence of $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \;\; \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \;\;\; s = 0 \text{ on } \mathbb{R}^{n-1},$$

and checking :

$$\|s\|_{W_0^{1,p}(\Omega)} \leq \|s\|_{W_1^{2,p}(\Omega)} \leq \ C\ \|f\|_{W_1^{0,p}(\Omega)}. \tag{34}$$

Then, thanks to [6], there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+ \quad z = g_1 \text{ on } \mathbb{R}^{n-1},$$

checking :

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq \ C\ \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{35}$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$ and we want to solve the following problem $(\mathcal{P}')$:

$$(\mathcal{P}')\ \ \Delta v = 0 \text{ in } \Omega,\ \ \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0,\ \ \ v = 0 \text{ on } \mathbb{R}^{n-1}.$$

Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. For almost any $(\boldsymbol{x'}, x_n) \in \Gamma_0$, we set

$$\pi\mu(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, x_n) - \mu(\boldsymbol{x'}, -x_n).$$

We notice that $\pi\mu \in W^{1-\frac{1}{p'},p'}(\Gamma_0)$, and we define

$$< g_\pi, \mu >:= < g, \pi\mu >_{\Gamma_0}\ .$$

It is obvious that $g_\pi \in W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and that g is the restriction of g_π to Γ_0. Moreover, we easily check that

$$< g_\pi, \xi >_{\widetilde{\Gamma}_0} = - < g_\pi, \mu >_{\widetilde{\Gamma}_0},$$

where $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$ with $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$. Thanks to [5], there exists a function $w \in W_0^{1,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$ solution of the following problem :

$$\Delta w = 0 \text{ in } \widetilde{\Omega},\ \ \frac{\partial w}{\partial \boldsymbol{n}} = g_\pi \text{ on } \widetilde{\Gamma}_0,$$

and checking :

$$\|w\|_{W_0^{1,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq \ C\ \|g_\pi\|_{W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)}.$$

Let w_0 be a solution of this problem. We set for almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$:

$$v_0(\boldsymbol{x'}, x_n) = -w_0(\boldsymbol{x'}, -x_n).$$

The function v_0 is in $W_0^{1,p}(\widetilde{\Omega})$ and like Δw_0 is nil in $\widetilde{\Omega}$, we easily check that Δv_0 is nil too. Thus $\dfrac{\partial v_0}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and we show, like done in the proof of Theorem 3.3 that $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$. So, the function v_0 is solution of the same problem that w_0 satisfies, which implies that $v_0 - w_0 \in \mathcal{P}_{[1-n/p]}$. We conclude like done in the proof of the previous theorem to show the existence of the solution $v \in W_0^{1,p}(\Omega)$ of the problem $(\mathcal{P}')$ checking :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}. \tag{36}$$

Finally, the function $u = z + s + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and the estimate (33) comes from (34), (35) and (36).

ii) We suppose $\dfrac{n}{p'} < 1$, $i.e$ $p < \dfrac{n}{n-1}$.

Thanks to the section 2, we know there exists a unique $z \in W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta z = 0 \text{ in } \Omega, \quad z = 0 \text{ on } \Gamma_0, \quad z = g_1 \text{ on } \mathbb{R}^{n-1},$$

checking the estimate :

$$\|z\|_{W_0^{1,p}(\Omega)} \leq C \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{37}$$

Like $\Delta z = 0 \in L^p(\Omega)$, $\eta = \dfrac{\partial z}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta$ and we want to find $v \in W_0^{1,p}(\Omega)$ solution of the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = f \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1}.$$

For this, we follow the same idea as the proof of the point ii) in Theorem 4.4 reversing only Γ_0 and $\mathbb{R}^{n-1}$ and noticing that, for $\mu \in W^{2-\frac{1}{p'},p'}(\Gamma_0)$, we know easily building $s \in W_1^{2,p'}(\Omega)$ such that

$$s = \mu \text{ and } \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad s = \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

checking

$$\|s\|_{W_1^{2,p'}(\Omega)} \leq C \|\mu\|_{W^{2-\frac{1}{p'},p'}(\Gamma_0)}$$

and that Ker $B' = \mathcal{F}_0^{p'}(\Omega) = \{0\}$. We have also the following estimate :

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \, (\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{38}$$

Finally, $u = z + v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}_{M_2})$ and we have the estimate (33) thanks to (37) and (38). □

Remark : We notice that when $p > \dfrac{n}{n-1}$ and when the data are more regular, the weak solution is more regular too ; it is in fact the solution of the theorem 5.2.

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