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RENEWAL SERIES AND SQUARE-ROOT BOUNDARIES FOR
BESSEL PROCESSES

NATHANAËL ENRIQUEZ, CHRISTOPHE SABOT, AND MARC YOR

Abstract. We show how a description of Brownian exponential functionals as a
renewal series gives access to the law of the hitting time of a square-root boundary
by a Bessel process. This extends classical results by Breiman and Shepp, concerning
Brownian motion, and recovers by different means, extensions for Bessel processes,
obtained independently by Delong and Yor.

Let \( B_t \) be the standard real valued Brownian motion and for \( \nu > 0 \), introduce the
geometric Brownian motion \( E^{(-\nu)}_t \) and its exponential functional \( A^{(-\nu)}_t \)
\[
E^{(-\nu)}_t := \exp(B_t - \nu t)
\]
\[
A^{(-\nu)}_t := \int_0^t (E^{(-\nu)}_s)^2 ds.
\]

Lamperti’s representation theorem [5] applied to \( E^{(-\nu)}_t \) states
\[
E^{(-\nu)}_t = R^{(-\nu)}_{A^{(-\nu)}_t}
\]
where \( (R^{(-\nu)}_u, u \leq T_0(R^{(-\nu)})) \) denotes the Bessel process of index \( (-\nu) \) (equivalently
of dimension \( \delta = 2(1 - \nu) \)), starting at 1, which is an \( \mathbb{R}_+ \)-valued diffusion with
infinitesimal generator \( \mathcal{L}^{(-\nu)} \) given by
\[
\mathcal{L}^{(-\nu)} f(x) = \frac{1}{2} f''(x) + \frac{1 - 2\nu}{2x} f'(x), \quad f \in C^2_b(\mathbb{R}_+^\star).
\]

Let us remark that, in the special case \( \nu = 1/2 \), equation (0.1) is nothing else but
the Dubins-Schwarz representation of the exponential martingale \( E^{(-1/2)}_t \) as Brownian
motion time changed with \( A^{(-1/2)}_t \).

For a short summary of relations between Bessel processes and exponentials of
Brownian motion, see e.g. Yor [10].

Let us consider now the following random variable \( Z \), which is often called a per-
petuity in the mathematical finance literature:
\[
Z := A^{(-\nu)}_\infty = \int_0^\infty (E^{(-\nu)}_s)^2 ds
\]
We deduce directly from (0.1) that
\[
A^{(-\nu)}_\infty = T_0(R^{(-\nu)}_1)
\]

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where \( T_0 := \inf\{u : X_u = 0\} \), and it is well-known (see [1], [2]), that

\[
\mathcal{A}_{\infty}^{(-\nu)} \overset{\text{law}}{=} \frac{1}{2\gamma_{\nu}}
\]

where \( \gamma_{\nu} \) is a gamma variable with parameter \( \nu \) (i.e. with density \( \frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x} \mathbf{1}_{x > 0} \)).

Our main result characterizes the law of the hitting time of a parabolic boundary by \( R_{\nu}^{(-\nu)} \) which corresponds to a Bessel process of dimension \( d < 2 \).

**Theorem 1.** Let \( 0 < b < c \), and \( \sigma := \inf\{u : (R_{\nu}^{(-\nu)})^2 = \frac{1}{c}(b + u)\} \) with \( R_0^{(-\nu)} = 1 \).

\[
E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_{\nu+s})^{-s}]}{E[(1 + 2c\gamma_{\nu+s})^{-s}]}, \text{ for any } s \geq 0
\]

Proof: using the strong Markov property and the stationarity of the increments of Brownian motion, we obtain that for any stopping time \( \tau \) of the Brownian motion

\[
\mathcal{A}_{\infty}^{(-\nu)} = Z = \mathcal{A}_{\tau}^{(-\nu)} + (\mathcal{E}_{\tau}^{(-\nu)})^2 Z'
\]

where \( Z' \) is independent of \( (\mathcal{A}_{\tau}^{(-\nu)}, \mathcal{E}_{\tau}^{(-\nu)}) \) and \( Z \overset{\text{law}}{=} Z' \).

This implies, by (0.1), that \( Z \) satisfies the following affine equation (see [3] for a survey about these equations)

\[
\mathcal{A}_{\infty}^{(-\nu)} = Z = \mathcal{A}_{\tau}^{(-\nu)} + (R_{\mathcal{A}_{\tau}^{(-\nu)}}^{(-\nu)})^2 Z'
\]

where \( Z' \) is independent of \( (\mathcal{A}_{\tau}^{(-\nu)}, R_{\mathcal{A}_{\tau}^{(-\nu)}}^{(-\nu)}) \) and \( Z \overset{\text{law}}{=} Z' \).

Obviously, \( \sigma < T_0(R_{\nu}^{(-\nu)}) \). Taking now:

\[
\tau = \inf\{t : (R_{\mathcal{A}_{\tau}^{(-\nu)}}^{(-\nu)})^2 = \frac{1}{c}(b + \mathcal{A}_{\tau}^{(-\nu)})\}
\]

we get \( \mathcal{A}_{\tau}^{(-\nu)} = \sigma \), and the identity in law

\[
b + Z \overset{\text{law}}{=} (b + \sigma)(1 + \frac{Z}{c})
\]

where the variables \( \sigma \) and \( Z \) on the right-hand side are independent.

As a result, we obtain the Mellin-Stieltjes transform of \( \sigma \):

\[
E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(b + Z)^{-s}]}{E[(c + Z)^{-s}]}
\]

But, from (0.2)

\[
E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(2\gamma_{\nu})^s \frac{1}{(1 + 2b\gamma_{\nu})^s}]}{E[(2\gamma_{\nu})^s \frac{1}{(1 + 2c\gamma_{\nu})^s}]}
\]

which gives the result.

One can now use the duality between the laws of Bessel processes of dimension \( d \) and \( 4 - d \) to get the analogous result of Theorem 1, and recover the result of Delong [4], [5], and Yor [6] which deals with the case \( d \geq 2 \).

**Theorem 2.** Let \( 0 < b < c \), and \( \sigma := \inf\{u : (R_{\nu}^{(\nu)})^2 = \frac{1}{c}(b + u)\} \) with \( R_0^{(\nu)} = 1 \).

\[
E[(b + \sigma)^{-s}] = c^{-s} \frac{E[(1 + 2b\gamma_{\nu})^{-s + \nu}]}{E[(1 + 2c\gamma_{\nu})^{-s + \nu}]}, \text{ for any } s \geq 0.
\]
Proof: it is based on the following classical relation between the laws of the Bessel processes with indices $\nu$ and $-\nu$:

$$\mathcal{P}_x^{(\nu)}|_{\mathcal{F}_t} = \frac{(X_{t\wedge T_0})^{2\nu}}{x^{2\nu}} \cdot \mathcal{P}_x^{(-\nu)}|_{\mathcal{F}_t}$$

which implies that

$$E_1^{(\nu)}[(b + \sigma)^{-s}] = E_1^{(-\nu)}[X^{2\nu}_\sigma(b + \sigma)^{-s}] = \frac{1}{c_{\nu}} E_1^{(-\nu)}[(b + \sigma)^{-s+\nu}]$$

Theorem 1 gives the result. \(\square\)

Finally, it is easily shown, thanks to the classical representations of the Whittaker functions (see Lebedev [3]), that the right-hand sides of (0.3) and (0.6) are expressed in terms of ratios of Whittaker functions.

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References

Laboratoire Modal’X, Université Paris 10, 200 Avenue de la République, 92000 Nanterre, France

E-mail address: nenriquez@u-paris10.fr

Université de Lyon, Université Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43, Boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: sabot@math.univ-lyon1.fr

Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR 7599, Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: deaproba@proba.jussieu.fr