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Abstract

From the study of a functional equation of Gibbs measures we calculate the limiting free energy of the Sherrington-Kirkpatrick spin glass model at a particular value of (low) temperature. This implies the following lower bound for the ground state energy ϵ_0

$$\epsilon_0 \geq -0.7833\cdots,$$

close to the replica symmetry breaking and numerical simulations values.

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1 Introduction and main result

During the last decade, mean field models of spin glasses have motivated increansingly many studies by physicists and mathematicians [1, 3, 4, 5, 7, 8, 10]. The rigorous understanding of the infinite volume limit of thermodynamic quantities remained quite insufficient until the recent breaktrough obtained by Guerra and Toninelli [4] on their existence and uniqueness. This major discovery followed by several important results [2, 3, 11] providing a mathematical interpretation of the original formulae proposed by Parisi [7] on the basis of heuristic arguments.

In this note, without making use of the replica approach, we calculate, for a particular value of the (low) temperature, the limiting free energy of the Sherrington-Kirkpatrick model and obtain a lower bound for the density of the ground state energy.

We first recall some basic definitions. Suppose that a finite set of n sites is given. Let $\sigma_i \in \{1, -1\}$ be the spin variable on the site i and σ a generic configuration in the configuration space $\Sigma_n = \{-1, 1\}^n$. The finite volume Hamiltonian of the model is given by the following real-valued function on Σ_n

$$H_n(\sigma) = -\frac{1}{\sqrt{n}} \sum_{1 \leq i < j \leq n} J_{ij} \sigma_i \sigma_j,$$

where the family of couplings J_{ij} are independed centered Gaussian random variables of variance 1.

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For a given inverse temperature $\beta > 0$, the disorder dependent partition function $Z_n(\beta)$, is defined by

$$Z_n(\beta, J) = \sum_\sigma \exp(-\beta H_n(\sigma, J)).$$

Moreover, if E_J denotes the expectation with respect to the randomness J_{ij}, one can easily check that $E_J Z_n(\beta, J) = 2^n e^{\frac{\beta^2}{4}(n-1)}$.

We denote by $\mu_{n,\beta}(\sigma|J)$, the conditionned on fixed randomness corresponding Gibbs probability measure, $\mu_{n,\beta}(\sigma|J) = e^{-\beta H_n(\sigma,J)}/Z_n(\beta, J)$. The entropy $S(\mu_{n,\beta}(\sigma|J))$ of $\mu_{n,\beta}(\sigma|J)$ is defined by

$$S(\mu_{n,\beta}(\sigma|J)) = -\sum_\sigma \mu_{n,\beta}(\sigma|J) \log \mu_{n,\beta}(\sigma|J).$$

For fixed randomness, the real functions

$$f_n(\beta) = \frac{1}{n} E_J \log Z_n(\beta, J)$$

and

$$\bar{f}_n(\beta) = \frac{1}{n} \log E_J Z_n(\beta, J),$$

define the quenched average of the free energy per site and the annealed specific free energy respectively.

The ground state energy density $-\epsilon_n(J)$ is defined by

$$-\epsilon_n(J) = \frac{1}{n} \inf_\sigma H_n(\sigma, J).$$

For the low temperatures region ($\beta > 1$), the J-almost sure existence of the infinite volume limits

$$\lim_{n\to\infty} f_n(\beta) = f_\infty(\beta),$$

$$-\lim_{n\to\infty} \epsilon_n(J) = \lim_{\beta\to\infty} \frac{f_\infty(\beta)}{\beta} = -\epsilon_0$$

was first proved by Guerra and Toninelli [4]. More recently, Guerra, and Aizenman, Sims and Starr [2] gave a clear mathematical interpretation of $f_\infty(\beta)$ in terms of the variational formula proposed by Parisi. The interest reader can find in [11] a review of the calculation of the free energy and the rigorous formulation of the Parisi formula.

In the following section we prove the

Proposition : *Let $\beta_* = 4\log 2 = 2.772588\cdots$ Almost surely,*

$$f_\infty(\beta_*) = \lim_{n\to\infty} \frac{1}{n} E_J \log Z_n(\beta_*, J) = \frac{\beta_*^2}{4} + \frac{1}{4}.$$

Lemma : *The ground state energy per site of the Sherrington-Kirkpatrick spin glass model is bounded almost surely by*

$$\epsilon_0 \geq = -0.7833\cdots.$$

2 Proof of the main result

Notice first that for all $\beta > 0$ the limit $f_\infty(\beta)$ exists and it is a convex function of β [4]. Let $\beta = 1$. From the high temperature results [1], we have that $f_\infty(1) = \log 2 + \frac{1}{4}$.
Our analysis will rely on the following easily verified relation:

$$\bar{f}_\infty(\beta_*) = \frac{\beta_*^2}{4} + \log 2 = \beta_*(\frac{\beta_*}{4} + \frac{1}{4}) = \beta_* f_\infty(\beta_1).$$

Indeed, we define the Gibbs probability measure $\mu_{n,\beta_*}(\sigma|J)$ by

$$\mu_{n,\beta_*}(\sigma|J) = \mu_{n,1}^{\beta_*}(\sigma|J)\frac{Z_n^{\beta_*}(1,J)}{Z_n(\beta_*,J)}.$$

Moreover, one can easily check that

$$\lim_{n\to\infty}\frac{1}{n}E_J \log \mu_{n,\beta_*}(\sigma|J) = \lim_{n\to\infty}\frac{\beta_*}{n}E_J \log \mu_{n,1}(\sigma|J) + \alpha_\infty(\beta_*),$$

where the limit $\alpha_\infty(\beta_*)$ gives the deviation of the free energy from its mean value:

$$\alpha_\infty(\beta_*) = \lim_{n\to\infty}\frac{\beta_*}{n}\log E_J Z_n(1,J) - \lim_{n\to\infty}\frac{1}{n}E_J \log Z_n(\beta_*,J) = \bar{f}_\infty(\beta_*) - f_\infty(\beta_*).$$

Now, from the fixed point theorem we have also that

$$\lim_{n\to\infty}\frac{1}{n}E_J \log \mu^*(\sigma|J) = \lim_{n\to\infty}\frac{\beta_*}{n}E_J \log \mu^*(\sigma|J) + \alpha_\infty(\beta_*),$$

where $\mu^*(\sigma|J)$ denotes the fixed point of the functional relation between the Gibbs probability measures $\mu_{n,1}(\sigma|J)$ and $\mu_{n,\beta_*}(\sigma|J)$.

In the following we estimate the limit $\lim_{n\to\infty}\frac{1}{n}E_J\log\mu^*(\sigma|J)$. We have, in particular, that

$$\lim_{n\to\infty}\frac{1}{n}E_J \log \mu_{n,1}(\sigma|J) = -f_\infty(1) + \frac{1}{4} = -\log 2,$$

i.e. the Gibbs measure $\mu_{n,1}(\sigma|J)$ behaves as the counting measure (in the case $\lim_{n\to\infty}\frac{1}{n}E_J\log e^{-H_n(\sigma,J)} = \lim_{n\to\infty}\frac{1}{n}\log E_J e^{-H_n(\sigma,J)} = \frac{1}{4}$), and,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log E_J e^{-\beta_* H(\sigma,J)} - \lim_{n\to\infty}\frac{1}{n}\log E_J Z_n(\beta_*,J) &= \frac{\beta_*^2}{4} - f_\infty(\beta_*) - \alpha_\infty(\beta_*) \\
&= \lim_{n\to\infty}\frac{1}{n}E_J \log \mu_{n,\beta_*}(\sigma|J) - \alpha_\infty(\beta_*) \\
&= -\log 2.
\end{aligned}$$

Then, we have following fixed point equation

$$\lim_{n\to\infty}\frac{1}{n}E_J \log \mu_n^*(\sigma|J) = \frac{\beta_*^2}{4} - f_\infty(\beta_*) = -\log 2 + \alpha_\infty(\beta_*).$$

Now, from the functional equation between the Gibbs measures $\mu_{n,1}(\sigma|J)$ and $\mu_{n,\beta_*}(\sigma\|J)$, we remark that since the limit $\frac{\beta_*^2}{4} - f_\infty(\beta_*)$ it corresponds, for $\beta = 1$, to the limit $\lim_{n\to\infty}\frac{1}{n}E_J\log\mu_n,1(\sigma|J) = \frac{\beta_*}{4} - f_\infty(1) = -\frac{1}{4}$, the fixed point is given by

$$\lim_{n\to\infty}\log\frac{1}{n}E_J \log \mu_n^*(\sigma|J) = -\frac{1}{4}.$$

This implies

$$\alpha_\infty(\beta_*) = \frac{\beta_*}{4} - \frac{1}{4} = \log 2 - \frac{1}{4},$$

and, consequently,

$$f_\infty(\beta_*) = \frac{\beta_*^2}{4} + \frac{1}{4}.$$

One can remark that the obtained value $\frac{\beta_*^2}{4} + \frac{1}{4} = 2.1718\cdots$ is close to the spherical bound value $(2.2058\cdots)$. In the context of large deviation theory, this result comes from Chebychev's inequality and it is detailed in [6].

We can now develop the lower bound for the ground state energy density $-\epsilon_n(J)$.

Notice firstly that

$$f_\infty(\beta_*) = \lim_{n\to\infty} \frac{1}{n} E_J \log Z_n(\beta_*, J) = \lim_{n\to\infty} \frac{1}{n} \sum_\sigma \mu_{n,\beta_*}(\sigma|J) \log e^{-\beta_* H_n(\sigma|J)} + \lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*}(\sigma|J)),$$

and, the limit

$$s(\mu_{\beta_*}) = \lim_{n\to\infty} \frac{1}{n} S(\mu_{n,\beta_*}(\sigma|J)),$$

gives the (mean) entropy of the Gibbs measure. Since $s(\mu_{\beta_*})$ is ≥ 0, one has that

$$\epsilon_0 \geq -\frac{\beta_*}{4} - \frac{1}{4\beta_*} = -0.7833\cdots.$$

3 Concluding remarks

In this note, we calculated, for a particular (low) temperature β_*, the value of the limiting free energy, without making use of the replica formula. One can easily check that β_* is given by $\beta_* = 2\beta_c^2$, where β_c is the critical temperature of the Random Energy Model.

The lower bound for the ground state energy density is established under the assumption of minimal entropy: $s(\mu_{\beta_*}) = 0$. Indeeed, one can show that the mean entropy vanishes at β_* and moreover the relative entropy of the Gibbs measure $\mu_{n,\beta_*}(\sigma|J)$ with respect to the counting measure is $\log 2$ [6].

References

[1] Aizenman, M., Lebowitz, J. L., Ruelle, D.: Some Rigorous Results on the Sherrington-Kirkpatrick Spin Glass Model. Commun. Math. Phys. **112**, 3–20 (1987).

[2] Aizenman, M., Sims, R., Starr, S.,L.: An extended variational principle for the SK spin-glass model. Phys. Rev. B, **6821**(21): 4403, (2003).

[3] Guerra, F.: Broken replica symmetry bounds in the mean field spin glass model. Comm. Math Phys. **233**(1), 1–12 (2003).

[4] Guerra, F., Toninelli, F.: The thermodynamic limit in mean field spin glass models. Commun. Math. Phys. **230**(1), 71–79 (2002).

[5] Koukiou, F.: The low temperature free energy of the Sherrington- Kirkpatrick spin glass model. Eur. Lett. **33**, 95–98 (1996).

[6] Koukiou, F.: Large deviations from the mean field behaviour in the spin glass model. To appear.

[7] Parisi, G.: A sequence of approximated solutions to the Sherrington-Kirkpatrick model for spin glasses. J. Phys. **A 13**, L115–L121 (1980). The order parameter for spin glasses: a function on the interval $[0,1]$. J. Phys. **A 13**, 1101–1112 (1980).

[8] Pastur, L., Shcherbina, M.V.: Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model. J. Stat. Phys. **62**, 1–19 (1991).

[9] Sherrington, D., Kirkpatrick, S.: Solvable model of a spin glass. Phys. Rev. Lett. **35**, 1792–1796 (1975). Infinite-ranged models of spin-glasses. Phys.Rev. **17**, 4384–4403 (1978).

[10] Talagrand, M.: *Spin Glasses: A Challenge for Mathematicians. Mean Field Models and Cavity Method* Springer-Verlag (2003).

[11] Talagrand, M.: The Parisi formula. Ann. of Mathematics **163**, 221–263 (2003).