# Self-adjointness via Hardy-like inequalities

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# SELF-ADJOINTNESS VIA PARTIAL HARDY-LIKE INEQUALITIES

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Abstract. Distinguished selfadjoint extensions of operators which are not semibounded can be deduced from the positivity of the Schur Complement (as a quadratic form). In practical applications this amounts to proving a Hardy-like inequality. Particular cases are Dirac-Coulomb operators where distinguished selfadjoint extensions are obtained for the optimal range of coupling constants.

*Keywords*: Relativistic quantum mechanics, Dirac operator, self-adjoint operator, self-adjoint extension, Schur complement.

## 1. Introduction.

In [4] we defined distinguished self-adjoint extensions of Dirac-Coulomb operators in the optimal range for the coupling constant. This was done by using a Hardy-like inequality which allowed the extension of one component of the operator by using the Friedrichs extension. Then, the remaining component could be extended by choosing the right domain for the whole operator. The method of proof used simple arguments of distributional differentiation. This work was the sequel of a series of papers where distinguished self-adjoint extensions of Dirac-Coulomb like operators were defined by different methods almost in the optimal range, without reaching the limit case (see [10, 9, 12, 13, 14, 7, 6]).

Here we present an abstract version of the method introduced in [4]. We believe that this will clarify the precise structure and hypotheses necessary to define distinguished self-adjoint extensions by this method.

The main idea in our method is that Hardy-like inequalities are fundamental to define distinguished (physically relevant) self-adjoint extensions *even* for operators that are not bounded below.

We are going to apply our method to operators $H$ defined on $\mathcal{D}_0^2$, where $\mathcal{D}_0$ is some dense subspace of a Hilbert space $\mathcal{H}_0$. The general structure taken into account here is:

$$H = \begin{pmatrix} P & Q \\ T & -S \end{pmatrix}, \tag{1}$$

where all the above operators satisfy $Q = T^*$, $P = P^*$, $S = S^*$ and $S \geq c_1 I > 0$. Moreover we assume that $P$, $Q$, $S$, , $T$, $S^{-1}T$ and $QS^{-1}T$ send $\mathcal{D}_0$ into $\mathcal{H}_0$.

In the Dirac-Coulomb case our choice was $\mathcal{H}_0 = L^2(\mathbb{R}^3, \mathbb{C}^2)$ and

$$P = V + 2 - \gamma,\ Q = T = -i\sigma\cdot\nabla,\ S = \gamma - V\,,$$

where $V$ is a potential bounded from above satisfying

$$\sup_{x\neq 0} |x||V(x)| \leq 1\,. \tag{2}$$

Moreover, $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices (see [4]) and $\gamma$ is a constant slightly above $\max_{\mathbb{R}^3} V(x)$. For $\mathcal{D}_0$ we chose $C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$. Note that in our paper [4], where we deal with Dirac-Coulomb like operators, there is an omission. We forgot to specify the conditions on the potential $V$ so that $QS^{-1}T$ is a symmetric operator on $C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$. The natural condition is that each component of

$$(\gamma - V)^{-2}\nabla V \tag{3}$$

is locally square integrable. This is easily seen to be true for the Coulomb-type potentials.

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In the general context of the operator $H$, as defined in (1), our main assumption is that there exists a constant $c_2 > 0$ such that for all $u \in \mathcal{D}_0$,

$$q_{c_2}(u,u) := \big((S+c_2)^{-1}Tu, Tu\big) + \big((P-c_2)u,u\big) \geq 0\,. \tag{4}$$

Note that since $\frac{d}{d\alpha}q_\alpha(u,u) \leq -(u,u)$, (4) implies in fact that for all $0 \leq \alpha \leq c_2$ and for all $u \in \mathcal{D}_0$,

$$q_\alpha(u,u) := \big((S+\alpha)^{-1}Tu, Tu\big) + \big((P-\alpha)u,u\big) \geq 0\,. \tag{5}$$

Antoher consequence of assumption (4) is that the quadratic form

$$q_0(u,u) = \big(S^{-1}Tu, Tu\big) + \big(Pu,u\big)\,, \tag{6}$$

defined for $u \in \mathcal{D}_0$, is positive definite:

$$q_0(u,u) = \big(S^{-1}Tu, Tu\big) + \big(Pu,u\big) \geq c_2(u,u). \tag{7}$$

Note that the operator $P + QS^{-1}T$ which is associated with the quadratic form $q_0$ is actually the Schur complement of $-S$. Note also that by our assumptions on $P$, $Q$, $T$ $S$ and by (4), for any $0 \leq \alpha \leq c_2$, $q_\alpha$ is the quadratic form associated with a positive symmetric operator. Therefore, by Thm. X.23 in [8]), it is closable and we denote its closure by $\widehat{q}_\alpha$ and its form domain, which is easily seen to be independent of $\alpha$ (see [4]) by $\mathcal{H}_{+1}$. Our main result states the following:

**Theorem 1.** *Assume the above hypotheses on the operators $P, Q, T, S$ and (4). Then there is a unique self-adjoint extension of $H$ such that the domain of the operator is contained in $\mathcal{H}_{+1} \times \mathcal{H}_0$.*

**Remark.** Note that what this theorem says that "in some sense" the Schur complement of $-S$ is positive, and therefore has a natural self-adjoint extension, then one can define a distinguished self-adjoint extension of the operator $H$ which is unique among those whose domain is contained in the form domain of the Schur complement of $-S$ times $\mathcal{H}_0$.

## 2. Intermediate results and proofs.

We denote by $R$ the unique selfadjoint operator associated with $\widehat{q}_0$: for all $u \in D(R) \subset \mathcal{H}_{+1}$,

$$\widehat{q}_0(u,u) = (u, Ru)\ . \tag{8}$$

$R$ is an isometric isomorphism from $\mathcal{H}_{+1}$ to its dual $\mathcal{H}_{-1}$. Using the second representation theorem in [5], Theorem 2.23, we know that $\mathcal{H}_{+1}$ is the operator domain of $R^{1/2}$, and

$$\widehat{q}_0(u,u) = (R^{1/2}u, R^{1/2}u)\ , \tag{9}$$

for all $u \in \mathcal{H}_{+1}$.

**Definition 2.** *We define the domain $\mathcal{D}$ of $H$ as the collection of all pairs $u \in \mathcal{H}_{+1}$, $v \in \mathcal{H}_0$ such that*

$$Pu + Qv\ , \quad Tu - Sv \in\ \mathcal{H}_0\ . \tag{10}$$

The meaning of these two expressions is in the weak (distributional) sense, i.e., the linear functional $(P\eta, u) + (Q^*\eta, v)$, which is defined for all test functions $\eta \in \mathcal{D}_0$, extends uniquely to a bounded linear functional on $\mathcal{H}_0$. Likewise the same for $(-S\eta, v) + (T^*\eta, v)$.

On the domain $\mathcal{D}$, we define the operator $H$ as

$$H\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Pu + Qv \\ Tu - Sv \end{pmatrix}. \tag{11}$$

Note that for all vectors $(u,v) \in \mathcal{D}$ the expected total energy is finite.

The following two results are important in the proof of Theorem 1.

**Proposition 3.** *Under the assumptions of Theorem 1*

$$\mathcal{H}_{+1} \subset \Big\{u \in \mathcal{H}_0\ :\ S^{-1}Tu \in \mathcal{H}_0\Big\}\,, \tag{12}$$

*where the embedding holds in the continuous sense. Therefore, we have the 'scale of spaces' $\mathcal{H}_{+1} \subset \mathcal{H}_0 \subset \mathcal{H}_{-1}$.*

*Proof.* Choose $c_2 \geq \alpha > 0$. Since $S \geq c_1 I$, we have for all $0 < \delta \leq \frac{c_1\alpha}{c_1+\alpha}$

$$S^{-1} - (S+\alpha)^{-1} \geq \delta\, S^{-2}\,, \tag{13}$$

and so, for all $u \in \mathcal{D}_0$,

$$q_0(u,u) \geq q_\alpha(u,u) + \alpha\,(u,u) + \delta\,(S^{-1}Tu, S^{-1}Tu) \geq \delta\,(u,u) + \delta\,(S^{-1}Tu, S^{-1}Tu)\ . \tag{14}$$

The proof can be finished by density arguments. □

**Lemma 4.** *For any $F$ in $\mathcal{H}_0$,*

$$QS^{-1}F \in \mathcal{H}_{-1}\ . \tag{15}$$

*Proof.* By our assumptions on $H$ and by Proposition 3, for every $\eta \in \mathcal{D}_0$,

$$\left|(S^{-1}T\eta,\, F)\right| \leq \delta^{-1/2}\,\|\eta\|_{\mathcal{H}_{+1}}\,\|F\|_2\ . \tag{16}$$

Hence, the linear functional

$$\eta \to (Q^*\eta, S^{-1}F) \tag{17}$$

extends uniquely to a bounded linear functional on $\mathcal{H}_{+1}$. □

*Proof of Theorem 1.* We shall prove Theorem 1 by showing that $H$ is symmetric and a bijection from its domain $\mathcal{D}$ onto $\mathcal{H}_0$. To prove the symmetry we have to show that for both pairs $(u,v)$, $(\tilde u,\tilde v)$ in the domain $\mathcal{D}$,

$$\left(H\begin{pmatrix}u\\ v\end{pmatrix},\begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix}\right) = (Pu+Qv,\,\tilde u) + (Tu - Sv,\,\tilde v) \tag{18}$$

equals

$$(u,\,P\tilde u + Q\tilde v) + (v,\,T\tilde u - S\tilde v) = \left(\begin{pmatrix}u\\ v\end{pmatrix},\,H\begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix}\right). \tag{19}$$

First, note that since $(u,v)$ is in the domain,

$$S(v - S^{-1}Tu) \in \mathcal{H}_0)\ . \tag{20}$$

We now claim that

$$(Pu+Qv,\,\tilde u) = (Ru,\tilde u) + (\,S(v-S^{-1}Tu),\,S^{-1}T\tilde u\,)\,. \tag{21}$$

Note that each term makes sense. The one on the left, by definition of the domain and the first on the right, because both $u,\tilde u$ are in $\mathcal{H}_{+1}$. The second term on the right side makes sense because of (20) above and Proposition 3. Moreover both sides coincide for $\tilde u$ chosen to be a test function and both are continuous in $\tilde u$ with respect to the $\mathcal{H}_{+1}$ -norm. Hence the two expressions coincide on the domain. Thus we get that

$$\left(H\begin{pmatrix}u\\ v\end{pmatrix},\begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix}\right) \tag{22}$$

equals

$$(Ru,\tilde u) - (\,S(v-S^{-1}Tu),\,\tilde v - S^{-1}T\tilde u\,)\,, \tag{23}$$

an expression which is symmetric in $(u,v)$ and $(\tilde u,\tilde v)$. To show that the operator is onto, pick any $F_1, F_2$ in $\mathcal{H}_0$. Since $R$ is an isomorphism, there exists a unique $u$ in $\mathcal{H}_{+1}$ such that

$$Ru = F_1 + QS^{-1}F_2\ . \tag{24}$$

Indeed, $F_1$ is in $\mathcal{H}_0$ and therefore in $\mathcal{H}_{-1}$. Moreover the second term is also in $\mathcal{H}_{-1}$ by Lemma 4.

Now define $v$ by

$$v = S^{-1}(Tu - F_2)\,, \tag{25}$$

which by Proposition 3 is in $\mathcal{H}_0$.

Now for any test function $\eta$ we have that

$$(P\eta,\,u) + (Q^*\eta,\,v) = (P\eta,\,u) + (T\eta,\,v) = (P\eta,\,u) + (\,T\eta,\,S^{-1}Tu) + (T\eta,\,(v - S^{-1}Tu)\,) \tag{26}$$

which equals

$$(\eta, Ru) + (T\eta,\,(v - S^{-1}Tu) = (\eta, F_1) \tag{27}$$

This holds for all test functions $\eta$, but since $F_1$ is in $\mathcal{H}_0$, the functional $\eta \to (P\eta,\, u) + (T\eta,\, v)$ extends uniquely to a linear continuous functional on $\mathcal{H}_0$ which implies that

$$Pu + Qv = F_1 \ . \tag{28}$$

Hence $(u, v)$ is in the domain $\mathcal{D}$ and the operator $H$ applied to $(u, v)$ yields $(F_1, F_2)$.

Let us now prove the injectivity of $H$. Assuming that

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{29}$$

we find by (24) and (25),

$$v = S^{-1}Tu \ , \qquad Ru = 0 \, .$$

Since $R$ is an isomorphism, this implies that $u = v = 0$.

It remains to show the uniqueness part in our theorem. By the bijectivity result proved above, for all $\binom{F_1}{F_2} \in {\mathcal{H}_0}^2$, there exists a unique pair $(\hat{u}\,, \hat{v}) \in \mathcal{H}_{+1} \times \mathcal{H}_0$ such that $H\binom{u}{v} = \binom{F_1}{F_2}$. Let us now pick any other self-adjoint extension with domain $\mathcal{D}'$ included in $\mathcal{H}_{+1} \times \mathcal{H}_0$. Then for all $(u, v) \in \mathcal{D}'$, $H\binom{u}{v}$ belongs to ${\mathcal{H}_0}^2$. Hence there exist a unique pair $(\hat{u}\,, \hat{v}) \in \mathcal{H}_{+1} \times \mathcal{H}_0$ such that $H\binom{\hat{u}}{\hat{v}} = H\binom{u}{v}$. But, by the above considerations on injectivity, $u = \hat{u}$ and $v = \hat{v}$. Therefore, $\mathcal{D}' \subset \mathcal{D}$ and so necessarily, $\mathcal{D}' = \mathcal{D}$. □

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