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Singularités au bord de solutions positives d’équations elliptiques non-linéaires

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Abstract
We study the behavior near $x_0$ of any positive solution of $(E)$ $-\Delta u = u^q$ in $\Omega$ which vanishes on $\partial\Omega \setminus \{x_0\}$, where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $q \geq (N+1)/(N-1)$ and $x_0 \in \partial\Omega$. Our results are based upon a priori estimates of solutions of $(E)$ and existence, non-existence and uniqueness results for solutions of some nonlinear elliptic equations on the upper-half unit sphere. To cite this article: M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).

Résumé
Nous étudions le comportement quand $x$ tend vers $x_0$ de toute solution positive de $(E)$ $-\Delta u = u^q$ dans $\Omega$ qui s’annule sur $\partial\Omega \setminus \{x_0\}$, où $\Omega \subset \mathbb{R}^N$ est un domaine régulier, $q \geq (N+1)/(N-1)$ et $x_0 \in \partial\Omega$. Nos résultats sont fondés sur des estimations a priori des solutions de $(E)$, et des résultats d’existence, de non existence et d’unicité de solutions de certaines équations elliptiques non linéaires sur la demi-sphère unité. Pour citer cet article : M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).

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Nous distinguerons les trois valeurs critiques de $q$ données par (4). Si $1 < q < q_1$, le comportement en 0 des solutions est décrit dans [4] ; aussi supposons-nous le plus souvent $q \geq q_1$. Si $u$ est une solution de (3) dans $\mathbb{R}_+^N$ de la forme $u(x) = u(r, \sigma) = r^{-2/(q-1)} \omega(\sigma)$, alors $\omega$ vérifie l’équation (6). Dans ce cas, nous avons le résultat suivant :

**Théorème 0.1**

(i) Si $1 < q \leq q_1$, le problème (3) n’admet aucune solution.

(ii) Si $q_1 < q < q_3$, (3) admet une unique solution, notée $\omega_0$.

(iii) Si $q \geq q_3$, (3) n’admet aucune solution.

Le résultat d’unicité décrit en (ii) est en fait un cas particulier d’un résultat plus général :

**Théorème 0.2** Pour tous $q > 1$ et $\lambda \in \mathbb{R}$, il existe au plus une solution positive de (7).

Ce résultat demeure si, dans (7), $S_{N-1}^+$ est remplacé par une boule dans $\mathbb{R}^N$, et $\Delta'$ par le laplacien ordinaire.

Par simplicité, nous pouvons supposer que $\partial \mathbb{R}^N_+$ est l’hyperplan tangent à $\Omega$ en 0. Le théorème ci-dessous donne une classification des singularités isolées du problème (3) :

**Théorème 0.3** Soit $q \geq q_1$, avec $q \neq q_2$. Supposons que la solution $u$ du problème (3) vérifie

\[ 0 \leq u(x) \leq C |x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \quad (1) \]

pour $C, a > 0$. Si $q_1 \leq q < q_3$, ou bien $u$ est continue en 0, ou bien

\[ u(r, \sigma) = \begin{cases} r^{-(N-1)} (\log (1/r))^{1-N} (kN \sigma_1 + o(1)) & \text{si } q = q_1, \\ r^{-2/(q-1)} (\omega_0(\sigma) + o(1)) & \text{si } q_1 < q < q_3, \end{cases} \quad (2) \]

lorsque $r \to 0$, uniformément par rapport à $\sigma \in S_{N-1}^+$ ; $k_N$ est une constante qui dépend seulement de $N$. Si $q \geq q_3$, $u$ est continue en 0.

L’estimation a priori (1) est obtenue pour $q_1 \leq q < q_2$ :

**Théorème 0.4** Si $q_1 \leq q < q_2$, toute solution $u$ de (3) vérifie (1) pour $C = C(N, q, \Omega) > 0$.

Les démonstrations détaillées sont présentées dans [2].

1. Introduction and main result

Let $\Omega$ be a smooth open subset of $\mathbb{R}^N$, $N \geq 2$, such that $0 \in \partial \Omega$ and let $q > 1$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

\[ \begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \setminus \{0\}. \end{cases} \quad (3) \]

Our goal in this paper is to describe the behavior of $u$ in a neighborhood of 0.

This problem has similar features with the case where $x_0 \in \Omega$, which has been studied by Gidas-Spruck [7]. In our case, we encounter three critical values of $q$ in describing the local behavior of $u$:

\[ q_1 := \frac{N+1}{N-1}, \quad q_2 := \frac{N+2}{N-2} \quad \text{if } N \geq 3 \quad \text{and} \quad q_3 := \frac{N+1}{N-3} \quad \text{if } N \geq 4. \quad (4) \]
When $1 < q < q_1$, it is proved in [4] that for every solution $u$ of (3) there exists $\alpha \geq 0$ (depending on $N$ and $u$) such that
\begin{equation}
  u(x) = \alpha |x|^{-N} \rho(x) \left(1 + o(1)\right) \quad \text{as } x \to 0,
\end{equation}
where $\rho(x) = \text{dist}(x, \partial \Omega)$, $\forall x \in \Omega$. For this reason, we shall mainly restrict ourselves to $q \geq q_1$.

Let us first consider the case where $\Omega = \mathbb{R}^N$ and we look for solutions of (3) of the form $u(x) = u(r, \sigma) = r^{-2/(q-1)} \omega(\sigma)$, where $r = |x|$ and $\sigma \in S^{N-1}_+$. An easy computation shows that $\omega$ must satisfy
\begin{equation}
  \begin{aligned}
    -\Delta' \omega &= \ell_{N,q} \omega + \omega^3 & \text{in } S^{N-1}_+,
    \\
    \omega &\geq 0 & \text{in } S^{N-1}_+,
    \\
    \omega &= 0 & \text{on } \partial S^{N-1}_+,
  \end{aligned}
\end{equation}
where $\Delta'$ denotes the Laplacian in $S^{N-1}$ and $\ell_{N,q} = \frac{2(N - q(N-2))}{(q-1)^2}$. Concerning equation (6), we prove

**Theorem 1.1** (i) If $1 < q \leq q_1$, then (6) admits no positive solution.

(ii) If $q_1 < q < q_3$, then (6) admits a unique positive solution.

(iii) If $q \geq q_3$, then (6) admits no positive solution.

One of the main ingredients in the proof of Theorem 1.1 (ii) is the following

**Theorem 1.2** If $q > 1$ and $\lambda \in \mathbb{R}$, then there exists at most one positive solution of
\begin{equation}
  \begin{aligned}
    -\Delta' v &= \lambda v + v^3 & \text{in } S^{N-1}_+,
    \\
    v &= 0 & \text{on } \partial S^{N-1}_+.
  \end{aligned}
\end{equation}

**Remark 1** We emphasize that in Theorem 1.2 we do not assume that $q$ is subcritical. The conclusion above remains valid if, in (7), $S^{N-1}_+$ is replaced by $B_1 \subset \mathbb{R}^N$ and $\Delta'$ by the usual Laplacian in $\mathbb{R}^N$. Theorem 1.2 extends a previous result of Kwong-Li [8].

We now return to the case where $\Omega \subset \mathbb{R}^N$ is an arbitrary smooth set such that $0 \in \partial \Omega$. For simplicity, we may assume that $\partial \mathbb{R}^N_+$ is the tangent hyperplane of $\Omega$ at 0. Using Theorem 1.2, we provide a classification of isolated singularities of solutions of (3):

**Theorem 1.3** Let $q \geq q_1$, $q \neq q_2$, and let $u$ be a solution of (3). Assume that $u$ satisfies
\begin{equation}
  0 \leq u(x) \leq C |x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0),
\end{equation}
for some $C, a > 0$. If $q_1 \leq q < q_3$, then either $u$ is continuous at 0 or
\begin{equation}
  u(r, \sigma) = \begin{cases}
    r^{-(N-1)} \left(\log (1/r)\right)^{\frac{1}{q-1}} \left(k_N \sigma_1 + o(1)\right) & \text{if } q = q_1, \\
    r^{-2/(q-1)} \left(\omega_0(\sigma) + o(1)\right) & \text{if } q_1 < q < q_3,
  \end{cases}
\end{equation}
as $r \to 0$, uniformly with respect to $\sigma \in S^{N-1}_+$. $k_N$ denotes a constant depending only on $N$ and $\omega_0$ is the unique positive solution of (6).

If $q \geq q_3$, then $u$ is continuous at 0.

**Remark 2** We do not know whether Theorem 1.3 is true when $q = q_2$. In this case, the equation is conformally invariant and thus other techniques are required. If $\Omega = \mathbb{R}^N_+$, then it can be proved that any solution of (3) depends only on the variables $r = |x|$ and $\theta = \cos^{-1}(x_1/|x|)$.

The next result establishes the existence of an *a priori* estimate for the solutions of (3). According to Theorem 1.4 below, assumption (8) is always fulfilled when $q_1 \leq q < q_2$.

**Theorem 1.4** Let $q_1 \leq q < q_2$ and let $u$ be a solution of (3). Then,
\begin{equation}
  0 \leq u(x) \leq C \rho(x) |x|^{-2/(q-1)-1} \quad \forall x \in \Omega \cap B_1(0),
\end{equation}
where $C$ depends on $N$, $q$ and $\Omega$.  


Remark 3 According to the Doob Theorem [6], any positive superharmonic function \( v \) in \( \Omega \) satisfies \( \int_\Omega |\nabla v| \rho < \infty \) and admits a boundary trace, which is a Radon measure on \( \partial \Omega \). If \( u \) is a solution of (3), then its trace must be of the form \( k\delta_{x_0} \), for some \( k \geq 0 \). We may have \( k > 0 \) if \( 1 < q < q_1 \) (see [1]), but \( k \) is necessarily equal to 0 if \( q \geq q_1 \). Indeed, by the maximum principle, \( u \) satisfies \( u \geq kP_{q_1}(x,0) \), where \( P_{q_1} \) denotes the Poisson potential of \( \Omega \). Since \( u \in L^3_q(\Omega) \) (by the Doob Theorem), we must have \( k = 0 \) if \( q \geq q_1 \).

Detailed proofs will appear in [2].

2. Sketch of the proofs

Proof of Theorem 1.1. Assertion (i) is proved by multiplying (6) by \( \phi(\sigma) = \sigma_1 \). Note that \( \phi \) is the first eigenfunction of \( -\Delta \) on \( S^{N-1}_+ \), with eigenvalue \( \lambda_1 = N - 1 \). Integrating the resulting expression over \( S^{N-1}_+ \), and using the fact that \( 1 < q \leq q_1 \implies \ell_{N,q} \geq \lambda_1 \), we obtain (i).

The existence part in (ii) is obtained by using the Mountain Pass Theorem; the uniqueness is a consequence of Theorem 1.2.

Assertion (iii) can be deduced from the following Pohožaev-type identity:

**Proposition 2.1** Assume \( N \geq 4 \) and \( q > 1 \). Then, any solution of (7) satisfies

\[
\frac{N-3}{q+1}(q-q_3) \int_{S^{N-1}_+} |\nabla' v|^2 \phi \, d\sigma - \frac{(N-1)(q-1)}{q+1} \int_{S^{N-1}_+} \phi \, d\sigma = \frac{N-1}{q-1} \int_{S^{N-1}_+} |\nabla' v|^2 \, d\sigma.
\]

This identity is obtained by computing the divergence of the vector field \( P = (\nabla' \phi, \nabla' v)\nabla' v \), where \( \nabla' \) is the gradient on \( S^{N-1}_+ \), and then using the fact that the first eigenfunction satisfies \( D^2 \phi + \phi g_0 = 0 \), where \( g_0 \) is the tensor of the standard metric on \( S^{N-1}_+ \). In order to establish (iii), it suffices to observe that \( \ell_{N,q} \leq \frac{N-1}{q-1} \iff q \geq q_3 \).

Proof of Theorem 1.2. We first notice that any positive solution of (7) depends only on the variable \( \theta = \cos^{-1}(x_1/|x|) \in [0, \pi/2] \); this follows from a straightforward adaptation of the Gidas-Ni-Nirenberg moving plane method to \( S^{N-1}_+ \) (see [9]). Thus, \( v \) satisfies

\[
\begin{align*}
v'' + (N - 2) \cot \theta v' + \lambda v + v^q &= 0 \quad \text{in } (0, \pi/2), \\
v'(0) &= 0, \quad v(\pi/2) = 0.
\end{align*}
\]

Let \( w(\theta) := \sin^\alpha \theta v(\theta) \), where \( \alpha > 0 \). By choosing \( \alpha = 2(N - 2)/(q + 3) \), then \( w \) satisfies

\[
(w'(\pi/2))^2 = \int_0^{\pi/2} G'(\theta) w^2(\theta) \, d\theta,
\]

where \( G \) is a function of the form \( G(\theta) = \sin^\beta \theta (\alpha_1 \sin^2 \theta + \alpha_2) \); the parameters \( \alpha_1, \alpha_2, \beta' \in \mathbb{R} \) can be explicitly computed in terms of \( \lambda, N \) and \( q \).

Assume, by contradiction, that \( v_1 \) and \( v_2 \) are two distinct solutions of (11). Then,

\[
\int_0^{\pi/2} v_1 v_2 (v_2^{q-1} - v_1^{q-1}) \, d\theta = 0.
\]

Therefore, their graphs must intersect at some \( \theta_0 \in (0, \pi/2) \). We claim that \( v_1 \) and \( v_2 \) intersect at least twice in \((0, \pi/2)\). If there is only one intersection point, then it can be shown that there exists \( \gamma \geq 0 \) such
that the function \( \theta \mapsto G'(\theta)(w_2^2(\theta) - \gamma w_1^2(\theta)) \) never vanishes in \((0, \pi/2)\). We then let \( L(t) := (t^2 - \gamma)^{-1} \), \( \forall t \in \mathbb{R} \setminus \{ \gamma \} \). By (12) and the Mean Value Theorem, there exists \( \theta_1 \in (0, \pi/2) \) such that

\[
L \left( \frac{w_2^2(\pi/2)}{w_1^2(\pi/2)} \right) = \frac{\int_0^{\pi/2} G'(\theta) w_2^2(\theta) \, d\theta}{\int_0^{\pi/2} G'(\theta) [w_2^2(\theta) - \gamma w_1^2(\theta)] \, d\theta} = L \left( \frac{w_2(\theta_1)}{w_1(\theta_1)} \right).
\]

Since \( L \) is injective in \( \mathbb{R}_+ \), this implies

\[
\frac{w_2^2(\pi/2)}{w_1^2(\pi/2)} = \frac{w_2(\theta_1)}{w_1(\theta_1)} \tag{14}
\]

On the other hand, by the Sturm-Liouville Theory, the function \( \theta \mapsto w_2(\theta)/w_1(\theta) \) is (strictly) monotone. L’Hôpital’s Rule yields a contradiction as we let \( \theta \to \pi/2 \). Therefore, \( v_1 \) and \( v_2 \) must intersect at least twice. This fact leads to another contradiction by using the Shooting Method (see [8]). Thus, \( v_1 = v_2 \) in \((0, \pi/2)\).

**Remark 4** The method above follows the lines of the proof of Kwong-Li [8]. The main difference is that we use an alternative argument based on the Mean Value Theorem in order to deduce (14). In [8], they have to assume that the exponent \( q \) is subcritical.

**Proof of Theorem 1.3.** It follows from methods developed in [7] and [3]. For simplicity, we shall assume that \( a = 1 \) and \( \partial \Omega \cap B_1 = \partial\mathbb{R}_+ \cap B_1 \). We set

\[
w(t, \sigma) = \rho^{2/(q-1)} w(r, \sigma), \quad t = \log (1/r) \in (0, \infty) \times S_{\mathbb{R}_+}^N := Q.
\]

Then, \( w \) satisfies

\[
w_{tt} - \left( N - 2 \frac{q+1}{q-1} \right) w_t + \Delta w + \ell N, q, w + w^q = 0 \quad \text{in } Q \tag{15}
\]

and \( w \) vanishes on \((0, \infty) \times \partial S_{\mathbb{R}_+}^N \). Since \( w \) is uniformly bounded on \( Q \), standard \textit{a priori} estimates for elliptic problems yield

\[
|\partial^k \nabla^j w| \leq M_{k,j} \quad \text{in } (1, \infty) \times S_{\mathbb{R}_+}^N
\]

for any integers \( k, j \geq 0 \), where \( \nabla^j \) stands for the covariant derivative on \( S_{\mathbb{R}_+}^N \). Thus, the trajectory \( T_w = \{ w(t, \cdot) : t \geq 1 \} \) is relatively compact in \( C^2(S_{\mathbb{R}_+}^{N-1}) \). Multiplying (15) by \( w_t \) and integrating over \( S_{\mathbb{R}_+}^{N-1} \), we obtain

\[
\frac{d}{dt} H(t) = \left( N - 2 \frac{q+1}{q-1} \right) \int_{S_{\mathbb{R}_+}^{N-1}} w_t^2 \, d\sigma,
\]

where

\[
H(t) := \frac{1}{2} \int_{S_{\mathbb{R}_+}^{N-1}} \left( w_t^2 - |\nabla w|^2 - \ell N, q, w^2 + \frac{2}{q+1} w^{q+1} \right) \, d\sigma.
\]

Since \( q \neq q_2 \), we know that \( N - 2(q+1)/(q-1) \neq 0 \). Thus, iterated energy estimates imply that \( w_t(t, \cdot), w_{tt}(t, \cdot) \to 0 \) in \( L^2(S_{\mathbb{R}_+}^{N-1}) \) as \( t \to \infty \). Therefore, the limit set \( \Gamma_w \) of \( T \) is a connected subset of the set of solutions of (6). By Theorem 1.1, we deduce that

\[
\Gamma_w = \begin{cases} 
\{0\} & \text{if } q = q_1 \text{ or } q = q_3, \\
\{0\} \text{ or } \{\omega_0\} & \text{if } q_1 < q < q_3.
\end{cases}
\]

Then, a linearization argument as in [3] leads to the conclusion if \( q > q_1 \).

We now consider the case \( q = q_1 \); we borrow some ideas from [1] and [11]. We first prove, by ODE techniques, that

\[
X(t) := \int_{S_{\mathbb{R}_+}^{N-1}} w(t, \cdot) \phi \, d\sigma \leq C t^{-(N-1)/2}.
\]

\[\tag{17}\]
Using (8) and the boundary Harnack inequality (see [5]), we derive

\[ 0 \leq w(t, \sigma) \leq C t^{-(N-1)/2} \quad \text{in} \quad (1, \infty) \times S^N_+ \tag{18} \]

Set \( \eta(t, \sigma) := t^{(N-1)/2} w(t, \sigma) \). We verify as above that the limit set \( \Gamma_\eta \) in \( C^2(S^N_+) \) of the trajectory \( T_\eta \) of \( \eta \) is an interval of the form \( \{ \kappa \phi : 0 \leq \kappa_0 \leq \kappa \leq \kappa_1 \} \). In order to show that \( T_\eta \) is reduced to a single point, we prove that \( \| r(t, \cdot) \|_{L^2} \leq C t^{-1} \), where \( r(t, \cdot) := \eta(t, \cdot) - z(t) \phi \) and \( z(t) = \int_{S^N_+ \eta(t, \cdot)} \phi \, d\sigma \).

Writing the equation satisfied by \( z \) as a non-homogeneous second order linear ODE, we prove that either \( z(t) \to 0 \), which implies that \( u \) is continuous at 0, or \( z(t) \to \tilde{k}_N \) as \( t \to \infty \), for some constant depending only on \( N \).

**Proof of Theorem 1.4.** It is an application of the Doubling Lemma Method introduced in [10], from which we derive the following local estimate:

**Lemma 2.1.** Let \( 1 < q < q_2 \) and let \( u \) be a solution of (3). Then, for every \( x_0 \in \partial \Omega \setminus \{0\} \) and \( 0 < R < |x_0| \), we have

\[ 0 \leq u(x) \leq C \left( R - |x - x_0| \right)^{-2/(q-1)} \quad \forall x \in B_R(x_0) \cap \Omega, \tag{19} \]

for some constant \( C > 0 \) depending only on \( \Omega \).

Apply this lemma with \( x_0 \in \partial \Omega \setminus \{0\} \) and \( R = |x_0|/2 \). Using elliptic regularity theory, we obtain

\[ 0 \leq u(x) \leq C \rho(x) |x|^{-2/(q-1)} \quad \forall x \in \Omega \text{ such that } 0 < \rho(x) < |x|/2. \]

If \( \rho(x) \geq |x|/2 \), then we use Gidas-Spruck’s internal estimates (see [7]). We thus obtain (10).

**References**


