Cluster categories for algebras of global dimension 2 and quivers with potential
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CLUSTER CATEGORIES FOR ALGEBRAS OF GLOBAL DIMENSION 2
AND QUIVERS WITH POTENTIAL

CLAIRE AMIOT

Abstract. Let \( k \) be a field and \( A \) a finite-dimensional \( k \)-algebra of global dimension \( \leq 2 \). We construct a triangulated category \( C_A \) associated to \( A \) which, if \( A \) is hereditary, is triangle equivalent to the cluster category of \( A \). When \( C_A \) is \( \text{Hom} \)-finite, we prove that it is 2-CY and endowed with a canonical cluster-tilting object. This new class of categories contains some of the stable categories of modules over a preprojective algebra studied by Geiss-Leclerc-Schröer and by Buan-Iyama-Reiten-Scott. Our results also apply to quivers with potential. Namely, we introduce a cluster category \( C_{(Q,W)} \) associated to a quiver with potential \((Q,W)\). When it is Jacobi-finite we prove that it is endowed with a cluster-tilting object whose endomorphism algebra is isomorphic to the Jacobian algebra \( J(Q,W) \).

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Introduction

The cluster category associated with a finite-dimensional hereditary algebra was introduced in [BMR’06] (and in [CCS06] for the $A_n$ case). It serves in the representation-theoretic approach to cluster algebras introduced and studied by Fomin and Zelevinsky in a series of articles (cf. [FZ02], [FZ03], [FZ07] and [BFZ05] with Berenstein). The link between cluster algebras and cluster categories is in the spirit of ‘categorification’. Several articles (e.g. [MRZ03], [BMR’06], [CK08], [CC06], [BMR07], [BMR08], [BMR-T07], [CK06]) deal with the categorification of the cluster algebra $A_Q$ associated with an acyclic quiver $Q$ using the cluster category $C_Q$ associated with the path algebra of the quiver $Q$. Another approach consists in categorifying cluster algebras by subcategories of the category of modules over a preprojective algebra associated to an acyclic quiver (cf. [GLS07a], [GLS06a], [GLS06b], [GLS07b], [BIRS07]). In both approaches the categories $C$ (or their associated stable categories) satisfy the following fundamental properties:

- $C$ is a triangulated category;
- $C$ is 2-Calabi-Yau (2-CY for short);
- there exist cluster-tilting objects.

It has been shown that these properties alone imply many of the most important theorems about cluster categories, respectively stable module categories over preprojective algebras (cf. [IY06], [KR06], [KR07], [Kel08a], [Pal], [Tab07]). In particular by [IY06], in a category $C$ with such properties it is possible to ‘mutate’ the cluster-tilting objects and there exist exchange triangles. This is fundamental for categorification.

Let $k$ be a field. In this article we want to generalize the construction of the cluster category replacing the hereditary algebra $kQ$ by a finite-dimensional algebra $A$ of finite global dimension. A candidate might be the orbit category $D^b(A)/\nu[-2]$, where $\nu$ is the Serre functor of the derived category $D^b(A)$. By [Kel05], such a category is triangulated if $A$ is derived equivalent to an hereditary category $\mathcal{H}$. However in general, it is not triangulated. Thus a more appropriate candidate is the triangulated hull $C_A$ of the orbit category $D^b(A)/\nu[-2]$. It is defined in [Kel05] as the stabilization of a certain dg category and contains the orbit category as a full subcategory. More precisely the category $C_A$ is a quotient of a triangulated category $\mathcal{T}$ by a thick subcategory $\mathcal{N}$ which is 3-CY. This leads us to the study of such quotients in full generality. We prove that the quotient is 2-CY if the objects of $\mathcal{T}$ are ‘limits’ of objects of $\mathcal{N}$ (Theorem 1.3). In particular by [IY06], in a category $C$ with such properties it is possible to ‘mutate’ the cluster-tilting objects and there exist exchange triangles. This is fundamental for categorification.

Let $k$ be a field. In this article we want to generalize the construction of the cluster category replacing the hereditary algebra $kQ$ by a finite-dimensional algebra $A$ of finite global dimension. A candidate might be the orbit category $D^b(A)/\nu[-2]$, where $\nu$ is the Serre functor of the derived category $D^b(A)$. By [Kel05], such a category is triangulated if $A$ is derived equivalent to an hereditary category $\mathcal{H}$. However in general, it is not triangulated. Thus a more appropriate candidate is the triangulated hull $C_A$ of the orbit category $D^b(A)/\nu[-2]$. It is defined in [Kel05] as the stabilization of a certain dg category and contains the orbit category as a full subcategory. More precisely the category $C_A$ is a quotient of a triangulated category $\mathcal{T}$ by a thick subcategory $\mathcal{N}$ which is 3-CY. This leads us to the study of such quotients in full generality. We prove that the quotient is 2-CY if the objects of $\mathcal{T}$ are ‘limits’ of objects of $\mathcal{N}$ (Theorem 1.3). In particular we deduce that the cluster category $C_A$ of an algebra of finite global dimension is 2-CY if it is Hom-finite (Corollary 1.5).

We study the particular case where the algebra is of global dimension $\leq 2$. Since $kQ$ is a cluster-tilting object of the category $C_Q$, the canonical candidate to be a cluster-tilting object in the category $C_A$ would be $A$ itself. Using generalized tilting theory (cf. [Kel94]), we give another construction of the cluster category. We find a triangle equivalence

$$C_A \xrightarrow{\sim} \text{per } \Pi / D^b \Pi$$

where $\Pi$ is a dg algebra in negative degrees which is bimodule 3-CY and homologically smooth. This equivalence sends the object $A$ onto the image of the free dg module $\Pi$ in the quotient. This leads us to the study of the categories $\text{per } \Gamma / D^b \Gamma$ where $\Gamma$ is a dg algebra with the above properties. We prove that if the zeroth cohomology of $\Gamma$ is finite-dimensional, then the category $\text{per } \Gamma / D^b \Gamma$ is 2-CY and the image of the free dg module $\Gamma$ is a cluster-tilting object (Theorem 2.1). We show that the algebra $H^0 \Gamma$ is finite-dimensional if and only if the quotient $\text{per } \Gamma / D^b \Gamma$ is Hom-finite. Thus we prove the existence of a cluster-tilting object in cluster categories $C_A$ associated with algebras of global dimension 2 which are Hom-finite (Theorem 4.10). Moreover,
this general approach applies to the Ginzburg dg algebras \cite{Gin06} associated with a quiver with potential. Therefore we introduce a new class of 2-CY categories \(C_{(Q,W)}\) endowed with a cluster-tilting object associated with a Jacobi-finite quiver with potential \((Q,W)\) (Theorem 3.6).

In \cite{GLS07b}, Geiss, Leclerc and Schröer construct subcategories \(C_M\) of \(\text{mod}\,\Lambda\) (where \(\Lambda = \Lambda_Q\) is a preprojective algebra of an acyclic quiver) associated with certain terminal \(kQ\)-modules \(M\). We show in the last part that the stable category of such a Frobenius category \(C_M\) is triangle equivalent to a cluster category \(C_A\) where \(A\) is the endomorphism algebra of a postprojective module over an hereditary algebra (Theorem 5.15). Another approach is given by Buan, Iyama, Reiten and Scott in \cite{BIRS07}. They construct 2-Calabi-Yau triangulated categories \(\text{Sub}_{\Lambda}/I_w\) where \(I_w\) is a two-sided ideal of the preprojective algebra \(\Lambda = \Lambda_Q\) associated with an element \(w\) of the Weyl group of \(Q\). For certain elements \(w\) of the Weyl group (namely those coming from preinjective tilting modules), we construct a triangle equivalence between \(\text{Sub}_{\Lambda}/I_w\) and a cluster category \(C_A\) where \(A\) is the endomorphism algebra of a postprojective module over a concealed algebra (Theorem 5.21).

**Plan of the paper.** The first section of this paper is devoted to the study of Serre functors in quotients of triangulated categories. In Section 2, we prove the existence of a cluster-tilting object in a 2-CY category coming from a bimodule 3-CY dg algebra. Section 3 is a direct application of these results to Ginzburg dg algebras associated with quivers with potential. In particular we give the definition of the cluster category \(C_{(Q,W)}\) of a Jacobi-finite quiver with potential \((Q,W)\). In section 4 we define cluster categories of algebras of finite global dimension. We apply the results of Sections 1 and 2 in subsection 4.3 to the particular case of global dimension \(\leq 2\). The last section links the categories introduced in \cite{GLS07b} and in \cite{BIRS07} with these new cluster categories \(C_A\).

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**Notations.** Throughout let \(k\) be a field. By triangulated category we mean \(k\)-linear triangulated category satisfying the Krull-Schmidt property. For all triangulated categories, we will denote the shift functor by \([1]\). For a finite-dimensional \(k\)-algebra \(A\) we denote by \(\text{mod}\,A\) the category of finite-dimensional right \(A\)-modules. More generally, for an additive \(k\)-category \(\mathcal{M}\) we denote by \(\text{mod}\,\mathcal{M}\) the category of finitely presented functors \(\mathcal{M}^{\text{op}} \to \text{mod}\,k\). Let \(D\) be the usual duality \(\text{Hom}_k(?,k)\). If \(A\) is a differential graded (\(=\text{dg}\)) \(k\)-algebra, we will denote by \(\mathcal{D} = DA\) the derived category of \(\text{dg}\,A\)-modules and by \(\mathcal{D}^b\) its full subcategory formed by the \(\text{dg}\,A\)-modules whose homology is of finite total dimension over \(k\). We write \(\text{per}\,A\) for the category of perfect \(\text{dg}\,A\)-modules, i.e. the smallest triangulated subcategory of \(DA\) stable under taking direct summands and which contains \(A\).

1. **Construction of a Serre functor in a quotient category**

1.1. **Bilinear form in a quotient category.** Let \(T\) be a triangulated category and \(\mathcal{N}\) a thick subcategory of \(T\) (i.e. a triangulated subcategory stable under taking direct summands). We assume that there is an auto-equivalence \(\nu\) in \(T\) such that \(\nu(\mathcal{N}) \subset \mathcal{N}\). Moreover we assume that there is a non degenerate bilinear form:}
$\beta_{N,X} : T(N,X) \times T(X,\nu N) \rightarrow k$

which is bifunctorial in $N \in \mathcal{N}$ and $X \in \mathcal{T}$.

Construction of a bilinear form in $\mathcal{T}/\mathcal{N}$. Let $X$ and $Y$ be objects in $\mathcal{T}$. The aim of this section is to construct a bifunctorial bilinear form:

$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X,Y) \times \mathcal{T}/\mathcal{N}(Y,\nu X[-1]) \rightarrow k.$

We use the calculus of left fractions [Ver77] in the triangle quotient $\mathcal{T}/\mathcal{N}$. Let $s^{-1} \circ f : X \rightarrow Y$ and $t^{-1} \circ g : Y \rightarrow \nu X[-1]$ be two morphisms in $\mathcal{T}/\mathcal{N}$. We can construct a diagram

where the cone of $s'$ is isomorphic to the cone of $s$. Denote by $N[1]$ the cone of $u$. It is in $\mathcal{N}$ since $\mathcal{N}$ is $\nu$-stable. Thus we get a diagram of the form:

where the two horizontal rows are triangles of $\mathcal{T}$. We define then $\beta'_{X,Y}$ as follows:

$\beta'_{X,Y}(s^{-1} \circ f, t^{-1} \circ g) = \beta_{N,Y}(v, w).$

Lemma 1.1. The form $\beta'$ is well-defined, bilinear and bifunctorial.

Proof. It is not hard to check that $\beta'$ is well-defined (cf. [Ami08]). Since $\beta$ is bifunctorial and bilinear, $\beta'$ satisfies the same properties. $\square$

1.2. Non-degeneracy. In this section, we find conditions on $X$ and $Y$ such that the bilinear form $\beta'_{X,Y}$ is non-degenerate.

Definition 1.2. Let $X$ and $Y$ be objects in $\mathcal{T}$. A morphism $p : N \rightarrow X$ is called a local $\mathcal{N}$-cover of $X$ relative to $Y$ if $N$ is in $\mathcal{N}$ and if it induces an exact sequence:

Let $Y$ and $Z$ be objects in $\mathcal{T}$. A morphism $i : Z \rightarrow N'$ is called a local $\mathcal{N}$-envelope of $Z$ relative to $Y$ if $N'$ is in $\mathcal{N}$ and if it induces an exact sequence:

Theorem 1.3. Let $X$ and $Y$ be objects of $\mathcal{T}$. If there exists a local $\mathcal{N}$-cover of $X$ relative to $Y$ and a local $\mathcal{N}$-envelope of $\nu X$ relative to $Y$, then the bilinear form $\beta'_{XY}$ constructed in the previous section is non-degenerate.
For a stronger version of this theorem see also [CR].

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{T}$ whose image in $\mathcal{T}/\mathcal{N}$ is in the kernel of $\beta'$. We have to show that it factorizes through an object of $\mathcal{N}$.

Let $p : N \to X$ be a local $\mathcal{N}$-cover of $X$ relative to $Y$, and let $X'$ be the cone of $p$. The morphism $f$ is in the kernel of $\beta'$, thus for each morphism $g : Y \to \nu N$ which factorizes through $\nu X'[-1]$, $\beta(fp, g)$ vanishes.

\[
\begin{array}{ccccccccc}
N & \xrightarrow{p} & X & \xrightarrow{f} & X' & \xrightarrow{i} & N[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nu X[-1] & \xrightarrow{g} & \nu X'[-1] & \xrightarrow{i} & \nu N & \xrightarrow{\nu i} & \nu X \\
\end{array}
\]

This means that the linear form $\beta(fp, ?)$ vanishes on the image of the morphism $\mathcal{T}(Y, \nu X'[-1]) \to \mathcal{T}(Y, \nu N)$. This image is canonically isomorphic to the kernel of the morphism $\mathcal{T}(Y, \nu N) \to \mathcal{T}(Y, \nu X)$.

Let $\nu i : \nu X \to \nu N'$ be a local $\mathcal{N}$-envelope of $\nu X$ relative to $Y$. The sequence

\[
\begin{array}{cccccc}
0 & \to & \mathcal{T}(Y, \nu X) & \xrightarrow{i} & \mathcal{T}(Y, \nu N') \\
\end{array}
\]

is then exact. Therefore, the form $\beta(fp, ?)$ vanishes on $\text{Ker}(\mathcal{T}(Y, \nu N) \to \mathcal{T}(Y, \nu N'))$.

\[
\begin{array}{ccccccccc}
N & \xrightarrow{p} & X & \xrightarrow{f} & X' & \xrightarrow{i} & N[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\nu X'[-1] & \xrightarrow{\nu i} & \nu N' & \xrightarrow{\nu i} & \nu N & \xrightarrow{\nu i} & \nu X & \xrightarrow{\nu i} & \nu X' \\
\end{array}
\]

Now $\beta$ is non-degenerate on

$\text{Coker}(\mathcal{T}(N', Y) \to \mathcal{T}(N, Y)) \times \text{Ker}(\mathcal{T}(Y, \nu N) \to \mathcal{T}(Y, \nu N'))$.

Thus the morphism $fp$ lies in $\text{Coker}(\mathcal{T}(N', Y) \to \mathcal{T}(N, Y))$, that is to say that $fp$ factorizes through $\nu i p$. Since $p : N \to X$ is a local $\mathcal{N}$-cover of $X$, $f$ factorizes through $N'$.

\[\square\]

Proposition 1.4. Let $X$ and $Y$ be objects in $\mathcal{T}$. If for each $N$ in $\mathcal{N}$ the vector spaces $\mathcal{T}(N, X)$ and $\mathcal{T}(Y, N)$ are finite-dimensional, then the existence of a local $\mathcal{N}$-cover of $X$ relative to $Y$ is equivalent to the existence of a local $\mathcal{N}$-envelope of $Y$ relative to $X$.

Proof. Let $g : N \to X$ be a local $\mathcal{N}$-cover of $X$ relative to $Y$. It induces an injection

\[
\begin{array}{cccccc}
0 & \to & \mathcal{T}(X, Y) & \xrightarrow{g^*} & \mathcal{T}(N, Y) \\
\end{array}
\]
The space $T(N, Y)$ is finite-dimensional by hypothesis. Fix a basis $(f_1, f_2, \ldots, f_r)$ of this space. This space is in duality with the space $T(Y, \nu N)$. Let $(f'_1, f'_2, \ldots, f'_r)$ be the dual basis of the basis $(f_1, f_2, \ldots, f_r)$. We show that the morphism

$$Y \xrightarrow{(f'_1, \ldots, f'_r)} \bigoplus_{i=1}^r \nu N$$

is a local $N$-envelope of $Y$ relative to $X$. We have a commutative diagram:

$$
\begin{array}{ccc}
T(X, Y) & \xrightarrow{(f'_1, \ldots, f'_r)_*} & T(X, \nu N) \\
\downarrow g^* & & \downarrow g^* \\
T(N, Y) & \xrightarrow{(f'_1, \ldots, f'_r)_*} & T(N, \nu N).
\end{array}
$$

If $f$ is in the kernel of $(f'_1, \ldots, f'_r)_*$, then for all $i = 1, \ldots, r$, the morphism $f'_i \circ f \circ g$ is zero. Thus $f \circ g$ is orthogonal on the vectors of the basis $f'_1, \ldots, f'_r$ and therefore vanishes. Since $g$ is a local $N$-cover of $X$ relative to $Y$, $f$ is zero, and the morphism $Y \xrightarrow{(f'_1, \ldots, f'_r)} \bigoplus_{i=1}^r \nu N$ is injective. Therefore, the morphism

$$Y \xrightarrow{(f'_1, \ldots, f'_r)} \bigoplus_{i=1}^r \nu N$$

is a local $N$-envelope of $Y$ relative to $X$. The proof of the converse is dual. □

**Examples.** Let $A$ be a finite-dimensional self-injective $k$-algebra. Denote by $T$ the derived category $D^b(\text{mod} \ A)$ and by $N$ the triangulated category $\text{per} \ A$. Since $A$ is finite-dimensional, there is an inclusion $N \subset T$. Moreover $A$ is self-injective so of infinite global dimension. Therefore the inclusion is strict. By [KV87], there is an exact sequence of triangulated categories:

$$0 \longrightarrow \text{per} \ A \longrightarrow D^b(\text{mod} \ A) \longrightarrow \text{mod} \ A \longrightarrow 0.$$  

The derived category $D^b(\text{mod} \ A)$ admits a Serre functor $\nu = \mathcal{L}_A^* \otimes_A D_A$ which stabilizes $\text{per} \ A$. Thus there is an induced functor in the quotient $\text{mod} \ A$ that we will also denote by $\nu$. Let $\Sigma$ be the suspension of the category $\text{mod} \ A$. One can easily construct (cf. [Ami08]) local $N$-covers and local $N$-envelopes, so we can apply theorem [1,3] and the functor $\Sigma^{-1} \circ \nu$ is a Serre functor for the stable category $\text{mod} \ A$.

An article of G. Tabuada [Tab07] gives an example of the converse construction. Let $C$ be an algebraic 2-Calabi-Yau category endowed with a cluster-tilting object. The author constructs a triangulated category $T$ and a triangulated 3-Calabi-Yau subcategory $N$ such that the quotient category $T/N$ is triangle equivalent to $C$. It is possible to show (cf. [Ami08]) that the categories $T$ and $N$ satisfy the hypotheses of theorem [1,3].

2. **Existence of a cluster-tilting object**

Let $A$ be a differential graded (=dg) $k$-algebra. We denote by $A^e$ the dg algebra $A^{op} \otimes A$. Suppose that $A$ has the following properties:

- $A$ is homologically smooth (i.e. the object $A$, viewed as an $A^e$-module, is perfect);
- for each $p > 0$, the space $H^p A$ is zero;
- the space $H^0 A$ is finite-dimensional;
• $A$ is bimodule 3-CY, i.e. there is an isomorphism in $\mathcal{D}(A^e)$
\[ \mathcal{R} \text{Hom}_{A^e}(A, A^e) \simeq A[-3]. \]

Since $A$ is homologically smooth, the category $\mathcal{D}^b A$ is a subcategory of $\text{per} A$ (see [Kel08a], lemma 4.1). We denote by $\pi$ the canonical projection functor $\pi : \text{per} A \to \mathcal{C} = \text{per} A/\mathcal{D}^b A$. Moreover, by the same lemma, there is a bifunctorial isomorphism
\[ \mathcal{D} \text{Hom}_D(L, M) \simeq \text{Hom}_D(M, L[3]) \]
for all objects $L$ in $\mathcal{D}^b A$ and all objects $M$ in $\text{per} A$. We call this property the CY property.

The aim of this section is to show the following result:

**Theorem 2.1.** Let $A$ be a dg $k$-algebra with the above properties. The category $\mathcal{C} = \text{per} A/\mathcal{D}^b A$ is Hom-finite and 2-CY. Moreover, the object $\pi(A)$ is a cluster-tilting object. Its endomorphism algebra is isomorphic to $H^0 A$.

2.1. t-structure on $\text{per} A$. The main tool of the proof of theorem 2.1 is the existence of a canonical t-structure in $\text{per} A$.

**t-structure on $\mathcal{D} A$.** Let $\mathcal{D}_{\leq 0}$ be the full subcategory of $\mathcal{D}$ whose objects are the dg modules $X$ such that $H^p X$ vanishes for all $p > 0$.

**Lemma 2.2.** The subcategory $\mathcal{D}_{\leq 0}$ is an aisle in the sense of Keller-Vossieck [KV88].

**Proof.** The canonical morphism $\tau_{\leq 0} \to A$ is a quasi-isomorphism of dg algebras. Thus we can assume that $A^p = 0$ for all $p > 0$. The full subcategory $\mathcal{D}_{\leq 0}$ is stable under $X \hookrightarrow X[1]$ and under extensions. We claim that the inclusion $\mathcal{D}_{\leq 0} \to \mathcal{D}$ has a right adjoint. Indeed, for each dg $A$-module $X$, the dg $A$-module $\tau_{\leq 0} X$ is a dg submodule of $X$, since $A$ is concentrated in negative degrees. Thus $\tau_{\leq 0}$ is a well-defined functor $\mathcal{D} \to \mathcal{D}_{\leq 0}$. One can check easily that $\tau_{\leq 0}$ is the right adjoint of the inclusion.

\[ \square \]

**Proposition 2.3.** Let $\mathcal{H}$ be the heart of the t-structure, i.e. $\mathcal{H}$ is the intersection $\mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. We have the following properties:

1. The functor $H^0$ induces an equivalence from $\mathcal{H}$ onto $\text{Mod} H^0 A$.
2. For all $X$ and $Y$ in $\mathcal{H}$, we have an isomorphism $\text{Ext}^1_{\mathcal{H}^0 A}(X, Y) \simeq \text{Hom}_D(X, Y[1])$.

Note that it is not true for general $i$ that $\text{Ext}^i_{\mathcal{H}}(X, Y) \simeq \text{Hom}_D(X, Y[i])$.

**Proof.** (i) We may assume that $A^p = 0$ for all $p > 0$. We then have a canonical morphism $A \to H^0 A$. The restriction along this morphism yields a functor $\Phi : \text{Mod} H^0 A \to \mathcal{H}$ such that $H^0 \Phi = \text{id}$. Thus the functor $H^0 : \mathcal{H} \to \text{Mod} H^0 A$ is full and essentially surjective. Moreover, it is exact and an object $N \in \mathcal{H}$ vanishes if and only if $H^0 N$ vanishes. Thus if $f : L \to M$ is a morphism of $\mathcal{H}$ such that $H^0(f) = 0$, then $\text{im} H^0(f) = 0$ implies that $H^0(\text{im} f) = 0$ and $\text{im} f = 0$, so $f = 0$. Thus $H^0 : \mathcal{H} \to \text{Mod} H^0 A$ is also faithful.

(ii) By section 3.1.7 of [BBD82] there exists a triangle functor $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}$ which yields for $X$ and $Y$ in $\mathcal{H}$ and for $n \leq 1$ an isomorphism (remark (ii) section 3.1.17 p.85)
\[ \text{Hom}_D(X, Y[n]) \simeq \text{Hom}_D(X, Y[n]). \]

Applying this for $n = 1$ and using (i), we get the result.

\[ \square \]
Hom-finiteness.

**Proposition 2.4.** The category per $A$ is Hom-finite.

**Lemma 2.5.** For each $p$, the space $H^pA$ is finite-dimensional.

*Proof.* By hypothesis, $H^pA$ is zero for $p > 0$. We prove by induction on $n$ the following statement: The space $H^{-n}A$ is finite-dimensional, and for all $p \geq n+1$ the space $\text{Hom}_D(\tau_{\leq -n}A, M[p])$ is finite-dimensional for each $M$ in $\text{mod} H^0A$.

For $n = 0$, the space $H^0A$ is finite-dimensional by hypothesis. Let $M$ be in $\text{mod} H^0A$. Since $\tau_{\leq 0}A$ is isomorphic to $A$, $\text{Hom}_D(\tau_{\leq 0}A, M[p])$ is isomorphic $H^0(M[p])$, and so is zero for $p \geq 1$.

Suppose that the property holds for $n$. Form the triangle:

$$
\begin{array}{ccc}
(H^{-n}A)[n-1] & \longrightarrow & \tau_{\leq -n-1}A \\
& \searrow & \downarrow \\
& & \tau_{\leq -n}A \\
& \longleftarrow & (H^{-n}A)[n]
\end{array}
$$

Let $p$ be an integer $\geq n+1$. Applying the functor $\text{Hom}_D(?, M[p])$ we get the long exact sequence:

\[ \cdots \longrightarrow \text{Hom}_D(\tau_{\leq -n}A, M[p]) \longrightarrow \text{Hom}_D(\tau_{\leq -n-1}A, M[p]) \longrightarrow \text{Hom}_D((H^{-n}A)[n-1], M[p]) \longrightarrow \cdots \]

By induction the space $\text{Hom}_D(\tau_{\leq -n}A, M[p])$ is finite-dimensional. Since $M[p]$ is in $\mathcal{D}^0A$ we can apply the CY property. If $p \geq n + 3$, we have isomorphisms:

\[ \text{Hom}_D((H^{-n}A)[n-1], M[p]) \cong \text{Hom}_D((H^{-n}A), M[p-n+1]) \]
\[ \cong D\text{Hom}_D(M[p-n-2], H^{-n}A). \]

Since $p-n-2$ is $\geq 1$, this space is zero.

If $p = n + 2$, we have the following isomorphisms.

\[ \text{Hom}_D((H^{-n}A)[n-1], M[n+2]) \cong \text{Hom}_D((H^{-n}A), M[3]) \]
\[ \cong D\text{Hom}_D(M, H^{-n}A) \]
\[ \cong D\text{Hom}_{H^0A}(M, H^{-n}A). \]

The last isomorphism comes from lemma 2.3 (i). By induction, the space $H^{-n}A$ is finite-dimensional. Thus for $p \geq n + 2$, the space $\text{Hom}_D((H^{-n}A)[n-1], M[p])$ is finite-dimensional.

If $p = n + 1$ we have the following isomorphisms:

\[ \text{Hom}_D((H^{-n}A)[n-1], M[n+1]) \cong \text{Hom}_D((H^{-n}A), M[2]) \]
\[ \cong D\text{Hom}_D(M, H^{-n}A[1]) \]
\[ \cong D\text{Ext}_H^1(M, H^{-n}A). \]

The last isomorphism comes from lemma 2.3 (ii). By induction, since $H^{-n}A$ is finite-dimensional, the space $\text{Hom}_D((H^{-n}A)[n-1], M[n+1])$ is finite-dimensional and so is $\text{Hom}_D(\tau_{\leq -n-1}A, M[n+1])$.

Now, look at the triangle

$$
\begin{array}{ccc}
\tau_{\leq -n-2}A & \longrightarrow & \tau_{\leq -n-1}A \\
\downarrow & & \downarrow \\
(H^{-n}A)[n+1] & \longrightarrow & (\tau_{\leq -n-2}A)[1].
\end{array}
$$
The spaces $\text{Hom}_D(\tau_{\leq -n-2}A, M[n+1])$ and $\text{Hom}_D((\tau_{\leq -n-2}A)[1], M[n+1])$ vanish since $M[n+1]$ is in $\mathcal{D}_{\geq -n-1}$. Thus we have

$$\text{Hom}_D(\tau_{\leq -n-1}A[n-1], M[n+1]) \simeq \text{Hom}_D((H^{-n-1}A)[n+1], M[n+1])$$

$$\simeq \text{Hom}_D(H^{-n-1}A, M)$$

$$\simeq \text{Hom}_{\text{per}A}(H^{-n-1}A, M).$$

This holds for all finite-dimensional $H^0A$-modules $M$. Thus it holds for the compact cogenerator $M = DH^0A$. The space $\text{Hom}_{\text{per}A}(H^{-n-1}A, DH^0A) \simeq DH^{-n-1}A$ is finite-dimensional, and therefore $H^{-(n+1)}A$ is finite-dimensional.

Proof. (of proposition 2.4) For each integer $p$, the space $\text{Hom}_D(A, A[p]) \simeq H^p(A)$ is finite-dimensional by lemma 2.5. The subcategory of $(\text{per}A)^{op} \times \text{per}A$ whose objects are the pairs $(X,Y)$ such that $\text{Hom}_D(X,Y)$ is finite-dimensional is stable under extensions and passage to direct factors.

**Restriction of the t-structure to per A.**

**Lemma 2.6.** For each $X$ in $\text{per}A$, there exist integers $N$ and $M$ such that $X$ belongs to $\mathcal{D}_{\leq N}$ and $\per A$.

Proof. The object $A$ belongs to $\mathcal{D}_{\leq 0}$. Moreover, since for $X$ in $\mathcal{D}A$, the space $\text{Hom}_D(A, X)$ is isomorphic to $H^0X$, the dg module $A$ belongs to $\per A$. Thus the property is true for $A$ for the same reasons, it is true for all shifts of $A$. Moreover, this property is clearly stable under taking direct summands and extensions. Thus it holds for all objects of $\text{per}A$.

This lemma implies the following result:

**Proposition 2.7.** The t-structure on $\mathcal{D}A$ restricts to $\text{per}A$.

Proof. Let $X$ be in $\text{per}A$, and look at the canonical triangle:

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{>0}X \longrightarrow (\tau_{\leq 0}X)[1].$$

Since $\text{per}A$ is Hom-finite, the space $H^pX \simeq \text{Hom}_D(A, X[p])$ is finite-dimensional for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$ by lemma 2.6. Thus the object $\tau_{>0}X$ is in $\mathcal{D}A$ and so in $\text{per}A$. Since $\text{per}A$ is a triangulated subcategory, it follows that $\tau_{\leq 0}X$ also lies in $\text{per}A$.

**Proposition 2.8.** Let $\pi$ be the projection $\pi : \text{per}A \rightarrow C$. Then for $X$ and $Y$ in $\text{per}A$, we have

$$\text{Hom}_C(\pi X, \pi Y) = \lim_{\longrightarrow} \text{Hom}_D(\tau_{\leq n}X, \tau_{\leq n}Y).$$

Proof. Let $X$ and $Y$ be in $\text{per}A$. An element of $\lim \text{Hom}_D(\tau_{\leq n}X, \tau_{\leq n}Y)$ is an equivalence class of morphisms $\tau_{\leq n}X \rightarrow \tau_{\leq n}Y$. Two morphisms $f : \tau_{\leq n}X \rightarrow \tau_{\leq n}Y$ and $g : \tau_{\leq m}X \rightarrow \tau_{\leq m}Y$ with $m \geq n$ are equivalent if there is a commutative square:

$$\begin{array}{ccc}
\tau_{\leq n}X & \xrightarrow{f} & \tau_{\leq n}Y \\
\downarrow & & \downarrow \\
\tau_{\leq m}X & \xrightarrow{g} & \tau_{\leq m}Y
\end{array}$$
where the vertical arrows are the canonical morphisms. If \( f \) is a morphism \( f : \tau_{\leq n}X \to \tau_{\leq n}Y \), we can form the following morphism from \( X \) to \( Y \) in \( C \):

\[
\begin{array}{ccc}
\tau_{\leq n}X & \xrightarrow{f} & \tau_{\leq n}Y \\
\downarrow & & \downarrow \\
X & \rightarrow & Y,
\end{array}
\]

where the morphisms \( \tau_{\leq n}X \to X \) and \( \tau_{\leq n}Y \to Y \) are the canonical morphisms. This is a morphism from \( \pi X \) to \( \pi Y \) in \( C \) because the cone of the morphism \( \tau_{\leq n}X \to X \) is in \( \mathcal{D}_{\leq 0}A \). Moreover, if \( f : \tau_{\leq n}X \to \tau_{\leq n}Y \) and \( g : \tau_{\leq m}X \to \tau_{\leq m}Y \) are equivalent, there is an equivalence of diagrams:

\[
\begin{array}{ccc}
\tau_{\leq n}X & \xrightarrow{f} & \tau_{\leq n}Y \\
\downarrow & & \downarrow \\
\tau_{\leq m}X & \xrightarrow{g} & \tau_{\leq m}Y,
\end{array}
\]

Thus we have a well-defined map from \( \lim_{\rightarrow} \text{Hom}_D(\tau_{\leq n}X, \tau_{\leq n}Y) \) to \( \text{Hom}_C(\pi X, \pi Y) \) which is injective.

Now let \( X' \xrightarrow{s} X \) be a morphism in \( \text{Hom}_C(\pi X, \pi Y) \). Let \( X'' \) be the cone of \( s \). It is an object of \( \mathcal{D}_{\leq 0}A \), and therefore lies in \( \mathcal{D}_{\geq n} \) for some \( n \ll 0 \). Thus there are no morphisms from \( \tau_{\leq n}X \) to \( X'' \) and there is a factorization:

\[
\begin{array}{ccc}
\tau_{\leq n}X & \xrightarrow{s} & X \\
\downarrow & 0 & \downarrow \\
X' & \rightarrow & X'' \rightarrow X'[1]
\end{array}
\]

We obtain an isomorphism of diagrams:

\[
\begin{array}{ccc}
X' & \xrightarrow{s} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tau_{\leq n}X} & Y
\end{array}
\]

The morphism \( f : \tau_{\leq n}X \to Y \) induces a morphism \( f' : \tau_{\leq n}X \to \tau_{\leq n}Y \) which lifts the given morphism. Thus the map from \( \lim_{\rightarrow} \text{Hom}_D(\tau_{\leq n}X, \tau_{\leq n}Y) \) to \( \text{Hom}_C(\pi X, \pi Y) \) is surjective. \( \square \)

2.2. Fundamental domain. Let \( \mathcal{F} \) be the following subcategory of \( \text{per} A \):

\[
\mathcal{F} = \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\leq -2} \cap \text{per} A.
\]

The aim of this section is to show:

**Proposition 2.9.** The projection functor \( \pi : \text{per} A \to C \) induces a \( k \)-linear equivalence between \( \mathcal{F} \) and \( C \).
add$(A)$-approximation for objects of the fundamental domain.

Lemma 2.10. For each object $X$ of $\mathcal{F}$, there exists a triangle

$$ P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1] $$

with $P_0$ and $P_1$ in add$(A)$.

Proof. For $X$ in per$A$, the morphism

$$ \text{Hom}_D(A, X) \to \text{Hom}_H(H^0A, H^0X) $$

$$ f \mapsto H^0(f) $$

is an isomorphism since $\text{Hom}_D(A, X) \simeq H^0X$. Thus it is possible to find a morphism $P_0 \to X$, with $P_0$ a free dg $A$-module, inducing an epimorphism $H^0P_0 \longrightarrow H^0X$. Now take $X$ in $\mathcal{F}$ and $P_0 \to X$ as previously and form the triangle

$$ P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow P_1[1]. $$

Step 1: The object $P_1$ is in $\mathcal{D}_{\leq 0} \cap \per \mathcal{D}_{\leq -1}$.

The objects $X$ and $P_0$ are in $\mathcal{D}_{\leq 0}$, so $P_1$ is in $\mathcal{D}_{\leq 1}$. Moreover, since $H^0P_0 \to H^0X$ is an epimorphism, $H^1(P_1)$ vanishes and $P_1$ is in $\mathcal{D}_{\leq 0}$.

Let $Y$ be in $\mathcal{D}_{\leq -1}$, and look at the long exact sequence:

$$ \cdots \longrightarrow \text{Hom}_D(P_0, Y) \longrightarrow \text{Hom}_D(P_1, Y) \longrightarrow \text{Hom}_D(X[-1], Y) \longrightarrow \cdots. $$

The space $\text{Hom}_D(X[-1], Y)$ vanishes since $X$ is in $\per \mathcal{D}_{\leq -2}$ and $Y$ is in $\mathcal{D}_{\leq -1}$. The object $P_0$ is free, and $H^0Y$ is zero, so the space $\text{Hom}_D(P_0, Y)$ also vanishes. Consequently, the object $P_1$ is in $\per \mathcal{D}_{\leq -1}$.

Step 2: $H^0P_1$ is a projective $H^0A$-module.

Since $P_1$ is in $\mathcal{D}_{\leq 0}$ there is a triangle

$$ \tau_{\leq -1}P_1 \longrightarrow P_1 \longrightarrow H^0P_1 \longrightarrow (\tau_{\leq -1}P_1)[1]. $$

Now take an object $M$ in the heart $\mathcal{H}$, and look at the long exact sequence:

$$ \cdots \longrightarrow \text{Hom}_D((\tau_{\leq -1}P_1)[1], M[1]) \longrightarrow \text{Hom}_D(H^0P_1, M[1]) \longrightarrow \text{Hom}_D(P_1, M[1]) \longrightarrow \cdots. $$

The space $\text{Hom}_D((\tau_{\leq -1}P_1)[1], M[1])$ is zero because $\text{Hom}_D(\mathcal{D}_{\leq -2}, \mathcal{D}_{\geq -1})$ vanishes in a $t$-structure. Moreover, the space $\text{Hom}_D(P_1, M[1])$ vanishes because $P_1$ is in $\per \mathcal{D}_{\leq -1}$. Thus $\text{Hom}_D(H^0P_1, M[1])$ is zero. But this space is isomorphic to the space $\text{Ext}^1_H(H^0P_1, M)$ by proposition 2.3. This proves that $H^0P_1$ is a projective $H^0A$-module.

Step 3: $P_1$ is isomorphic to an object of add$(A)$.

As previously, since $H^0P_1$ is projective, it is possible to find an object $P$ in add$(A)$ and a morphism $P \to P_1$ inducing an isomorphism $H^0P \to H^0P_1$. Form the triangle

$$ Q \overset{u}{\longrightarrow} P \overset{v}{\longrightarrow} P_1 \overset{w}{\longrightarrow} Q[1] $$

Since $P$ and $P_1$ are in $\mathcal{D}_{\leq 0}$ and $H^0(v)$ is surjective, the cone $Q$ lies in $\mathcal{D}_{\leq 0}$. But then $w$ is zero since $P_1$ is in $\per \mathcal{D}_{\leq -1}$. Thus the triangle splits, and $P$ is isomorphic to the direct sum $P_1 \oplus Q$. Therefore we have a short exact sequence:

$$ 0 \longrightarrow H^0Q \longrightarrow H^0P \longrightarrow H^0P_1 \longrightarrow 0,$$
and $H^0Q$ vanishes. The object $Q$ is in $D_{\leq -1}$, the triangle splits, and there is no morphism between $P$ and $D_{\leq -1}$, so $Q$ is the zero object.

$\square$

**Equivalence between the shifts of $\mathcal{F}$.**

**Lemma 2.11.** The functor $\tau_{\leq -1}$ induces an equivalence from $\mathcal{F}$ to $\mathcal{F}[1]$

**Proof. Step 1:** The image of the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is in $\mathcal{F}[1]$.

Recall that $\mathcal{F}$ is $D\leq 0 \cap \per A$ so $\mathcal{F}[1]$ is $D_{\leq -1} \cap \per A$. Let $X$ be an object in $\mathcal{F}$. By definition, $\tau_{\leq -1}X$ lies in $D_{\leq -1}$ and there is a canonical triangle:

$$\tau_{\leq -1}X \xrightarrow{f} X \xrightarrow{X} H^0X \xrightarrow{\tau_{\leq -1}X[1]} .$$

Now let $Y$ be an object in $D_{\leq -3}$ and form the long exact sequence

$$\cdots \longrightarrow \text{Hom}_D(X,Y) \longrightarrow \text{Hom}_D(\tau_{\leq -1}X,Y) \longrightarrow \text{Hom}_D((H^0X)[-1],Y) \longrightarrow \cdots$$

Since $X$ is in $D_{\leq -2}$, the space $\text{Hom}_D(X,Y)$ vanishes. The object $H^0X[-1]$ is of finite total dimension, so by the CY property, we have an isomorphism

$$\text{Hom}_D(H^0X[-1],Y) \simeq D\text{Hom}_D(Y,H^0X[2]).$$

But since $\text{Hom}_D(D_{\leq -3},D_{\leq -2})$ is zero, the space $\text{Hom}_D((H^0X)[-1],Y)$ vanishes and $\tau_{\leq -1}X$ lies in $D_{\leq -3}$.

**Step 2:** The functor $\tau_{\leq -1} : \mathcal{F} \rightarrow \mathcal{F}[1]$ is fully faithful.

Let $X$ and $Y$ be two objects in $\mathcal{F}$ and $f : \tau_{\leq -1}X \rightarrow \tau_{\leq -1}Y$ be a morphism.

$$H^0X[-1] \longrightarrow \tau_{\leq -1}X \xrightarrow{f} X \longrightarrow H^0X$$

The space $\text{Hom}_D(H^0X[-1],Y)$ is isomorphic to $D\text{Hom}_D(Y,H^0X[2])$ by the CY property. Since $Y$ is in $D_{\leq -2}$, this space is zero, and the composition $i \circ f$ factorizes through the canonical morphism $\tau_{\leq -1}X \rightarrow X$. Therefore, the functor $\tau_{\leq -1}$ is full.

Let $X$ and $Y$ be objects of $\mathcal{F}$ and $f : X \rightarrow Y$ a morphism satisfying $\tau_{\leq -1}f = 0$. It induces a morphism of triangles:

$$\begin{array}{ccc}
H^0X[-1] & \longrightarrow & \tau_{\leq -1}X \\
\downarrow & & \downarrow \\
H^0Y[-1] & \longrightarrow & \tau_{\leq -1}Y \\
\end{array}$$

The composition $f \circ i$ vanishes, so $f$ factorizes through $H^0X$. But by the CY property the space of morphisms $\text{Hom}_D(H^0X,Y)$ is isomorphic to $D\text{Hom}_D(Y,H^0X[3])$ which is zero since $Y$ is in $D_{\leq -2}$. Thus the functor $\tau_{\leq -1}$ restricted to $\mathcal{F}$ is faithful.

**Step 3:** The functor $\tau_{\leq -1} : \mathcal{F} \rightarrow \mathcal{F}[1]$ is essentially surjective.

Let $X$ be in $\mathcal{F}[1]$. By the previous lemma there exists a triangle

$$P_1[1] \longrightarrow P_0[1] \longrightarrow X \longrightarrow P_1[2]$$
with $P_0$ and $P_1$ in $\text{add}(A)$. Denote by $\nu$ the Nakayama functor on the projectives of $\text{mod}H^0A$. Let $M$ be the kernel of the morphism $\nu H^0P_1 \to \nu H^0P_0$. It lies in the heart $\mathcal{H}$.

**Substep (i): There is an isomorphism of functors:** $\text{Hom}(?, X[1])|_\mathcal{H} \simeq \text{Hom}_\mathcal{H}(?, M)$

Let $L$ be in $\mathcal{H}$. We then have a long exact sequence:

$$\cdots \to \text{Hom}_\mathcal{D}(L, P_0[2]) \to \text{Hom}_\mathcal{D}(L, X[1]) \to \text{Hom}_\mathcal{D}(L, P_1[3]) \to \text{Hom}_\mathcal{D}(L, P_0[3]) \to \cdots$$

The space $\text{Hom}_\mathcal{D}(L, P_0[2])$ is isomorphic to $D\text{Hom}_\mathcal{D}(P_0, L[1])$ by the CY property, and vanishes because $P_0$ is in $^{\perp}D_{\leq -1}$. Moreover, we have the following isomorphisms:

$$\text{Hom}_\mathcal{D}(L, P_1[3]) \simeq D\text{Hom}_\mathcal{D}(P_1, L)$$

$$\simeq D\text{Hom}_\mathcal{H}(H^0P_1, L)$$

$$\simeq \text{Hom}_\mathcal{H}(L, \nu H^0P_1).$$

Thus $\text{Hom}_\mathcal{D}(?, X[1])|_\mathcal{H}$ is isomorphic to the kernel of $\text{Hom}_\mathcal{H}(?, \nu H^0P_1) \to \text{Hom}_\mathcal{H}(?, \nu H^0P_0)$, which is just $\text{Hom}_\mathcal{H}(?, M)$.

**Substep (ii): There is a monomorphism of functors:** $\text{Ext}_\mathcal{H}^1(?, M) \hookrightarrow \text{Hom}_\mathcal{D}(?, X[2])|_\mathcal{H}$.

For $L$ in $\mathcal{H}$, look at the following long exact sequence:

$$\cdots \to \text{Hom}_\mathcal{D}(L, P_1[3]) \to \text{Hom}_\mathcal{D}(L, P_1[3]) \to \text{Hom}_\mathcal{D}(L, X[2]) \to \text{Hom}_\mathcal{D}(L, P_1[4]) \to \cdots$$

The space $\text{Hom}_\mathcal{D}(L, P_1[4])$ is isomorphic to $D\text{Hom}_\mathcal{D}(P_1[1], L)$ which is zero since $P_1[1]$ is in $D_{\leq -1}$ and $L$ is in $D_{\geq 0}$. Thus the functor $\text{Hom}_\mathcal{D}(?, X[2])|_\mathcal{H}$ is isomorphic to the cokernel of $\text{Hom}_\mathcal{H}(?, \nu H^0P_1) \to \text{Hom}_\mathcal{H}(?, \nu H^0P_0)$. By definition, $\text{Ext}_\mathcal{H}^1(?, M)$ is the first homology of a complex of the form:

$$\cdots \to 0 \to \text{Hom}_\mathcal{H}(?, \nu H^0P_1) \to \text{Hom}_\mathcal{H}(?, \nu H^0P_0) \to \text{Hom}_\mathcal{H}(?, I) \to \cdots$$

where $I$ is an injective $H^0A$-module. Thus we get the canonical injection:

$$\text{Ext}_\mathcal{H}^1(?, M) \hookrightarrow \text{Hom}_\mathcal{D}(?, X[2])|_\mathcal{H}.$$

Now form the following triangle:

$$X \to Y \to M \to X[1].$$

**Substep (iii): $Y$ is in $\mathcal{F}$ and $\tau_{\leq -1}Y$ is isomorphic to $X$.**

Since $X$ and $M$ are in $D_{\leq 0}$, $Y$ is in $D_{\leq 0}$. Let $Z$ be in $D_{\leq -2}$ and form the following long exact sequence:

$$\cdots \to \text{Hom}_\mathcal{D}(X[1], Z) \to \text{Hom}_\mathcal{D}(M, Z) \to \text{Hom}_\mathcal{D}(Y, Z) \to \text{Hom}_\mathcal{D}(X, Z) \to \text{Hom}_\mathcal{D}(M[-1], Z) \cdots$$

By the CY property and the fact that $Z[2]$ is in $D_{\leq 0}$, we have isomorphisms

$$\text{Hom}_\mathcal{D}(M[-1], Z) \simeq D\text{Hom}_\mathcal{D}(Z[-2], M)$$

$$\simeq D\text{Hom}_\mathcal{H}(H^{-2}Z, M).$$

Moreover, since $X$ is in $^{\perp}D_{\leq -3}$, we have

$$\text{Hom}_\mathcal{D}(X, Z) \simeq \text{Hom}_\mathcal{D}(X, (H^{-2}Z)[2])$$

$$\simeq D\text{Hom}_\mathcal{H}(H^{-2}Z, X[1]).$$
By substep (i) the functors $\Hom_M(?, M)$ and $\Hom_M(?, X[1])$, are isomorphic. Therefore we deduce that the morphism $\Hom_D(X, Z) \to \Hom_D(M[-1], Z)$ is an isomorphism.

Now look at the triangle
\[
\tau_{\leq -3}Z \to Z \to H^{-2}Z[2] \to (\tau_{\leq -3}Z)[1]
\]
and form the commutative diagram
\[
\begin{array}{ccc}
\Hom_D(M, \tau_{\leq -3}Z) & \to & \Hom_D(M, Z) \\
a & b & \downarrow c \\
\Hom_D(X[1], \tau_{\leq -3}Z) & \to & \Hom_D(X[1], Z)
\end{array}
\]

By the CY property and the fact that $(\tau_{\leq -3}Z)[-3]$ is in $D_{\leq 0}$, we have isomorphisms
\[
\Hom_D(M[-1], \tau_{\leq -3}Z)[-1]) \cong D\Hom_D(\tau_{\leq -3}Z[-3], M) \\
\cong D\Hom_H(H^{-3}Z, M).
\]

Since $X$ is in $\perp D_{\leq -3}$, we have
\[
\Hom_D(X, (\tau_{\leq -3}Z)[-1]) \cong \Hom_D(X, H^{-3}Z[-2]) \\
\cong D\Hom_H(H^{-3}Z, X[1]).
\]

Now we deduce from substep (i) that $a[-1]$ is an isomorphism.

The space $\Hom_D(X[1], \tau_{\leq -3}Z[1])$ is zero because $X$ is in $\perp D_{\leq -3}$. Moreover there are isomorphisms
\[
\Hom_D(M, H^{-2}Z[2]) \cong D\Hom_D(H^{-2}Z, M[1]) \\
\cong D\Ext_H^1(H^{-2}Z, M).
\]

The space $\Hom_D(X[1], H^{-2}Z[2])$ is isomorphic to $D\Hom_D(H^{-2}Z, X[2])$. And by substep (ii), the morphism $\Ext_H(?, M) \to \Hom_M(?, X[2])$ is injective, so $c$ is surjective. Therefore using a weak form of the five-lemma we deduce that $b$ is surjective.

Finally, we have the following exact sequence:
\[
\Hom_D(X[1], Z) \to \Hom_D(M, Z) \to \Hom_D(Y, Z) \to \Hom_D(X, Z) \to \Hom_D(M[-1], Z)
\]

Thus the space $\Hom_D(M, Z)$ is zero, and $Z$ is in $\perp D_{\leq -2}$.

It is now easy to see that there is an isomorphism of triangles:
\[
\begin{array}{ccc}
\tau_{\leq -1}Y & \to & Y \\
\downarrow & & \downarrow \\
X & \to & Y \to M \\
\downarrow & & \downarrow \\
& & \tau_{\leq -1}Y[1] \\
\end{array}
\]

\[
\begin{array}{ccc}
\tau_{\leq n}X & \to & \tau_{\leq n}Y \\
X & \to & Y.
\end{array}
\]

**Proof of proposition 2.9. Step 1: The functor $\pi$ restricted to $\mathcal{F}$ is fully faithful.**

Let $X$ and $Y$ be objects in $\mathcal{F}$. By proposition 2.3 (iii), the space $\Hom_C(\pi X, \pi Y)$ is isomorphic to the direct limit $\lim \Hom_D(\tau_{\leq n}X, \tau_{\leq n}Y)$. A morphism between $X$ and $Y$ in $C$ is a diagram of the form
\[
\begin{array}{ccc}
\tau_{\leq n}X & \to & \tau_{\leq n}Y \\
X & \to & Y.
\end{array}
\]
The canonical triangle

\[(\tau_{>n}X)[-1] \rightarrow \tau_{\leq n}X \rightarrow X \rightarrow \tau_{>n}X\]

yields a long exact sequence:

\[
\cdots \rightarrow \text{Hom}_D(\tau_{>n}X, Y) \rightarrow \text{Hom}_D(X, Y) \rightarrow \text{Hom}_D(\tau_{\leq n}X, Y) \rightarrow \text{Hom}_D((\tau_{>n}X)[-1], Y) \rightarrow \cdots
\]

The space \(\text{Hom}_D((\tau_{>n}X)[-1], Y)\) is isomorphic to the space \(D\text{Hom}_D(Y, (\tau_{>n}X)[2])\). The object \(X\) is in \(D_{\leq 0}\), thus so is \(\tau_{>n}X\), and the space \(D\text{Hom}_D(Y, (\tau_{>n}X)[2])\) vanishes. For the same reasons, the space \(\text{Hom}_D(\tau_{>n}X, Y)\) vanishes. Thus there are bijections

\[
\text{Hom}_D(\tau_{\leq n}X, \tau_{\leq n}Y) \sim \text{Hom}_D(\tau_{\leq n}X, Y) \sim \text{Hom}_D(X, Y)
\]

Therefore, the functor \(\pi : \mathcal{F} \rightarrow \mathcal{C}\) is fully faithful.

**Step 2:** For \(X\) in \(\text{per} \, A\), there exists an integer \(N\) and an object \(Y\) of \(\mathcal{F}[-N]\) such that \(\pi X\) and \(\pi Y\) are isomorphic in \(\mathcal{C}\).

Let \(X\) be in \(\text{per} \, A\). By lemma 2.11, there exists an integer \(N\) such that \(X\) is in \(\perp D_{\leq N-2}\). For an object \(Y\) in \(D_{\leq N-2}\), the space \(\text{Hom}_D((\tau_{>n}X)[-1], Y)\) is isomorphic to \(D\text{Hom}_D(Y, (\tau_{>n}X)[2])\) and thus vanishes. Therefore, \(\tau_{\leq N}X\) is still in \(\perp D_{\leq N-2}\), and thus is in \(\mathcal{F}[-N]\). Since \(\tau_{>N}X\) is in \(D^bA\), the objects \(\tau_{\leq N}X\) and \(X\) are isomorphic in \(\mathcal{C}\).

**Step 3:** The functor \(\pi\) restricted to \(\mathcal{F}\) is essentially surjective.

Let \(X\) be in \(\text{per} \, A\) such that \(\tau_{\leq N}X\) in \(\mathcal{F}[-N]\). By lemma 2.11, \(\tau_{<1}\) induces an equivalence between \(\mathcal{F}\) and \(\mathcal{F}[1]\). Thus since the functor \(\pi \circ \tau_{<1} : \text{per} \, A \rightarrow \mathcal{C}\) is isomorphic to \(\pi\), there exists an object \(Y\) in \(\mathcal{F}\) such that \(\pi(Y)\) and \(\pi(X)\) are isomorphic in \(\mathcal{C}\). Therefore, the functor \(\pi\) restricted to \(\mathcal{F}\) is essentially surjective.

**Proposition 2.12.** If \(X\) and \(Y\) are objects in \(\mathcal{F}\), there is a short exact sequence:

\[
0 \rightarrow \text{Ext}^1_D(X, Y) \rightarrow \text{Ext}^1_\mathcal{C}(X, Y) \rightarrow D\text{Ext}^1_D(Y, X) \rightarrow 0.
\]

**Proof.** Let \(X\) and \(Y\) be in \(\mathcal{F}\). The canonical triangle

\[
\tau_{<0}X \rightarrow X \rightarrow \tau_{\geq 0}X \rightarrow (\tau_{<0}X)[1]
\]

yields the long exact sequence:

\[
\text{Hom}_D((\tau_{\geq 0}X)[-1], Y[1]) \rightarrow \text{Hom}_D(\tau_{<0}X, Y[1]) \rightarrow \text{Hom}_D(X, Y[1]) \rightarrow \text{Hom}_D(\tau_{\geq 0}X, Y[1]).
\]

The space \(\text{Hom}_D(X[-1], Y[1])\) is zero because \(X\) is in \(\perp D_{\leq -2}\) and \(Y\) is in \(D_{\leq 0}\). Moreover, the space \(\text{Hom}_D(\tau_{\geq 0}X, Y[1])\) is zero because of the CY property. Thus this long sequence reduces to a short exact sequence:

\[
0 \rightarrow \text{Ext}^1_D(X, Y) \rightarrow \text{Hom}_D(\tau_{<0}X, Y[1]) \rightarrow \text{Hom}_D((\tau_{\geq 0}X)[-1], Y[1]) \rightarrow 0.
\]

**Step 1:** There is an isomorphism \(\text{Hom}_D((\tau_{\geq 0}X)[-1], Y) \simeq D\text{Ext}^1_D(Y, X)\).

The space \(\text{Hom}_D((\tau_{\geq 0}X)[-1], Y[1])\) is isomorphic to \(D\text{Hom}_D(Y, \tau_{\geq 0}X[1])\) by the CY property.
But since $\text{Hom}_D(Y, (\tau_{<0} X)[1])$ and $\text{Hom}_D(Y, (\tau_{<0} X)[2])$ are zero, we have an isomorphism
\[
\text{Hom}_D(\tau_{\geq 0} X[-1], Y) \cong D\text{Ext}^1_D(Y, X).
\]

**Step 2:** There is an isomorphism $\text{Ext}^1_C(\pi X, \pi Y) \cong \text{Hom}_D(\tau_{\leq -1} X, Y[1])$.

By lemma $2.11$, the object $\tau_{<0} X$ belongs to $\mathcal{F}[1]$ and clearly $Y[1]$ belongs to $\mathcal{F}[1]$. By proposition $2.9$ (applied to the shifted $t$-structure), the functor $\pi : \text{per} A \rightarrow C$ induces an equivalence from $\mathcal{F}[1]$ to $C$ and clearly we have $\pi(\tau_{<0} X, Y[1]) \cong \pi(X)$. We obtain bijections
\[
\text{Hom}_D(\tau_{<0} X, Y[1]) \cong \text{Hom}_D(\pi \tau_{<0} X, \pi Y[1]) \cong \text{Hom}_D(\pi X, \pi Y[1]).
\]

\[\square\]

**Proof of theorem $2.7$.**

**Step 1:** The category $C$ is $\text{Hom}$-finite and 2-CY.

The category $\mathcal{F}$ is obviously $\text{Hom}$-finite, hence so is $C$ by proposition $2.3$. The categories $\mathcal{T} = \text{per} A$ and $\mathcal{N} = D^b A \subset \text{per} A$ satisfy the hypotheses of section 1. By $[\text{Kel08a}]$, thanks to the CY property, there is a bifunctorial non degenerate bilinear form:
\[
\beta_{N,X} : \text{Hom}_D(N, X) \times \text{Hom}_D(X, N[3]) \rightarrow k
\]
for $N$ in $D^b A$ and $X$ in $\text{per} A$. Thus, by section 1, there exists a bilinear bifunctorial form
\[
\beta'_{X,Y} : \text{Hom}_C(X, Y) \times \text{Hom}_C(Y, X[2]) \rightarrow k
\]
for $X$ and $Y$ in $C = \text{per} A / D^b A$. We would like to show that it is non degenerate. Since $\text{per} A$ is $\text{Hom}$-finite, by theorem $1.3$ and proposition $1.4$, it is sufficient to show the existence of local $\mathcal{N}$-envelopes. Let $X$ and $Y$ be objects of $\text{per} A$. Therefore by lemma $2.10$, $X$ is in $\per D_{\leq N}$. Thus there is an injection
\[
0 \longrightarrow \text{Hom}_D(X, Y) \longrightarrow \text{Hom}_D(X, \tau_{>N} Y)
\]
and $Y \rightarrow \tau_{>N} Y$ is a local $\mathcal{N}$-envelope relative to $X$. Therefore, $C$ is 2-CY.

**Step 2:** The object $\pi A$ is a cluster-tilting object of the category $C$.

Let $A$ be the free dg $A$-module in $\text{per} A$. Since $H^1 A$ is zero, the space $\text{Ext}^1_D(A, A)$ is also zero. Thus by the short exact sequence
\[
0 \longrightarrow \text{Ext}^1_D(A, A) \longrightarrow \text{Ext}^1_C(\pi A, \pi A) \longrightarrow D\text{Ext}^1_D(A, A) \longrightarrow 0
\]
of proposition $2.12$, $\pi(A)$ is a rigid object of $C$. Now let $X$ be an object of $C$. By proposition $2.9$, there exists an object $Y$ in $\mathcal{F}$ such that $\pi Y$ is isomorphic to $X$. Now by lemma $2.10$, there exists a triangle in $\text{per} A$
\[
P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow P_1[1]
\]
with $P_1$ and $P_0$ in $\text{add}(A)$. Applying the triangle functor $\pi$ we get a triangle in $C$:
\[
\pi P_1 \longrightarrow \pi P_0 \longrightarrow X \longrightarrow \pi P_1[1]
\]
with $\pi P_1$ and $\pi P_0$ in $\text{add}(\pi A)$. If $\text{Ext}^1_C(\pi A, X)$ vanishes, this triangle splits and $X$ is a direct factor of $\pi P_0$. Thus, the object $\pi A$ is a cluster-tilting object in the 2-CY category $C$. 

3. Cluster categories for Jacobi-finite quivers with potential

3.1. Ginzburg dg algebra. Let $Q$ be a finite quiver. For each arrow $a$ of $Q$, we define the cyclic derivative with respect to $a$ $\partial_a$ as the unique linear map

$$\partial_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path $p$ to the sum $\sum_{p=uvu} vu$ taken over all decompositions of the path $p$ (where $u$ and $v$ are possibly idempotents $e_i$ associated to a vertex $i$ of $Q$).

An element $W$ of $kQ/[kQ, kQ]$ is called a potential on $Q$. It is given by a linear combination of cycles in $Q$.

**Definition 3.1** (Ginzburg). Let $Q$ be a finite quiver and $W$ a potential on $Q$. Let $\hat{Q}$ be the graded quiver with the same vertices as $Q$ and whose arrows are

- the arrows of $Q$ (of degree 0),
- an arrow $a^* : j \to i$ of degree $-1$ for each arrow $a : i \to j$ of $Q$,
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

The Ginzburg dg algebra $\Gamma(Q, W)$ is a dg $k$-algebra whose underlying graded algebra is the graded path algebra $k\hat{Q}$. Its differential is the unique linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p udv,$$

for all homogeneous $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\hat{Q}$:

- $da = 0$ for each arrow $a$ of $Q$,
- $d(a^*) = \partial_a W$ for each arrow $a$ of $Q$,
- $d(t_i) = e_i(\sum a[a, a^*])e_i$ for each vertex $i$ of $Q$ where $e_i$ is the idempotent associated to $i$ and the sum runs over all arrows of $Q$.

The strictly positive homology of this dg algebra clearly vanishes. Moreover B. Keller showed the following result:

**Theorem 3.2** (Keller). Let $Q$ be a finite quiver and $W$ a potential on $Q$. Then the Ginzburg dg algebra $\Gamma(Q, W)$ is homologically smooth and bimodule 3-CY.

3.2. Jacobian algebra.

**Definition 3.3.** Let $Q$ be a finite quiver and $W$ a potential on $Q$. The Jacobian algebra $J(Q, W)$ is the zeroth homology of the Ginzburg algebra $\Gamma(Q, W)$. This is the quotient algebra

$$kQ/\langle \partial_a W, a \in Q_1 \rangle$$

where $\langle \partial_a W, a \in Q_1 \rangle$ is the two-sided ideal generated by the $\partial_a W$.

Remark: We follow the terminology of H. Derksen, J. Weyman and A. Zelevinsky ([DWZ07], definition 3.1).

In recent works, B. Keller ([Kel08]) and A. Buan, O. Iyama, I. Reiten and D. Smith ([BIRS08]) have shown independently the following result:

**Theorem 3.4** (Keller, Buan-Iyama-Reiten-Smith). Let $T$ be a cluster-tilting object in the cluster category $C_Q$ associated to an acyclic quiver $Q$. Then there exists a quiver with potential $(Q', W)$ such that $\text{End}_{C_Q}(T)$ is isomorphic to $J(Q', W)$.
3.3. Jacobi-finite quiver with potentials. The quiver with potential \((Q, W)\) is called Jacobi-finite if the Jacobian algebra \(J(Q, W)\) is finite-dimensional.

**Definition 3.5.** Let \((Q, W)\) be a Jacobi-finite quiver with potential. Denote by \(\Gamma\) the Ginzburg dg algebra \(\Gamma(Q, W)\). Let \(\text{per} \Gamma\) be the thick subcategory of \(D \Gamma\) generated by \(\Gamma\) and \(D^b \Gamma\) the full subcategory of \(D \Gamma\) of the dg \(\Gamma\)-modules whose homology is of finite total dimension. The cluster category \(\mathcal{C}_{(Q, W)}\) associated to \((Q, W)\) is defined as the quotient of triangulated categories \(\text{per} \Gamma/ D^b \Gamma\).

Combining theorem 2.1 and theorem 3.2 we get the result:

**Theorem 3.6.** Let \((Q, W)\) be a Jacobi-finite quiver with potential. The cluster category \(\mathcal{C}_{(Q, W)}\) associated to \((Q, W)\) is Hom-finite and 2-CY. Moreover the image \(T\) of the free module \(\Gamma\) in the quotient \(\text{per} \Gamma/ D^b \Gamma\) is a cluster-tilting object. Its endomorphism algebra is isomorphic to the Jacobian algebra \(J(Q, W)\).

As a direct consequence of this theorem we get the corollary:

**Corollary 3.7.** Each finite-dimensional Jacobi algebra \(J(Q, W)\) is 2-CY-tilted in the sense of I. Reiten (cf. [Rei07]), i.e. it is the endomorphism algebra of some cluster-tilting object of a 2-CY category.

**Definition 3.8.** Let \((Q, W)\) and \((Q', W')\) be two quivers with potential. A triangular extension between \((Q, W)\) and \((Q', W')\) is a quiver with potential \((\bar{Q}, \bar{W})\) where
- \(\bar{Q}_0 = Q_0 \cup Q'_0\);
- \(\bar{Q}_1 = Q_1 \cup Q'_1 \cup \{a_i, i \in I\}\), where for each \(i\) in the finite index set \(I\), the source of \(a_i\) is in \(Q_0\) and the tail of \(a_i\) is in \(Q'_0\);
- \(\bar{W} = W + W'\).

**Proposition 3.9.** Denote by \(\mathcal{J} \mathcal{F}\) the class of Jacobi-finite quivers with potential. The class \(\mathcal{J} \mathcal{F}\) satisfies the properties:

1. it contains all acyclic quivers (with potential 0);
2. it is stable under mutation of quivers with potential defined in [DWZ07];
3. it is stable under triangular extensions.

**Proof.**

1. This is obvious since the Jacobi algebra \(J(Q, 0)\) is isomorphic to \(kQ\).
2. This is corollary 6.6 of [DWZ07].
3. Let \((Q, W)\) and \((Q', W')\) be two quivers with potential in \(\mathcal{J} \mathcal{F}\) and \((\bar{Q}, \bar{W})\) a triangular extension. Let \(\bar{Q}_1 = Q_1 \cup Q'_1 \cup F\) be the set of arrows of \(Q\). We have then
   \[
   k\bar{Q} = kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ
   \]
   where \(R\) is the semi-simple algebra \(kQ_0\) and \(R'\) is \(kQ'_0\). Let \(\bar{W}\) be the potential \(W + W'\) associated to the triangular extension. If \(a\) is in \(Q_1\), then \(\partial_a \bar{W} = \partial_a W\), if \(a\) is in \(Q'_1\) then \(\partial_a \bar{W} = \partial_a W'\) and if \(a\) is in \(F\), then \(\partial_a \bar{W} = 0\). Thus we have isomorphisms
   \[
   J(\bar{Q}, \bar{W}) = k\bar{Q}/(\partial_a \bar{W}, a \in \bar{Q}_1) \\
   \simeq kQ' \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/(\partial_a W, a \in Q_1, \partial_a W', b \in Q'_1) \\
   \simeq kQ'/ (\partial_a W', b \in Q'_1) \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R kQ/(\partial_a W, a \in Q_1) \\
   \simeq J(Q', W') \otimes_{R'} (R' \oplus kF \oplus R) \otimes_R J(Q, W).
   \]

Thus if \(J(Q', W')\) and \(J(Q, W)\) are finite-dimensional, \(J(\bar{Q}, \bar{W})\) is finite-dimensional since \(F\) is finite.
In a recent work \cite{KY08}, B. Keller and D. Yang proved the following:

**Theorem 3.10 (Keller-Yang).** Let \((Q, W)\) be a Jacobi-finite quiver with potential. Assume that \(Q\) has no loops nor two-cycles. For each vertex \(i\) of \(Q\), there is a derived equivalence
\[
\mathcal{D}\Gamma(\mu_i(Q, W)) \simeq \mathcal{D}\Gamma(Q, W),
\]
where \(\mu_i(Q, W)\) is the mutation of \((Q, W)\) at the vertex \(i\) in the sense of \cite{DWZ07}.

Remark: in fact Keller and Yang proved this theorem in a more general setting. This also true if \((Q, W)\) is not Jacobi-finite, but then there is a derived equivalence between the completions of the Ginzburg dg algebras.

Another link between mutation of quivers with potential and mutations of cluster-tilting objects is given in \cite{BIRS08} (theorem 5.1):

**Theorem 3.11 (Buan-Iyama-Reiten-Smith).** Let \(C\) be a 2-CY triangulated category with a cluster-tilting object \(T\). If the endomorphism algebra \(\text{End}_C(T)\) is isomorphic to the Jacobian algebra \(J(Q, W)\) for some quiver with potential \((Q, W)\), and if no 2-cycles start in the vertex \(i\) of \(Q\), then we have an isomorphism
\[
\text{End}_C(\mu_i(T)) \simeq J(\mu_i(Q, W)).
\]

Combining these two theorems with theorem 3.6, we get the corollary:

**Corollary 3.12.**
1. If \(Q\) is an acyclic quiver, and \(W = 0\), the cluster category \(\mathcal{C}_{(Q,W)}\) is canonically equivalent to the cluster category \(\mathcal{C}_Q\).
2. Let \(Q\) be an acyclic quiver and \(T\) a cluster-tilting object of \(\mathcal{C}_Q\). If \((Q', W)\) is the quiver with potential associated with the cluster-tilted algebra \(\text{End}_C(T)\) (cf. \cite{Kel08b} \cite{BIRS08}), then the cluster category \(\mathcal{C}_{(Q,W)}\) is triangle equivalent to the cluster category \(\mathcal{C}_{Q'}\).

**Proof.**
(1) The cluster category \(\mathcal{C}_{(Q,0)}\) is a 2-CY category with a cluster-tilting object whose endomorphism algebra is isomorphic to \(kQ\). Thus by \cite{KR07}, this category is triangle equivalent to \(\mathcal{C}_Q\).

(2) In a cluster category, all cluster-tilting objects are mutation equivalent. Thus there exists a sequence of mutations which links \(kQ\) to \(T\). Moreover the quiver of a cluster-tilted algebra has no loops nor 2-cycles. Thus by theorem 5.1 of \cite{BIRS08}, the quiver with potential \((Q, W)\) is mutation equivalent to \((Q', 0)\). Then the theorem of Keller and Yang \cite{KY08} applies and we have an equivalence
\[
\mathcal{D}\Gamma(Q, W) \simeq \mathcal{D}\Gamma(Q', 0).
\]

Thus the categories \(\mathcal{C}_{(Q,W)}\) and \(\mathcal{C}_{(Q',0)}\) are triangle equivalent. By (1) we get the result.

4. Cluster categories for non hereditary algebras

4.1. **Definition and results of Keller.** Let \(A\) be a finite-dimensional \(k\)-algebra of finite global dimension. The category \(\mathcal{D}^bA\) admits a Serre functor \(\nu_A = \mathbf{L} \otimes_A DA\) where \(D\) is the duality \(\text{Hom}_k(\cdot, k)\) over the ground field. The orbit category
\[
\mathcal{D}^bA/\nu_A \circ [-2]
\]
is defined as follows:

□
the objects are the same as those of $\mathcal{D}^b A$;

- if $X$ and $Y$ are in $\mathcal{D}^b A$ the space of morphisms is isomorphic to the space
  \[ \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D} A}(X, (\nu_A Y[-2i]). \]

By Theorem 1 of [Kel05], this category is triangulated if $A$ is derived equivalent to an hereditary category. This happens if $A$ is the endomorphism algebra of a tilting module over an hereditary algebra, or if $A$ is a canonical algebra (cf. [HR02], [Hap01]).

In general it is not triangulated and we define its triangulated hull as the algebraic triangulated category $C_A$ with the following universal property:

- There exists an algebraic triangulated functor $\pi : \mathcal{D}^b A \to C_A$.
- Let $B$ be a dg category and $X$ an object of $\mathcal{D}^b (A^{\text{op}} \otimes B)$. If there exists an isomorphism in $\mathcal{D}(A^{\text{op}} \otimes B)$ between $DA \otimes_A [X-2]$ and $X$, then the triangulated algebraic functor $\pi^L : \mathcal{D}^b A \to DB$ factorizes through $\pi$.

Let $B$ be the dg algebra $A \oplus DA[-3]$. Denote by $p : B \to A$ the canonical projection. It induces a triangulated functor $p_* : \mathcal{D}^b A \to \mathcal{D}^b B$. Let $\langle A \rangle_B$ be the thick subcategory of $\mathcal{D}^b B$ generated by the image of $p_*$. By Theorem 2 of [Kel05] (cf. also [Kel08c]), the triangulated hull of the orbit category $\mathcal{D}^b A/\nu_A \circ [-2]$ is the category $C_A = \langle A \rangle_B / \text{per} B$.

We call it the cluster category of $A$. Note that if $A$ is the path algebra of an acyclic quiver, there is an equivalence $C_Q = \mathcal{D}^b (k Q)/\nu \circ [-2] \simeq \langle k Q \rangle_B / \text{per} B$.

4.2. 2-Calabi-Yau property. The dg $B$-bimodule $DB$ is clearly isomorphic to $B[3]$, so it is not hard to check the following lemma:

**Lemma 4.1.** For each $X$ in $\text{per} B$ and $Y$ in $\mathcal{D}^b B$ there is a functorial isomorphism $D\text{Hom}_{DB}(X, Y) \simeq \text{Hom}_{DB}(Y, X[3])$.

So we can apply results of section 1 and construct a bilinear bifunctorial form:

\[ \beta'_{XY} : \text{Hom}_{\mathcal{C} A}(X, Y) \times \text{Hom}_{\mathcal{C} A}(Y, X[2]) \to k. \]

**Theorem 4.2.** Let $X$ and $Y$ be objects in $\mathcal{D} = \mathcal{D}^b B$. If the spaces $\text{Hom}_{\mathcal{D}}(X, Y)$ and $\text{Hom}_{\mathcal{D}}(Y, X[3])$ are finite-dimensional, then the bilinear form $\beta'_{XY}$ is non-degenerate.

Before proving this theorem, we recall some results about inverse limits of sequences of vector spaces that we will use in the proof. Let $\cdots \to V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \to V_1 \xrightarrow{\varphi} V_0$ be an inverse system of vector spaces (or vector space complexes) inverse system. We then have the following exact sequence

\[ 0 \to \lim V_p \to \prod_{p} V_p \xrightarrow{\Phi} \prod_{q} V_q \to \lim^1 V_p \to 0 \]

where $\Phi$ is defined by $\Phi(v_p) = v_p - \varphi(v_p) \in V_p \oplus V_{p-1}$ where $v_p$ is in $V_p$.

Recall two classical lemmas due to Mittag-Leffler:
Lemma 4.3. If, for all $p$, the sequence of vector spaces $W_i = \text{Im}(V_{p+i} \to V_p)$ is stationary, then $\varprojlim V_p$ vanishes.

This happens in particular when all vector spaces $V_p$ are finite-dimensional.

Lemma 4.4. Let $\cdots \to V_p \xrightarrow{\varphi} V_{p-1} \to \cdots \to V_1 \xrightarrow{\varphi} V_0$ be an inverse system of finite-dimensional vector spaces such that $V_\infty = \varprojlim V_p$ is also finite-dimensional. Let $V'_p$ be the image of $V_\infty$ in $V_p$. The sequence $V'_p$ is stationary and we have $V'_p = \varprojlim V'_p = V'_\infty$.

Proof. (of theorem 4.2) Let $X$ and $Y$ be objects of $\mathcal{D}^bB$ such that $\text{Hom}_{\mathcal{D}^B}(X, Y)$ is finite-dimensional. We will prove that there exists a local $\text{per} B$-cover of $X$ relative to $Y$.

Let $P_\bullet: \cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_0$ be a projective resolution of $X$. The complex $P_\bullet$ has components in $\text{per} B$, and its homology vanishes in all degrees except in degree zero, where it is $X$. Let $P_{\leq n}$ and $P_{>n}$ be the natural truncations, and denote by $\text{Tot}(P)$ the total complex associated to $P_\bullet$. For all $n \in \mathbb{N}$, there is an exact sequence of $\text{dg} B$-modules:

$$0 \to \text{Tot}(P_{\leq n}) \to \text{Tot}(P) \to \text{Tot}(P_{>n}) \to 0$$

The complex $\text{Tot}(P)$ is quasi-isomorphic to $X$, and the complex $\text{Tot}(P_{\leq n})$ is in $\text{per} B$. Moreover, $\text{Tot}(P)$ is the colimit of $\text{Tot}(P_{\leq n})$. Thus by definition, we have the following equalities

$$\text{Hom}^*_B(\text{Tot}(P), Y) = \text{Hom}^*_B(\text{colim} \text{Tot}(P_{\leq n}), Y) = \varprojlim \text{Hom}^*_B(\text{Tot}(P_{\leq n}), Y).$$

Denote by $V_\bullet$ the complex $\text{Hom}^*_B(\text{Tot}(P), Y)$. In the inverse system

$$\cdots \to V_p \xrightarrow{\varphi} V_{p-1} \xrightarrow{\varphi} \cdots \to V_1 \xrightarrow{\varphi} V_0,$$

all the maps are surjective, so by lemma 4.3, there is a short exact sequence

$$0 \to V_\infty \to \prod V_p \xrightarrow{\Phi} \prod V_q \to 0$$

which induces a long exact sequence in cohomology

$$\cdots \to \prod H^{-1} V_q \xrightarrow{\Phi} H^0(V_\infty) \xrightarrow{\prod H^0 V_p} \prod H^0 V_q \to \lim H^{-1} V_p \to \lim H^0 V_p \to \cdots$$

We have the equalities

$$H^0(V_\infty) = H^0(\text{Hom}^*_B(\text{Tot}(P), Y)) = \text{Hom}_{\mathcal{D}}(\text{Tot}(P), Y) = \text{Hom}_{\mathcal{D}}(X, Y).$$

Denote by $W_p$ the complex $\text{Hom}_{\mathcal{D}}(\text{Tot}(P_{\leq p}), Y)$ and by $U_p$ the complex $H^{-1}(V_p) = \text{Hom}_{\mathcal{D}}(\text{Tot}(P_{\leq p}), Y)$. The spaces $(U_p)_p$ are finite-dimensional, so by lemma 4.3, $\varprojlim U_p$ vanishes and we have an isomorphism

$$H^0(\varprojlim V_p) = H^0(V_\infty) \simeq \varprojlim H^0(V_p).$$

The system $(W_p)_p$ satisfies the hypothesis of lemma 4.4. In fact, for each integer $p$, the space $\text{Hom}_{\mathcal{D}}(\text{Tot}(P_{\leq p}), Y)$ is finite-dimensional because $\text{Tot}(P_{\leq p})$ is in $\text{per} B$. Moreover, by the last two
equalities \( W_\infty = \lim W_p \) is isomorphic to \( \text{Hom}_D(X, Y) \) which is finite-dimensional by hypothesis. By lemma 4.4, the system \((W'_p)_p\) formed by the image of \( W_\infty \) in \( W_p \) is stationary. More precisely, there exists an integer \( n \) such that \( W'_n = \lim W''_p \). Moreover \( W'_n \) is a subspace of \( W_n = \text{Hom}_D(Tot(P_{\leq n}), Y) \) and there is an injection

\[
\text{Hom}_D(X, Y) \hookrightarrow \text{Hom}_D(Tot(P_{\leq n}), Y).
\]

This yields a local per \( B \)-cover of \( X \) relative to \( Y \).

The spaces \( \text{Hom}_D(N, X) \) and \( \text{Hom}_D(X, N) \) are finite-dimensional for \( N \) in per \( B \) and \( X \) in \( \mathcal{D}^b \). Thus by proposition 4.4, there exists local per \( B \)-envelopes. Therefore theorem 1.3 applies and \( \beta' \) is non-degenerate.

\[ \square \]

**Corollary 4.5.** Let \( A \) be a finite-dimensional \( k \)-algebra with finite global dimension. If the cluster category \( \mathcal{C}_A \) is \( \text{Hom} \)-finite, then it is 2-CY as a triangulated category.

**Proof.** Denote by \( p_* : \mathcal{D}^b A \to \mathcal{D}^b B \) the restriction of the projection \( p : B \to A \).

Let \( X \) and \( Y \) be in \( \mathcal{D}^b(A) \). By hypothesis, the vector spaces

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b A}(X, \nu^p Y[-2p]) \quad \text{and} \quad \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b A}(Y, \nu^p X[-2p + 3])
\]

are finite-dimensional. But by \([\text{Kel05}]\), the space \( \text{Hom}_{\mathcal{D}^b B}(p_* X, p_* Y) \) is isomorphic to

\[
\bigoplus_{p \geq 0} \text{Hom}_{\mathcal{D}^b A}(X, \nu^p Y[-2p]),
\]

so is finite-dimensional. For the same reasons, the space \( \text{Hom}_{\mathcal{D}^b B}(Y, X[3]) \) is also finite-dimensional. Applying theorem 4.4, we get a non-degenerate bilinear form \( \beta'_{p_* X, p_* Y} \). The non-degeneracy property is extension closed, so for each \( M \) and \( N \) in \( (A)_B \) the form \( \beta_{MN} \) is non-degenerate.

\[ \square \]

**4.3. Case of global dimension 2.** In this section we assume that \( A \) is a finite-dimensional \( k \)-algebra of global dimension \( \leq 2 \).

**Criterion for Hom-finiteness.** The canonical \( t \)-structure on the derived category \( \mathcal{D} = \mathcal{D}^b A \) satisfies the property:

**Lemma 4.6.** We have the following inclusions \( \nu(\mathcal{D}_{\geq 0}) \subset \mathcal{D}_{\geq -2} \) and \( \nu^{-1}(\mathcal{D}_{\leq 0}) \subset \mathcal{D}_{\leq 2} \). Moreover, the space \( \text{Hom}_\mathcal{D}(U, V) \) vanishes for all \( U \) in \( \mathcal{D}_{\geq 0} \) and all \( V \) in \( \mathcal{D}_{\leq -3} \).

**Proposition 4.7.** Let \( X \) be the \( A \)-\( A \)-bimodule \( \text{Ext}^2_A(DA, A) \). The endomorphism algebra \( \bar{A} = \text{End}_A(A) \) is isomorphic to the tensor algebra \( T_A X \) of \( X \) over \( A \).

**Proof.** By definition, the endomorphism space \( \text{End}_A(A) \) is isomorphic to

\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_\mathcal{D}(A, \nu^p A[-2p])
\]

For \( p \geq 1 \), the object \( \nu^p A[-2p] \) is in \( \mathcal{D}_{\geq 2} \) since \( \nu A \) is in \( \mathcal{D}_{\geq 0} \). So since \( A \) is in \( \mathcal{D}_{\leq 0} \), the space \( \text{Hom}_\mathcal{D}(A, \nu^p A[-2p]) \) vanishes.
The functor $\nu = L \otimes_A DA$ admits an inverse $\nu^{-1} = - L \otimes_A R\text{Hom}_A(DA, A)$. Since the global dimension of $A$ is $\leq 2$, the homology of the complex $R\text{Hom}_A(DA, A)$ is concentrated in degrees 0, 1 and 2:

$$H^0(R\text{Hom}_A(DA, A)) = \text{Hom}_D(DA, A)$$
$$H^1(R\text{Hom}_A(DA, A)) = \text{Ext}^1_A(DA, A)$$
$$H^2(R\text{Hom}_A(DA, A)) = \text{Ext}^2_A(DA, A).$$

Let us denote by $Y$ the complex $R\text{Hom}_A(DA, A)[2]$. We then have


Therefore we get the following equalities

$$\text{Hom}_{DA}(A, S^{-p}[-2p]) = \text{Hom}_{DA}(A, Y^{-p})$$
$$= H^0(Y^{-p}).$$

Since $H^0(Y) = X$, we conclude using the following easy lemma. \qed

**Lemma 4.8.** Let $M$ and $N$ be two complexes of $A$-modules whose homology is concentrated in negative degrees. Then there is an isomorphism

$$H^0(M \otimes_A N) \simeq H^0(M) \otimes_A H^0(N).$$

**Proposition 4.9.** Let $A$ be a finite-dimensional algebra of global dimension 2. The following properties are equivalent:

1. The cluster category $\mathcal{C}_A$ is $\text{Hom}$-finite;
2. The functor $? \otimes_A \text{Ext}^2(DA, A)$ is nilpotent;
3. The functor $\text{Tor}_2^A(?, DA)$ is nilpotent;
4. There exists an integer $N$ such that there is an inclusion $\Phi^N(D_{\geq 0}) \subset D_{\geq 1}$ where $\Phi$ is the autoequivalence $\nu_A[-2]$ of the category $D = D^bA$ and $D_{\geq 0}$ is the right aisle of the natural $t$-structure of $D^bA$.

**Proof.** 1 $\Rightarrow$ 2: It is obvious by proposition [4.7].

2 $\Leftrightarrow$ 3 $\Leftrightarrow$ 4: Let $\Phi$ be the autoequivalence $\nu_A[-2]$ of $D^bA$. The functor $\text{Tor}_2^A(?, DA)$ is isomorphic to $H^0 \circ \Phi$ and $? \otimes_A \text{Ext}^2_A(DA, A)$ is isomorphic to $H^0 \circ \Phi^{-1}$.

Thus it is sufficient to check that there are isomorphisms

$$H^0 \Phi \circ H^0 \Phi \simeq H^0 \Phi^2 \text{ and } H^0 \Phi^{-1} \circ H^0 \Phi^{-1} \simeq H^0 \Phi^{-2}.$$

This is easy using Lemma [4.8] since the algebra $A$ has global dimension $\leq 2$.

4 $\Rightarrow$ 1: Suppose that there exists some $N \geq 0$ such that $\Phi^N(D_{\geq 0})$ is included in $D_{\leq 1}$. For each object $X$ in $\mathcal{C}_A$, the class of the objects $Y$ such that the space $\text{Hom}_{\mathcal{C}_A}(X, Y)$ (resp. $\text{Hom}_{\mathcal{C}_A}(Y, X)$) is finite-dimensional, is extension closed. Therefore, it is sufficient to show that for all simples $S$, $S'$, and each integer $n$, the space $\text{Hom}_{\mathcal{C}_A}(S, S'[n])$ is finite-dimensional.

There exists an integer $p_0$ such that for all $p \geq p_0$ $\Phi^p(S')$ is in $D_{\geq p+1}$. Therefore, because of the defining properties of the $t$-structure, the space

$$\bigoplus_{p \geq p_0} \text{Hom}_{D}(S, \Phi^p(S')[n])$$
vanishes. Similarly, there exists an integer $q_0$ such that for all $q \geq q_0$, we have $\Phi^q(S) \in D_{\geq -n+3}$. Since the algebra $A$ is of global dimension $\leq 2$, the space
\[
\bigoplus_{q \geq q_0} \text{Hom}_D(\Phi^q(S), S'[n])
\]vanishes. Thus the space
\[
\bigoplus_{p \in \mathbb{Z}} \text{Hom}_D(S, \Phi^p(S')[n]) = \bigoplus_{p=-q_0}^{p_0} \text{Hom}_D(S, \Phi^p(S')[n])
\]is finite-dimensional.

\[\square\]

**Cluster-tilting object.** In this section we prove the following theorem:

**Theorem 4.10.** Let $A$ be a finite-dimensional $k$-algebra of global dimension $\leq 2$. If the functor $\text{Tor}^A_2(?, DA)$ is nilpotent, then the cluster category $\mathcal{C}_A$ is $\text{Hom}$-finite, 2-CY and the object $A$ is a cluster-tilting object.

We denote by $\Theta$ a cofibrant resolution of the dg $A$-bimodule $R\text{Hom}^\bullet_A(DA, A)$. Following [Kel08a] and [Kel08b], we define the 3-derived preprojective algebra as the tensor algebra
\[\Pi_3(A) = T_A(\Theta[2]).\]
The complex $R\text{Hom}^\bullet_A(DA, A)[2]$ has its homology concentrated in degrees $-2$, $-1$ and $0$, and we have
\[H^{-2}(\Theta[2]) \simeq \text{Hom}_{DA}(DA, A), \quad H^{-1}(\Theta[2]) \simeq \text{Ext}^1_A(DA, A)\]
and $H^0(\Theta[2]) \simeq \text{Ext}^2_A(DA, A)$.

Thus the homology of the dg algebra $\Pi_3(A)$ vanishes in strictly positive degrees and we have
\[H^0\Pi_3A = T_A(\text{Ext}^1_A(DA, A)) = \tilde{A}.
\]
By proposition [4.9], it is finite-dimensional. Moreover, Keller showed that $\Pi_3(A)$ is homologically smooth and bimodule 3-CY [Kel08b]. Thus we can apply theorem 2.1 and we have the following result:

**Corollary 4.11.** The category $\mathcal{C} = \text{per} \Pi_3A/D^b\Pi_3A$ is 2-CY and the free dg module $\Pi_3A$ is a cluster-tilting object in $\mathcal{C}$.

To complete the proof of Theorem 4.10, we now construct a triangle equivalence between $\mathcal{C}_A$ and $\mathcal{C}$ sending $A$ to $\Pi_3A$.

Let us recall a theorem of Keller ([Kel94], or theorem 8.5, p.96 [AHHK07]):

**Theorem 4.12.** [Keller] Let $B$ be a dg algebra, and $T$ an object of $DB$. Denote by $C$ the dg algebra $R\text{Hom}^\bullet_B(T, T)$. Denote by $\langle T \rangle_B$ the thick subcategory of $DB$ generated by $T$. The functor $R\text{Hom}^\bullet_B(T, ?) : DB \rightarrow DC$ induces an algebraic triangle equivalence
\[R\text{Hom}^\bullet_B(T, ?) : \langle T \rangle_B \sim \text{per} C.\]

Let us denote by $\mathcal{H}o(dgalg)$ the homotopy category of dg algebras, i.e. the localization of the category of dg algebras at the class of quasi-isomorphisms.

**Lemma 4.13.** In $\mathcal{H}o(dgalg)$, there is an isomorphism between $\Pi_3A$ and $R\text{Hom}_B(A_B, A_B)$. 

Proof. The dg algebra $B$ is $A \oplus (DA)[-3]$. Denote by $X$ a cofibrant resolution of the dg $A$-bimodule $DA[-2]$. Now look at the dg submodule of the bar resolution of $B$ seen as a bimodule over itself (see the proof of theorem 7.1 in [Kel05]):

\[
\text{bar}(X, B) : \cdots \longrightarrow B \otimes_A X \otimes A^2 \otimes_A B \longrightarrow B \otimes_A X \otimes_A B \longrightarrow B \otimes_A B \longrightarrow 0
\]

This is a cofibrant resolution of the dg $B$-bimodule $B$. Thus $A \otimes_B \text{bar}(X, B)$ is a cofibrant resolution of the dg $B$-module $A$. Therefore, we have the following isomorphisms

\[
R\text{Hom}^\bullet_B(A_B, A_B) \cong \text{Hom}^\bullet_B(A \otimes_B \text{bar}(X, B), A) \\
\cong \prod_{n \geq 0} \text{Hom}^\bullet_B(A \otimes_A X \otimes A^n \otimes_A B, A_B) \\
\cong \prod_{n \geq 0} \text{Hom}^\bullet_A(X \otimes A^n, \text{Hom}_B(B, A_B)) \\
\cong \prod_{n \geq 0} \text{Hom}^\bullet_A(X \otimes A^n, A_A),
\]

where the differential on the last complex is induced by that of $X \otimes A^n$. Note that

\[
\text{Hom}^\bullet_A(X, A) = R\text{Hom}^\bullet_A(DA[-2], A) = R\text{Hom}^\bullet_A(DA, A)[2] = \Theta[2].
\]

We can now use the following lemma:

Lemma 4.14. Let $A$ be a dg algebra, and $L$ and $M$ dg $A$-bimodules such that $M_A$ is perfect as right dg $A$-module. There is an isomorphism in $\mathcal{D}(A^{op} \otimes A)$

\[
R\text{Hom}^\bullet_A(L, A) \otimes_A R\text{Hom}^\bullet_A(M, A) \simeq R\text{Hom}^\bullet_A(M \otimes_A L, A).
\]

Proof. Let $X$ and $M$ be dg $A$-bimodules. The following morphism of $\mathcal{D}(A^{op} \otimes A)$

\[
X \otimes_A R\text{Hom}_A(M, A) \longrightarrow R\text{Hom}_A(M, X)
\]

is clearly an isomorphism for $M = A$. Thus it is an isomorphism if $M$ is perfect as a right dg $A$-module. Applying this to the right dg $A$-module $R\text{Hom}_A(L, A)$, we get an isomorphism of dg $A$-bimodules

\[
R\text{Hom}_A(L, A) \otimes_A R\text{Hom}_A(M, A) \simeq R\text{Hom}_A(M, R\text{Hom}_A(L, A)).
\]

Finally, by adjunction we get an isomorphism of dg $A$-bimodules

\[
R\text{Hom}_A(L, A) \otimes_A R\text{Hom}_A(M, A) \simeq R\text{Hom}_A(M \otimes_A L, A).
\]

Therefore, the dg $A$-bimodule $\text{Hom}^\bullet_A(X \otimes A^n, A_A)$ is isomorphic to $(\Theta[2]) \otimes A^n$, and there is an isomorphism of dg algebras

\[
R\text{Hom}^\bullet_B(A_B, A_B) \cong \bigoplus_{n \geq 0} (\Theta[2]) \otimes A^n = \Pi_3(A)
\]

because for each $p \in \mathbb{Z}$, the group $H^p(\Theta[2] \otimes A^n)$ vanishes for all $n \gg 0$. \hfill \square
By theorem 4.12, the functor \( R\text{Hom}^*_B(A_B, ?) \) induces an equivalence between the thick subcategory \(<A>_B\) of \( DB \) generated by \( A \), and \( \text{per} \Pi_3(A) \). Thus we get a triangle equivalence that we will denote by \( F \):

\[
F = R\text{Hom}^*_B(A_B, ?) : <A>_B \xrightarrow{\sim} \text{per} \Pi_3A
\]

This functor sends the object \( A_B \) of \( DB \) onto the free module \( \Pi_3A \) and the free \( B \)-module \( B \) onto \( R\text{Hom}^*_B(A_B, B) \simeq R\text{Hom}^*_B(A_B, DB[-3]) \), that is to say onto \( (DA)[-3]_{\Pi_3A} \). So \( F \) induces an equivalence

\[
F : \text{per} B = <B>_B \xrightarrow{\sim} <DA[-3]>_{\Pi_3A} = <A>_{\Pi_3A}.
\]

**Lemma 4.15.** The thick subcategory \(<A>_{\Pi_3A}\) of \( DB_{\Pi_3A} \) generated by \( A \) is \( DB_{\Pi_3A} \).

**Proof.** The algebra \( A \) is finite-dimensional, thus \(<A>_{\Pi_3A} \) is obviously included in \( DB_{\Pi_3A} \). Moreover, the category \( DB_{\Pi_3A} \) equals \( \langle \text{mod} H^0(\Pi_3A) \rangle_{\Pi_3A} \) by the existence of the \( t \)-structure. The dg algebra \( \Pi_3A \) is the tensor algebra \( T_A(\theta[2]) \) thus there is a canonical projection \( \Pi_3A \to A \) which yields a restriction functor \( DB_A \to DB(\Pi_3A) \) respecting the \( t \)-structure:

\[
\text{mod} H^0 \Pi_3A = \mathcal{H} \longmapsto DB(\Pi_3A)
\]

This restriction functor induces a bijection in the set of isomorphism classes of simple modules because the kernel of the map \( H^0(\Pi_3A) \to A \) is a nilpotent ideal (namely the sum of the tensor powers over \( A \) of the bimodule \( \text{Ext}^2_A(DA, A) \)). Thus each simple of \( \text{mod} H^0 \Pi_3A \) is in \(<A>_{\Pi_3A}\) and we have

\[
<A>_{\Pi_3A} \simeq \langle \text{mod} H^0(\Pi_3A) \rangle_{\Pi_3A} \simeq DB_{\Pi_3A}.
\]

**Proof. (of theorem 4.10)** By proposition 4.9 and corollary 4.5, the cluster category is \( \text{Hom} \)-finite and 2-CY. Furthermore, the functor \( F = R\text{Hom}^*_B(A_B, ?) \) induces the following commutative square:

\[
F : <A>_B \xrightarrow{\sim} \text{per} \Pi_3A
\]

\[
\text{per} B \xrightarrow{\sim} DB_{\Pi_3A}
\]

Thus \( F \) induces a triangle equivalence

\[
\mathcal{C}_A = <A>_B/\text{per} B \xrightarrow{\sim} \text{per} \Pi_3A/DB_{\Pi_3A} = C
\]

sending the object \( A \) onto the free module \( \Pi_3A \). By theorem 2.7, \( A \) is therefore a cluster-tilting object of the cluster category \( \mathcal{C}_A \).

**Quiver of the endomorphism algebra of the cluster-tilting object.** Let \( A = kQ/I \) be a finite-dimensional \( k \)-algebra of global dimension \( \leq 2 \). Suppose that \( I \) is an admissible ideal generated by a finite set of minimal relations \( r_i, i \in J \) where for each \( i \in J \), the relation \( r_i \) starts at the vertex \( s(r_i) \) and ends at the vertex \( t(r_i) \). Let \( \tilde{Q} \) be the following quiver:

- the set of the vertices of \( \tilde{Q} \) equals that of \( Q \);
• the set of arrows of $\tilde{Q}$ is obtained from that of $Q$ by adding a new arrow $\rho_i$ with source $t(r_i)$ and target $s(r_i)$ for each $i$ in $J$.

We then have the following proposition, which has essentially been proved by I. Assem, T. Brüstle and R. Schiffler [ABS06] (thm 2.6). The proposition is also proved in [Kel08b].

**Proposition 4.16.** If the algebra $\text{End}_{C_A}(A) = \tilde{A}$ is finite-dimensional, then its quiver is $\tilde{Q}$.

**Proof.** Let $B$ be a finite-dimensional algebra. The vertices of its quiver are determined by the quotient $B/rad(B)$ and the arrows are determined by $rad(B)/rad^2(B)$. Denote by $X$ the $A$-$A$-bimodule $\text{Ext}^2_A(DA, A)$. Since $X \otimes_A X$ is in $rad^2(B)$, the quiver of $\tilde{A} = T_A X$ is the same as the quiver of the algebra $A \times X$. The proof is then exactly the same as in [ABS06] (thm 2.6).

**Example.** We refer to [GLS07a] for this example. Let $Q$ be a Dynkin quiver. Let $A$ be its Auslander algebra. The algebra $A$ is of global dimension $\leq 2$. The category $\text{mod} A$ is equivalent to the category $\text{mod} (\text{mod} kQ)$ of finitely presented functors $(\text{mod} kQ)^{op} \rightarrow \text{mod} k$. The projective indecomposables of $\text{mod} A$ are the representable functors $U^\perp = \text{Hom}_{kQ}(?, U)$ where $U$ is an indecomposable $kQ$-module. Let $S$ be a simple $A$-module. Since $A$ is finite-dimensional, this simple is associated to an indecomposable $U$ of $\text{mod} kQ$. If $U$ is not projective, then it is easy to check that in $D^b(A)$ the simple $S_U$ is isomorphic to the complex:

$$\cdots \longrightarrow 0 \longrightarrow (\tau U)^\perp \longrightarrow E^\perp \longrightarrow U^\perp \longrightarrow 0 \longrightarrow \cdots$$

where $0 \longrightarrow \tau U \longrightarrow E \longrightarrow U \longrightarrow 0$ is the Auslander-Reiten sequence associated to $U$. Thus $\Phi(S_U) = \nu S_U[-2]$ is the complex:

$$\cdots \longrightarrow 0 \longrightarrow (\tau U)^\vee \longrightarrow E^\vee \longrightarrow U^\vee \longrightarrow 0 \longrightarrow \cdots$$

where $U^\vee$ is the injective $A$-module $D\text{Hom}_{kQ}(U, ?)$. It follows from the Auslander-Reiten formula that this complex is quasi-isomorphic to the simple $S_{\tau U}$.

If $U$ is projective, then $S_U$ is isomorphic in $D^b(A)$ to

$$\cdots \longrightarrow 0 \longrightarrow (rad U)^\perp \longrightarrow U^\perp \longrightarrow 0 \longrightarrow \cdots,$$

and then $\Phi(S_U)$ is in $D_{\geq 1}$. Since for each indecomposable $U$ there is some $N$ such that $\tau^N U$ is projective, there is some $M$ such that $\Phi^M(D_{\geq 0})$ is included in $D_{\geq 1}$. By proposition 4.9, the cluster category $C_A$ is $\text{Hom}$-finite, and $2$-CY by corollary 4.3.

The quiver of $A$ is the Auslander-Reiten quiver of $\text{mod} kQ$. The minimal relations of the algebra $A$ are given by the mesh relations. Thus the quiver of $\tilde{A}$ is the same as that of $A$ in which arrows $\tau x \rightarrow x$ are added for each non-projective indecomposable $x$.

For instance, if $Q$ is $A_4$ with the orientation $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$, then the quiver of the algebra $\tilde{A}$ is the following:

```
\begin{center}
\begin{tikzpicture}
  \foreach \i in {1,...,4} {\node (v\i) at (90 + \i*90:2) {$\bullet$};}
  \draw (v1) -- (v2) -- (v3) -- (v4) -- (v1);
  \draw (v1) -- (v3);
  \draw (v2) -- (v4);
\end{tikzpicture}
\end{center}
```
5. Stable module categories as cluster categories

5.1. Definition and first properties. Let $B$ be a concealed algebra \[Rin84\], i.e. the endomorphism algebra of a preinjective tilting module over a finite-dimensional hereditary algebra. Let $H$ be a postprojective slice of $\text{mod} \ B$. We denote by $\text{add}(H)$ the smallest subcategory of $\text{mod} \ B$ which contains $H$ and which is stable under taking direct summands. Let $Q$ be the quiver such that $\text{End}_B(H)$ is the path algebra $kQ$ and let $Q_0 = \{1, \cdots, n\}$ be its set of vertices. By Happel \[Hap87\], there is a triangle equivalence:

\[
D^b(B) \xrightarrow{\text{DRHom}_B(?,H)} D^b(kQ) \xleftarrow{(D?)^L \otimes_{kQ} H} D^b(\text{mod} \ B).
\]

Notice that these functors induce quasi-inverse equivalences between $\text{add}(H)$ and the subcategory of finite-dimensional injective $kQ$-modules.

Define $M$ as the following subcategory of $\text{mod} \ B$:

\[
M = \{X \in \text{mod} \ B \mid \text{Ext}^1_B(X, H) = 0\} = \{X \in \text{mod} \ B \mid X \text{ is cogenerated by } H\}
\]

We denote by $\tau_B$ the AR-translation of the category $\text{mod} \ B$ and by $\tau_A$ the AR-translation of $D^b(B)$.

The following proposition is a classical result in tilting theory (see for example \[Rin84\]).

**Proposition 5.1.**

1. For each $X$ in $M$ there exists a triangle

\[
X \xrightarrow{} H_0 \xrightarrow{} H_1 \xrightarrow{} X[1]
\]

in $D^b(\text{mod} \ B)$ functorial in $X$ with $H_0$ and $H_1$ in $\text{add}(H)$;

2. $M \subset \text{mod} \ B$ is closed under kernels so in particular, $M$ is closed under $\tau_B$;

3. for each indecomposable $X$ in $M$ there exists a unique $q \geq 0$ such that $\tau_B^{-q}X$ is in $\text{add}(H)$;

4. the category $M$ has finitely many indecomposables.

**Hom-finiteness.** Let $\overline{M}$ be the quotient $M/\text{add}(H)$. Denote by $p : M \to \overline{M}$ the canonical projection. Since $H$ is a slice, we have the following properties.

**Proposition 5.2.**

1. The category $\overline{M}$ is equivalent to the full subcategory of $M$ whose objects do not have non zero direct factors in $\text{add}(H)$. We denote by $i : \overline{M} \to M$ the associated inclusion.

2. The category $\overline{M} \subset \text{mod} \ B$ is closed under kernels, and hence under $\tau_B$.

3. The right exact functor $i : \text{mod} \overline{M} \to \text{mod} M$ induced by $i : \overline{M} \to M$ is isomorphic to the restriction along $p$.

**Proposition 5.3.** Let $A$ be the endomorphism algebra $\text{End}_B(\bigoplus_{M \in \text{ind} \overline{M}} M)$. The global dimension of $A$ is at most 2.

**Proof.** There is an equivalence of categories between $\text{mod} A$ and $\text{mod} \overline{M}$. Since $\overline{M}$ is stable under kernels, the global dimension of $A$ is $\leq 2$. \hfill $\Box$

**Theorem 5.4.** The cluster category $\mathcal{C}_A$ is a Hom-finite, 2-CY category, and the object $A$ is a cluster-tilting object in $\mathcal{C}_A$.

**Proof.** Using corollary 4.3 and theorem 4.9, we just have to check that the functor $\text{Tor}_A^2(?, DA)$ is nilpotent. Since there are finitely many indecomposables in $\overline{M}$, the proof is the same as for an Auslander algebra (cf. the examples of section 4.3). \hfill $\Box$
Construction of the functor $F : \text{mod} \mathcal{M} \to \text{f.l.} \Lambda$. Denote by $\mathcal{I}(kQ)$ the subcategory of the preinjective modules of $\text{mod} kQ$.

**Proposition 5.5.** There exists a $k$-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ unique up to isomorphism such that

- $P$ restricted to subcategory of the injective $kQ$-modules is isomorphic to the restriction of the functor $D(?) \otimes_{kQ} H$;
- for each indecomposable $X$ in $\mathcal{I}(kQ)$ such that $P(X)$ is not projective, the image

$$0 \longrightarrow P(\tau_D X) \xrightarrow{P_i} P(E) \xrightarrow{P_p} P(X) \longrightarrow 0$$

of an Auslander-Reiten sequence in $\text{mod} kQ$ ending at $X$

$$0 \longrightarrow \tau_D X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

is an Auslander-Reiten sequence in $\text{mod} B$ ending at $P(X)$.

Moreover, the functor $P$ is full, essentially surjective, and satisfies $P \circ \tau_D \simeq \tau_B \circ P$.

**Proof.** The Auslander-Reiten quivers $\Gamma_\mathcal{I}$ of $\mathcal{I}(kQ)$ and $\Gamma_\mathcal{M}$ of $\mathcal{M}$ are connected translation quivers. Each vertex of $\Gamma_\mathcal{I}$ is of the form $\tau^q_D x$ with $q \geq 0$ and $x$ indecomposable injective. Each vertex of $\Gamma_\mathcal{M}$ is of the form $\tau^q_B x$ where $x$ is in $\text{add}(H)$ ((3) of proposition 5.1). Moreover, there is a canonical isomorphism of quivers $\bar{P} : \Gamma_{kQ} \to \Gamma_{\text{add}(H)}$. Thus we can inductively construct a morphism of quivers (that we will still denote by $\bar{P}$) $\bar{P} : \Gamma_\mathcal{I} \to \Gamma_\mathcal{M}$ extending $\bar{P}$ such that:

- $\bar{P}(\tau_D x) = \tau_B \bar{P}(x)$ for each vertex $x$ of $\Gamma_\mathcal{I}$;
- $\bar{P}(\sigma_D \alpha) = \sigma_B \bar{P}(\alpha)$ for each arrow $\alpha : x \to y$ of $\Gamma_\mathcal{I}$, where $\sigma_D \alpha$ (resp. $\sigma_B \beta$) denotes the arrow $\tau_D y \to x$ (resp. $\tau_B y \to x$) such that the mesh relations in $\Gamma_\mathcal{I}$ (resp. in $\Gamma_\mathcal{M}$) are of the form $\sum_{\alpha} \tau_D(\alpha) \alpha$ (resp. $\sum_{\beta} \tau_B(\beta) \beta$).

Clearly, this morphism of translation quivers induces surjections in the sets of vertices and the sets of arrows.

The categories $\mathcal{I}(kQ)$ and $\mathcal{M}$ are standard, i.e., $k$-linearly equivalent to the mesh categories of their Auslander-Reiten quivers. Up to isomorphism, an equivalence $k(\Gamma_\mathcal{I}) \to \mathcal{I}(kQ)$ is uniquely determined by its restriction to a slice. Thus there exists a $k$-linear functor $P : \mathcal{I}(kQ) \to \mathcal{M}$ unique up to isomorphism which is equal to $D(?) \otimes_{kQ} H$ on the slice of the injectives and such that the square

$$\begin{array}{ccc}
k(\Gamma_\mathcal{I}) & \sim & \mathcal{I}(kQ) \\
\bar{P} & & p \\
k(\Gamma_\mathcal{M}) & \sim & \mathcal{M}
\end{array}$$

is commutative. This functor $P$ sends Auslander-Reiten sequences

$$0 \longrightarrow \tau_D X \xrightarrow{i} E \xrightarrow{p} X \longrightarrow 0$$

to Auslander-Reiten sequences

$$0 \longrightarrow \tau_B P(X) \xrightarrow{P_i} P(E) \xrightarrow{P_p} P(X) \longrightarrow 0$$

if $P(X)$ is not projective. Since $\bar{P}$ is surjective, $P$ is full and essentially surjective. \qed
Lemma 5.6. Let $X$ and $Y$ be indecomposables in $\mathcal{I}(kQ)$. The kernel of the map $\text{Hom}_{kQ}(X, Y) \to \text{Hom}_{B}(PX, PY)$ is generated by compositions of the form $X \to Z \to Y$ where $Z$ is indecomposable and $P(Z)$ is zero.

Proof. If $P(X)$ or $P(Y)$ is zero this is obviously true. Suppose they are not. The mesh relations are minimal relations of the $k$-linear category $\mathcal{M}$ and $P$ is full. Thus the kernel of the functor $P$ is the ideal generated by the morphisms of the form $U \xrightarrow{g} V \xrightarrow{h} W$ where $0 \to P(U) \to P(V) \to P(W) \to 0$ is an Auslander-Reiten sequence in $\mathcal{M}$. Since $P(U)$ is isomorphic to $\tau B P(W)$, the indecomposable $U$ is isomorphic to $\tau D(W)$. By the construction of $P$, $V$ is a direct factor of the middle term of the Auslander-Reiten sequence ending at $W$, and we can ‘complete’ the composition $\tau D W \xrightarrow{g} V \xrightarrow{h} W$ into an Auslander-Reiten sequence

$$0 \to \tau D W \xrightarrow{\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)} V \oplus V' \xrightarrow{(h \; h')} W \to 0$$

with $P(V') = 0$ and $P(g') = P(h') = 0$. Thus the morphism $hg = -h'g'$ factors through an object in the kernel of $P$. \hfill \Box

Now let $\Lambda$ be the preprojective algebra associated to the acyclic quiver $Q$. It is defined as the quotient $kQ/(c)$ where $Q$ is the double quiver of $Q$ which is obtained from $Q$ by adding to each arrow $a : i \to j$ an arrow $a^* : j \to i$ pointing in the opposite direction, and where $(c)$ is the ideal generated by the element

$$c = \sum_{a \in Q_1} (a^* a + a a^*)$$

where $Q_1$ is the set of arrows of $Q$. We denote by $e_i$ the idempotent of $\Lambda$ associated with the vertex $i$. We then have a natural functor

$$\text{proj} \Lambda \quad \to \quad \mathcal{I}^\Pi(kQ)$$

$$e_i \Lambda \quad \mapsto \quad \prod_{p \geq 0} \tau_B^p I_i$$

where $\mathcal{I}^\Pi(kQ)$ is the closure of $\mathcal{I}(kQ)$ under countable products. Composing this functor with the natural extension of $P$ to $\mathcal{I}^\Pi(kQ)$, we get a functor:

$$\text{proj} \Lambda \quad \to \quad \mathcal{M}$$

$$e_i \Lambda \quad \mapsto \quad \bigoplus_{p \geq 0} \tau_B^p H_i.$$

Therefore the restriction along this functor yields a functor $F : \text{mod} \mathcal{M} \to \text{mod} \Lambda$. Moreover, since $\mathcal{M}$ has finitely many indecomposables, the functor $F$ takes its values in the full subcategory $\text{f.i.} \Lambda$ formed by the $\Lambda$-modules of finite length.

This is an exact functor since it is a restriction. If $M$ is an $\mathcal{M}$-module, then the vector space $F(M)e_j$ is isomorphic to $\bigoplus_{p \geq 0} M(\tau_B^p H_j)$. For $X$ in $\overline{\mathcal{M}}$, there exists $i \in Q_0$ and $q \geq 0$ such that $\tau^q H_i = X$. It is then easy to check that the image $F(S_X)$ of the simple associated to $X$ is the simple $\Lambda$-module $S_i$.

**Fundamental propositions.**

**Proposition 5.7.** For $X$ in $\overline{\mathcal{M}}$, there exists a functorial sequence in $\text{mod} \Lambda$ of the form

$$0 \to F \circ i_*(X^\wedge) \to F(H_0^\wedge) \to F(H_1^\wedge) \to F \circ i_*(X^V) \to 0$$
where \(i_* : \text{mod} \overline{\mathcal{M}} \to \text{mod} \mathcal{M}\) is the right exact functor induced by \(i : \overline{\mathcal{M}} \to \mathcal{M}\), and where \(H_0\) and \(H_1\) are in \(\text{add}(H)\).

**Proof.** Let \(X\) be in \(\overline{\mathcal{M}}\), and \(iX\) its image in \(\mathcal{M}\). By (1) of proposition 5.8, there exists a triangle functorial in \(X\):

\[
\begin{array}{c}
iX \longrightarrow H_0 \longrightarrow H_1 \longrightarrow (iX)[1]\end{array}
\]

with \(H_0\) and \(H_1\) in \(\text{add}(H)\). It yields a long exact sequence in \(\text{mod} \mathcal{M}\):

\[
0 \longrightarrow (iX) \longrightarrow H_0 \longrightarrow H_1 \longrightarrow \text{Ext}^1_B(?, iX)_{\vert \mathcal{M}} \longrightarrow \text{Ext}^1_B(?, H_0)_{\vert \mathcal{M}} \longrightarrow \cdots.
\]

By definition, the functor \(\text{Ext}^1_B(?, iX)_{\vert \mathcal{M}}\) is zero. The Auslander-Reiten formula gives us an isomorphism

\[
\text{Ext}^1_B(?, iX)_{\vert \mathcal{M}} \cong D\text{Hom}_B(\tau_B^{-1}iX, ?)_{\vert \mathcal{M}}/\text{proj} B.
\]

Since \(F\) is an exact functor, we get the following exact sequence in \(\text{f.l.} \Lambda\):

\[
0 \longrightarrow F((iX)^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow F((\tau_B^{-1}iX)^\vee)/\text{proj} B \longrightarrow 0
\]

By definition, we have \(F((iX)^\wedge) \cong (F \circ i_*)(X^\wedge)\). For \(j = 1, \ldots, n\), we have an isomorphism:

\[
F((\tau_B^{-1}iX)^\vee)/\text{proj} B)e_j \cong \bigoplus_{p \geq 0} D\text{Hom}_B(\tau_B^{-1}iX, \tau_B^{-1}i_B^p H_j)/\text{proj} B.
\]

For \(p \geq 0\), we have \(\tau_B^p(H_j) = \tau_B^{-1}(\tau_B^{p+1}H_j)\) if and only if \(\tau_B^p H_j\) is not projective. Thus we have a vector space isomorphism

\[
F((\tau_B^{-1}iX)^\vee)/\text{proj} B)e_j \cong \bigoplus_{p \geq 0} D\text{Hom}_B(\tau_B^{-1}iX, \tau_B^{-1}i_B^p H_j)/\text{proj} B.
\]

A morphism \(f : \tau^{-1}X \to \tau^{-1}Y\) factorizes through a projective object if and only if \(\tau(f) : X \to Y\) is not zero. Thus we have:

\[
F((\tau_B^{-1}iX)^\vee)/\text{proj} B)e_j \cong \bigoplus_{p \geq 1} D\text{Hom}_B(iX, \tau_B^p H_j)
\]

\[
\cong \bigoplus_{p \geq 0} D\text{Hom}_B(X, \tau_B^p H_j)/[\text{add}(H)]
\]

\[
\cong (F \circ p^\vee)(X^\vee)e_j \cong (F \circ i_*)(X^\vee)e_j.
\]

Therefore we get this exact sequence in \(\text{f.l.} \Lambda\), functorial in \(X\):

\[
0 \longrightarrow (F \circ i_*)(X^\wedge) \longrightarrow F(H_0^\wedge) \longrightarrow F(H_1^\wedge) \longrightarrow (F \circ i_*)(X^\vee) \longrightarrow 0
\]

\(\square\)

**Proposition 5.8.** Let \(U\) and \(V\) be indecomposables in \(\overline{\mathcal{M}}\). We have an isomorphism

\[
\text{Hom}_{\text{C}_{\Lambda}}(U^\wedge, V^\wedge) \cong \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p U, V)/[\text{add} \tau_B^p H]
\]

where \(\mathcal{M}(\tau_B^p U, V)/[\text{add} \tau_B^p H]\) is the cokernel of the composition map

\[
\mathcal{M}(\tau_B^p U, \tau_B^p H) \otimes \mathcal{M}(\tau_B^p H, V) \longrightarrow \mathcal{M}(\tau_B^p U, V).
\]

We first show the following:
Lemma 5.9. Let $e_U$ and $e_V$ be the idempotents of $A$ associated to the indecomposables $U$ and $V$. We have an isomorphism
\[ e_U \text{Ext}^2_A(DA, A)e_V \simeq \mathcal{M}(\tau_B U, V)/[\text{add} \tau_B H] \]
where $\mathcal{M}(\tau_B U, V)/[\text{add} \tau_B H]$ is the cokernel of the composition map
\[ \mathcal{M}(\tau_B U, \tau_B H) \otimes \mathcal{M}(\tau_B H, V) \longrightarrow \mathcal{M}(\tau_B U, V). \]

Proof. We have the following isomorphisms:
\[ e_U \text{Ext}^2_A(DA, A)e_V = \text{Ext}^2_A(D(e_U A), e_V) \simeq \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(?,?), \overline{\mathcal{M}}(?, V)[2]). \]
Denote by $\overline{\mathcal{M}}$ the category $\mathcal{M}/\text{proj} B$. The functor $\tau_B$ induces an equivalence of $k$-linear categories $\tau_B : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$. Thus we get the following isomorphisms
\[ \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U,?), \overline{\mathcal{M}}(?, V)[2]) \simeq \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(\tau_B^{-1} U, \tau_B^{-1}?), \overline{\mathcal{M}}(\tau_B^{-1}?, \tau_B^{-1}V)[2]) \]
\[ \simeq \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(\tau_B^{-1} U, ?), \overline{\mathcal{M}}(? , \tau_B^{-1}V)[2]) \]
\[ \simeq \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(\tau_B^{-1} U, ?)/\text{proj} B, \overline{\mathcal{M}}(? , \tau_B^{-1}V)/\text{proj} B[2]). \]
But by the previous lemma, we know a projective resolution in $\text{mod} \mathcal{M}$ of the module $D\mathcal{M}(\tau_B^{-1} U, ?)/\text{proj} B$. Namely, there exists an exact sequence in $\text{mod} \mathcal{M}$ of the form:
\[ 0 \longrightarrow \mathcal{M}(?, U) \longrightarrow \mathcal{M}(?, H_0) \longrightarrow \mathcal{M}(?, H_1) \longrightarrow D\mathcal{M}(\tau_B^{-1} U, ?)/\text{proj} B \longrightarrow 0 \]
where $H_0$ and $H_1$ are in $\text{add}(H)$. Thus we get (using Yoneda’s lemma)
\[ \text{Hom}_{D(\overline{\mathcal{M}})}(D\overline{\mathcal{M}}(U,?), \overline{\mathcal{M}}(?, V)[2]) \simeq \text{Hom}_{D(\overline{\mathcal{M}})}(\mathcal{M}(?, U), \mathcal{M}(?, \tau_B^{-1}V)/\text{proj} B)/[\text{add} \mathcal{M}(?, H)] \]
\[ \simeq \mathcal{M}(U, \tau_B^{-1}V)/[\text{add} H] \]
\[ \simeq \mathcal{M}(\tau_B U, V)/[\text{add} \tau_B H]. \]

Since $V$ is in $\overline{\mathcal{M}}$, a non zero morphism of $\mathcal{M}(\tau_B U, V)$ cannot factorize through $\text{add}(H)$. Thus we get $\overline{\mathcal{M}}(\tau_B U, V)/[\text{add} \tau_B H] \simeq \mathcal{M}(\tau_B U, V)/[\text{add} \tau_B H].$

□

Proof. (of proposition 5.8) In this proof, for simplicity we denote $\tau_B$ by $\tau$. Let $\tilde{A}$ be the algebra $\text{End}_C(A)$. By proposition 4.7, we have a vector space isomorphism
\[ e_U A e_V \simeq e_U A e_V \oplus e_U \text{Ext}^2_A(DA, A)e_V \oplus e_U \text{Ext}^2_A(DA, A)^{\otimes 2}e_V \oplus \ldots \]

We prove by induction that
\[ e_U \text{Ext}^2_A(DA, A)^{\otimes p}e_V \simeq \mathcal{M}(\tau^p U, V)/[\text{add} \tau^p H]. \]
For $p = 0$, $e_U A e_V$ is isomorphic to $\overline{\mathcal{M}}(U, V)$ by Yoneda’s lemma, and so to $\mathcal{M}(U, V)/[\text{add}(H)]$. Suppose the proposition holds for an integer $p - 1 \geq 0$. We then have
\[ e_U \text{Ext}^2_A(DA, A)^{\otimes p}e_V \simeq \sum_{W \in \text{ind}(\overline{\mathcal{M}})} e_U \text{Ext}^2_A(DA, A)^{\otimes p-1}e_W \otimes e_W \text{Ext}^2_A(DA, A)e_V. \]
The sum means here the direct sum modulo the mesh relations of the category $\overline{\mathcal{M}}$. Thus this vector space is the sum over the indecomposables $W$ of $\overline{\mathcal{M}}$ of
\[ \mathcal{M}(\tau^{p-1} U, W)/[\text{add} \tau^{p-1} H] \otimes \mathcal{M}(\tau W, V)/[\text{add}(\tau H)] \]
modulo the mesh relations of $\mathcal{M}$. This is isomorphic to the cokernel of the map $\varphi^{p-1}_{\tau^p U,W} \otimes 1_{W,V} + 1_{\tau^{p-1} U,W} \otimes \varphi^1_{W,V}$ where

$$\varphi^i_{X,Y} : \mathcal{M}(X, \tau^i H) \otimes \mathcal{M}(\tau^i H, Y) \rightarrow \mathcal{M}(X, Y)$$

is the composition map and where $1_{X,Y} : \mathcal{M}(X, Y) \rightarrow \mathcal{M}(X, Y)$ is the identity. The cokernel of this map is isomorphic to the cokernel of the map $\varphi^p_{\tau^p U, \tau W} \otimes 1_{W,V} + 1_{U,\tau W} \otimes \varphi^1_{W,V}$. But we have an isomorphism

$$\sum_{W \in \text{ind} \mathcal{M}} \mathcal{M}(\tau^p U, \tau W) \otimes \mathcal{M}(\tau W, V) \simeq \mathcal{M}(\tau^p U, V).$$

Finally we get

$$\text{Coker} \left( \sum_{W \in \text{ind} \mathcal{M}} \varphi^p_{\tau^p U, \tau W} \otimes 1_{W,V} + 1_{U,\tau W} \otimes \varphi^1_{W,V} \right) \simeq \text{Coker}(\varphi^p_{\tau^p U, V} + \varphi^1_{\tau^p U, V}).$$

Furthermore, a morphism in $\mathcal{M}(\tau^p U, V)$ which factorizes through $\tau H$ factorizes through $\tau^p H$ since $H$ is a slice and $U$ is in $\mathcal{M}$. Thus this cokernel is in fact isomorphic to the cokernel of $\varphi^p_{\tau^p U, V}$ that is to say to the space

$$\mathcal{M}(\tau^p U, V)/[\text{add} \tau^p H].$$

\[ \square \]

5.2. Case where $B$ is hereditary.

Results of Geiss, Leclerc and Schröer. Let $Q$ be a finite connected quiver without oriented cycles with $n$ vertices. Denote by $\mathcal{P}$ the postprojective component of the Auslander-Reiten quiver of $\text{mod} kQ$, and by $P_1, \ldots, P_n$ the indecomposable projectives.

**Definition 5.10** (Geiss-Leclerc-Schröer, [GLS07b]). A $kQ$-module $M = M_1 \oplus \cdots \oplus M_r$, where the $M_i$ are pairwise non isomorphic indecomposables, is called initial if the following conditions hold:

- for all $i = 1, \ldots, r$, $M_i$ is postprojective;
- if $X$ is an indecomposable $kQ$-module with $\text{Hom}_{kQ}(X, M) \neq 0$, then $X$ is in $\text{add}(M)$;
- and $P_i \in \text{add}(M)$ for each indecomposable projective $kQ$-module $P_i$.

We define the integers $t_i$ as

$$t_i = \max\{j \geq 0| \tau^{-j}(P_i) \in \text{add}(M) - \{0\}\}.$$  

Denote by $\Lambda$ the preprojective algebra associated to $Q$. There is a canonical embedding of algebras $kQ \hookrightarrow \Lambda$. Denote by $\pi_Q : \text{mod} \Lambda \rightarrow \text{mod} kQ$ the corresponding restriction functor.

**Theorem 5.11** (Geiss-Leclerc-Schröer, [GLS07b]). Let $M$ be an initial $kQ$-module, and let $\mathcal{C}_M = \pi^{-1}_Q(\text{add}(M))$ be the subcategory of all $\Lambda$-modules $X$ with $\pi_Q(X) \in \text{add}(M)$. The following holds:

(i) the category $\mathcal{C}_M$ is a Frobenius category with $n$ projective-injectives;
(ii) the stable category $\mathcal{L}_M$ is a 2-CY triangulated category.
Recall from Ringel [Rin98] that the category $\text{mod} \Lambda$ can be seen as $\text{mod} kQ(\tau^{-1}, 1)$. The objects are pairs $(X, f)$ where $X$ is in $\text{mod} kQ$ and $f : \tau^{-1}X \to X$ is a morphism in $\text{mod} kQ$. The morphisms $\varphi$ between $(X, f)$ and $(Y, g)$ are commutative squares:

\[
\begin{array}{ccc}
\tau^{-1}X & \xrightarrow{f} & X \\
\tau^{-1} \varphi & \downarrow & \downarrow \varphi \\
\tau^{-1}Y & \xrightarrow{g} & Y
\end{array}
\]

The image of an object $(X, f)$ under $\pi_Q : \text{mod} \Lambda \to \text{mod} kQ$ is then the module $X$.

Let $X = \tau^{-l}P_i$ be an indecomposable summand of an initial module $M$. Let $R_X = (Y, f)$ be the following object in $\text{mod} kQ(\tau^{-1}, 1) \simeq \text{mod} \Lambda$:

\[
Y = \bigoplus_{j=0}^l \tau^{-j}P_i \quad \text{and} \quad f : \bigoplus_{j=1}^{l+1} \tau^{-j}P_i \to \bigoplus_{j=0}^l \tau^{-j}P_i
\]

is given by the matrix

\[
f = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 0
\end{pmatrix}.
\]

**Proposition 5.12** (Geiss-Leclerc-Schröer [GLS07b]). The category $C_M$ has a canonical maximal rigid object $R = \bigoplus_{X \in \text{ind add}(M)} R_X$. The projective-injectives of $C_M$ are the $R_{\tau^{-i}P_i}$, $i = 1, \ldots, n$. Therefore, $R$ is a cluster-tilting object in $C_M$.

**Endomorphism algebra of the cluster-tilting object.** Let $Q$ be a connected quiver without oriented cycles and denote by $B$ the path algebra $kQ$. Let $M$ be an initial $B$-module. Let $H$ be the following postprojective slice $H = \bigoplus_{i=1}^n \tau^{-i}P_i$ of $\text{mod} B$. Let $Q'$ be the quiver such that $\text{End}_B(H)$ is isomorphic to $kQ'$.

Let us define, as in the previous section, the subcategory $\mathcal{M}$ of $D^b(\text{mod} kQ)$ as

\[
\mathcal{M} = \{ X \in \text{mod} kQ / \text{Ext}^1_B(X, H) = 0 \}.
\]

It is then obvious that $\mathcal{M} = \text{add}(M)$. As previously, we denote by $\Lambda$ the preprojective algebra associated with $Q'$. It is isomorphic to the one associated with $Q$ because $Q$ and $Q'$ have the same underlying graph. Recall that we have $\overline{\mathcal{M}} = \mathcal{M}/\text{add}(H)$, and that $A = \text{End}_B(\overline{\mathcal{M}})$ is an algebra of global dimension 2. Note that in this case $\tau_B$ and $\tau_D$ coincide on the objects of $\text{mod} B$ which have no projective direct summands since $B$ is hereditary. We will denote it by $\tau$ in this section.

**Lemma 5.13.** Let $U$ and $V$ be indecomposables in $\overline{\mathcal{M}}$. We have

\[
\text{Hom}_A(R_U, R_V) \simeq \bigoplus_{j \geq 0} \mathcal{M}(\tau^jU, V).
\]

**Proof.** Let $P$ and $Q$ be projective indecomposables such that $U = \tau^{-q}Q$ and $V = \tau^{-p}P$. 


Case 1: $p \leq q$

An easy computation gives the following equalities

\[ \text{Hom}_A(R_U, R_V) \simeq \bigoplus_{j=0}^{p} \mathcal{M}(Q, \tau^{-j}P) \simeq \bigoplus_{j=0}^{p} \mathcal{M}(\tau^{-p+j}Q, \tau^{-p}P) \]

\[ \simeq \bigoplus_{j=0}^{p} \mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q), \tau^{-p}P) \simeq \bigoplus_{j=q-p}^{q} \mathcal{M}(\tau^{-j}U, V). \]

Since $\mathcal{M}(\tau^kU, V)$ vanishes for $k \leq q - p + 1$ and since $\tau^kU$ vanishes for $k \geq q + 1$ we get an isomorphism

\[ \text{Hom}_A(R_U, R_V) \simeq \bigoplus_{j \geq 0} \mathcal{M}(\tau^{-j}U, V). \]

Case 2: $p > q$

In this case, a morphism from $R_U$ to $R_V$ is given by morphisms $a_j \in \mathcal{M}(Q, \tau^{-j}P)$, with $j = 0, \ldots, p$ such that $\tau^{-q+1}a_j = 0$ for $j = 0, \ldots, p - q - 1$. But since $\tau^{-q+1}P$ is not zero for $j = 0, \ldots, p - q - 1$, the morphism $\tau^{-q+1}a_j : \tau^{-q+1}Q \to \tau^{-q+1}P$ vanishes if and only if $a_j$ vanishes. Thus we get

\[ \text{Hom}_A(R_U, R_V) \simeq \bigoplus_{j=p-q}^{p} \mathcal{M}(Q, \tau^{-j}P) \simeq \bigoplus_{j=p-q}^{p} \mathcal{M}(\tau^{-p+j}Q, \tau^{-p}P) \]

\[ \simeq \bigoplus_{j=p-q}^{p} \mathcal{M}(\tau^{-p+j+q}(\tau^{-q}Q), \tau^{-p}P) \simeq \bigoplus_{j=0}^{q} \mathcal{M}(\tau^{-j}U, V). \]

Since $\tau^{-j}U$ vanishes for $j \geq q + 1$ we get

\[ \text{Hom}_A(R_U, R_V) \simeq \bigoplus_{j \geq 0} \mathcal{M}(\tau^{-j}U, V). \]

\[ \square \]

**Corollary 5.14.** Let $U$ and $V$ be indecomposable objects in $\overline{\mathcal{M}}$. We have

\[ \text{Hom}_{\mathcal{C}_M}(R_U, R_V) \simeq e_U \tilde{A}e_V \]

and therefore the algebras $\tilde{A}$ and $\text{End}_{\overline{\mathcal{M}}}(R)$ are isomorphic.

**Proof.** The projective-injectives in the category $\mathcal{C}_M$ are the $R_{H_i}$ with $i = 1, \ldots, n$. Denote by $R_H$ the sum $\bigoplus_{i=1}^{n} R_{H_i}$. Thus $\text{Hom}_{\mathcal{C}_M}(R_U, R_V)$ is the cokernel of the composition map

\[ \text{Hom}_{\mathcal{C}_M}(R_U, R_{H_I}) \otimes \text{Hom}_{\mathcal{C}_M}(R_{H_I}, R_V) \longrightarrow \text{Hom}_{\mathcal{C}_M}(R_U, R_V). \]

By the previous lemma this map is isomorphic to the following

\[ \bigoplus_{i,j \geq 0} \mathcal{M}(\tau^i U, H) \otimes \mathcal{M}(\tau^j H, V) \xrightarrow{\Phi} \bigoplus_{p \geq 0} \mathcal{M}(\tau^p U, V) \]

Given two morphisms $f \in \mathcal{M}(\tau^i U, H)$ and $\mathcal{M}(\tau^j H, V)$, $\Phi(f \otimes g)$ is the composition $\tau^i f \circ g \in \mathcal{M}(\tau^{i+j} U, V)$. Thus the cokernel of this map is the cokernel of the map

\[ \bigoplus_{p \geq 0} \bigoplus_{i=0}^{p} \mathcal{M}(\tau^i U, \tau^j H) \otimes \mathcal{M}(\tau^i H, V) \xrightarrow{\Phi} \bigoplus_{p \geq 0} \mathcal{M}(\tau^p U, V). \]
By the universal property of the orbit category, we have the factorization
\[ \text{Hom}(D_{\text{ind}}(R_U, R_V)) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau^p U, V)/[\text{add}\tau^p H], \]
and we conclude using proposition \[5.3\].

**Triangle equivalence.**

**Theorem 5.15.** The functor \( F \circ i_* : \text{mod} \overline{\mathcal{M}} \to \text{f.l.} \Lambda \) yields a triangle equivalence between \( \mathcal{C}_{\overline{\mathcal{M}}} \) and \( \mathcal{C}_{\mathcal{M}} \).

**Proof.** Let \( X = \tau_B^{-i} P_i \) be an indecomposable of \( \mathcal{M} \). Let \( X^\wedge \) be the projective \( \mathcal{M} \)-module \( \text{Hom}_B(?, X)_{\mathcal{M}} \). The underlying vector space of \( F(X^\wedge) \) is
\[
F(X^\wedge) \simeq \bigoplus_{q \geq 0} \text{Hom}_B(\tau_B^q H, \tau_B^{-i} P_i) \simeq \bigoplus_{q \geq 0} \tau_B^q B, \tau_B^{-i} P_i
\]
\[
\simeq \bigoplus_{q \geq 0} \text{Hom}_B(B, \tau_B^q P_i) \simeq \bigoplus_{q \geq 0} \tau_B^q P_i.
\]

It is then not hard to see that \( F(X^\wedge) \) is equal to \( R_X \). Thus each projective \( X^\wedge \) is sent onto an object of \( \mathcal{C}_\mathcal{M} \). Therefore \( F \) induces a functor \( F : \mathcal{D}^b(\mathcal{M}) \to \mathcal{D}^b(\mathcal{C}_\mathcal{M}) \). Moreover for \( i = 1, \ldots, n \), \( F(\tau_i^{-1} P_i) \) is equal to \( R_{\tau_i^{-1} P_i} \), i.e. a projective-injective of \( \mathcal{C}_\mathcal{M} \). We have the following composition:
\[
\mathcal{D}^b(\overline{\mathcal{M}}) \simeq \mathcal{D}^b(\Lambda) \xrightarrow{i_*} \mathcal{D}^b(\mathcal{M}) \xrightarrow{F} \mathcal{D}^b(\mathcal{C}_\mathcal{M}) \xrightarrow{\pi} \mathcal{D}^b(\mathcal{C}_\mathcal{M})/\text{per} \mathcal{C}_\mathcal{M} \simeq \mathcal{C}_\mathcal{M}
\]
The functor \( F \circ i_* \) is clearly isomorphic to the left derived tensor product with the \( \Lambda \)-\( \Lambda \)-bimodule \( R = F \circ i_*(A) \). By proposition \[5.7\], for \( X \) in \( \overline{\mathcal{M}} \), we have the following exact sequence, functorial in \( X \):
\[
0 \longrightarrow F \circ i_*(X^\wedge) \longrightarrow F(H^\wedge_0) \longrightarrow F(H^\wedge_1) \longrightarrow F \circ i_*(X^\wedge) \longrightarrow 0
\]
with \( H_0 \) and \( H_1 \) in \( \text{add}(H) \). It yields a morphism
\[
F \circ i_*(DA) \to F \circ i_*(A)[2]
\]
in the derived category of \( \Lambda \)-\( \Lambda \)-bimodules. Since the objects \( F(H^\wedge_0) \) and \( F(H^\wedge_1) \) vanish in the stable category \( \mathcal{C}_\mathcal{M} \), the image
\[
F \circ i_*(DA) \to F \circ i_*(A)[2]
\]
of this morphism in the category of \( \Lambda \)-\( \Lambda \)-bimodules is invertible, where \( \Lambda \) is a dg category whose perfect derived category is algebraically equivalent to the stable category \( \mathcal{C}_\mathcal{M} \). In other words, in the derived category \( \mathcal{D}(\Lambda^{\text{op}} \otimes \Lambda) \), we have an isomorphism
\[
DA \otimes_{\Lambda} \pi F i_*(A) \simeq \pi F i_*(A)[-2].
\]
By the universal property of the orbit category, we have the factorization
\[
\mathcal{D}^b(\overline{\mathcal{M}}) \xrightarrow{\tau^L_{\otimes_{\Lambda} B}} \mathcal{C}_{\overline{\mathcal{M}}} \xrightarrow{=} \mathcal{C}_\mathcal{M}.
\]
This factorization is an algebraic functor between 2-CY categories which sends the cluster-tilting object \(A\) onto the cluster-tilting object \(\bar{R}\). Moreover by corollary 5.14, it yields an equivalence between the categories \(\text{add}(A)\) and \(\text{add}(\bar{R})\). Thus it is an algebraic triangle equivalence.

\[\square\]

Note that if \(M\) is the initial module \(kQ \oplus \tau^{-1}kQ\), Geiss, Leclerc and Schröer proved, using a result of Keller and Reiten [KR06], that the 2-CY category \(\mathcal{C}_M\) is triangle equivalent to the cluster category \(\mathcal{C}_Q\). Here, \(H\) is \(\tau^{-1}kQ\) and then \(\mathcal{M}\) is \(kQ\), so we get another proof of this fact.

5.3. Relation with categories \(\text{Sub}\Lambda/\mathcal{I}_w\).

Results of Buan, Iyama, Reiten and Scott. Let \(Q\) be a finite connected quiver without oriented cycles and \(\Lambda\) the associated preprojective algebra. We denote by \(\{1, \ldots, n\}\) the set of vertices of \(Q\). For a vertex \(i\) of \(Q\), we denote by \(I_i\) the ideal \(\Lambda(1-e_i)\Lambda\) of \(\Lambda\). We denote by \(W\) the Coxeter group associated to the quiver \(Q\). The group \(W\) is defined by the generators \(1, \ldots, n\) and the relations:

- \(i^2 = 1\) for all \(i\) in \(\{1, \ldots, n\}\);
- \(ij = ji\) if there are no arrows between the vertices \(i\) and \(j\);
- \(iji = jij\) if there is exactly one arrow between \(i\) and \(j\).

Let \(w = i_1i_2\ldots i_r\) be a \(W\)-reduced word. For \(m \leq r\), let \(\mathcal{I}_{w_m}\) be the following ideal:

\[\mathcal{I}_{w_m} = I_{i_m}\cdots I_{i_2}I_{i_1}.\]

For simplicity we will denote \(\mathcal{I}_{w_m}\) by \(\mathcal{I}_w\). The category \(\text{Sub}\Lambda/\mathcal{I}_w\) is the subcategory of \(\text{f.l.}\Lambda\) generated by the sub-\(\Lambda\)-modules of \(\Lambda/\mathcal{I}_w\).

**Theorem 5.16** (Buan-Iyama-Reiten-Scott [BIRS07]). The category \(\text{Sub}\Lambda/\mathcal{I}_w\) is a Frobenius category and its stable category \(\text{Sub}\Lambda/\mathcal{I}_w\) is 2-CY. The object \(T_w = \bigoplus_{m=1}^r e_{i_m}\Lambda/\mathcal{I}_{w_m}\) is a cluster-tilting object.

Note that this theorem is written only for non Dynkin quivers in [BIRS07], but the Dynkin case is an easy consequence of theorem II.2.8 and corollary II.3.5 of [BIRS07].

**Construction of a reduced word.** Let \(B\) be a concealed algebra, and \(H\) a postprojective slice in \(\text{mod} B\). Let \(Q\) the quiver of \(\text{End}_B(H)\). It is a finite quiver without oriented cycles. We denote by \(\{1, \ldots, n\}\) its set of vertices and by \(\Lambda\) its preprojective algebra. We define as previously \(\mathcal{M} = \{X \in \text{mod} B/\text{Ext}_B^2(X, H) = 0\}\).

Let us order the indecomposables \(X_1, \ldots, X_N\) of \(\mathcal{M}\) in such a way: if the morphism space \(\text{Hom}_B(X_i, X_j)\) does not vanish, \(i\) is smaller than \(j\). This is possible since \(Q\) has no oriented cycles.

By proposition 5.1, for \(X_i \in \mathcal{M}\) there exists a unique \(q \geq 0\) such that \(\tau_B^{-q}X_i \simeq H_{\varphi(i)}\) for a certain integer \(\varphi(i)\). So we get a function \(\varphi : \{1, \ldots, N\} \rightarrow \{1, \ldots, n\}\). Let \(w\) be the word \(\varphi(1)\varphi(2)\ldots\varphi(N)\).

**Proposition 5.17.** The word \(w\) is \(W\)-reduced.

**Proof.** The proof is in several steps:

*Step 1:* For two integers \(i < j\) in \(\{1, \ldots, N\}\), we have \(\varphi(i) = \varphi(j)\) if and only if there exists a positive integer \(p\) such that \(X_i = \tau_B^pX_j\).
Step 2: The element \( w \) of the Coxeter group does not depend on the order on the indecomposables of \( \mathcal{M} \).

Let \( i \) be in \( \{1, \ldots, N-1\} \). Assume there is an arrow \( \varphi(i) \rightarrow \varphi(i+1) \) in \( Q \). We show that there is an arrow \( X_i \rightarrow X_{i+1} \) in the Auslander-Reiten quiver of \( \mathcal{M} \). By proposition 5.1, there exist positive integers \( p \) and \( q \) such that \( X_i = \tau_B^p X_{\varphi(i)} \) and \( X_{i+1} = \tau_B^q X_{\varphi(i+1)} \). By hypothesis there is an arrow between \( H_{\varphi(i)} \) and \( H_{\varphi(i+1)} \). Thus we want to show that \( p \) is equal to \( q \).

Suppose that \( p \geq q+1 \), then since \( \mathcal{M} \) is closed under \( \tau_B \), the objects \( \tau_B^p X_{\varphi(i)} \) and \( \tau_B^{q+1} X_{\varphi(i+1)} \) are non zero and are in \( \mathcal{M} \). Let \( l \) be the integer in \( \{1, \ldots, N\} \) such that \( X_l = \tau_B^{q+1} H_{\varphi(i+1)} \). We have an arrow

\[
X_i = \tau_B^p X_{\varphi(i)} \rightarrow \tau_B^q X_{\varphi(i+1)} = \tau_B^{-1} X_l.
\]

Thus, by the property of the AR-translation, there is an arrow between \( X_i \rightarrow X_l \). Thus \( i \) should be strictly greater than \( l \). But by step 1, and the hypothesis \( p \geq q+1 \), we have \( i+1 \leq l \). This is a contradiction.

The cases \( q \geq p+1 \), and \( \varphi(i+1) \rightarrow \varphi(i) \) in \( Q \) can be solved in the same way.

Step 3: It is not possible to have \( \varphi(i) = \varphi(i+1) \).

Suppose we have \( \varphi(i) = \varphi(i+1) \). By step 1 there exists a positive integer \( p \) such that \( X_i = \tau_B^p X_{i+1} \). Suppose that \( p \geq 2 \), then \( \tau_B X_{i+1} = \tau_B^{p+1} X_i \) is in \( \mathcal{M} \), it is isomorphic to an \( X_k \) for an integer \( k \) with \( \varphi(k) = \varphi(i) \). But \( k \) must be strictly greater than \( i \) and strictly smaller than \( i+1 \) which is clearly impossible. Thus \( p \) is equal to 1. There should exist an \( X_l \) in \( \mathcal{M} \) such that \( \text{Hom}(X_i, X_l) \neq 0 \) and \( \text{Hom}(X_l, X_{i+1}) \neq 0 \). Thus \( l \) must be strictly between \( i \) and \( i+1 \) which is impossible.

Step 4: It is not possible to have \( \varphi(i) = \varphi(i+2) \) and \( \varphi(i+1) = \varphi(i+3) \) with exactly one arrow in \( Q \) between \( \varphi(i) \) and \( \varphi(i+1) \).

In this case we have, by step 1, \( X_i = \tau_B^p X_{i+2} \) and \( X_{i+1} = \tau_B^q X_{i+3} \). By the same argument as in step 3, \( p \) and \( q \) have to be equal to 1. Thus the AR quiver of \( \mathcal{M} \) has locally the following form:

![Diagram](https://via.placeholder.com/150)

The module \( X_{i+1} \) is the unique direct predecessor of \( X_{i+2} \). Indeed, suppose there is an \( X_k \) with an arrow \( X_k \rightarrow X_{i+2} \). Thus there is an arrow \( \tau_B X_{i+2} = X_i \rightarrow X_k \) and \( k \) must be strictly between \( i \) and \( i+2 \). By the same argument, there is only one arrow with tail \( X_{i+3} \), one arrow with source \( X_i \) and one arrow with source \( X_{i+1} \). Thus we have the following AR sequences in \( \text{mod } B \):

\[
0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow X_{i+2} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow X_{i+1} \rightarrow X_{i+2} \rightarrow X_{i+3} \rightarrow 0
\]

which is clearly impossible.

Step 5: There is no subsequence of type \( jkjlkl \) in \( w \) with an arrow between \( j \) and \( k \) and an arrow between \( k \) and \( l \).

Suppose we have \( \varphi(i) = \varphi(i+2) = j, \varphi(i+1) = \varphi(i+4) = k \) and \( \varphi(i+3) = \varphi(i+5) = l \). As previously, we have \( X_i = \tau_B X_{i+2} \), \( X_{i+1} = \tau_B X_{i+4} \) and \( X_{i+3} = \tau_B X_{i+5} \). There is an arrow \( X_{i+1} \rightarrow X_{i+2} \) so there is an arrow \( X_{i+2} \rightarrow X_{i+4} \). There is an arrow \( X_{i+3} \rightarrow X_{i+4} \) thus there is
an arrow $X_{i+1} \to X_{i+3}$. As in step 4 it is easy to see that the AR quiver of $\mathcal{M}$ locally looks like:

```
\[ \begin{array}{c}
X_i & \longrightarrow & X_{i+1} & \longrightarrow & X_{i+2} & \longrightarrow & X_{i+3} & \longrightarrow & X_{i+4} & \longrightarrow & X_{i+5} & \longrightarrow & \cdots \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
\end{array} \]
```

Thus we have the 3 following AR sequences in $\text{mod } B$:

\[ 0 \to X_i \to X_{i+1} \to X_{i+2} \to 0 \quad 0 \to X_{i+3} \to X_{i+4} \to X_{i+5} \to 0 \]

and

\[ 0 \to X_{i+1} \to X_{i+3} \oplus X_{i+2} \to X_{i+4} \to 0 \]

A simple argument of dimension permits us to conclude that $X_i$ and $X_{i+5}$ must be zero, that is a contradiction.

By the second step, we know that using the relation of commutativity is the same as changing the order on the indecomposables of $\mathcal{M}$. Moreover we just saw that locally we can not reduce the word $w$. Thus it is reduced.

\[ \square \]

**Image of the cluster-tilting object.** Let $F : \text{mod } \mathcal{M} \to \text{f.l.}\Lambda$ be the functor constructed in section \[5.\].

**Proposition 5.18.** For $i = 1, \ldots, N$, we have an isomorphism in f.l.$\Lambda$:

\[ F(X_i^\wedge) \simeq e_{\varphi(i)}\Lambda/I_{w_i} \]

where $w_i$ is the word $\varphi(1) \cdots \varphi(i)$.

**Proof.** The functor $F$ is right exact and sends the simple functor $S_{X_i}$ onto the simple $S_{\varphi(i)}$. Since $F(X_i^\wedge)$ surjects onto $F(S_{X_i})$, there is a morphism $e_{\varphi(i)}\Lambda \to F(X_i^\wedge)$. Explicitly, we will take the morphism given in this way:

The object $X_i$ is of the form $\tau_B^q H_{\varphi(i)}$ for $q \geq 0$. If $j$ is in $\{1, \ldots, n\}$, the vector space $e_{\varphi(i)}\Lambda e_j$ is isomorphic to $\prod_{p \geq 0} \text{Hom}_{kQ} (\tau_B^p I_j, \tau_B^p I_{\varphi(i)})$ where $I_j$ is the injective indecomposable module of $\text{mod } kQ$ corresponding to the vertex $j$. Let $f$ be a morphism in $\text{Hom}_{kQ} (\tau_B^p I_j, \tau_B^p I_{\varphi(i)})$, then $\tau_B^p(f)$ is a morphism in $\text{Hom}_{kQ} (\sigma_B^{p+q} I_j, \sigma_B^{p+q} I_{\varphi(i)})$, and then $P(\tau_B^p f) = \tau_B^p P(f)$ is a morphism in $\mathcal{M}$ from $\sigma_B^{p+q} H_j$ to $\tau_B^p H_{\varphi(i)} = X_i$, thus is in $F(X_i^\wedge)e_j$.

**Step 1:** The morphism $e_{\varphi(i)}\Lambda \to F(X_i^\wedge)$ vanishes on the ideal $I_{w_i}$.

A word $j_1 j_2 \cdots j_r$ will be called a subword of $w_i$ if there exist integers $1 \leq l_1 < l_2 < \cdots < l_r \leq i$ such that $j_1 j_2 \cdots j_r = \varphi(l_1) \varphi(l_2) \cdots \varphi(l_r)$. It is easy to check that the vector space $e_{\varphi(i)}I_{w_i}e_j$ is generated by the paths from $j$ to $\varphi(i)$ such that there exists a factorization

\[ j \rightsquigarrow j_1 \rightsquigarrow j_2 \rightsquigarrow \cdots \rightsquigarrow j_r \rightsquigarrow \varphi(i) \]

with $j j_1 j_2 \cdots j_r \varphi(i)$ not a subword of $w_i$.
Let $f$ be a morphism $\tau_B^q I_j \rightarrow I_{\phi(i)}$ in $\mathcal{I}(kQ)$ given by such a path. Assume that the image $P(\tau_B^q f)$ of $f$ in $F(X_i^n)$ is non zero. Let 
\[
\tau_B^p I_j \xrightarrow{f_0} \tau_B^p I_{j_1} \xrightarrow{f_1} \ldots \tau_B^p I_{j_s} \xrightarrow{f_s} \tau_B^p I_{j_{s+1}} \rightarrow \ldots \rightarrow \tau_B^p I_{j_r} \xrightarrow{f_r} I_{\phi(i)}
\]
be the factorization of $f$ given by the above factorization of the path. Therefore $P(\tau_B^q f)$ is equal to the composition 
\[
\tau_B^{p+q} H_j \xrightarrow{f_0} \tau_B^{p+q} H_{j_1} \xrightarrow{f_1} \ldots \tau_B^{p+q} H_{j_s} \xrightarrow{f_s} \tau_B^{p+q} H_{j_{s+1}} \rightarrow \ldots \rightarrow \tau_B^{p+q} H_{j_r} \xrightarrow{f_r} \tau_B^{q} H_{\phi(i)} = X_i.
\]
Since $P(\tau_B^q f)$ is not zero, all morphisms $P(\tau_B^q f_i)$ are not zero, and all objects $\tau_B^{p+q} H_{j_k}$ are non zero. Thus the objects $\tau_B^{p+q} H_{j_k}$ are of the form $X_{h_k}$ with $h_0 < h_1 < \ldots < h_r < i$. Furthermore, we have $\phi(h(1)) = j_1$. This contradiction shows that the image of $f$ in $F(X_i^n)$ must be zero.

**Step 2:** The morphism $e_{\phi(i)} \Lambda \rightarrow F(X_i^n)$ is surjective.

Let $f$ be a morphism $\tau_B^{p+q} H_j \rightarrow \tau_B^q H_{\phi(i)} = X_i$ in $\mathcal{M}$. Hence $\tau_B^{-q} f$ is a morphism $\tau_B^q H_j \rightarrow H_{\phi(i)}$ in $\mathcal{M}$. Since $P$ is full (cf. prop. 5.3), there exists a morphism $g : \tau_B^q I_j \rightarrow I_{\phi(i)}$ such that $P(g) = \tau_B^{-q} f$. Thus we have $P(\tau_B^q g) = \tau_B^q P(g) = f$.

**Step 3:** The morphism $e_{\phi(i)} \Lambda / \mathcal{I}_{w_i} \rightarrow F(X_i^n)$ is injective.

Let $f$ be a non zero morphism $\tau_B^q I_j \rightarrow I_{\phi(i)}$ in $\mathcal{I}(kQ)$ such that $P(\tau_B^q f)$ is zero. By lemma 5.4, we can assume that there exists a factorization of $\tau_B^q f$ of the form 
\[
\tau_B^{q+p} I_j \xrightarrow{h} Y \xrightarrow{g} \tau_B^q I_{\phi(i)}
\]
with $Y$ indecomposable and $P(Y) = 0$. The object $Y$ is of the form $\tau_B^q I_h$ with $h \geq q$ and we have $\tau_B^h H_l = 0$.

The morphism $g$ is a sum of compositions of irreducible morphisms between indecomposables.

Let 
\[
\tau_B^q I_l \xrightarrow{g_0} Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \ldots \xrightarrow{g_s} Y_s \xrightarrow{g_s} \tau_B^q I_{\phi(i)}
\]
be such a summand of $g$. The objects $Y_k$, $1 \leq k \leq s$ are indecomposable and so are of the form $\tau_B^q I_{j_k}$, and the morphisms $g_k$, $0 \leq k \leq s$ are irreducible. We will show that the word $l_{j_1 j_2 \ldots j_s} \phi(i)$ is not a subword of $w_i$. Without loss of generality, we may assume that for $1 \leq k \leq s$, $P(Y_k)$ is not zero, so there exist integers $l_k$ such that $P(Y_k) = X_{l_k}$. Since the morphisms $g_k$ are irreducible, $P(g_k)$ does not vanish, and we have $1 \leq l_1 < l_2 < \ldots < l_s < i$. The word $j_{j_1 j_2 \ldots j_s} \phi(i)$ is equal to the word $\phi(l_1) \phi(l_2) \cdots \phi(l_s) \phi(i)$, so $j_{j_1 j_2 \ldots j_s} \phi(i)$ is a subword of $w_i$.

**Substep 1:** The sequence $1 \leq l_1 < l_2 < \cdots < l_s < i$ is the maximal element of the set 
\[
\{1 \leq l_1 < l_2 < \cdots < l_s < i \mid \phi(l_1) = j_1, \ldots, \phi(l_s) = j_s, \phi(l_{s+1}) = \phi(i)\}
\]
for the lexicographic order.

We prove by decreasing induction that $l_k$ is the maximal integer with $l_k < l_{k+1}$ and $\phi(l_k) = j_k$. For $k = s+1$ it is obvious. Now suppose there exists an integer $i_k$ such that $\phi(l_k) = \phi(i_k) = j_k$ and $l_k < i_k < l_{k+1}$. Thus by step 1 of proposition 5.17, there exists an integer $r \geq 1$ such that $X_{l_k} = \tau_B^p X_{l_k}$. The morphism $P(g_k) : X_{l_k} \rightarrow X_{l_{k+1}}$ is irreducible, so there exists a non zero
irreducible morphism $X_{l_k+1} \to \tau_B^{-1}X_{l_k}$. The object $\tau_B^{-1}X_{l_k}$ is in $\mathcal{M}$ since $X_{l_k}$ and $\tau_B^{-r}X_{l_k} = X_{l_k}$ are in $\mathcal{M}$. It is of the form $X_t$, and we have $l_{k+1} < t$. Since $r$ is $\geq 1$, $t$ is $\leq i_k$ by step 1 of proposition 5.17. This implies $l_{k+1} < i_k$ which is a contradiction.

Substep 2: $l$ does not belong to the set $\{\varphi(1), \varphi(2), \ldots, \varphi(l_1 - 1)\}$.
Suppose that there exists an integer $1 \leq k \leq N$ such that $\varphi(k)$ is equal to $l$. Thus there exists an integer $r \geq 0$ such that $X_k$ is equal to $\tau_B^r H_l$. Since $\tau_B^r H_l = P(\tau_B^r I_l)$ is zero, $r$ is $\leq h - 1$.
Since the morphism $g_0 : \tau_B^r I_l \to Y_1$ is an irreducible morphism of $\mathcal{I}(kQ)$, there exists an irreducible morphism $Y_1 \to \tau_B^{r+1} I_l$ in $\mathcal{I}(kQ)$. Thus there exists an irreducible morphism $\tau_B^{-r+1} Y_1 \to \tau_B^{r} I_l$ in $\mathcal{I}(kQ)$. The object $P(\tau_B^{r} I_l) = \tau_B^{r} H_l = X_k$ is not zero and lies in $\mathcal{M}$, so the object $P(\tau_B^{-r+1} Y_1) = \tau_B^{-r+1} X_{l_1}$ is not zero and lies in $\mathcal{M}$ since $\mathcal{M}$ is stable by kernel. Thus there is an irreducible morphism $\tau_B^{-r+1} X_{l_1} = X_t \to X_k$ in $\mathcal{M}$. Therefore $t$ has to be $< k$. Moreover since $r - h + 1 \leq 0$, $l_1$ is $\leq s$ by step 1 of proposition 5.17. Finally we get $l_1 < k$.
Combining substep 1 and substep 2, we can prove that $l_j, l_{j+1} \ldots \varphi(i)$ can not be a subword of $w_i$. Indeed, assume $l_j, l_{j+1} \ldots \varphi(i)$ is a subword of $w_i$. There exist $1 \leq i_0 < i_1 < \ldots < i_s < i_{s+1} \leq i$ such that $\varphi(i_0) \varphi(i_1) \ldots \varphi(i_{s+1}) = l_j, l_{j+1} \ldots \varphi(i)$. In particular, the word $j_1, j_2 \ldots j_s \varphi(i)$ is a subword of $w_i$ and $1 \leq i_1 < \ldots < i_s < i_{s+1} \leq i$ is in the set of substep 1. Thus by substep 1, $i_1$ has to be $\leq l_1$. By substep 2, $i_0$ can not exist.

Endomorphism algebra of the cluster-tilting object.

Lemma 5.19. Let $X_i$ and $X_j$ be indecomposables of $\mathcal{M}$. We have an isomorphism of vector spaces

$$\text{Hom}_\Lambda(e_{\varphi(j)} \Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)} \Lambda/\mathcal{I}_{w_i}) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i).$$

Proof. Case 1: $j \geq i$
By [BIRS07] (lemma II.1.14) we have an isomorphism

$$\text{Hom}_\Lambda(e_{\varphi(j)} \Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)} \Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)} \Lambda/\mathcal{I}_{w_i} e_{\varphi(j)}.$$

By proposition 5.18, this is isomorphic to the space

$$\bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i).$$

By definition of the function $\varphi$, there exists some $q \geq 1$ such that $X_j = \tau_B^q H_{\varphi(j)}$. Thus we can write the sum

$$\bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) = \bigoplus_{p=1}^q \mathcal{M}(\tau_B^{-p} X_j, X_i) \oplus \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i)$$

Since $j \geq i$, there is no morphism from $\tau_B^{-p} X_j$ to $X_i$ for $p \geq 1$, and the first summand is zero. Therefore we get the result.

Case 2: $j < i$
By [BIRS07] (lemma II.1.14) we have an isomorphism

$$\text{Hom}_\Lambda(e_{\varphi(j)} \Lambda/\mathcal{I}_{w_j}, e_{\varphi(i)} \Lambda/\mathcal{I}_{w_i}) \simeq e_{\varphi(i)} (\mathcal{I}_{\varphi(i)} \ldots \mathcal{I}_{\varphi(j+1)}/\mathcal{I}_{w_i}) e_{\varphi(j)}.$$
By proposition 5.18, this space is a subspace of the space
\[ \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) \simeq \bigoplus_{p \geq 1} \mathcal{M}(\tau_B^p X_j, X_i) \oplus \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i). \]

**Step 1:** If \( f \) is a non zero morphism in \( \mathcal{M}(\tau_B^p X_j, X_i) \) with \( p \geq 1 \), then \( f \) is not in the space \( e_{\varphi(i)} I_{\varphi(i)} \cdots I_{\varphi(j+1)} e_{\varphi(j)} \).

Let \( X_{l_0} \) be the indecomposable \( \tau_B^p X_j \). Since \( p \geq 1 \) then \( l_0 \) is \( \leq j + 1 \). The morphism is a sum of composition of the form
\[ X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_l \]
with the \( X_{l_k} \) indecomposables. Since \( f \) is not zero, we have \( j + 1 \leq l_0 < l_1 < \ldots < l_r < i \).

Thus the word \( \varphi(l_0) \varphi(l_1) \cdots \varphi(l_r) \varphi(i) \) is a subword of \( \varphi(j+1) \varphi(j+2) \cdots \varphi(i) \). Since it holds for each factorization of \( f \), the morphism \( f \) is not in the space \( e_{\varphi(i)} I_{\varphi(i)} \cdots I_{\varphi(j+1)} e_{\varphi(j)} \).

**Step 2:** If \( f \) is a morphism in \( \mathcal{M}(\tau_B^p X_j, X_i) \) with \( p \geq 0 \) then \( f \) is in the space \( e_{\varphi(i)} I_{\varphi(i)} \cdots I_{\varphi(j+1)} e_{\varphi(j)} \).

Let \( X_{l_0} \) be the indecomposable \( \tau_B^p X_j \). Since \( p \geq 0 \), we have \( l_0 \leq j \). Let us show that if \( f \) is a composition of irreducible morphisms
\[ X_{l_0} \longrightarrow X_{l_1} \longrightarrow \cdots \longrightarrow X_{l_r} \longrightarrow X_{l_{r+1}} = X_l \]
then the word \( \varphi(l_0) \varphi(l_1) \cdots \varphi(l_r) \varphi(i) \) is not a subword of \( \varphi(j+1) \varphi(j+2) \cdots \varphi(i) \).

We have \( l_0 < l_1 < \cdots < l_r < i \). Since \( l_0 \) is \( < j + 1 \), and \( i \) is \( \leq j + 1 \), there exists \( 1 \leq k \leq r+1 \) such that \( l_{k-1} < j+1 \leq l_k \). Therefore \( \varphi(l_k) \cdots \varphi(l_r) \varphi(i) \) is a subword of \( \varphi(j+1) \varphi(j+2) \cdots \varphi(i) \), and the sequence \( l_k < l_{k+1} < \cdots < l_r < i \) is the maximal element of the set
\[ \{ j+1 \leq i_k < \cdots < l_{r+1} \leq i \} \quad | \quad \varphi(i_k) = \varphi(l_k), \ldots, \varphi(i_r) = \varphi(l_r), \varphi(l_{r+1}) = \varphi(i) \}

for the lexicographic order (exactly for the same reasons as in substep 1 of proposition 5.18).

Now we can prove exactly as in substep 2 of proposition 5.18 that \( \varphi(l_{k-1}) \) does not belong to the set \( \{ \varphi(j+1), \ldots, \varphi(l_k - 1) \} \). Thus \( \varphi(l_{k-1}) \varphi(l_k) \cdots \varphi(l_r) \varphi(i) \) can not be a subword of \( \varphi(j+1) \varphi(j+2) \cdots \varphi(i) \).

Finally, let \( f = f_1 + f_2 \) be a morphism in
\[ \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p H_{\varphi(j)}, X_i) \simeq \bigoplus_{p \geq 1} \mathcal{M}(\tau_B^p X_j, X_i) \oplus \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i). \]

By step 2, \( f_2 \) is in the space \( e_{\varphi(i)} I_{\varphi(i)} \cdots I_{\varphi(j+1)} e_{\varphi(j)} \). By step 1 the morphism \( f \) is in \( e_{\varphi(i)} I_{\varphi(i)} \cdots I_{\varphi(j+1)} e_{\varphi(j)} \) if and only if \( f - 1 \) is zero. Thus we get an isomorphism
\[ \operatorname{Hom}_\Lambda(e_{\varphi(j)} \Lambda/I_w^j, e_{\varphi(i)} \Lambda/I_w^i) \simeq \bigoplus_{p \geq 0} \mathcal{M}(\tau_B^p X_j, X_i). \]

\[ \square \]

**Corollary 5.20.** If \( X_i \) and \( X_j \) are indecomposables of \( \overline{\mathcal{M}} \), then we have
\[ \operatorname{Hom}_{\operatorname{Sub}/I_w} (e_{\varphi(j)} \Lambda/I_w^j, e_{\varphi(i)} \Lambda/I_w^i) \simeq e_X \tilde{A} e_X. \]

**Proof.** The proof is exactly the same as the proof of corollary 5.14. \( \square \)
Triangle equivalence.

Theorem 5.21. The functor $F \circ i_* : \text{mod} \mathcal{M} \rightarrow \text{f.l.} \Lambda$ induces an algebraic triangle equivalence between $\mathcal{C}_{\mathcal{M}}$ and $\text{Sub} \Lambda/\mathcal{T}_w$.

Proof. By proposition 5.18, the functor $F$ sends the projectives of $\text{mod} \mathcal{M}$ onto the summands of the cluster-tilting object $T_w$ of the category $\text{Sub} \Lambda/\mathcal{T}_w$. For $i = 1, \ldots, n$, the projective $H_i^\downarrow$ is sent to the projective-injective $\Lambda/\mathcal{T}_w e_i$. Furthermore, by corollary 5.20, $F \circ i_*$ induces an equivalence between the subcategories $\text{add}(A)$ and $\text{add}(T_w)$. Thus we can conclude as in the proof of theorem 5.13. $\square$

5.4. Example. We refer to [Ami08] for more examples. Let $Q$ be the following quiver: $1 \rightarrow 2 \rightarrow 3$

The preinjective component of $\text{mod} kQ$ looks as follows:

Here we denote the $kQ$-modules by their dimension vectors in order to lighten the writing. For example the module $[1 4 2]$ has the following composition series: $2 2 2 2$.

If we mutate the tilting object $[2 6 3] \oplus [1 4 2] \oplus [1 1 0]$ in the direction $[1 4 2]$, we stay in the preinjective component. We get the tilting object:

$$T = [2 6 3] \oplus [3 8 4] \oplus [1 1 0].$$

The algebra $B = \text{End}_{kQ}(T)$ is a concealed algebra and is given by the quiver:

$$a \quad 2 \quad b$$

with the relation $ba + b'a' = 0$.

The functor $R\text{Hom}_{kQ}(T, ?)$ yields an equivalence between $\mathcal{D}^b(kQ)$ and $\mathcal{D}^b B$. Denote by $H$ the image of $D(kQ)$ through this equivalence. This is a postprojective slice of $\text{mod} B$. Moreover, this equivalence restricts to an equivalence between the category $\mathcal{M} = \{X \in \text{mod} B \mid \text{Ext}^1_B(X, H) = 0\}$ and the category $\mathcal{M}' = \{X \in \text{mod} kQ \mid \text{Ext}^1_{kQ}(T, X) = 0\}$. The indecomposable objects of $\mathcal{M}'$ are

$[3 8 4], [2 6 3], [1 4 2], [1 1 0], [0 2 1], \text{and } [0 1 0].$

The quiver of $\mathcal{M}'$ with an admissible ordering is the following:

The dotted arrows represent the Auslander translation $\tau_B$. The projective indecomposables of $\text{mod} \mathcal{M}$ have the following dimension vectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 3 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 8 & 6 & 2 \\ 4 & 1 & 1 \end{pmatrix}$$
Now let Λ be the preprojective associated to the quiver $Q$. The functor $F : \text{mod} \mathcal{M} \rightarrow \text{mod} \Lambda$ sends the simples $\mathcal{M}$-modules $S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on the simple $\Lambda$-module $S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It sends the simple $\mathcal{M}$-modules $S_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ on the simple $\Lambda$-module $S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the simple $\mathcal{M}$-module $S_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ on the simple $\Lambda$-module $S_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since it is exact, it preserves the composition series and then it is easy to compute the image of the indecomposable projective $\mathcal{M}$-modules. We get

\[ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

The projectives of the preprojective algebra associated to $Q$ have the following composition series:

\[ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 2 \\ 3 \\ 1 \\ 3 \\ 3 \\ 2 \\ 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \\ 2 \\ 2 \\ 1 \\ 3 \\ 3 \\ 2 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \]

The word $w$ associated with the ordering is $w = 232132$. Thus the maximal rigid object of the category $\text{Sub} \Lambda / \mathcal{I}_w$ is

\[ R = 2 \oplus_2 \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 0 \end{bmatrix} \oplus_2 \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \oplus_2 \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \]

It is easy to check that $R$ is the image by $F$ of the projective indecomposable $\mathcal{M}$-modules. The last three summands are the projective-injectives of the Frobenius category $\text{Sub} \Lambda / \mathcal{I}_w$. This confirms proposition 5.13.

Now take the module $X = 1$ in $\mathcal{M}$. It corresponds to the module $[3 \ 8 \ 4]$ in $\text{mod} kQ$. We have the injective resolution in $\text{mod} kQ$:

\[ 0 \rightarrow [3 \ 8 \ 4] \rightarrow [0 \ 2 \ 1]^4 \oplus [1 \ 1 \ 0]^3 \rightarrow [0 \ 1 \ 0]^3 \rightarrow 0 \]

Thus the short exact sequence in $\mathcal{M}$: $0 \rightarrow X \rightarrow H_0 \rightarrow H_1 \rightarrow 0$ is the following:

\[ 0 \rightarrow 1 \rightarrow 4^3 \oplus 5^4 \rightarrow 6^3 \rightarrow 0 \]

Therefore, the sequence $0 \rightarrow X^\vee \rightarrow H_0^\vee \rightarrow H_1^\vee \rightarrow (\tau^{-1}X)^\vee / \text{proj} B \rightarrow 0$ in $\text{mod} \mathcal{M}$ becomes:

\[ 0 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 \oplus \begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}^4 \rightarrow \begin{bmatrix} 8 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}^3 \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix} \rightarrow 0 \]

where $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 \end{bmatrix}$ is the quotient of $(\tau_B^{-1}1)^\vee = 3^\vee = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 \end{bmatrix}$ by the projectives. Applying the projection functor we get the exact sequence in $\text{mod} \Lambda$:

\[ 0 \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}^3 \oplus \begin{bmatrix} 4 \\ 0 \end{bmatrix}^4 \rightarrow \begin{bmatrix} 8 \\ 1 \end{bmatrix}^3 \rightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow 0 \]
The algebra $A$ is the endomorphism algebra of the direct sum of the indecomposables of $\mathcal{M} = \mathcal{M}/addH \simeq \mathcal{M'}/addD(kQ)$. Thus the algebra $A$ is given by the quiver

$$
\begin{array}{ccc}
1 & \to & 2 \\
\downarrow & & \downarrow \\
1 & \to & 3
\end{array}
$$

and the relation $ba + b'a' = 0$.

By Theorem 5.3 the cluster category $\mathcal{C}_A$ associated with the algebra $A$ is 2-Calabi-Yau, Hom-finite and $A \in \mathcal{C}_A$ is a cluster-tilting object. Moreover by proposition 4.16, the quiver of the cluster-tilted algebra $\tilde{A} = \text{End}_{\mathcal{C}_A}(A)$ has the form:

$$
\begin{array}{ccc}
2 & \to & 3 \\
\downarrow & & \downarrow \\
1 & \to & 3
\end{array}
$$

The injective $A$-module $I_1 = 1^\vee_{M'/M}$ has dimension vector $[1 \ 2 \ 3] = [3 \ 2 \ 1 \ 3 \ 3 \ 2 \ 1 \ 3]$. Its image by $i^*$ is the $M$-module $[1 \ 2 \ 0 \ 0 \ 3 \ 0 \ 0 \ 0]$. Its image through $F$ is the same as the image of the $M$-module $[0 \ 0 \ 1 \ 2 \ 3 \ 0 \ 1 \ 3]$, indeed we have $F \circ i^*(1^\vee_{M'/M}) = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$. By the exact sequence above, there is an isomorphism in $\text{Sub}_{\Lambda/I_w}$ between $F \circ i^*(I_1)$ and $F \circ i^*(P_1)[2]$ where $P_1$ is the projective $A$-module with vector dimension $[1 \ 0 \ 0]$.

**References**


On algebraic Calabi-Yau categories, Ph.D. in preparation.

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