HAL
open science

Local time steps for a finite volume scheme

Isabelle Faille, Frédéric Nataf, Françoise Willien, Sylvie Wolf

To cite this version:

Isabelle Faille, Frédéric Nataf, Françoise Willien, Sylvie Wolf. Local time steps for a finite volume scheme. 2008. hal-00277186

HAL Id: hal-00277186

https://hal.science/hal-00277186

Preprint submitted on 5 May 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

hal-00277186, version 1 - 5 May 2008

Local time steps for a finite volume scheme

I. Faille, * F. Nataf† F. Willien ‡ S. Wolf‡

May 5, 2008

Contents

1 Introduction **3**

2 Description of the local time stepping strategy **4**
2.1 Discretization 6
2.2 Solution method 8

3 Finite volume discretization **9**
3.1 Mesh and definition 9
3.2 Cell centered finite volume scheme in the subdomains 11
3.3 Finite volume on the interface 13

4 A general stability result **14**

5 Well posedness **22**

6 Error estimate **23**

7 Numerical results **28**

8 Conclusion **31**

*Institut Franais du Pétrole, Rueil-Malmaison, France. isabelle.faille@ifp.fr
†Laboratoire J.L. Lions, CNRS UMR7598, Université Pierre et Marie Curie, France. nataf@ann.jussieu.fr
‡Institut Franais du Pétrole, Rueil-Malmaison, France. francoise.willien@ifp.fr
‡Laboratoire de Tectonique - CNRS UMR 7072, Université Pierre & Marie Curie - Paris VI, 4 place Jussieu, 75252 Paris Cedex 05, France. sylvie.wolf@upmc.fr

Abstract

We present a strategy for solving time-dependent problems on grids with local refinements in time using different time steps in different regions of space. We discuss and analyze two conservative approximations based on finite volume with piecewise constant projections and domain decomposition techniques. Next we present an iterative method for solving the composite-grid system that reduces to solution of standard problems with standard time stepping on the coarse and fine grids. At every step of the algorithm, conservativity is ensured. Finally, numerical results illustrate the accuracy of the proposed methods.

1 Introduction

In many physical applications, there are special features which greatly affect the solution globally as well as locally. One important example is the local spatial and temporal behavior of multiphase fluid flow around a production well in the petroleum recovery applications. To capture this local behavior, spatial local refinement is necessary. However, it requires a reduction of the time step, compared to the one used with a coarse mesh, in order to get a solution accurate enough in the refined zone and to avoid convergence problems when solving the non linear discretized equations. When applied uniformly on all the simulation domain, this reduced time step leads to unacceptable cpu-time making the use of local time steps highly desirable. To be efficient, a local time-stepping strategy (numerical scheme and solution method) must

- ensure accuracy of the solution i.e. the solution has to be more accurate than the one obtained with a global coarse mesh,
- ensure stability without any too restrictive condition on the time step,
- lead to reduced cpu-time compared to the one obtained when using a small time step on the whole domain.

In the framework of reservoir simulation where local grid refinement is necessary to represent correctly important local phenomena in the wells vicinity, the corresponding numerical scheme must also be locally conservative in order to be applicable to multi-phase flow simulations where a coupled system of parabolic and hyperbolic equations has to be solved.

For parabolic equations, different approaches have been proposed in the past which extend the classical implicit finite difference scheme to local refinement in time. In [6], the scheme is written as a cell centered Finite Volume scheme. At the interface between the coarse time step zone and the refined time step one, the flux over the coarse step is taken equal to the integral over the corresponding refined steps of the flux computed from the refined zone. This refined flux approximation requires values of the unknowns in the coarse zone at small time steps which are computed using piecewise constant or linear interpolation from the coarse unknowns. The scheme is conservative. Stability and error estimation are obtained for the piecewise constant interpolation. On the contrary, for the linear interpolation, stability is obtained under a sufficient condition which is as restrictive as the time step limitation obtained for an explicit scheme. In [3], Dawson et al proposed to couple classical implicit finite difference schemes in the refined and coarse zones using an explicit approximation at the interface on a larger mesh size in order to attenuate the time step limitation due to the explicit approximation. Although interesting for its simplicity, this approach can not be retained due to its time step limitation. In [5], Ewing and Lazarov proposed an implicit non conservative approach. The scheme for the coarse nodes is straightforward while the fine grid nodes located at the interface between the coarse and fine regions require "slave" points at small time steps on the coarse grid

side, which are not grid points. As in [6], the values of the unknown at these slave points are obtained by linear interpolation in time between the corresponding nodes of the coarse grid, and the set of discretized equations involves all the unknowns between two coarse time levels. Stability and error analysis are performed. The solution method uses an iterative method associated to a coarse grid preconditionner of the Schur complement of the system where the refined region unknowns have been eliminated. In the more applied framework of compositional multiphase flow, [4] introduced an implicit timestepping method. For each global time step, the problem is solved implicitly in the whole domain but using a linear approximation of the model in the refined regions which avoids any convergence problem of the non linear solver due to refined mesh. Then, the refined zones are solved using a local time step and taking as boundary conditions the fluxes computed during the first stage at the interface between the refined and coarse zones. This approach ensures that the method is conservative. It is moreover rather efficient as, compared to the cpu-time necessary to solve the problem with a large timestep on the whole domain, it only requires additional cpu-time to solve the equations once in the refined zone. However, the accuracy of the solution is not controlled. Looking for an efficient solution method, [12] used the same finite difference scheme as [5] for linear parabolic equations but proposed a predictor-corrector method. In the predictor stage, the solution is computed at the coarse time step on all the domain and in the correction step, the solution is computed in the refined grid at small time steps using values at slave nodes interpolated from the coarse nodes solution obtained in the first stage. They show that the predictor corrector approach preserves the maximum principle satisfied by the solution of the scheme.

Our paper proposes a local time step strategy based on the domain decomposition framework. It extends the approach introduced in [6] by generalizing the interface conditions used to couple the coarse and refined time-step domains. The method is conservative. Stability and error estimates, which are different from that obtained in [6], are presented. A solution method which improves the predictor-corrector methods of [4] and [12] is proposed. In order to simplify the presentation and to concentrate on the difficulties arising from the local refinement in the time direction, we will first explain the approach in the case of a one-dimensional spatial problem in section 2. Stability and error estimates are then proven in the more general case of nD spatial grids in sections 3 to 6. Finally, some numerical results are presented in section 7.

2 Description of the local time stepping strategy

We consider the following problem: Let $T > 0$ and Ω be an open bounded domain of $\mathbb{R}^d$, $d \geqslant 1$, $p_0 : \Omega \mapsto \mathbb{R}$ and $f : \Omega \times (0, T) \mapsto \mathbb{R}$ be given functions. Find $p : \Omega \times [0, T] \mapsto \mathbb{R}$ such

that:

$$\left.\begin{aligned} \frac{\partial p}{\partial t}(x,t) - \Delta p(x,t) = f(x,t) \quad \forall x \in \Omega \quad \forall t \in [0,T] \\ p(0,x) = p_0(x) \quad \forall x \in \Omega \\ p(x,t) = 0 \quad \forall x \in \partial\Omega \quad \forall t \in [0,T] \end{aligned}\right\} \tag{1}$$

In order to explain the scheme, we consider the $d = 1$ case with the cell centered grid shown in Figure 1 and a time step which is variable in space. Namely, the domain Ω is decomposed into two non overlapping subdomains Ω_1 and Ω_2 where two different time-step sizes are used : the coarser time step is denoted δt_2 (in Ω_2) and the finer time step is denoted δt_1 (in Ω_1) such that $\mathcal{K}\delta t_1 = \delta t_2$ with $\mathcal{K} \in \mathbb{N}^*$ (Figure 2).

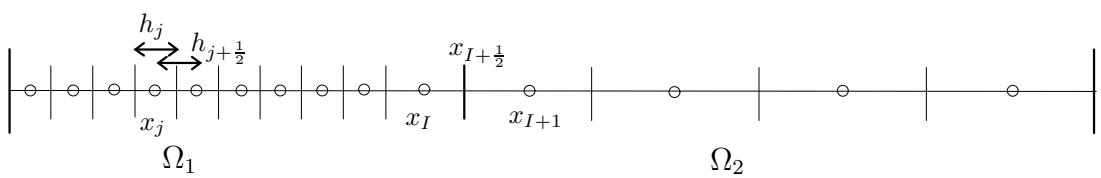

Figure 1: 1D cell-centered grid

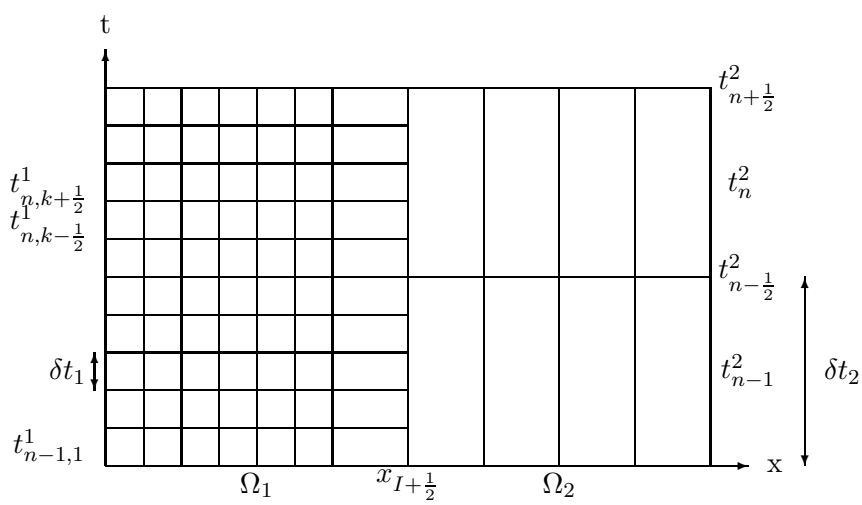

Figure 2: Time-space discretization.

2.1 Discretization

In each subdomain, the equation is discretised using a classical cell centered finite volume implicit scheme:

$$\frac{h_j}{\delta t_1}(p_j^{n,k+1} - p_j^{n,k}) - (u_{j+\frac{1}{2}}^{n,k+1} - u_{j-\frac{1}{2}}^{n,k+1}) = h_j f_j^{n,k+1} \text{ for } j \leq I \tag{2a}$$

$$\frac{h_j}{\delta t_2}(p_j^{n+1} - p_j^{n}) - (u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1}) = h_j f_j^{n+1} \text{ for } j > I \tag{2b}$$

where p_j^n is an approximation of the unknown in the space-time cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}})$, $p_j^{n,k}$ in the cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (t_{n,k-\frac{1}{2}}, t_{n,k+\frac{1}{2}})$ (see Figures 1 and 2),

$$f_j^{n+1} = \frac{1}{h_j \delta t_1} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} f(x,t) dx dt$$

and

$$f_j^{n,k+1} = \frac{1}{h_j \delta t_2} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t_{n,k-\frac{1}{2}}}^{t_{n,k+\frac{1}{2}}} f(x,t) dx dt\,.$$

Except for the boundary nodes (which are handled classically and not precised here) and interface node $x_{I+\frac{1}{2}}$, the flux approximation $u_{j+\frac{1}{2}}^s$ is given by:

$$u_{j+\frac{1}{2}}^s = \frac{p_{j+1}^s - p_j^s}{h_{j+\frac{1}{2}}} \text{ for } s = n \text{ if } j > I \text{ or } s = n, k \text{ if } j < I \tag{3}$$

For the approximation of the fluxes on the interface $u_{I+\frac{1}{2}}^{n+1}$ and $u_{I+\frac{1}{2}}^{n,k}$, we consider the space-time domain decomposition framework and introduce $p_{I+\frac{1}{2}}^{n+1}$ and $p_{I+\frac{1}{2}}^{n,k}$ the unknown approximations on the interface $x_{I+\frac{1}{2}}$. The flux approximations are classically obtained as

$$u_{I+\frac{1}{2}}^{n,k} = \frac{p_{I+\frac{1}{2}}^{n,k} - p_I^{n,k}}{0.5 h_I} \tag{4}$$

$$u_{I+\frac{1}{2}}^{n+1} = \frac{p_{I+1}^{n+1} - p_{I+\frac{1}{2}}^{n+1}}{0.5 h_{I+1}} \tag{5}$$

Discretizations in the two domains are linked by interface conditions which enforce flux and unknown continuity on the interface $x_{I+\frac{1}{2}} \times (t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}})$. We consider the following two possibilities:

$$\delta t_2 u^{n+1}_{I+\frac{1}{2}} = \sum_{k=1}^{\mathcal{K}} \delta t_1 u^{n,k}_{I+\frac{1}{2}} \tag{6a}$$

$$p^{n,k}_{I+\frac{1}{2}} = p^{n+1}_{I+\frac{1}{2}} \quad k = 1 \cdots \mathcal{K} \tag{6b}$$

or:

$$u^{n,k}_{I+\frac{1}{2}} = u^{n+1}_{I+\frac{1}{2}} \quad k = 1 \cdots \mathcal{K} \tag{7a}$$

$$\delta t_2 p^{n+1}_{I+\frac{1}{2}} = \sum_{k=1}^{\mathcal{K}} \delta t_1 p^{n,k}_{I+\frac{1}{2}} \tag{7b}$$

These interface conditions can be rewritten in terms of the L2 orthogonal projections on sets of piecewise constant functions in time (see section 3.3). Both sets of conditions ensure local conservation for the coarse time step.

Another way to couple the fine and coarse grid which is somewhat more natural, is to introduce the unknown approximations $p^{n+1}_{I+\frac{1}{2}}$ and $p^{n,k}_{I+\frac{1}{2}}$ not on the interface but rather in the neighbouring cells p^{n+1}_{I} and $p^{n,k}_{I+1}$. The flux approximations on the interface are then directly expressed as for an interior edge and instead of (4),(5), we have:

$$u^{n,k}_{I+\frac{1}{2}} = \frac{p^{n,k}_{I+1} - p^{n,k}_{I}}{h_{I+\frac{1}{2}}} \tag{8}$$

$$u^{n+1}_{I+\frac{1}{2}} = \frac{p^{n+1}_{I+1} - p^{n+1}_{I}}{h_{I+\frac{1}{2}}} \tag{9}$$

and the interface conditions become:

$$\delta t_2 u^{n+1}_{I+\frac{1}{2}} = \sum_{k=1}^{\mathcal{K}} \delta t_1 u^{n,k}_{I+\frac{1}{2}} \tag{10a}$$

$$p^{n,k}_{I+1} = p^{n+1}_{I+1} \quad k = 1 \cdots \mathcal{K} \tag{10b}$$

or:

$$u^{n,k}_{I+\frac{1}{2}} = u^{n+1}_{I+\frac{1}{2}} \quad k = 1 \cdots \mathcal{K} \tag{11a}$$

$$\delta t_2 p^{n+1}_{I} = \sum_{k=1}^{\mathcal{K}} \delta t_1 p^{n,k}_{I} \tag{11b}$$

We have thus four possible coupling schemes: (4),(5),(6)–(4),(5),(7)–(8),(9),(10)–(8),(9),(11) which are analysed in the sequel. Equations (8),(9),(10) are the approximations proposed in [6].

2.2 Solution method

To solve the system of algebraic equations for the unknowns values of the approximate solution between two coarse time levels, which includes all intermediate local time levels, we propose a method which combines the attractive feature of predictor-corrector approaches with the accuracy of domain decomposition type iterative algorithms. The method includes a predictor stage which corresponds to the computation of the solution on the coarse grid time and an iterative corrector stage where refined and coarse grids unknowns are solved alternatively, until interface conditions are satisfied, using a Schwarz multiplicative Dirichlet Neumann algorithm [11]. If we consider (8),(9),(10) interface conditions, the algorithm is:

- Predictor stage: computation of an approximate solution at coarse time step on the whole grid: $\tilde{p}_j^{n+1}$ for all j

$$\frac{h_j}{\delta t_2}(\tilde{p}_j^{n+1} - p_j^n) - (\tilde{u}_{j+\frac{1}{2}}^{n+1} - \tilde{u}_{j-\frac{1}{2}}^{n+1}) = h_j f_j^{n+1} \text{ for all } j \tag{12}$$

- Corrector iterative stage if interface conditions (10) are used:
  Solve alternatively the equations in domain Ω_1 using (10b) interface condition and in domain Ω_2 using (10a), until both interface conditions are satisfied simultaneously.

If (8),(9),(11) interface conditions are used, the corrector iterative stage is :

- Corrector iterative stage:
  Solve alternatively the equations in domain Ω_1 using (11a) interface condition and in domain Ω_2 using (11b), until both interface conditions are satisfied simultaneously.

These algorithms can also be written for (4),(5),(6) or (4),(5),(7) interface conditions. We can notice that it is not necessary to iterate the corrector stage until convergence to obtain a conservative solution. It is sufficient to stop the process after the resolution of the domain where Neumann interface conditions are imposed. The algorithm proposed in [4] consists in the predictor stage (12) and in the first iteration of the corrective stage with (11a) interface condition. This solution method is not limited to linear discrete equations and can be extended to the resolution of the non linear equations that arise in petroleum recovery applications. Following the idea of [4], the predictor stage would then use a linear approximation of the problem in the refined domain in order to avoid convergence problems of the Newton algorithm, while the iterative corrector stage would consider the non linear problem.

3 Finite volume discretization

Problem (1) is rewritten as a domain decomposition problem. The domain Ω is decomposed into two non overlapping subdomains Ω_1 and Ω_2 ($\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and $\Omega_1 \cap \Omega_2 = \emptyset$). The interface is denoted by $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Problem (1) is equivalent to:

Find $p^1 : \Omega_1 \times [0,T] \mapsto \mathbb{R}$ and $p^2 : \Omega_2 \times [0,T] \mapsto \mathbb{R}$ such that:

$$\frac{\partial p^i}{\partial t}(x,t) - \Delta p^i(x,t) = f(x,t) \quad \forall x \in \Omega_i \quad \forall t \in [0,T] \quad \forall i \in \{1,2\} \tag{13a}$$

$$p^i(x,0) = p_0(x) \quad \forall x \in \Omega_i \quad \forall i \in \{1,2\} \tag{13b}$$

$$p^i(x,t) = 0 \quad \forall x \in \partial\Omega_i \cap \partial\Omega \quad \forall t \in [0,T] \quad \forall i \in \{1,2\} \tag{13c}$$

$$\frac{\partial p^1}{\partial n_1}(x,t) + \frac{\partial p^2}{\partial n_2}(x,t) = 0 \quad \forall x \in \Gamma \quad \forall t \in [0,T] \tag{13d}$$

$$p^1(x,t) = p^2(x,t) \quad \forall x \in \Gamma \quad \forall t \in [0,T] \tag{13e}$$

where n_i is the outward normal to domain Ω_i, $i = 1,2$.

Problem (13) is discretized using a cell centered finite volume scheme in each subdomain [7]. We choose this scheme as an example but other schemes would be possible as well.

3.1 Mesh and definition

For $i = 1,2$, let $\mathcal{T}_i$ be a set of closed polygonal subsets associated with Ω_i such that $\bar{\Omega}_i = \cup_{K \in \mathcal{T}_i} K$. We shall denote $h = \max_{i \in \{1,2\}, K \in \mathcal{T}_i} diam(K)$ its mesh size. We shall use the following notation for all $i = 1,2$.

- $\mathcal{E}_{\Omega_i}$ is the set of faces of $\mathcal{T}_i$.
- $\mathcal{E}_{iD}$ is the set of faces such that $\partial\Omega_i \cap \partial\Omega = \cup_{\epsilon \in \mathcal{E}_{iD}} \epsilon$ (let us recall that a Dirichlet boundary condition will be imposed on $\partial\Omega_i \cap \partial\Omega$).
- $\mathcal{E}_i$ is the set of faces such that $\partial\Omega_i \backslash \partial\Omega = \cup_{\epsilon \in \mathcal{E}_i} \epsilon$. Grids are matching on the interface so that
$$\mathcal{E} := \mathcal{E}_1 = \mathcal{E}_2 \,.$$
- $\forall K \in \mathcal{T}_i$,

  $\mathcal{E}(K)$ denotes the set of faces of K.

  $\mathcal{E}_{iD}(K) = \mathcal{E}(K) \cap \mathcal{E}_{iD}$ is the set of faces of K which are on $\partial\Omega_i \cap \partial\Omega$.

  $\mathcal{E}_i(K) = \mathcal{E}(K) \cap \mathcal{E}_i$ is the set of faces of K which are on $\partial\Omega_i \backslash \partial\Omega$.

  $\mathcal{N}_i(K) = \{K' \in \mathcal{T}_i | K \cap K' \in \mathcal{E}_{\Omega_i}\}$ is the set of the control cells adjacent to K in Ω_i.

  $K_i(\varepsilon)$ denotes the cell of $\mathcal{T}_i$ adjacent to $\varepsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$.

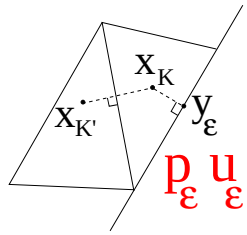

Figure 3: Assumption 3.1

- the time step in subdomain Ω_i is denoted by δt_i, and N_i denotes the number of time steps of the simulation so that $N_1\delta t_1 = N_2\delta t_2 = T$. Parameters δt_i, N_i satisfy

$$\frac{N_1}{N_2} = \frac{\delta t_2}{\delta t_1} = \mathcal{K} \in \mathbb{N}^* \tag{14}$$

- Let $[0,T]_{\delta t_i}$ denote the discretization of the time interval $[0,T]$ in each subdomain Ω_i : $[0,T]_{\delta t_i} = (t_n^i)_{n=1,\dots,N_i}$, with $t_n^i = (n-\frac{1}{2})\delta t_i$; since the time discretization in Ω_1 is a refinement of that in Ω_2, we shall also write : $t_{n,k}^1 = (n-1)\delta t_2 + (k-\frac{1}{2})\delta t_1$, $n = 1,\dots,N_2$, $k = 1,\dots,\mathcal{K}$

We make the following geometrical assumptions on the global mesh : $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$

Assumption 3.1. $\mathcal{T}$ is a finite volume admissible mesh, i.e., $\mathcal{T}$ is a set of closed subsets of dimension d such that

- for any $(K,K') \in \mathcal{T}^2$ with $K \neq K'$, one has either $[K\,K'] := K \cap K' \in \mathcal{E}_{\Omega_1} \cup \mathcal{E}_{\Omega_2}$ or $\dim(K \cap K') < d-1$
- for $i = 1,2$, there exist points $(y_\epsilon)_{\varepsilon\in\mathcal{E}_{\Omega_i}}$ on the faces and points $(x_K)_{K\in\mathcal{T}_i}$ inside the cells such that (see figure 3)
  - for any adjacent cells K and K', the straight line $[x_K, x_{K'}]$ is perpendicular to the face $[K\,K']$ and $[x_K, x_{K'}] \cap [K\,K'] = \{y_\epsilon\}$
  - for any face $\epsilon \in \mathcal{E}_{iD}$, let $K(\epsilon) \in \mathcal{T}_i$ be such that $\epsilon \subset K$: then the straight line $[x_{K(\epsilon)}, y_\varepsilon]$ is perpendicular to ϵ
- Each mesh $\mathcal{T}_i$, $i = 1,2$ has at least one interior cell.

Notation 3.2. *For all $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$, $i = 1,2$, d_ϵ^i denotes the distance between $x_{K_i(\epsilon)}$ and y_ϵ.*

3.2 Cell centered finite volume scheme in the subdomains

The unknowns of the scheme and what they aim to approximate are ($i = 1, 2$):

$$\begin{aligned} p_K^{i,n} &\simeq p^i(x_K, t_n),\ K \in \mathcal{T}_i \\ p_\varepsilon^{i,n} &\simeq p^i(y_\varepsilon, t_n),\ \varepsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \\ u_\varepsilon^{i,n} &\simeq \frac{\partial p^i}{\partial n_i}(y_\varepsilon, t_n),\ \varepsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \end{aligned}$$

The numerical flux is defined by:

$$u_\varepsilon^{i,n} = \frac{p_\varepsilon^{i,n} - p_{K_i(\varepsilon)}^{i,n}}{d_\varepsilon^i} \quad \forall \varepsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \quad \forall n \in \{0..N_i - 1\} \quad \forall i \in \{1, 2\} \tag{15}$$

The scheme is defined by (see e.g. [7] for its derivation):

$$\begin{aligned} &\frac{p_K^{i,n+1} - p_K^{i,n}}{\delta t_i}\text{meas}(K) - \sum_{K' \in \mathcal{N}_i(K)} \frac{p_{K'}^{i,n+1} - p_K^{i,n+1}}{d(x_K, x_{K'})}\text{meas}([KK']) \\ &\quad - \sum_{\varepsilon \in \mathcal{E}_i(K) \cup \mathcal{E}_{iD}(K)} u_\varepsilon^{i,n+1}\text{meas}(\varepsilon) = f_K^{i,n+1}\text{meas}(K) \quad \forall K \in \mathcal{T}_i \quad \forall n \in 0..N_i - 1 \end{aligned} \tag{16}$$

where $d(x_K, x_{K'})$ is the distance between x_K and $x_{K'}$ and $f_K^{i,n}$ is an approximation to $\frac{1}{\delta t_i}\int_{t_{n-1/2}^i}^{t_{n+1/2}^i} \frac{1}{\text{meas}(K)} \int_K f$. The initial and boundary conditions are discretized by:

$$p_K^{i,0} = p_0(x_K) \quad \forall K \in \mathcal{T}_i \quad \forall i \in \{1, 2\} \tag{17}$$

$$p_\varepsilon^{i,n+1} = 0 \quad \forall \varepsilon \in \mathcal{E}_{iD} \quad \forall n \in 0..N_i - 1 \quad \forall i \in \{1, 2\} \tag{18}$$

When there is no domain decomposition, this scheme has been analyzed in [7] in the more general case of discontinuous coefficients, and it is proven to be of order 1 for a discrete H^1-norm.

In order to define the domain decomposition discretization scheme, we shall define in section 3.3 the matching conditions for the diffusive fluxes.

Discrete spaces

- $$\begin{aligned} P_0(\mathcal{T}_i \times [0, T]_{\delta t_i}) = \big\{ p : \bar{\Omega}_i \times [0, T] \mapsto \mathbb{R} \backslash\ \forall n \in \{0, \ldots, N_i - 1\} \\ \forall K \in \mathcal{T}_i \ \ p_{|K \times (t_{n-1/2}^i, t_{n+1/2}^i)} \equiv C^t \\ \text{and } \forall \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}\ \ ,\ p_{|\epsilon \times (t_{n-1/2}^i, t_{n+1/2}^i)} \equiv C^t \big\} \end{aligned} \tag{19}$$

- Similarly, $P_0(\mathcal{E}\times[0,T]_{\delta t_i})$ is the space of piecewise constant functions on the interface for the time mesh of subdomain Ω_i.
- $P_0([0,T]_{\delta t_i}) = \Big\{ p : [0,T] \mapsto \mathbb{R}\backslash \ \ \forall n \in \{0,\dots,N_i-1\}\ p_{|(t^i_{n-1/2},t^i_{n+1/2})} \equiv C^t \ \Big\}$

These are spaces of piecewise constant functions.

Let $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, we denote its restriction

- to $\bar{\Omega}_i \times \{t^i_n\}$ by $p^{i,n}$, for $n \in \{0,\dots,N_i-1\}$
- to $\epsilon \times [0,T]_{\delta t_i}$ by p^i_ϵ for $\epsilon \in \mathcal{E}$
- to $\mathcal{E} \times [0,T]_{\delta t_i}$ by $p^i_{\mathcal{E}}$

We introduce the following norms and semi-norms for any $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$:

$$\|p^{i,n}\|^2_{L^2(\Omega_i)} = \sum_{K\in\mathcal{T}_i} (p^{i,n}_K)^2 \mathrm{meas}(K)$$

$$\|p^i\|^2_{L^2(0,T;L^2(\Omega_i))} = \sum_{n=0}^{N_i} \delta t_i \|p^{i,n}\|^2_{L^2(\Omega_i)}$$

and

$$\begin{aligned} |p^{i,n}|^2_{1,\mathcal{T}_i} = \sum_{K\in\mathcal{T}_i} \Bigg(& \sum_{K'\in\mathcal{N}_i(K)} \frac{(p^{i,n}_{K'} - p^{i,n}_K)^2}{d(x_K,x_{K'})} \mathrm{meas}([KK']) \\ & + \sum_{\varepsilon\in\mathcal{E}_i(K)} \frac{(p^{i,n}_\varepsilon - p^{i,n}_K)^2}{d(x_K,y_\varepsilon)} \mathrm{meas}(\varepsilon) + \sum_{\varepsilon\in\mathcal{E}_{iD}(K)} \frac{(p^{i,n}_K)^2}{d(x_K,y_\varepsilon)} \mathrm{meas}(\varepsilon) \Bigg) \end{aligned} \tag{20}$$

and

$$|p^i|^2_{1,\mathcal{T}_i,\delta t_i} = \sum_{n=0}^{N_i} \delta t_i |p^{i,n}|^2_{1,\mathcal{T}_i} \tag{21}$$

Definition 3.1. *Let $p^i, u^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, $i=1,2$, we define a discrete scalar product by:*

$$\sum_{i=1}^{2} \left\langle u^i, p^i \right\rangle_{L^2(0,T;L^2(\Gamma))} := \sum_{i=1}^{2} \sum_{n=0}^{N_i-1} \delta t_i \sum_{\varepsilon\in\mathcal{E}} u^{i,n+1}_\varepsilon p^{i,n+1}_\varepsilon meas(\varepsilon)$$

Notation 3.3. *Let $i=1,2$, $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, for $\epsilon \in \mathcal{E}$, $u^i_\epsilon(p)$ denotes the associated numerical flux defined by* (15)*. Very often, we will simply write u^i_ϵ and $u^i_{\mathcal{E}} = (u_\varepsilon(p^i))_{\varepsilon\in\mathcal{E}_i}$.*

3.3 Finite volume on the interface

In order to enforce the weak continuity of the primary unknown p and of its normal derivative (denoted by u) across the interface $\Gamma \times [0,T]$, we introduce $\mathcal{Q}_i$ the L^2 orthogonal projection onto $P_0([0,T]_{\delta t_i})$. We have the following compatibility condition:

Lemma 3.4. For all $u_i \in P_0([0,T]_{\delta t_i})$, $i = 1, 2$,

$$\langle \mathcal{Q}_1(u_2), u_1 \rangle_{L^2([0,T])} = \langle u_2, \mathcal{Q}_2(u_1) \rangle_{L^2([0,T])}$$

As in mortar methods [2], we consider that one subdomain enforces the weak continuity of the primary unknown which is interpreted as the Dirichlet interface condition. This subdomain is called the master. The other subdomain is called the slave. It enforces the weak continuity of the normal derivative which corresponds to a Neumann interface condition.

Since here the interfaces are non matching only in the time direction, it is possible to define matching conditions locally on each interface face $\varepsilon \in \mathcal{E}$.

Interface Scheme (IS$_1$) **based on interface unknowns** The subscript m will denote the master subdomain and e the slave ($\{m, e\} = \{1, 2\}$), the interface conditions on $\Gamma \times [0,T]$ read:

$$\begin{cases} u_\varepsilon^m = \mathcal{Q}_m(-u_\varepsilon^e) \\ p_\varepsilon^e = \mathcal{Q}_e(p_\varepsilon^m) \end{cases} \quad \forall \varepsilon \in \mathcal{E} \tag{IS$_1$}$$

Overlapping interface scheme (IS$_2$) The Dirichlet boundary condition is modified but not the Neumann one:

$$\begin{cases} u_\varepsilon^m = \mathcal{Q}_m(-u_\varepsilon^e) \\ p_\varepsilon^e + d_\varepsilon^m u_\varepsilon^e = \mathcal{Q}_e(p_\varepsilon^m - d_\varepsilon^m u_\varepsilon^m) \end{cases} \quad \forall \varepsilon \in \mathcal{E} \tag{IS$_2$}$$

The modified Dirichlet interface condition comes from the following relations:

$$\begin{aligned} u_\varepsilon^e &= \frac{\mathcal{Q}_e(p^m_{K_m(\varepsilon)}) - p^e_{K_e(\varepsilon)}}{d_\varepsilon^m + d_\varepsilon^e} \\ &= \frac{\mathcal{Q}_e(p_\varepsilon^m - d_\varepsilon^m u_\varepsilon^m) - p_\varepsilon^e + d_\varepsilon^e u_\varepsilon^e}{d_\varepsilon^m + d_\varepsilon^e} \end{aligned}$$

where $d_\varepsilon^i = d(x_{K_i(\varepsilon)}, y_\varepsilon)$. The first line of the above formula is somewhat natural. When writing the finite volume scheme for a cell $K_e(\epsilon)$ adjacent to the interface in the "slave" subdomain, it is necessary to approximate the flux on the face ϵ. This is done using pressure values from both sides of the interface: the pressure in the "slave" subdomain and pressures values in the neighboring "master" subdomain. These last values are made compatible with the "slave" unknowns by using the transmission operator $\mathcal{Q}_e$. Finally, all

quantities are expressed in terms of interface values in order to ease a comparison with (IS_1).

Due to the fact that the large time step δt_2 is a multiple of the small time step δt_1, we have a simple form for the L^2 projection operators.

Lemma 3.5. *We have* $\mathcal{Q}_2 : P_0([0,T]_{\delta t_1}) \mapsto P_0([0,T]_{\delta t_2})$ *and* $\mathcal{Q}_1 : P_0([0,T]_{\delta t_2}) \mapsto P_0([0,T]_{\delta t_1})$

$$\begin{cases} \text{For } v_2 \in P_0([0,T]_{\delta t_2}),\ \mathcal{Q}_1(v_2)_{|(t^1_{n-1/2},t^1_{n+1/2})} = v_{2|(t^1_{n-1/2},t^1_{n+1/2})} & \forall\, n \in \{0,\dots,N_1-1\} \\ \text{For } v_1 \in P_0([0,T]_{\delta t_1}),\ \mathcal{Q}_2(v_1))_{|(t^2_{n-1/2},t^2_{n+1/2})} = \dfrac{1}{\delta t_2}\displaystyle\int_{t^2_{n-1/2}}^{t^2_{n+1/2}} v_1 & \forall\, n \in \{0,\dots,N_2-1\} \end{cases} \tag{22}$$

We also need the following technical assumptions. For the family of meshes we consider, the mesh close to the interface is not too stretched:

Assumption 3.6. *There exists a constant* $\alpha > 0$ *such that* $d^m_\varepsilon \leqslant \alpha d^e_\varepsilon$ *for all* $\varepsilon \in \mathcal{E}$.

We also need a geometric assumption:

Assumption 3.7. *For all* $\varepsilon \in \mathcal{E}$, y_ε *is the barycenter of the face* ε *and for* $i = 1, 2$,

$$\frac{diam(\varepsilon)^2}{d(x_{K_i(\varepsilon)}, y_\varepsilon)} = O(h)$$

4 A general stability result

We prove a stability result for (16) modified by the introduction of discretization error terms $R^{i,n}_K$, $R^{i,n}_{KK'}$ and $R^{i,n}_\varepsilon$ as well as $F^{i,n}_K$ which will be defined precisely in the sequel ($i \in \{1,2\}$):

$$\left(\frac{p^{i,n+1}_K - p^{i,n}_K}{\delta t_i} - R^{i,n+1}_K\right)\mathrm{meas}(K) - \sum_{K' \in \mathcal{N}_i(K)} \left(\frac{p^{i,n+1}_{K'} - p^{i,n+1}_K}{d(x_K, x_{K'})} - R^{i,n+1}_{KK'}\right)\mathrm{meas}([KK'])$$
$$- \sum_{\varepsilon \in \mathcal{E}_i(K) \cup \mathcal{E}_{iD}(K)} (u^{i,n+1}_\varepsilon - R^{i,n+1}_\varepsilon)\mathrm{meas}(\varepsilon) = F^{i,n+1}_K \mathrm{meas}(K) \quad \forall K \in \mathcal{T}_i \quad \forall n \in \{0,\dots,N_i-1\}\,. \tag{23}$$

Formula (15) is modified as well by introducing error terms at each time step n

$$U^{i,n}_\varepsilon = u^{i,n}_\varepsilon - \frac{p^{i,n}_\varepsilon - p^{i,n}_{K_i(\varepsilon)}}{d^i_\varepsilon} \quad \forall \varepsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \quad \forall n \in \{1,\dots,N_i\} \quad \forall i \in \{1,2\} \tag{24}$$

(with $p_\varepsilon^{i,n} = 0$ for all $\varepsilon \in \mathcal{E}_{iD}$).

The interface conditions (IS_1) are modified in the following manner by the terms $\mathcal{P}_\varepsilon \in P_0(\mathcal{E} \times [0,T]_{\delta t_e})$ and $\mathcal{U}_\varepsilon \in P_0(\mathcal{E} \times [0,T]_{\delta t_m})$ which will be defined in the sequel:

$$\begin{cases} p_\varepsilon^e = \mathcal{Q}_e(p_\varepsilon^m) + \mathcal{P}_\varepsilon^e \\ u_\varepsilon^m = \mathcal{Q}_m(-u_\varepsilon^e) + \mathcal{U}_\varepsilon^m \end{cases} \quad \forall \varepsilon \in \mathcal{E} \tag{IS$_1$'}$$

Interface conditions(IS_2) are similarly modified for all $\varepsilon \in \mathcal{E}$:

$$\begin{aligned} u_\varepsilon^e =& \frac{\mathcal{Q}_e(p_{K_m(\varepsilon)}^m) + \mathcal{P}_\varepsilon - p_{K_e(\varepsilon)}^e}{d_\varepsilon^m + d_\varepsilon^e} \\ =& \frac{\mathcal{Q}_e(p_\varepsilon^m - d_\varepsilon^m u_\varepsilon^m + d_\varepsilon^m U_\varepsilon^m) + \mathcal{P}_\varepsilon - p_\varepsilon^e + d_\varepsilon^e u_\varepsilon^e - d_\varepsilon^e U_\varepsilon^e}{d_\varepsilon^m + d_\varepsilon^e} \end{aligned}$$

The interface conditions (IS_2) are thus modified in the following manner:

$$\begin{cases} u_\varepsilon^m = \mathcal{Q}_m(-u_\varepsilon^e) + \mathcal{U}_\varepsilon \\ p_\varepsilon^e + d_\varepsilon^m u_\varepsilon^e = \mathcal{Q}_e(p_\varepsilon^m - d_\varepsilon^m u_\varepsilon^m + d_\varepsilon^m U_\varepsilon^m) + \mathcal{P}_\varepsilon - d_\varepsilon^e U_\varepsilon^e \end{cases} \quad \forall \varepsilon \in \mathcal{E} \tag{IS$_2$'}$$

We make some additional assumptions.

Assumption 4.1. $R_{KK'} + R_{K'K} = 0 \quad \forall K \in \mathcal{T}_i \quad \forall K' \in \mathcal{N}_i(K) \quad \forall i \in \{1,2\}$

Assumption 4.2. *We suppose that the following estimates are satisfied by the consistency errors:*
for all $i \in \{1,2\}$ and all $n \in \{1,\dots,N_i\}$, we have:

$$\begin{aligned} U_\varepsilon^{i,n} =& O(h^{\eta_1} + \delta t_i^{\gamma_1}) \quad \forall K \in \mathcal{T}_i \quad \forall \varepsilon \in \mathcal{E}_i(K) \cup \mathcal{E}_{iD}(K) \\ R_K^{i,n} =& O(h^{\eta_2} + \delta t_i^{\gamma_2}) \quad \forall K \in \mathcal{T}_i \\ R_{KK'}^{i,n} =& O(h^{\eta_3} + \delta t_i^{\gamma_3}) \quad \forall K \in \mathcal{T}_i \quad \forall K' \in \mathcal{N}_i(K) \\ R_\varepsilon^{i,n} =& O(h^{\eta_4} + \delta t_i^{\gamma_4}) \quad \forall K \in \mathcal{T}_i \quad \forall \varepsilon \in \mathcal{E}_i(K) \cup \mathcal{E}_{iD}(K) \\ F_K^{i,n} =& O(h^{\eta_5} + \delta t_i^{\gamma_5}) \quad \forall K \in \mathcal{T}_i \\ \mathcal{P}_\varepsilon^{i,n} =& O(h^{\eta_6} + \delta t_2^{\gamma_6}) \quad \forall \varepsilon \in \mathcal{E} \\ \mathcal{U}_\varepsilon^{i,n} =& O(h^{\eta_7} + \delta t_1^{\gamma_7}) \quad \forall \varepsilon \in \mathcal{E} \end{aligned}$$

We first prove an estimate for (23)-(24) not taking into account the interface conditions:

Lemma 4.3. *Let $(p^i, u^i(p^i))$ satisfy* (23)-(24) *and suppose that assumptions 3.1 and 4.1 are satisfied.*

Then :

$$\frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_i}(p_K^{i,N_i})^2 meas(K)+\frac{1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p_{K'}^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,x_{K'})}meas([KK'])$$
$$+\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}meas(\varepsilon)$$
$$\leqslant\sum_{i=1}^{2}\left\langle u^i,p^i\right\rangle_{L^2(0,T;L^2(\Gamma))}+\frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_i}(p_K^{i,0})^2 meas(K)$$
$$+\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}(F_K^{i,n+1}+R_K^{i,n+1})p_K^{i,n+1}meas(K)$$
$$+\frac{1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}R_{KK'}^{i,n+1}(p_{K'}^{i,n+1}-p_K^{i,n+1})meas([KK'])$$
$$-\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}R_\varepsilon^{i,n+1}p_K^{i,n+1}meas(\varepsilon)$$
$$-\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}U_\varepsilon^{i,n+1}(p_\varepsilon^{i,n+1}-p_K^{i,n+1})meas(\varepsilon)\quad(25)$$

Proof. In each subdomain Ω_i, we multiply (23) by $\delta t_i p_K^{i,n+1}$ and we sum over cells K and time step n and make use of the following formula:

$$\begin{aligned}\sum_{n=0}^{N_i-1}(p_K^{i,n+1}-p_K^{i,n})p_K^{i,n+1}=&\frac{1}{2}\sum_{n=0}^{N_i-1}\left[(p_K^{i,n+1})^2-(p_K^{i,n})^2+(p_K^{i,n+1}-p_K^{i,n})^2\right]\\=&\frac{1}{2}\left[(p_K^{i,N_i})^2-(p_K^{i,0})^2+\sum_{n=0}^{N_i-1}(p_K^{i,n+1}-p_K^{i,n})^2\right]\\\geqslant&\frac{1}{2}\left[(p_K^{i,N_i})^2-(p_K^{i,0})^2\right]\end{aligned}$$

$$\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{p_{K'}^{i,n}-p_K^{i,n}}{d(x_K,x_{K'})}p_K^{i,n}\text{meas}([KK'])=-\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p_{K'}^{i,n}-p_K^{i,n})^2}{d(x_K,x_{K'})}\text{meas}([KK'])$$

Using the fact that $R_{KK'}+R_{K'K}=0$ (Assumption 4.1), we have :

$$\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}R_{KK'}^{i,n}p_K^{i,n}\text{meas}([KK'])=-\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}R_{KK'}^{i,n}(p_{K'}^{i,n}-p_K^{i,n})\text{meas}([KK'])$$

Then, we get:

$$\frac{1}{2}\sum_{K\in\mathcal{T}_i}\Big((p_K^{i,N_i})^2-(p_K^{i,0})^2\Big)\,\mathrm{meas}(K)+\frac{1}{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p_{K'}^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,x_{K'})}\mathrm{meas}([KK'])$$
$$-\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}u_\varepsilon^{i,n+1}p_K^{i,n+1}\mathrm{meas}(\varepsilon)\leqslant\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}F_K^{i,n+1}p_K^{i,n+1}\mathrm{meas}(K)$$
$$+\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}R_K^{i,n+1}p_K^{i,n+1}\mathrm{meas}(K)+\frac{1}{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}R_{KK'}^{i,n+1}(p_{K'}^{i,n+1}-p_K^{i,n+1})\mathrm{meas}([KK'])$$
$$-\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}R_\varepsilon^{i,n+1}p_K^{i,n+1}\mathrm{meas}(\varepsilon)$$

We sum over the subdomains and make use of the following formula for any $\varepsilon\in\mathcal{E}_i\cup\mathcal{E}_{iD}$ and $K=K(\varepsilon)$,

$$\begin{aligned}-u_\varepsilon^{i,n}p_K^{i,n}&=-u_\varepsilon^{i,n}p_\varepsilon^{i,n}-u_\varepsilon^{i,n}(p_K^{i,n}-p_\varepsilon^{i,n})\\&=-u_\varepsilon^{i,n}p_\varepsilon^{i,n}+\left(\frac{p_K^{i,n}-p_\varepsilon^{i,n}}{d(x_K,y_\varepsilon)}-U_\varepsilon^{i,n}\right)(p_K^{i,n}-p_\varepsilon^{i,n})\\&=-u_\varepsilon^{i,n}p_\varepsilon^{i,n}+\frac{(p_K^{i,n}-p_\varepsilon^{i,n})^2}{d(x_K,y_\varepsilon)}-(p_K^{i,n}-p_\varepsilon^{i,n})U_\varepsilon^{i,n}\end{aligned}$$

Then, (25) follows from $p_\varepsilon^{i,n}=0$ for all $\varepsilon\in\mathcal{E}_{iD}$. □

We focus now on the interface terms on Γ that appear in the first term of the right hand side of (25). We first consider the interface matching conditions (IS_1').

Lemma 4.4. *Let* (p,u) *satisfy* (23), (24), (IS_1'). *Then, we have:*

$$\sum_{i=1}^{2}\langle u^i,p^i\rangle_{L^2(0,T;L^2(\Gamma))}=\langle p^m,\mathcal{U}_\mathcal{E}\rangle_{L^2(0,T;L^2(\Gamma))}+\langle u^e,\mathcal{P}_\mathcal{E}\rangle_{L^2(0,T;L^2(\Gamma))}\tag{26}$$

Proof. From the interface conditions (IS$_1$'), we have:

$$\begin{aligned}
&\sum_{n=0}^{N_m-1} \delta t_m \sum_{\varepsilon\in\mathcal{E}} u_\varepsilon^{m,n+1} p_\varepsilon^{m,n+1} \mathrm{meas}(\varepsilon) + \sum_{n=0}^{N_e-1} \delta t_e \sum_{\varepsilon\in\mathcal{E}} u_\varepsilon^{e,n+1} p_\varepsilon^{e,n+1} \mathrm{meas}(\varepsilon) \\
&\quad = \langle u^m, p^m\rangle_{L^2(0,T;L^2(\Gamma))} + \langle u^e, p^e\rangle_{L^2(0,T;L^2(\Gamma))} \\
&\quad = \langle \mathcal{Q}_m(-u^e) + \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} + \langle u^e, \mathcal{Q}_e(p^m) + \mathcal{P}_\mathcal{E}\rangle_{L^2(0,T;L^2(\Gamma))} \\
&\quad = \underbrace{\langle -\mathcal{Q}_m(u^e), p^m\rangle_{L^2(0,T;L^2(\Gamma))} + \langle u^e, \mathcal{Q}_e(p^m)\rangle_{L^2(0,T;L^2(\Gamma))}}_{=0 \text{ (Lemma 3.4)}} \\
&\qquad + \langle \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} + \langle u^e, \mathcal{P}_\mathcal{E}\rangle_{L^2(0,T;L^2(\Gamma))}
\end{aligned}$$

□

We consider now the interface scheme (IS$_2$')

Lemma 4.5. *Let* (p, u) *satisfy* (23), (24), (IS$_2$'). *Then, we have*

$$\sum_{i=1}^{2} \left\langle p^i, u^i\right\rangle_{L^2(0,T;L^2(\Gamma))} \leqslant \langle p^m, \mathcal{U}_\mathcal{E}\rangle_{L^2(0,T;L^2(\Gamma))} + \langle u^e, \mathcal{P}_\mathcal{E} - d_\mathcal{E}^e U_\mathcal{E}^e - d_\mathcal{E}^m(\mathcal{U}_\mathcal{E} - U_\mathcal{E}^m)\rangle_{L^2(0,T;L^2(\Gamma))} \tag{27}$$

where $d_\mathcal{E}^i$ *is the piecewise constant function on* Γ *such that* $d_\mathcal{E}^i(x) = d_\varepsilon^i$ *for all* $x \in \varepsilon \in \mathcal{E}$.

Proof. Using the interface conditions (IS$_2$'), we have:

$$\begin{aligned}
&\textstyle\sum_{i=1}^2 \left\langle p^i, u^i\right\rangle_{L^2(0,T;L^2(\Gamma))} = \langle -\mathcal{Q}_m(u^e) + \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} \\
&+ \langle u^e, \mathcal{Q}_e(p^m - d_\mathcal{E}^m u_\mathcal{E}^m + d_\mathcal{E}^m U_\mathcal{E}^m) + \mathcal{P}_\mathcal{E} - d_\mathcal{E}^m u_\mathcal{E}^e - d_\mathcal{E}^e U_\mathcal{E}^e\rangle_{L^2(0,T;L^2(\Gamma))}
\end{aligned}$$

Since we use L^2 projection, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^2 \left\langle p^i, u^i\right\rangle_{L^2(0,T;L^2(\Gamma))} = \langle -u^e + \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} \\
&+ \langle u^e, p^m - d_\mathcal{E}^m u_\mathcal{E}^m + d_\mathcal{E}^m U_\mathcal{E}^m + \mathcal{P}_\mathcal{E} - d_\mathcal{E}^m u_\mathcal{E}^e - d_\mathcal{E}^e U_\mathcal{E}^e\rangle_{L^2(0,T;L^2(\Gamma))}
\end{aligned}$$

Simplifying the relation and using again the first equation of (IS$_2$'),

$$\begin{aligned}
&\textstyle\sum_{i=1}^2 \left\langle p^i, u^i\right\rangle_{L^2(0,T;L^2(\Gamma))} = \langle \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} \\
&+ \langle u^e, -d_\mathcal{E}^m \mathcal{Q}_m(-u^e) - d_\mathcal{E}^m \mathcal{U}_\mathcal{E} + d_\mathcal{E}^m U_\mathcal{E}^m + \mathcal{P}_\mathcal{E} - d_\mathcal{E}^m u_\mathcal{E}^e - d_\mathcal{E}^e U_\mathcal{E}^e\rangle_{L^2(0,T;L^2(\Gamma))}
\end{aligned}$$

Since $\mathcal{Q}_m$ is a L^2 projection, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^2 \left\langle p^i, u^i\right\rangle_{L^2(0,T;L^2(\Gamma))} \leq \langle \mathcal{U}_\mathcal{E}, p^m\rangle_{L^2(0,T;L^2(\Gamma))} \\
&+ \langle u^e, -d_\mathcal{E}^m \mathcal{U}_\mathcal{E} + d_\mathcal{E}^m U_\mathcal{E}^m + \mathcal{P}_\mathcal{E} - d_\mathcal{E}^e U_\mathcal{E}^e\rangle_{L^2(0,T;L^2(\Gamma))}
\end{aligned}$$

□

Theorem 4.6. *Suppose assumptions 3.1, 4.1 and 4.2 hold. Let* (p,u) *satisfy* (23), (24). *If one of the two conditions is satisfied,*

i) (p,u) *satisfies transmission conditions* (IS_1'),

ii) (p,u) *satisfies transmission conditions* (IS_2') *and assumption 3.6 holds*

Then, we have the following estimate:

$$\sum_{i=1}^{2} |p^i|^2_{1,\mathcal{T}_i,\delta t_i} + 2\sum_{i=1}^{2} \|p^{i,N_i}\|^2_{L^2(\Omega_i)} \leqslant 2\sum_{i=1}^{2} \|p^{i,0}\|^2_{L^2(\Omega_i)} + O(h)^\eta + O(\delta t_2)^\gamma$$

where $\eta = 2\min(\eta_j)_{j=1,\dots,7}$ *et* $\gamma = 2\min(\gamma_j)_{j=1,\dots,7}$.

Proof. The proof consists in estimating the terms in formula (25) of lemma 4.3. We often use the relation

$$|ab| \leqslant \frac{C}{2}a^2 + \frac{1}{2C}b^2 \quad \forall (a,b) \in \mathbb{R}^2 \quad \forall C \in \mathbb{R}^*_+$$

with various constants C_i which are independent of the parameters of the mesh size. For the terms that are classical in the finite volume theory, we simply write the estimate.

We begin with the estimate of the classical term

$$\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i} F_K^{i,n+1} p_K^{i,n+1} \mathrm{meas}(K) \leqslant \frac{C_5}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i} (p_K^{i,n+1})^2 \mathrm{meas}(K) + O(h^{2\eta_5} + \delta t_i^{2\gamma_5})$$

We also have the term

$$\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i} R_K^{i,n+1} p_K^{i,n+1} \mathrm{meas}(K) \leqslant \frac{C_2}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i} (p_K^{i,n+1})^2 \mathrm{meas}(K) + O(h^{2\eta_2} + \delta t_i^{2\gamma_2})$$

and the term

$$-\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)} R_\varepsilon^{i,n+1} p_\varepsilon^{i,n+1} \mathrm{meas}(\varepsilon) \leqslant \frac{C_4}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1} \delta t_i \sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)} (p_\varepsilon^{i,n+1})^2 \mathrm{meas}(\varepsilon) + O(h^{2\eta_4} + \delta t_i^{2\gamma_4})$$

We consider now

$$
\begin{aligned}
&-\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}(p_\varepsilon^{i,n+1}-p_K^{i,n+1})U_\varepsilon^{i,n+1}\mathrm{meas}(\varepsilon)\\
&\qquad=-\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{p_\varepsilon^{i,n+1}-p_K^{i,n+1}}{\sqrt{d(x_K,y_\varepsilon)}}U_\varepsilon^{i,n+1}\sqrt{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\\
&\qquad\leqslant\frac{C_1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\\
&\qquad\quad+\frac{1}{2C_1}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}(U_\varepsilon^{i,n+1})^2d(x_K,y_\varepsilon)\mathrm{meas}(\varepsilon)\\
&\qquad\leqslant\frac{C_1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\\
&\qquad\quad+\frac{1}{2C_1}O(h^{2\eta_1}+\delta t_i^{2\gamma_1})\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}d(x_K,y_\varepsilon)\mathrm{meas}(\varepsilon)\\
&\qquad\leqslant\frac{C_1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)+O(h^{2\eta_1}+\delta t_i^{2\gamma_1})
\end{aligned}
$$

where we have used the following formula (see [7])

$$
\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}d(x_K,x_{K'})\,meas([K,K']+\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}d(x_K,y_\varepsilon)\,\mathrm{meas}(\varepsilon)\leq d\,meas(\Omega_i)
$$

In a classical way, we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}R_{KK'}^{i,n+1}(p_K^{i,n+1}-p_{K'}^{i,n+1})\mathrm{meas}([KK'])\\
&\qquad\leqslant\frac{C_3}{4}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p_K^{i,n+1}-p_{K'}^{i,n+1})^2}{d(x_K,x_{K'})}\mathrm{meas}([KK'])+O(h^{2\eta_3}+\delta t_i^{2\gamma_3})
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}R_\varepsilon^{i,n+1}(p_\varepsilon^{i,n+1}-p_K^{i,n+1})\mathrm{meas}(\varepsilon)\\
&\qquad\leqslant\frac{C_4'}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)+O(h^{2\eta_4}+\delta t_i^{2\gamma_4})
\end{aligned}
$$

We now focus on the interface terms in (25) starting with transmission scheme (IS_1'). By (26) of lemma 4.4 :

$$\begin{aligned}\sum_{i=1}^{2}\left\langle u^i,p^i\right\rangle_{L^2(0,T;L^2(\Gamma))} &= \langle p^m,\mathcal{U}_{\mathcal{E}}\rangle_{L^2(0,T;L^2(\Gamma))} + \langle U^e_{\mathcal{E}},\mathcal{P}_{\mathcal{E}}\rangle_{L^2(0,T;L^2(\Gamma))}\\ &\quad+\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}}\frac{p^{e,n+1}_{\varepsilon}-p^{e,n+1}_{K}}{d(x_K,y_\varepsilon)}\mathcal{P}^{n+1}_{\varepsilon}\mathrm{meas}(\varepsilon)\\ &\leqslant\frac{C_5}{2}\sum_{n=0}^{N_m-1}\delta t_m\sum_{\varepsilon\in\mathcal{E}}(p^{m,n+1}_{\varepsilon})^2\mathrm{meas}(\varepsilon)+\frac{C_6}{2}\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}}\frac{(p^{e,n+1}_{\varepsilon}-p^{e,n+1}_{K})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\\ &\quad+O(h)^{\min(\eta_1+\eta_6,2\eta_6,2\eta_7)}+O(\delta t_2)^{\min(\gamma_1+\gamma_6,2\gamma_6,2\gamma_7)}\end{aligned}$$

The analysis of the interface term in (25) with the transmission scheme (IS_2') is more involved. By (27) of lemma 4.5, we have the following additional term:

$$\begin{aligned}-\langle u^e,d^e_{\mathcal{E}}U^e_{\mathcal{E}}+d^m_{\mathcal{E}}(\mathcal{U}_{\mathcal{E}}-U^m_{\mathcal{E}})\rangle_{L^2(0,T;L^2(\Gamma))} &= \underbrace{-\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}d^e_\varepsilon(U^{e,n}_\varepsilon)^2\mathrm{meas}(\varepsilon)}_{=O(h^{2\eta_1}+\delta t_e^{2\gamma_1})}\\ &\underbrace{-\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}d^e_\varepsilon\frac{p^{e,n}_\varepsilon-p^{e,n}_{K_e(\varepsilon)}}{d^e_\varepsilon}U^{e,n}_\varepsilon\mathrm{meas}(\varepsilon)}_{\leqslant\frac{C_1'}{2}\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}\frac{(p^{e,n}_\varepsilon-p^{e,n}_{K_e(\varepsilon)})^2}{d^e_\varepsilon}\mathrm{meas}(\varepsilon)+O(h^{2\eta_1}+\delta t_e^{2\gamma_1})}\\ &\underbrace{-\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}d^m_\varepsilon U^{e,n}_\varepsilon(\mathcal{U}^n_\varepsilon-U^{m,n}_\varepsilon)\mathrm{meas}(\varepsilon)}_{=O(h^{2\min(\eta_1,\eta_7)}+\delta t_m^{2\min(\gamma_1,\gamma_7)})}\\ &\underbrace{-\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}d^m_\varepsilon\frac{p^{e,n}_\varepsilon-p^{e,n}_{K_e(\varepsilon)}}{d^e_\varepsilon}(\mathcal{U}^n_\varepsilon-U^{m,n}_\varepsilon)\mathrm{meas}(\varepsilon)}_{\leqslant\frac{C_2'}{2}\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}\frac{(p^{e,n}_\varepsilon-p^{e,n}_{K_e(\varepsilon)})^2}{d^e_\varepsilon}\mathrm{meas}(\varepsilon)+O(h^{2\min(\eta_1,\eta_7)}+\delta t_m^{2\min(\gamma_1,\gamma_7)})\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}\frac{(d^m_\varepsilon)^2}{d^e_\varepsilon}\mathrm{meas}(\varepsilon)}\end{aligned}$$

By assumption 3.6, the last term is under control since $\sum_{n=0}^{N_e-1}\delta t_e\sum_{\varepsilon\in\mathcal{E}_e(K)}\frac{(d^m_\varepsilon)^2}{d^e_\varepsilon}\mathrm{meas}(\varepsilon)\leqslant\alpha T\mathrm{meas}(\Omega_m)$.

Summing up all these estimates, we have:

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_i}(p^{i,N_i}_K)^2\mathrm{meas}(K)+\left(\frac{1}{2}-\frac{C_3}{4}\right)\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p^{i,n+1}_{K'}-p^{i,n+1}_K)^2}{d(x_K,x_{K'})}\mathrm{meas}([KK'])\\ &+\left(1-\frac{C_1}{2}-\frac{C_1'}{2}-\frac{C_2'}{2}-\frac{C_4'}{2}-\frac{C_6}{2}\right)\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p^{i,n+1}_\varepsilon-p^{i,n+1}_K)^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\end{aligned}$$

$$\leqslant \frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_i}(p_K^{i,0})^2\mathrm{meas}(K) + \frac{C_2+C_5}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}(p_K^{i,n+1})^2\mathrm{meas}(K)$$
$$+ \frac{C_4+C_7}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)}(p_\varepsilon^{i,n+1})^2\mathrm{meas}(\varepsilon)$$
$$+ O(h)^{\min((2\eta_j)_{j=1\dots7},\eta_1+\eta_6)} + O(\delta t_2)^{\min((2\gamma_j)_{j=1\dots7},\gamma_1+\gamma_6)}$$

where C_1' and C_2' are zero for the scheme (IS_1') In other words, we have:

$$\frac{1}{2}\sum_{i=1}^{2}\|p^{i,N_i}\|^2_{L^2(\Omega_i)} + \min\left(\frac{1}{2}-\frac{C_3}{4}, 1-\frac{C_1}{2}-\frac{C_1'}{2}-\frac{C_2'}{2}-\frac{C_4'}{2}-\frac{C_6}{2}\right)\sum_{i=1}^{2}|p^i|^2_{1,\mathcal{T}_i,\delta t_i}$$
$$\leqslant \frac{1}{2}\sum_{i=1}^{2}\|p^{i,0}\|^2_{L^2(\Omega_i)} + \frac{C_2+C_5}{2}\sum_{i=1}^{2}\|p^i\|^2_{L^2(0,T;L^2(\Omega_i))} + \frac{C_4+C_7}{2}\sum_{i=1}^{2}\|p^i\|^2_{L^2(0,T;L^2(\Gamma))}$$
$$+ O(h)^{\min((2\eta_j)_{j=1\dots7},\eta_1+\eta_6)} + O(\delta t_2)^{\min((2\gamma_j)_{j=1\dots7},\gamma_1+\gamma_6)}$$

We notice that $\min(2\eta_1, 2\eta_6, \eta_1+\eta_6) = 2\min(\eta_1,\eta_6)$. Moreover, using a result in [7] (discrete Poincaré inequality, lemma 3.1), we have:

$$\|p^i\|_{L^2(0,T;L^2(\Omega_i))} \leqslant \mathrm{diam}(\Omega_i)|p^i|_{1,\mathcal{T}_i,\delta t_i} \quad \forall i\in\{1,2\} \tag{28}$$

We also use the discrete trace estimate proved in [8] :

$$\|p^i\|^2_{L^2(0,T;L^2(\partial\Omega_i))} \leqslant C(\Omega_i)\left(\|p^i\|^2_{L^2(0,T;L^2(\Omega_i))} + |p^i|^2_{1,\mathcal{T}_i,\delta t_i}\right) \quad \forall i\in\{1,2\} \tag{29}$$

Taking small enough constants (independently of h) suffices to end the proof of Theorem 4.6. □

5 Well posedness

Theorem 5.1. *We assume that assumption 3.1 holds.*
Then, the problem defined by (16)-(15) *and interface scheme either* (IS_1) *or* (IS_2) *is well-posed.*

Proof. In both cases, we have a square linear system. It is thus sufficient to prove that the only solution with a zero right hand side and initial condition is zero, i.e.

$$U_\varepsilon^{i,n} = R_K^{i,n} = R_{KK'}^{i,n} = R_\varepsilon^{i,n} = F_K^{i,n} = p_K^{i,0} = \mathcal{P}_\varepsilon = \mathcal{U}_\varepsilon = 0 \quad \forall i\in\{1,2\} \quad \forall n\in\{0\dots N_i\} \quad \forall K\in\mathcal{T}_i\,.$$

Then, by Lemma 4.3 and Lemma 4.4 (resp. 4.5) for transmission scheme (IS_1) (resp. (IS_2)), we have

$$\frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_i}(p_K^{i,N_i})^2\mathrm{meas}(K)+\frac{1}{2}\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}\frac{(p_{K'}^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,x_{K'})}\mathrm{meas}([KK'])$$
$$+\sum_{i=1}^{2}\sum_{n=0}^{N_i-1}\delta t_i\sum_{K\in\mathcal{T}_i}\sum_{\varepsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)}\frac{(p_\varepsilon^{i,n+1}-p_K^{i,n+1})^2}{d(x_K,y_\varepsilon)}\mathrm{meas}(\varepsilon)\leqslant 0$$

that is for all $i=1,2$, $1\leq n\leq N_i$, $p^{i,n}$ is a constant. By equation (16) in every subdomain, we then have that the value of the constant is independent of n. Since the initial condition is zero, the constant is actually zero. □

6 Error estimate

Let p^1, p^2 be the solution to the continuous problem (13a)-(13d). We define the interpolation on the mesh $\mathcal{T}_1\cup\mathcal{T}_2$ at time t_n by:

$$\left.\begin{aligned}
&\tilde{p}_K^{i,n}=p^i(x_K,t_n)\quad\forall K\in\mathcal{T}_i\\
&\tilde{p}_\varepsilon^{i,n}=\frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}}\frac{1}{\mathrm{meas}(\varepsilon)}\int_\varepsilon p^i\quad\forall\varepsilon\in\mathcal{E}_i\\
&\tilde{p}_\varepsilon^{i,n}=0\quad\forall\varepsilon\in\mathcal{E}_{iD}\\
&\tilde{u}_\varepsilon^{i,n}=\frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}}\frac{1}{\mathrm{meas}(\varepsilon)}\int_\varepsilon\frac{\partial p^i}{\partial n_i}\quad\forall\varepsilon\in\mathcal{E}_i\cup\mathcal{E}_{iD}
\end{aligned}\right\}\quad\forall n\in\{0,\dots,N_i\}\quad\forall i\in\{1,2\}$$

We have to estimate the error terms $e_K^i=p_K^i-\tilde{p}_K^i$, $e_\varepsilon^i=p_\varepsilon^i-\tilde{p}_\varepsilon^i$ and $q_\varepsilon^i=u_\varepsilon^i-\tilde{u}_\varepsilon^i$.

Theorem 6.1. *We suppose that the solution has the following regularity:*

$$p\in C^1(0,T;C^2(\bar{\Omega}))\tag{30}$$

and that the numerical right hand side is such that:

$$\frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}}\frac{1}{meas(K)}\int_K\left(f_K^{i,n}-f\right)=O(diam(K)+\delta t_i)\quad\forall n\in\{0,\dots,N_i\}\quad\forall i\in\{1,2\}\tag{31}$$

We assume that Assumptions 3.1, 3.7 hold and that transmission scheme (IS_1) *is used. Then, we have the following estimate:*

$$\sum_{i=1}^{2}|e^i|_{1,\mathcal{T}_i,\delta t_i}+\sum_{i=1}^{2}\|e^{i,Ni}\|_{L^2(\Omega_i)}=O(h+\delta t)$$

where $\delta t = \max(\delta t_1, \delta t_2)$.
The same estimate holds for transmission scheme (IS$_2$) *if in addition Assumption 3.6 holds.*

Proof. It is easy to check that the errors e_K, e_ε et q_ε satisfy (23)-(24) with error terms defined by

- $F_K^{i,n} = \dfrac{1}{\delta t_i} \displaystyle\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \dfrac{1}{\mathrm{meas}(K)} \int_K \left(f_K^{i,n} - f \right)$
- $R_K^{i,n} = \dfrac{1}{\delta t_i} \displaystyle\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \dfrac{1}{\mathrm{meas}(K)} \int_K \dfrac{\partial p^i}{\partial t} - \dfrac{\tilde{p}_K^{i,n} - \tilde{p}_K^{i,n-1}}{\delta t}$
- $R_{KK'}^{i,n} = \dfrac{1}{\delta t_i} \displaystyle\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \dfrac{1}{\mathrm{meas}([KK'])} \int_{[KK']} \dfrac{\partial p^i}{\partial n_i} - \dfrac{p^i(x_{K'}, t_n) - p^i(x_K, t_n)}{d(x_K, x_{K'})}$
- $R_\varepsilon^{i,n} = \dfrac{1}{\delta t_i} \displaystyle\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \dfrac{1}{\mathrm{meas}(\varepsilon)} \int_\varepsilon \dfrac{\partial p^i}{\partial n_i} - \tilde{u}_\varepsilon^{i,n}$
- $U_\varepsilon^{i,n} = -(\tilde{u}_\varepsilon^{i,n} - \dfrac{\tilde{p}_\varepsilon^{i,n} - \tilde{p}_{K_i(\varepsilon)}}{d(x_{K_i(\varepsilon)}, y_\varepsilon)})$
- $\mathcal{U}_{\varepsilon,T} = \mathcal{Q}_m(-\tilde{u}^e_{\varepsilon,T}) - \tilde{u}^m_{\varepsilon,T}$
- For the scheme (IS$_1$): $\mathcal{P}_{\varepsilon,T} = \mathcal{Q}_e(\tilde{p}^m_{\varepsilon,T}) - \tilde{p}^e_{\varepsilon,T}$
- For the scheme (IS$_2$): $\mathcal{P}_{\varepsilon,T} = \mathcal{Q}_e\left(\tilde{p}^m_{K_m(\varepsilon)}\right) - \tilde{p}^e_{K_e(\varepsilon)} - (d^m_\varepsilon + d^e_\varepsilon)\tilde{u}^e_{\varepsilon,T}$

The derivation of the formula for $\mathcal{P}_{\varepsilon,T}$ and $\mathcal{U}_{\varepsilon,T}$ are made explicit in the sequel when these terms are estimated. Let us remark that we have by construction $e_K^{i,0} = 0$ for all $K \in \mathcal{T}_i$, $i = 1, 2$.

By assumption and by using Taylor expansion, it is classical to check that the error terms R_K, $R_{KK'}$ and F_K satisfy Assumption 4.2 with $\eta_i = \gamma_i = 1$ for $i = 2, 3, 5$. As regards the term U_ε, we proceed as in [1]. From Assumption (30) on the regularity of the solution,

we have:

$$
\begin{aligned}
U_\varepsilon^{i,n} =& \frac{\frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^i - p^i(x_{K_i(\varepsilon)}, t_n)}{d(x_{K_i(\varepsilon)}, y_\varepsilon)} - \frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^i}{\partial n_i} \\
=& \frac{\frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^i - p^i(y_\varepsilon, t_n)}{d(x_{K_i(\varepsilon)}, y_\varepsilon)} + \left(\frac{p^i(y_\varepsilon, t_n) - p^i(x_{K_i(\varepsilon)}, t_n)}{d(x_{K_i(\varepsilon)}, y_\varepsilon)} - \frac{\partial p^i}{\partial n_i}(y_\varepsilon, t_n)\right) \\
&+ \left(\frac{\partial p^i}{\partial n_i}(y_\varepsilon, t_n) - \frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^i}{\partial n_i}\right) \\
=& \frac{O(\text{diam}(\varepsilon))^2}{d(x_{K_i(\varepsilon)}, y_\varepsilon)} + O(d(x_{K_i(\varepsilon)}, y_\varepsilon)) + O(\text{diam}(\varepsilon)) = O(h) \text{ by assumption 3.7}
\end{aligned}
$$

thus, $\eta_1 = 1$ and $\gamma_1 \geq 1$. Since,

$$
R_\varepsilon^{i,n} = \frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \left(\frac{\partial p^i}{\partial n_i} - \frac{1}{\delta t_i}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^i}{\partial n_i}\right) = 0\,,
$$

we have $\eta_4, \gamma_4 \geq 1$.
We now consider the non classical consistency error terms $\mathcal{P}_\varepsilon$ et $\mathcal{U}_\varepsilon$. For the transmission condition (IS_1) we have:

$$
\begin{aligned}
q_\varepsilon^m =& u_\varepsilon^m - \tilde{u}_\varepsilon^m = \mathcal{Q}_m(-u_\varepsilon^e) - \tilde{u}_\varepsilon^m \\
=& \mathcal{Q}_m(-q_\varepsilon^e - \tilde{u}_\varepsilon^e) - \tilde{u}_\varepsilon^m = \mathcal{Q}_m(-q_\varepsilon^e) + \underbrace{\mathcal{Q}_m(-\tilde{u}_\varepsilon^e) - \tilde{u}_\varepsilon^m}_{\mathcal{U}_\varepsilon}
\end{aligned}
$$

The error on the transmission condition on the interface reads:

$$
\begin{aligned}
e_\varepsilon^e =& p_\varepsilon^e - \tilde{p}_\varepsilon^e = \mathcal{Q}_e(p_\varepsilon^m) - \tilde{p}_\varepsilon^e \\
=& \mathcal{Q}_e(e_\varepsilon^m + \tilde{p}_\varepsilon^m) - \tilde{p}_\varepsilon^e = \mathcal{Q}_e(e_\varepsilon^m) + \underbrace{\mathcal{Q}_e(\tilde{p}_\varepsilon^m) - \tilde{p}_\varepsilon^e}_{\mathcal{P}_\varepsilon}
\end{aligned}
$$

If $(m, e) = (1, 2)$, we get :

$$
\begin{aligned}
\mathcal{P}_\varepsilon^n =& \left[\mathcal{Q}_2(\tilde{p}^1_{\varepsilon,T})\right]^n - \tilde{p}_\varepsilon^{2,n} \\
=& \frac{\delta t_1}{\delta t_2}\sum_{k=1}^{K}\frac{1}{\delta t_1}\int_{t_{n,k-1/2}}^{t_{n,k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^1 - \frac{1}{\delta t_2}\int_{t_{n-1/2}}^{t_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^2 = 0 \\
\mathcal{U}_\varepsilon^{n,k} =& \left[\mathcal{Q}_1(-\tilde{u}_\varepsilon^2)^{n,k} - \tilde{u}_\varepsilon^1\right]^{n,k} \\
=& -\tilde{u}_\varepsilon^{2,n} - \tilde{u}_\varepsilon^{1,n,k} \\
=& -\frac{1}{\delta t_2}\int_{t_{n-1/2}}^{t_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^2}{\partial n_2} - \frac{1}{\delta t_1}\int_{t_n^{k-1/2}}^{t_n^{k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^1}{\partial n_1} = O(\delta t_2)
\end{aligned}
$$

If $(m,e)=(2,1)$, we have

$$\begin{aligned}
\mathcal{P}_\varepsilon^{n,k} &= \left[\mathcal{Q}_1(\tilde{p}^2_{\varepsilon,T})\right]^{n,k} - \tilde{p}^{1,n,k}_\varepsilon \\
&= \tilde{p}^{2,n}_\varepsilon - \tilde{p}^{1,n,k}_\varepsilon \\
&= \frac{1}{\delta t_2}\int_{t_{n-1/2}}^{t_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^2 - \frac{1}{\delta t_1}\int_{t_n^{k-1/2}}^{t_n^{k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon p^1 = O(\delta t_2) \\
\mathcal{U}_\varepsilon^n &= \left[\mathcal{Q}_2(-\tilde{u}^1_\varepsilon) - \tilde{u}^2_\varepsilon\right]^n \\
&= -\frac{\delta t_1}{\delta t_2}\sum_{k=1}^{\mathcal{K}} \tilde{u}^{1,n,k}_\varepsilon - \tilde{u}^{2,n}_\varepsilon \\
&= \frac{\delta t_1}{\delta t_2}\sum_{k=1}^{\mathcal{K}} \frac{1}{\delta t_1}\int_{t_n^{k-1/2}}^{t_n^{k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^1}{\partial n_1} - \frac{1}{\delta t_2}\int_{t_{n-1/2}}^{t_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^2}{\partial n_2} = 0
\end{aligned}$$

We consider now transmission scheme (IS_2): $\mathcal{U}_\varepsilon$ is left unchanged and $\mathcal{P}_\varepsilon$ now reads :

$$\begin{aligned}
e^e_\varepsilon + d^m_\varepsilon q^e_\varepsilon &= p^e_\varepsilon - \tilde{p}^e_\varepsilon + d^m_\varepsilon(u^e_\varepsilon - \tilde{u}^e_\varepsilon) \\
&= \mathcal{Q}_e(p^m_\varepsilon - d^m_\varepsilon u^m_\varepsilon) - \tilde{p}^e_\varepsilon - d^m_\varepsilon \tilde{u}^e_\varepsilon \\
&= \mathcal{Q}_e(e^m_\varepsilon + \tilde{p}^m_\varepsilon - d^m_\varepsilon q^m_\varepsilon - d^m_\varepsilon \tilde{u}^m_\varepsilon) - \tilde{p}^e_\varepsilon - d^m_\varepsilon \tilde{u}^e_\varepsilon \\
&= \mathcal{Q}_e(e^m_\varepsilon - d^m_\varepsilon q^m_\varepsilon + d^m_\varepsilon U^m_\varepsilon) - d^e_\varepsilon U^e_\varepsilon + \underbrace{\mathcal{Q}_e\left(\tilde{p}^m_{K_m(\varepsilon)}\right) - \tilde{p}^e_{K_e(\varepsilon)} - (d^m_\varepsilon + d^e_\varepsilon)\tilde{u}^e_\varepsilon}_{\mathcal{P}_\varepsilon}
\end{aligned}$$

If $(m,e)=(1,2)$, we have

$$\begin{aligned}
\mathcal{P}^n_\varepsilon &= \left[\mathcal{Q}_2\left(\tilde{p}^1_{K_1(\varepsilon)}\right)\right]^n - \tilde{p}^{2,n}_{K_2(\varepsilon)} - (d^1_\varepsilon + d^2_\varepsilon)\tilde{u}^{2,n}_\varepsilon \\
&= \frac{\delta t_1}{\delta t_2}\sum_{k=1}^{\mathcal{K}} p^1(x_{K_1(\varepsilon)}, t^k_n) - p^2(x_{K_2(\varepsilon)}, t_n) - (d^1_\varepsilon + d^2_\varepsilon)\frac{1}{\delta t_2}\int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \frac{\partial p^2}{\partial n_2} \\
&= \frac{1}{\delta t_2}\sum_{k=1}^{\mathcal{K}} \int_{t_n^{k-1/2}}^{t_n^{k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \left(p^1(x_{K_1(\varepsilon)}, t^k_n) - p^2(x_{K_2(\varepsilon)}, t_n) - d(x_{K_1(\varepsilon)}, x_{K_2(\varepsilon)})\frac{\partial p^2}{\partial n_2}\right) \\
&= O(h+\delta t)
\end{aligned}$$

If $(m,e)=(2,1)$, $\mathcal{P}_\varepsilon$ reads :

$$\begin{aligned}
\mathcal{P}^{n,k}_\varepsilon &= \left[\mathcal{Q}_1\left(\tilde{p}^2_{K_2(\varepsilon)}\right)\right]^{n,k} - \tilde{p}^{1,n,k}_{K_1(\varepsilon)} - (d^1_\varepsilon + d^2_\varepsilon)\tilde{u}^{1,n,k}_\varepsilon \\
&= \frac{1}{\delta t_1}\int_{t_n^{k-1/2}}^{t_n^{k+1/2}} \frac{1}{\text{meas}(\varepsilon)}\int_\varepsilon \left(p^2(x_{K_2(\varepsilon)}, t_n) - p^1(x_{K_1(\varepsilon)}, t^k_n) - d(x_{K_1(\varepsilon)}, x_{K_2(\varepsilon)})\frac{\partial p^1}{\partial n_1}\right) \\
&= O(h+\delta t)
\end{aligned}$$

For all cases, we have thus $\eta_6, \eta_7, \gamma_6, \gamma_7 \geq 1$. Then the error estimate follows from Theorem 4.6. $\square$

7 Numerical results

In this part, we illustrate the method with a parabolic equation coming from a previous article of Ewing and Lazarov [5]. We consider the (IS_2) interface conditions, i.e. equation (10) in one dimension, which are more natural. We solve the following model problem :

$$\frac{\partial}{\partial t}p(x,t) - \frac{\partial^2 p}{\partial x^2}(x,t) = f(x,t) \quad \forall t \in [0.,0.1] \quad \forall x \in [0.,1.] \tag{32}$$

$$p(x,t) = 0, \forall x \in \partial\Omega \tag{33}$$

$$p(x,0) = 0 \tag{34}$$

The following function is used as an exact solution :

$$p(x,t) = exp(20(t-t^2) - 37x^2 + 8x - 1)$$

This function represents a bump with a maximum value near the position $x = 0.15$.

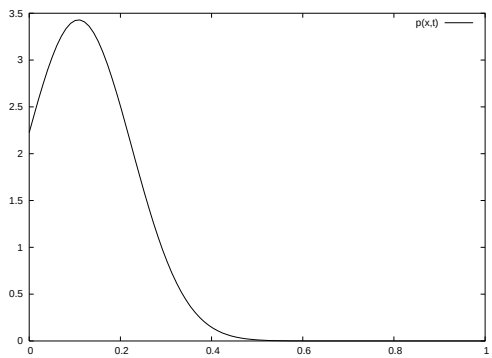

Figure 4: Exact solution for t=0.1

In the interval $[0.5, 1.]$, the function is close to 0. In this interval, the function changes negligibly in time. In contrast, the function changes rapidly in time in the interval $[0., 0.5]$ and simulates a local behavior.

We use two different time step sizes :

- a fine time step $\delta t_1 = 0.002$ in the subdomain $\Omega_1 = [0., 0.25]$, discretized with a fine grid $\delta x_1 = 0.01$
- a coarse time step $\delta t_2 = 0.02$ in the subdomain $\Omega_2 = [0.25, 1.]$, discretized with a coarse grid $\delta x_2 = 0.05$

The interface is placed at $x = 0.25$. This is a worst case since the domain with local refinement only partially covers the interval $[0., 0.25]$ where the solution changes quickly.

We consider two cases. The first one (coarse master) is when the coarse domain enforces the Dirichlet condition, see equation (10). The second one (fine master) is when the refined domain enforces the Dirichlet condition, see equation (11). We make a comparison with the algorithm given by Mlacnik and Heinemann [9, 10]. In the following pictures, we plot the evolution of the errors in space and the time evolution of the L^2 norm of the error between the exact solution, the two local time step methods and the solution with the fine or coarse time step on the whole domain.

At each coarse time step, we solve the set of discretized equations using the iterative algorithm explained in section 2.2 with the stopping criterium $\varepsilon = 10^{-5}$. In the following figures, we plot the error between these two solutions and the exact solution. For completeness, we also plot the error for a computation with either the coarse or the fine time step on the whole domain. The number of iterations needed to reach the convergence is quite small; it is about 6 for the fine master method and about 8 for the coarse master method.

We notice that for both cases, the error is significantly smaller than the one of the coarse time step. Morever, for the fine master method, see equation(11), the error is close to the fine time step error in the refined zone, see figure 5.

As explained in section 2.2, it is not necessary to iterate until convergence the algorithm to obtain a conservative method. In figure 6, we plot the error after only one iteration of the corrector stage. As expected,the errors are larger than with the converged solutions. Let us recall, that the method proposed by Mlacnik corresponds to the fine master curve in figure 6.

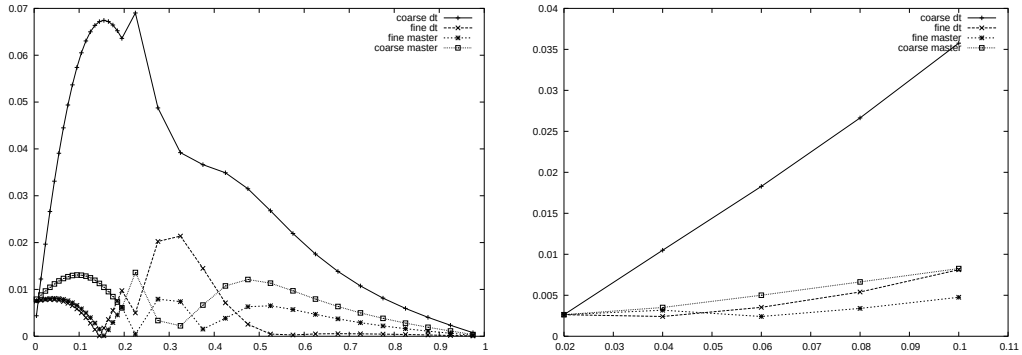

Figure 5: After convergence, on left : error in space with the exact solution at time $t = 0.1$, on right : time evolution of the L^2 norm of the error.

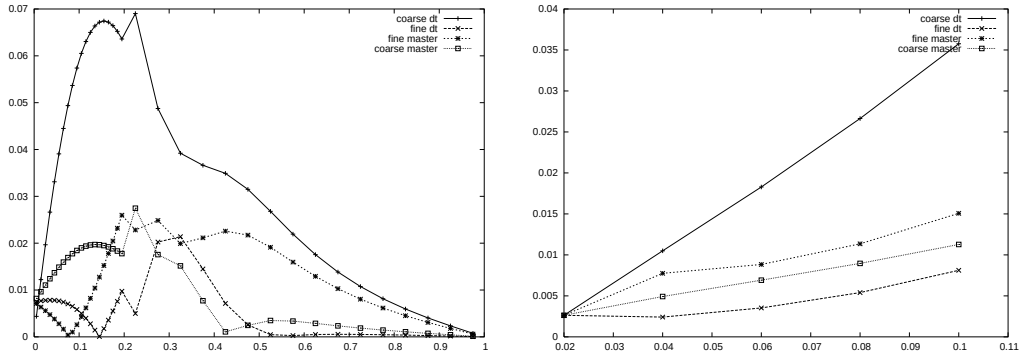

Figure 6: After one iteration, on left : error in space with the exact solution at time $t = 0.1$, on right : time evolution of the L^2 norm of the error.

8 Conclusion

We have proposed a local time step strategy for solving problems on grids with different time steps in different regions. We have analyzed two schemes: (IS_1) and (IS_2). In (IS_1), the coupling involves additional interface unknowns. Scheme (IS_2) is written only in terms of "classical" finite volume unknowns. Both schemes are conservative, of order one in space and time. The assumptions are more restrictive for (IS_2) than for (IS_1). We have presented an iterative solution method for solving the composite grid system. Its main feature is that at every stage, conservativity is ensured. Numerical tests on a toy problem confirm the capabilities of the method. The scheme is being implemented in a multiphase three dimensional simulation code.

:

References

[1] Yves Achdou, Caroline Japhet, Yvon Maday, and Frédéric Nataf. A new cement to glue non-conforming grids with Robin interface conditions: the finite volume case. *Numer. Math.*, 92(4):593–620, 2002.

[2] Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[3] C.N. Dawson, Q. Du, and T.F. Dupont. A finite difference domaine decomposition algorithm for numerical solution of the heat equation. *Math. Comp.*, 57(195):63–71, 1991.

[4] F.X. Deimbacher, F. Komlosi, and Z.E. Heinemann. Fundamental concepts and potential applications of the windowing technique in reservoir simulation. In *Paper SPE 29851 presented at the SPE Middle East Oil Show, Bahrain*, paper SPE 29851, March 1995.

[5] R.E. Ewing and R.D. Lazarov. Approximation of parabolic problems on grids locally refined in time and space. *Applied Numerical Mathematics*, 14(1-3):199–211, 1994.

[6] R.E. Ewing, R.D. Lazarov, and P.S. Vassilevski. Finite difference schemes on grids with local refinement in time and space for parabolic problems I. Derivation, stability, and error analysis. *Computing*, 45(3):193–215, 1990.

[7] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. Ciarlet and J.-L. Lions, editors, *Handbook of numerical analysis*, volume 7, pages 713–1020. North Holland edition, 2000.

[8] T. Gallouët, R. Herbin, and M.H. Vignal. Error estimates on the approximate finite volume solution of convection diffusion equations with general boundary conditions. *J. Numer. Anal.*, 37:1935–1972, 2000.

[9] M.J. Mlacnik. *Using well windows in full field reservoir simulation.* PhD thesis, University of Leoben, Austria, 2002.

[10] M.J. Mlacnik and Z.E. Heinemann. Using well windows in full field reservoir simulation. In *Proc. of the SPE Reservoir Simulation Symposium, Houston, TX, U.S.A.*, paper SPE 66371, February 2001.

[11] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. OXFORD University Press, 1999.

[12] G. I. Shishkin and P. N. Vabishchevich. Interpolation finite difference schemes on grids locally refined in time. *Comput. Methods Appl. Mech. Engrg.*, 190(8-10):889–901, 2000.