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We denote by $x$ a real variable and by $n$ a positive integer variable. The reference measure on the real line $\mathbb{R}$ is the Lebesgue measure. In this note we will use only basic properties of the Lebesgue measure and integral on $\mathbb{R}$.

It is well known that the fact that a function tends to zero at infinity is a condition neither necessary nor sufficient for this function to be integrable. However, we have the following result.

**Theorem 1.** Let $f$ be an integrable function on the real line $\mathbb{R}$. For almost all $x \in \mathbb{R}$, we have

$$(1) \quad \lim_{n \to \infty} f(nx) = 0.$$ 

**Remark 1.** It is too much hope in Theorem 1 for a result for all $x$ because we consider an integrable function $f$, which can take arbitrary values on a set of zero measure. Even if we consider only continuous functions, the result does not hold for all $x$. Indeed a classical result, using a Baire category argument, tells us that if $f$ is a continuous function on $\mathbb{R}$ such that for all nonzero $x$, $\lim_{n \to \infty} f(nx) = 0$, then $\lim_{x \to \pm \infty} f(x) = 0$. Thus for a continuous integrable function $f$ which does not tend to zero at infinity, property (1) is true for almost all $x$ and not for all $x$.

**Remark 2.** Let $f$ be an integrable and nonnegative function on $\mathbb{R}$. We have $\int f(nx) \, dx = \frac{1}{n} \int f(x) \, dx$. Hence for any nonnegative real sequence $(\varepsilon_n)$ such that $\sum_n \varepsilon_n / n < +\infty$, we have

$$\sum_n \int \varepsilon_n f(nx) \, dx < +\infty,$$

and the monotone convergence theorem (or Fubini’s theorem) ensures that the function $x \mapsto \sum_n \varepsilon_n f(nx)$ is integrable, hence almost everywhere finite. In particular, for almost all $x$, we have $\lim_{n \to \infty} \varepsilon_n f(nx) = 0$. This argument is not sufficient to prove Theorem 1.

Now we will state that, in a sense, Theorem 1 gives an optimal result. The strength of the following theorem lies in the fact that the sequence $(a_n)$ can tend to infinity arbitrarily slowly.

**Theorem 2.** Let $(a_n)$ be a real sequence which tends to $+\infty$. There exists a continuous and integrable function $f$ on $\mathbb{R}$ such that, for almost all $x$,

$$\lim_{n \to \infty} \sup_{n \to \infty} a_n f(nx) = +\infty.$$
Moreover, there exists an integrable function $f$ on $\mathbb{R}$ such that, for all $x$,
\[
\limsup_{n \to \infty} a_n f(nx) = +\infty.
\]

**Question.** Under the hypothesis of Theorem 2, does there exist a continuous and integrable $f$ such that, for all $x$, $\limsup_{n \to \infty} a_n f(nx) = +\infty$? We do not know the answer to this question, and we propose it to the reader. However, the next remark shows that the answer is positive under a slightly more demanding hypothesis.

**Remark 3.** If the sequence $(a_n)$ is nondecreasing and satisfies $\sum_n \frac{1}{n a_n} < +\infty$, then there exists a continuous and integrable function $f$ on $\mathbb{R}$ such that for all $x$, $\limsup_{n \to \infty} a_n f(nx) = +\infty$.

**Remark 4.** In Theorem 2 we cannot replace the hypothesis $\lim_{n \to \infty} a_n = +\infty$ by $\limsup_{n \to \infty} a_n = +\infty$. Indeed, by a simple change of variable we can deduce from Theorem 1 the following result: for all integrable function $f$ on $\mathbb{R}$, $\lim_{n \to \infty} n f(n^2 x) = 0$ for almost all $x$.

(Apply Theorem 1 to the function $x \mapsto x f(x^2)$.) Thus the conclusion of Theorem 2 is false for the sequence $(a_n)$ defined by $a_n = \begin{cases} \sqrt{n} & \text{if } n \text{ is a square of integer}, \\ 0 & \text{if not}. \end{cases}$

In the remainder of this note, we give proofs of the two theorems and of Remark 3.

**Proof of Theorem 1.** The function $f$ is integrable on $\mathbb{R}$. Let us fix $\varepsilon > 0$ and denote by $E$ the set of points $x > 0$ such that $|f(x)| \geq \varepsilon$. We know that $E$ has finite measure. We are going to show that, for almost all $x \in [0,1]$, we have $n x \in E$ for only finitely many $n$’s. (If $A$ is a measurable subset of $\mathbb{R}$, we denote by $|A|$ its Lebesgue measure.)

For each integer $m \geq 1$, let $E_m := E \cap (m - 1, m]$. Let us fix $a \in (0, 1)$. For each integer $n \geq 1$, we consider the set
\[
F_n := \left( \frac{1}{n} E \right) \cap [a, 1) = \left( \frac{1}{n} \bigcup_{m \geq 1} E_m \right) \cap [a, 1) = \frac{1}{n} \bigcup_{m \geq 1} (E_m \cap [na, n)) .
\]

We have
\[
\sum_{n=1}^{+\infty} |F_n| = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{n} |E_m \cap [na, n)| .
\]

In this doubly indexed sum of positive numbers, we can invert the order of summation. Moreover, noticing that $E_m \cap [na, n) = \emptyset$ if $n > m/a$ or $n \leq m - 1$, we obtain
\[
\sum_{n=1}^{+\infty} |F_n| = \sum_{m=1}^{+\infty} \sum_{n=m}^{+\infty} \frac{1}{n} |E_m \cap [na, n)| \leq \sum_{m=1}^{+\infty} |E_m| \sum_{n=m}^{+\infty} \frac{[m/a]}{n} .
\]
By comparison of the discrete sum with an integral, we see that, for all \( m \geq 1 \),
\[
\sum_{n=m}^{\infty} \frac{1}{n} \leq (1 - \ln a).
\]
Thus we have
\[
\sum_{n=1}^{\infty} |F_n| \leq (1 - \ln a) \sum_{m=1}^{\infty} |E_m| = (1 - \ln a)|E| < +\infty.
\]
This implies that almost every \( x \) belongs to only finitely many sets \( F_n \). (This statement is the Borel-Cantelli lemma, which has a one line proof:
\[
\sum \mathbb{1}_{F_n}(x) \, dx = \sum \mathbb{1}_{F_n}(x) < +\infty.
\]

Returning to the definition of \( F_n \), we conclude that, for almost all \( x \in [a, 1] \), for all large enough \( n \), \( x \notin F_n \), i.e. \( nx \notin E \). Since \( a \) is arbitrary, we have in fact: for almost all \( x \in [0, 1] \), for all large enough \( n \), \( nx \notin E \).
We have proved that, for all \( \varepsilon > 0 \), for almost all \( x \in [0, 1] \), for all large enough \( n \), \( |f(nx)| \leq \varepsilon \). Since we have to consider only countably many \( \varepsilon \)'s, we can invert for all \( \varepsilon > 0 \) and for almost all \( x \in [0, 1] \). We conclude that, for almost all \( x \in [0, 1] \), \( \lim_{n \to \infty} f(nx) = 0 \). It is immediate, by a linear change of variable (for example), that this result extends to almost all \( x \in \mathbb{R} \).

**Proof of Theorem 2.** We will utilize the following theorem, a fundamental result in the metric theory of Diophantine approximation [4], Theorem 32.

**Khinchin’s Theorem.** Let \( (b_n) \) be a sequence of positive real numbers such that the sequence \( (nb_n) \) is nonincreasing and the series \( \sum_n b_n \) diverges. For almost all real numbers \( x \), there are infinitely many integers \( n \) such that \( \text{dist}(nx, \mathbb{Z}) < b_n \).

We will also make use of the following lemma, which will be proved in the sequel.

**Lemma 1.** Let \( (c_n) \) be a sequence of nonnegative real numbers going to zero. There exists a sequence of positive real numbers \( (b_n) \) such that the sequence \( (nb_n) \) is nonincreasing, \( \sum_n b_n = +\infty \), and \( \sum_n b_n c_n < +\infty \).

Let us prove Theorem 2.

Replacing if necessary \( a_n \) by \( \inf_{k \geq n} a_k \), we can suppose that the sequence \( (a_n) \) is nondecreasing. Applying the preceding lemma to the the sequence \( c_n = 1/\sqrt{n} \), we obtain a sequence \( (b_n) \) such that the sequence \( (nb_n) \) is nonincreasing, \( \sum_n b_n = +\infty \), and \( \sum_n b_n/\sqrt{n} < +\infty \). The sequence \( (b_n) \) tends to zero, and we can impose the additional requirement that \( b_n < 1/2 \) for all \( n \).

We consider the function \( f_1 \) defined on \( \mathbb{R} \) by
\[
f_1(x) = \begin{cases} 1/\sqrt{n} & \text{if } |x-n| \leq b_n \text{ for an integer } n \geq 1, \\ 0 & \text{if not.} \end{cases}
\]
This function is integrable, due to the last condition imposed on \( (b_n) \).

By Khinchin’s theorem, for almost all \( x > 0 \), there exist pairs of positive integers \( (n, k(n)) \), with arbitrarily large \( n \), such that
\[
|nx - k(n)| \leq b_n.
\]
Let us consider one fixed such \( x \) in the interval \((0, 1)\). We have \( \lim_{n \to \infty} k(n) = +\infty \) and, since \( \lim_{n \to +\infty} b_n = 0 \), we have \( k(n) \leq n \) for all large enough \( n \). For such
procedure extends this property to almost all real numbers. We consider the function 
example, we can choose an increasing sequence of numbers (\(a_n\)) such that the sequence 
\(\sum_{n=0}^{\infty} g_n(x)\). This function \(g\) is nonnegative and integrable on \(\mathbb{R}\). It is locally a step function. For almost all \(x\) between 0 and 1, we have

\[
\limsup_{n \to \infty} a_n f_1(n x) = +\infty.
\]

From this, it is not difficult to construct a continuous and integrable function \(f\) on \(\mathbb{R}\) such that, for all \(m > 0\), there exists \(A_m > 0\) with \(f \geq f_m\) on \([A_m, +\infty)\). For example, we can choose an increasing sequence of numbers \((A_m)\) such that

\[
\int_{A_m}^{+\infty} f_1(x) + f_2(x) + \cdots + f_m(x) \, dx \leq \frac{1}{m^2};
\]

then we define \(g = f_1 + f_2 + \cdots + f_m\) on the interval \([A_m, A_{m+1})\). Since

\[
\sum_{m} \int_{A_m}^{A_{m+1}} f_1(x) + f_2(x) + \cdots + f_m(x) \, dx < \infty,
\]

this function \(g\) is integrable. Then we just have to find a continuous and integrable function \(f\) which dominates \(g\); this can be achieved since the function \(g\) is locally a step function: choose \(f\) to be zero on \((-\infty, 0]\) and continuous on \(\mathbb{R}\) such that \(g \leq f\) and, for all \(m > 0\), \(\int_{-\infty}^{m} f(x) - g(x) \, dx \leq 1/m^2\), so that \(\int_{0}^{+\infty} f(x) - g(x) \, dx < +\infty\).

For almost all \(x \geq 0\), we have \(\limsup_{n \to \infty} a_n f_1(n x) = +\infty\). A symmetrization procedure extends this property to almost all real numbers.

The first part of Theorem 3 is proved. The second part is a direct consequence. We consider the function \(f\) constructed above, and we denote by \(F\) the set of \(x\) such that the sequence \((a_n f_1(n x))\) is bounded. The set \(\{n x \mid x \in F, n \in \mathbb{N}\}\) has zero measure. We modify the function \(f\) on this set, choosing for example the value 1. The new function is integrable and satisfies, for all \(x\), \(\limsup_{n \to \infty} a_n f(n x) = +\infty\). \(\square\)

**Proof of Lemma 4.** The sequence \((e_n)\) is given, and it goes to zero. We will construct by induction an increasing sequence of integers \((n_k)\) and a nonincreasing sequence of positive numbers \((d_k)\), and we will define \(b_n = d_k / n\) for \(n_{k-1} \leq n < n_k\). The numbers \(d_k\) will be chosen so that \(\sum_{i=n_{k-1}}^{n_k-1} b_i = 1\); thus we require that

\[
d_k := \left(\sum_{i=n_{k-1}}^{n_k-1} \frac{1}{i}\right)^{-1}.
\]
We start from $n_0 = 1$, and then we choose $n_1 > n_0$ such that, for all $n \geq n_1$, $|c_n| \leq 1/2$. In the next step, we choose $n_2 > n_1$ such that $d_2 \leq d_1$ and, for all $n \geq n_2$, $|c_n| \leq 1/4$. More generally, if $n_1, n_2, \ldots, n_{k-1}$ have been constructed, we choose $n_k > n_{k-1}$ such that $d_k \leq d_{k-1}$ and, for all $n \geq n_k$, $|c_n| \leq 2^{-k}$. (Of course, this is possible because $\lim_{n \to +\infty} \left( \sum_{i=n_{k-1}}^{n} \frac{1}{n} \right) = 0$.)

This defines the sequence $(b_n)$ by blocks. The sequence $(nb_n)$ is nonincreasing and, for all $k \geq 1$, we have

$$\sum_{i=n_{k-1}}^{n_k-1} b_i = 1 \quad \text{and} \quad \sum_{i=n_{k-1}}^{n_k-1} b_i c_i \leq 2^{1-k}.$$ 

This guarantees that $\sum_n b_n = +\infty$ and $\sum_n b_n c_n < +\infty$. The lemma is proved. □

About Remark 3. Dirichlet’s lemma in Diophantine approximation (based on the pigeon-hole principle) concerns the particular case $b_n = 1/n$ in Khinchin’s theorem and it gives a result for all $x$.

Lemma 2 (Dirichlet’s Lemma). For all real numbers $x$, there exist infinitely many integers $n$ such that $\text{dist}(nx, \mathbb{Z}) \leq \frac{1}{n}$.

Now, we justify Remark 3. We consider a nondecreasing sequence of positive real numbers $(a_n)$ such that

$$\sum_n \frac{1}{nad_n} < +\infty.$$ 

We claim that there exists a sequence of positive real numbers $(b_n)$ such that

$$b_n a_n \to +\infty \quad \text{and} \quad \sum_n \frac{b_n}{n} < +\infty.$$ 

Here is a proof of this claim: for each $k \geq 1$, there exists $n(k)$ such that

$$\sum_{n \geq n(k)} \frac{1}{nad_n} \leq \frac{1}{k^2}.$$ 

We have

$$\sum_n \text{card}\{k \mid n(k) \leq n\} \frac{1}{nad_n} = \sum_{k \geq 1} \sum_{n \geq n(k)} \frac{1}{nad_n} < +\infty,$$

and we can define $b_n := \text{card}\{k \mid n(k) \leq n\}/a_n$.

Given this sequence $(b_n)$, we consider the function $f$ defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} b_k & \text{if } |x - k| \leq 1/k, \ k \text{ an integer}, \ k \geq 2, \\ 0 & \text{if not}. \end{cases}$$

This function is integrable.

Using Dirichlet’s lemma, we have the following: for each fixed $x$ in $(0, 1)$, there exist pairs of positive integers $(n, k(n))$, with $n$ arbitrarily large, such that $|nx - k(n)| \leq 1/n$. We have $\lim_{n \to +\infty} k(n) = +\infty$ and, for all large enough $n$, $k(n) \leq n$. Hence there exist infinitely many $n$’s such that

$$|nx - k(n)| \leq \frac{1}{k(n)}$$ 

and so $f(nx) = b_{k(n)}$. 

For such an \( n \), we have
\[
a_n f(nx) = a_n b_{k(n)} \geq a_{k(n)} b_{k(n)}.
\]
(We used here the fact that the sequence \( (a_n) \) is nondecreasing.) This proves that
\[
\limsup_{n \to \infty} a_n f(nx) = +\infty.
\]
This result obtained for all numbers \( x \) between 0 and 1 extends to all real numbers by the same argument as the one used in the proof of Theorem 2. We can also replace the local step function by a continuous one as we did before.

Theorem 1 answers a question asked by Aris Danilidis.

References


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