Symmetry groups in nonlinear elasticity: An exercise in vintage mathematics

Annie Raoult

SYMMETRY GROUPS IN NONLINEAR ELASTICITY:
An exercise in vintage mathematics

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Abstract. This manuscript aims at characterizing energy densities and constitutive laws of transversely isotropic materials, orthotropic elastic materials and materials with non orthogonal families of fibers. It makes explicit references to results that are scattered over the literature and, although said to be well-known, are not always easy to locate. Direct proofs that are thought to be new and simplified expressions of constitutive laws for materials with two preferred directions are given.

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1 Material symmetry group. Axiom of material indifference. Isotropy.

Let a reference configuration Ω of an elastic body be given and let us denote by $\mathbb{M}_3^+$ the set of 3×3 matrices with positive determinant. Let $\hat{T} : \Omega \times \mathbb{M}_3^+$ be the constitutive law of the Cauchy stress tensor. The symmetry group of the material at point $x \in \Omega$ is defined by

$$\mathcal{G}_x = \{H \in \mathbb{M}_3^+;\ \forall F \in \mathbb{M}_3^+,\ \hat{T}(x, FH) = \hat{T}(x, F)\}. \tag{1}$$

Let SL(3) be the set of matrices with determinant 1, also known as the proper unimodular group. It can be shown [2] that for physical reasons $\mathcal{G}_x$ is necessarily a subset of SL(3). The following classification is classical:

- If for any $x \in \Omega, \mathcal{G}_x = \mathrm{SL}(3)$, the material is said to be an elastic fluid. Any deformation that does not change the volume preserves the stresses.
- If for any $x \in \Omega, \mathcal{G}_x \subset \mathrm{SO}(3)$, the material is said to be an elastic solid. In such a case, any non rigid deformation changes the stresses.
- If $\mathcal{G}_x = \mathrm{SO}(3)$, the elastic body is isotropic at point x, otherwise it is anisotropic.

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We recall that the constitutive law $\hat{T}_R$ of the first Piola-Kirchhoff stress tensor is defined in terms of the Cauchy stress law by the formula

$$\forall x \in \Omega, \forall F \in \mathbb{M}_3^+,\ \hat{T}_R(x,F) = \hat{T}(x,F)\mathrm{Cof\,F}. \tag{2}$$

Then,

$$\mathcal{G}_x = \{H \in \mathbb{M}_3^+;\ \forall F \in \mathbb{M}_3^+, \hat{T}_R(x,FH) = \hat{T}_R(x,F)\,\mathrm{Cof\,H}\}, \tag{3}$$

and if we restrict to elastic solids, which will be the case from now on,

$$\mathcal{G}_x = \{Q \in \mathrm{SO}(3);\ \forall F \in \mathbb{M}_3^+,\ \hat{T}_R(x,FQ) = \hat{T}_R(x,F)Q\}. \tag{4}$$

When dealing with hyperelastic materials that are materials such that $\hat{T}_R$ is the derivative with respect to F of a scalar-valued function W, we can express material properties in terms of the stored energy density W. This makes some reasonings much simpler than their counterparts on the tensor-valued constitutive laws. Let us first translate the definition of the symmetry group.

Proposition 1 - *The symmetry group of a hyperelastic material with stored energy density W is given by*

$$\mathcal{G}_x = \{Q \in \mathrm{SO}(3);\ \forall F \in \mathbb{M}_3^+,\ W(x,FQ) = W(x,F)\}. \tag{5}$$

The proof of this result is classical and can be taken for instance from [1].

Proof - The variable x can be omitted in the sequel. Let us first prove that if $Q \in \mathrm{SO}(3)$ is such that $W(x,FQ) = W(x,F)$ for all $F \in \mathbb{M}_3^+$, then Q belongs to $\mathcal{G}_x$. Differentiating the equality, we obtain

$$\forall H \in \mathbb{M}_3^+, W'(FQ)(HQ) = W'(F)(H),$$

which means that

$$\forall H \in \mathbb{M}_3^+, \hat{T}_R(FQ) : (HQ) = \hat{T}_R(F) : H$$

which in turn reads

$$\hat{T}_R(FQ)Q^T = \hat{T}_R(F),\ \text{or equivalently, } \hat{T}_R(FQ) = \hat{T}_R(F)Q.$$

Conversely, let $Q \in \mathrm{SO}(3)$ be such that for all $F \in \mathbb{M}_3^+$, one has $\hat{T}_R(F\,Q) = \hat{T}_R(F)\,Q$. Then, we infer from the previous calculations that the mapping $F \in \mathbb{M}_3^+ \mapsto W(FQ) - W(F) \in \mathbb{R}$ is constant. In other terms, there exists $W_0(Q) \in \mathbb{R}$ such that

$$\forall F \in \mathbb{M}_3^+,\ W(FQ) - W(F) = W_0(Q).$$

This equality implies that

$$\forall n \in \mathbb{N},\ W(Q^n) = W(I) + nW_0(Q).$$

The continuous function W must remain bounded on the set $\{Q^n, n \in \mathbb{N}\}$ which is a subset of the compact set SO(3), therefore $W_0(Q)$ is necessarily equal to 0. $\square$

A necessary requirement for a stored energy density is to satisfy the axiom of material indifference and we will suppose from now on that this requirement is fulfilled. From all classical books on nonlinear elasticity, we know that this means that the stored energy density is a function of its argument F only through F^TF. In other words, there exists a mapping $\tilde{W} : \Omega \times \mathbb{S}_3^+$ such that

$$\forall x \in \Omega, \forall F \in \mathbb{M}_3^+, \ W(x, F) = \tilde{W}(x, F^TF)$$

where $\mathbb{S}_3^+$ denotes the set of 3×3 symmetric matrices with positive determinant. Proving the following proposition is straightforward.

Proposition 2 - *The symmetry group at point x is given by*

$$\mathcal{G}_x = \{Q \in \mathrm{SO}(3); \ \forall C \in \mathbb{S}_3^+, \ \tilde{W}(x, Q^TCQ) = \tilde{W}(x, C)\}. \tag{6}$$

By the local inversion theorem, the mapping $C \in \mathbb{S}_3^+ \mapsto C^{1/2} \in \mathbb{S}_3^+$ can be shown to be differentiable and, since $\tilde{W}(C) = W(C^{1/2})$, the mapping $\tilde{W}$ is differentiable. It is then classical that the second Piola-Kirchhoff stress tensor is given by

$$\tilde{\Sigma}(x, C) = 2\,\frac{\partial \tilde{W}}{\partial C}(x, C).$$

1.1 Isotropic densities.

We are now interested in determining restrictions that material symmetry properties impose on the form of the stored energy densities. For isotropic materials, the result is given by the following representation theorem which can be found in [1], [2], [4], [9] and other books. Its more intricate version for tensor-valued functions was first published in [7]. We denote by ι^3 the $\mathbb{R}^3$-valued function $\iota^3 = (\iota_1, \iota_2, \iota_3)$ and by $\iota^3(\mathbb{S}_3^+) \subset (\mathbb{R}^{+*})^3$ its range set.

Proposition 3 - *The material is isotropic at point x in Ω if and only if there exists a mapping $w(x, \cdot) : \iota^3(\mathbb{S}_3^+) \mapsto \mathbb{R}$ such that*

$$\forall C \in \mathbb{S}_3^+, \ \tilde{W}(x, C) = w(x, \iota^3(C)). \tag{7}$$

For the sake of completeness let us recall the proof of this well-known result.

Proof - We can omit the variable x. Obviously, if (7) is satisfied, then W is isotropic. Let us now prove the converse assertion. Let W be isotropic. Then,

$$\forall Q \in \mathrm{SO}(3), \ \forall C \in \mathbb{S}_3^+, \ \tilde{W}(Q^TCQ) = \tilde{W}(C), \tag{8}$$

from which we want to derive that if C and C' in $\mathbb{S}_3^+$ are such that $\iota(C) = \iota(C')$, then $\tilde{W}(C) = \tilde{W}(C')$. Two symmetric matrices C and C' such that $\iota(C) = \iota(C')$ can be diagonalized under the form

$$C = R\Delta R^T,\ C' = R'\Delta' R'^T,\ R \in \mathrm{O}(3),\ R' \in \mathrm{O}(3)$$

where Δ and Δ' are two diagonal matrices that are equal up to a permutation of their diagonal entries. Therefore, there exists $O \in \mathrm{O}(3)$ such that $\Delta' = O\Delta O^T$ which implies that $C' = R'\Delta' R'^T = R' O \Delta O^T R'^T = (R' O R^T) C (R O^T R'^T)$. This reads $C' = Q^T C Q$ with $Q = R O^T R'^T$ an orhogonal matrix. Either Q belongs to $\mathrm{SO}(3)$, and from (8) we have that $\tilde{W}(C) = \tilde{W}(C')$. Or $-Q$ belongs to $\mathrm{SO}(3)$ and from $C' = (-Q)^T C(-Q)$ we derive the same result. As a consequence the mapping $\tilde{W}$ can be factorized through the mapping ι^3, which ends the proof. $\square$

Remark 1 - *The above proposition states that an isotropic energy does not depend of six real numbers (the entries of the symmetric matrix C), but of three real numbers only. The result is optimal in the sense that the number of invariants cannot be further reduced.*

Proof - Indeed, if the number of invariants could be reduced, then any of the three principal invariants could be expressed in terms of the two other ones. This would mean that from the knowledge of the values of two elementary symmetric functions of three numbers, one would derive these three numbers, which is untrue. $\square$

Remark 2 - *The range set $\iota^3(\mathbb{S}_3^+)$ is a strict subset of $(\mathbb{R}^{+*})^3$ and it is not an open subset.*

Proof - Choose $\lambda_1 \in \mathbb{R}^{+*}$ and choose $k \in \mathbb{R}^*$. Let $\jmath = (\lambda_1, k^2, \lambda_1 k^2)$. Obviously $\jmath$ belongs to $(\mathbb{R}^{+*})^3$. But it does not belong to $\iota^3(\mathbb{S}_3^+)$; otherwise, the roots of the polynomial $\lambda^3 - \lambda_1\lambda^2 + k^2\lambda - \lambda_1 k^2$ which are λ_1 and $\pm ik$, would be positive real numbers. This proves that $\iota^3(\mathbb{S}_3^+)$ is not equal to $(\mathbb{R}^{+*})^3$. The fact that it is not an open set is mentioned in [1]. Let us anyway give a proof of this result. Let $\lambda_1, a \in \mathbb{R}^{+*}$ and define $\jmath = (\lambda_1 + 2a, 2a\lambda_1 + a^2, \lambda_1 a^2)$. Obviously, $\jmath$ belongs to $\iota^3(\mathbb{S}_3^+)$, actually $(\jmath_1, \jmath_2, \jmath_3)$ are the symmetric functions associated with the roots of the polynomial $(\lambda - \lambda_1)(\lambda - a)^2$. For $\varepsilon > 0$, let $\jmath(\varepsilon) = (\lambda_1 + 2a, 2a\lambda_1 + a^2 + \varepsilon^2, \lambda_1(a^2 + \varepsilon^2))$. Its entries $\jmath_1(\varepsilon), \jmath_2(\varepsilon), \jmath_3(\varepsilon)$ are the symmetric functions of the roots of the polynomial $(\lambda - \lambda_1)(\lambda - (a + i\varepsilon))(\lambda - (a - i\varepsilon))$. Since $\jmath(\varepsilon)$ goes to $\jmath$ when ε goes to 0, we see that any neighbourhood of j contains points that do not belong to $\iota^3(\mathbb{S}_3^+)$. $\square$

1.2 Isotropic constitutive laws.

We recall that for any hyperelastic material the constitutive law $\hat{\Sigma}$ of the second Piola-Kirchhoff stress tensor is given in terms of the elastic energy density by

$$\hat{\Sigma}(x, F) = \tilde{\Sigma}(x, F^T F) = 2\,\frac{\partial \tilde{W}}{\partial C}(x, F^T F), \tag{9}$$

where, as already mentioned, the differentiability of $\tilde{W}$ is a consequence of the differentiability of W. Our aim is to derive from (7) the form of isotropic constitutive laws. We have seen in Remark 2 that the set $\iota^3(\mathbb{S}_3^+)$ is not open, and for the sake of simplicity, we assume that w can be extended in a function still denoted by w defined on $\Omega \times (\mathbb{R}^{+*})^3$ and differentiable with respect to its second variable ι^3. Then, from formula (7), we have

$$\forall F \in \mathbb{M}_3^+, \hat{\Sigma}(x, F) = 2 \sum_{i=1}^{3} \frac{\partial w}{\partial \iota_i}(x, \iota(F^T F))\iota_i'(F^T F) \tag{10}$$

where $\iota_i', i = 1, 2, 3$ denotes the derivative of $\iota_i : \mathbb{S}_3^+ \mapsto \mathbb{R}$. As $\iota_1(C) = \mathrm{tr}(C)$ and $\iota_3(C) = \det(C)$, obviously

$$\iota_1'(C) = Id, \ \iota_3'(C) = \mathrm{Cof}\, C.$$

From $\iota_2(C) = \mathrm{tr}(\mathrm{Cof}(C)) = \frac{1}{2}((\mathrm{tr}C)^2 - \mathrm{tr}(C^2))$, we derive that

$$\iota_2'(C)(D) = \iota_1(C)\mathrm{tr}D - \mathrm{tr}(CD) = (\iota_1(C)Id - C^T) : D.$$

Using the symmetry of C, those formulas read

$$\iota_1'(C) = Id, \ \iota_2'(C) = \iota_1(C)Id - C, \ \iota_3'(C) = \iota_3(C)C^{-1}. \tag{11}$$

It follows that $\hat{\Sigma}(x, F)$ can be decomposed along Id, $C = F^T F$ and C^{-1} with the following coefficients

$$\hat{\Sigma}(x, F) = 2[\frac{\partial w}{\partial \iota_1} + \iota_1(C)\frac{\partial w}{\partial \iota_2}]Id - 2\frac{\partial w}{\partial \iota_2}C + 2\iota_3(C)\frac{\partial w}{\partial \iota_3}C^{-1}, \tag{12}$$

where $\frac{\partial w}{\partial \iota_k}$ stands for $\frac{\partial w}{\partial \iota_k}(x, \iota(C))$. By using the Cayley-Hamilton theorem, an analogous formula can be written with C^{-1} replaced by C^2. It is classically written under the form

$$\hat{\Sigma}(x, F) = \gamma_0(x, \iota(C))Id + \gamma_1(x, \iota(C))C + \gamma_2(x, \iota(C))C^2. \tag{13}$$

As for the Cauchy stress tensor T, it is related to Σ by $\hat{T}(x, F) = (\det F)^{-1} F\, \hat{\Sigma}(x, F)F^T$ and it can be expressed as

$$\hat{T}(x, F) = 2\iota_3(B)^{-1/2} \left[\iota_3(B)\frac{\partial w}{\partial \iota_3}Id + [\frac{\partial w}{\partial \iota_1} + \iota_1(B)\frac{\partial w}{\partial \iota_2}]B - \frac{\partial w}{\partial \iota_2}B^2 \right] \tag{14}$$

where $B = FF^T$ and where $\frac{\partial w}{\partial \iota_k}$ stands for $\frac{\partial w}{\partial \iota_k}(x, \iota(B))$. This formula is classically written under the form

$$\hat{T}(x, F) = \beta_0(x, \iota(B))Id + \beta_1(x, \iota(B))B + \beta_2(x, \iota(B))B^2. \tag{15}$$

From a practical point of view, the above results show that the matrices $\hat{\Sigma}(x, F)$ and $\hat{T}(x, F)$ that belong to the six-dimensional vector space $\mathbb{S}_3$ can actually be decomposed along a set of three matrices that vary with F. Coefficients in the decomposition depend on F by means of the three principal invariants.

2 Transverse isotropy.

For the sake of formula conciseness, we drop from now on, unless necessary, the variable x in energy densities and constitutive laws.

Transverse isotropy is introduced in the engineering literature in the following way: transversely isotropic materials are materials that have the same properties in one plane and different properties in the direction normal to this plane. This usually corresponds to the description of materials reinforced by one family of fibers. As said in [3], their stiffness is greater in the fiber direction τ and the material response along directions orthogonal to this preferred directions is isotropic. For a fiber-reinforced material described in this way, performing a rotation around the preferred direction before applying a homogeneous deformation should not change the energy. Moreover, performing a rotation with axis orthogonal to the preferred direction and with angle π before applying a homogeneous deformation should not change the energy either. Indeed, all these rotations are exactly those which leave the plane orthogonal to the preferred direction invariant.

Definition 1 - *A material is said to be transversely isotropic at point $x \in \Omega$ with respect to a unit vector τ if and only if $\{Q \in \mathrm{SO}(3);\, Q\tau = \pm\tau\} \subset \mathcal{G}_x$.*

This definition coincides with one of the choices proposed in [5], namely choice g_4. In terms of the stored energy density, transverse isotropy with respect to τ reads

$$\forall Q \in \mathrm{SO}(3) \text{ such that } Q\tau = \pm\tau,\ \forall C \in \mathbb{S}_3^+,\ \tilde{W}(Q^T C Q) = \tilde{W}(C). \tag{16}$$

Choose R in $\mathrm{O}(3) \setminus \mathrm{SO}(3)$ such that $R\tau = \pm\tau$. Then $Q = -R$ belongs to $\mathrm{SO}(3)$ and satisfies $Q\tau = \pm\tau$. Therefore property (16) can be extended to all orthogonal matrices and the energy of a transversely isotropic material satisfies

$$\forall Q \in \mathrm{O}(3) \text{ such that } Q\tau = \pm\tau,\ \forall C \in \mathbb{S}_3^+,\ \tilde{W}(Q^T C Q) = \tilde{W}(C). \tag{17}$$

This is called choice g_5 in [5].

2.1 Tranversely isotropic densities

We are now in a position to prove a representation theorem. As transverse isotropy is less restrictive than isotropy, more freedom is left to the stored energy density which is shown to be a function of five scalar variables. As customary, we set

$$\iota_4(C) = \tau \cdot C\tau,\ \iota_5(C) = \tau \cdot C^2\tau,$$

although a rigorous notation should be $\iota_4(C,\tau), \iota_5(C,\tau)$. In addition, we define $\iota^5 : \mathbb{S}_3^+ \mapsto (\mathbb{R}^{+*})^5$ by $\iota^5 = (\iota_1, \iota_2, \iota_3, \iota_4, \iota_5)$. Clearly,

$$\forall Q \in \mathrm{O}(3) \text{ such that } Q\tau = \pm\tau,\ \forall C \in \mathbb{S}_3^+,\ \iota^5(Q^T C Q) = \iota^5(C). \tag{18}$$

Note that the range of ι^5 is a strict subset of $(\mathbb{R}^{+*})^5$. Indeed,

$$\iota_5(C) = C\tau \cdot C\tau = \|C\tau\|^2 \geq \iota_4(C),$$

which suffices to prove that $\iota^5(\mathbb{S}_3^+) \neq (\mathbb{R}^{+*})^5$.

It is shown in [5] that, from a transverse isotropic density, one can construct a function $\tilde{Z} : \mathbb{S}_3^+ \times \{p \in \mathbb{R}^3; \|p\| = 1\} \mapsto \mathbb{R}$ that is isotropic in terms of the pair (C,p). Then one can make use of the so-called functional bases given in [10], [11]. We give here a proof that does not call to this clever trick. Although more complicate, this proof is in the spirit of the proof of Rivlin-Ericksen theorem given in [1].

Proposition 4 - *A material is transversely isotropic at point x in Ω with respect to τ if and only if there exists a mapping $w(x,\cdot) : \iota^5(\mathbb{S}_3^+) \mapsto \mathbb{R}$ such that*

$$\forall C \in \mathbb{S}_3^+,\ \tilde{W}(x,C) = w(x, \iota^5(C)). \tag{19}$$

Proof - Obviously, if (19) is satisfied, then W is transversely isotropic with respect to τ. Let us now prove the converse assertion. Let W be transversely isotropic with respect to τ. Let C and C' in $\mathbb{S}_3^+$ such that $\iota^5(C) = \iota^5(C')$. We need to prove that there exists Q in $\mathrm{O}(3)$ satisfying $Q\tau = \pm\tau$ and such that $C' = Q^T C Q$. We will actually prove that this obtained with a Q such that $Q\tau = \tau$. From the identity of the three principal invariants, we know that there exists P in $\mathrm{O}(3)$ such that $C' = P^T C P$. The other two identities read

$$\tau \cdot C\tau = P\tau \cdot CP\tau \quad \text{and } \tau \cdot C^2\tau = P\tau \cdot C^2 P\tau. \tag{20}$$

Let $(u_i), i = 1,2,3$, be an orthonormal basis of eigenvectors of C with associated eigenvalues $\lambda_i, i = 1,2,3$. Then,

$$C = \lambda_1 u_1 \otimes u_1 + \lambda_2 u_2 \otimes u_2 + \lambda_3 u_3 \otimes u_3 \text{ and } C^2 = \lambda_1^2 u_1 \otimes u_1 + \lambda_2^2 u_2 \otimes u_2 + \lambda_3^2 u_3 \otimes u_3$$

from which we derive that

$$\tau \cdot C\tau = \sum_i \lambda_i (u_i \cdot \tau)^2,\ P\tau \cdot CP\tau = \sum_i \lambda_i (u_i \cdot P\tau)^2,$$

and

$$\tau \cdot C^2\tau = \sum_i \lambda_i^2 (u_i \cdot \tau)^2,\ P\tau \cdot C^2 P\tau = \sum_i \lambda_i^2 (u_i \cdot P\tau)^2.$$

Equations (20) prove that

$$\sum_i \lambda_i [(u_i \cdot \tau)^2 - (u_i \cdot P\tau)^2] = 0 \quad \text{and} \ \sum_i \lambda_i^2 [(u_i \cdot \tau)^2 - (u_i \cdot P\tau)^2] = 0. \tag{21}$$

From $\|\tau\|^2 = \|P\tau\|^2$, we also have

$$\sum_i [(u_i \cdot \tau)^2 - (u_i \cdot P\tau)^2] = 0. \tag{22}$$

The above three equations make a van der Monde linear system.

Point 1. Let us first concentrate on the case when the three eigenvalues are distinct. Then the van der Monde matrix is invertible and for any $i = 1,2,3$, one has $u_i \cdot \tau = \varepsilon_i (u_i \cdot P\tau)$ with $\varepsilon_i = \pm 1$, or equivalently, $\tau \cdot u_i = \varepsilon_i (\tau \cdot P^T u_i)$. Let us split the reasoning into several cases.

(i) suppose that for any i, $u_i \cdot \tau = u_i \cdot P\tau$. Then $P\tau = \tau$. We can choose $Q = P$.

(ii) suppose that for two indices, say $\alpha = 1,2$, $\varepsilon_\alpha = 1$ and that $\varepsilon_3 = -1$. In other words,

$$\tau \cdot u_\alpha = \tau \cdot P^T u_\alpha,\ \alpha = 1,2,\ \tau \cdot u_3 = -\tau \cdot P^T u_3. \tag{23}$$

Define Q by

$$Q^T u_\alpha = P^T u_\alpha,\ \alpha = 1,2,\ Q^T u_3 = -P^T u_3.$$

As P^T maps an orthonormal basis onto an orthonormal basis, Q is an orthogonal matrix. Let us check that $Q\tau = \tau$. It suffices to show that for any $i = 1,2,3$, $\tau \cdot u_i = \tau \cdot Q^T u_i$ which follows immediately from (23). Moreover

$$C' = P^T C P = \sum_i \lambda_i P^T u_i \otimes P^T u_i = \sum_i \lambda_i Q^T u_i \otimes Q^T u_i = Q^T C Q.$$

Therefore we have identified an orthogonal matrix Q such that $Q\tau = \tau$ and $C' = Q^T C Q$. An example of this situation is given by τ the unit vector along $u_2 + u_3$, P^T the rotation around u_1 with angle $\pi/2$ and Q^T the symmetry with respect to the plane generated by u_1 and $u_2 + u_3$.

(iii) suppose that for two indices, say $\alpha = 1,2$, $\varepsilon_\alpha = -1$ and that $\varepsilon_3 = 1$. By letting $P' = -P$, we are back to the previous case.

(iv) finally, if for any i, $u_i \cdot \tau = -u_i \cdot P\tau$. Then $P\tau = -\tau$. We can choose $Q = -P$.

Point 2. We now turn to the case when two eigenvalues of C are equal, say $\lambda_1 = \lambda_2$. Equations (21)-(22) no longer define an invertible system. They are equivalent to

$$(\tau \cdot u_1)^2 + (\tau \cdot u_2)^2 = (\tau \cdot P^T u_1)^2 + (\tau \cdot P^T u_2)^2,\ (\tau \cdot u_3)^2 = (\tau \cdot P^T u_3)^2. \tag{24}$$

We are looking again for $Q \in \mathrm{O}(3)$ satisfying $Q\tau = \tau$ and such that $C'Q^T = Q^T C$, which means that for all $i = 1,2,3$, $C'Q^T u_i = \lambda_i Q^T u_i$. In other words, we search for $Q \in \mathrm{O}(3)$ such that on the one hand $Q\tau = \tau$ and that on the other hand $Q^T u_\alpha, \alpha = 1,2$, is an eigenvector of C' associated with λ_1 and $Q^T u_3$ is an eigenvector of C' associated with λ_3. We recall that

$$C' = \lambda_1 (P^T u_1 \otimes P^T u_1 + P^T u_2 \otimes P^T u_2) + \lambda_3 P^T u_3 \otimes P^T u_3. \tag{25}$$

Let us split the reasoning into several cases.

(i) suppose that $\tau \cdot u_3 = \tau \cdot P^T u_3$. We define Q^T along u_3 by $Q^T u_3 = P^T u_3$. As for the action of Q^T on u_1 and u_2, we require that Q^T maps the plane generated by u_1 and u_2 onto the plane generated by $P^T u_1$ and $P^T u_2$, is an orthogonal transformation and finally satisfies $\tau \cdot u_\alpha = \tau \cdot Q^T u_\alpha, \alpha = 1, 2$. The first two conditions are satisfied as soon that $Q^T u_1$ and $Q^T u_2$ are given by

$$Q^T u_1 = \cos\theta(P^T u_1) + \sin\theta(P^T u_2), \quad Q^T u_2 = -\sin\theta(P^T u_1) + \cos\theta(P^T u_2),$$

where θ is arbitrary. At this time, we know that Q is orthogonal and that $C'Q^T = Q^T C$. The third condition is fulfilled if

$$\tau \cdot u_1 = \cos\theta(\tau \cdot P^T u_1) + \sin\theta(\tau \cdot P^T u_2), \quad \text{and} \quad \tau \cdot u_2 = -\sin\theta(\tau \cdot P^T u_1) + \cos\theta(\tau \cdot P^T u_2).$$

Such a θ exists as a consequence of the first assertion in (24). Then, for any $i = 1, 2, 3$, $\tau \cdot u_i = \tau \cdot Q^T u_i$ and $Q\tau = \tau$.

(i) suppose that $\tau \cdot u_3 = -\tau \cdot P^T u_3$. Let $P' = -P$. The first equation in (24) and equation (25) hold with P' in place of P. It suffices to make the same reasoning as before with P replaced by P' to obtain the result.

Point 3. When all three eigenvalues are equal to λ, then $C' = C = \lambda d$. We can choose $Q = Id$.

From points 1, 2 and 3, we derive that $\tilde{W}$ can be factorized through the mapping ι^5. This means that there exists a function $w : \iota^5(\mathbb{S}_3^+) \mapsto \mathbb{R}$ such that $\tilde{W} = w \circ \iota^5$. The function w can be extended to $(\mathbb{R}^{+*})^5$ if needed. □

Remark 3 - *Equation (19) shows that in addition to the usual three principal invariants which account for the local mean change of length of curves that are tangent to the eigenvectors of C, the local mean change of area of surface elements that are orthogonal to the eigenvectors of C and the local change of volume, the deformed energy depends on the fourth and fifth invariants. Since $\iota_4(C) = \tau \cdot C\tau = \|F\tau\|^2$, this invariant gives the value of the change of length along τ. Let us try and give an interpretation of the fifth invariant $\iota_5(C) = \|C\tau\|^2$: Complete τ in an orthonormal basis $(\tau, \tau', \tau \wedge \tau')$, then*

$$\iota_5(C) = \|C\tau\|^2 = (C\tau \cdot \tau)^2 + (C\tau \cdot \tau')^2 + (C\tau \cdot \tau \wedge \tau')^2 = (F\tau \cdot F\tau)^2 + (F\tau \cdot F\tau')^2 + (F\tau \cdot F(\tau \wedge \tau'))^2.$$

This quantity can be measured by experiments. It measures how an orthonormal basis with one vector equal to τ deforms under the action of F. It mixes changes of lengths and changes of angles.

We can now give a statement analogous to Remark 1.

Remark 4 - *The above proposition states that a transversely isotropic energy does not depend of six real numbers (the entries of the symmetric matrix C), but of five real numbers only. The result is optimal in the sense that none of the five invariants $\iota_k, k = 1, 2, 3, 4, 5$ is a function of the other four ones.*

Proof - In all cases to be examined, we introduce two matrices C and C'. We leave it to the reader to check that they are symmetric definite positive, which can be done by computing their principal minors.

- ι_5 is not a function of $(\iota_1, \iota_2, \iota_3, \iota_4)$: Consider the two matrices C and C' defined by

$$C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; C' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

  Easy computations give $\iota_1(C) = \iota_1(C') = 5$, $\iota_2(C) = \iota_2(C') = 6$, $\iota_3(C) = \iota_3(C') = 1$, and $\iota_4(C) = \iota_4(C') = 2$, But, $\iota_5(C) = 6$, $\iota_5(C') = 5$.

- ι_4 is not a function of $(\iota_1, \iota_2, \iota_3, \iota_5)$: Choose

$$C = \begin{pmatrix} 1 & 1 & \sqrt{3} \\ 1 & 2 & 0 \\ \sqrt{3} & 0 & 20 \end{pmatrix}, \; C' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & 20 \end{pmatrix}.$$

  One easily checks that $\iota_1(C) = \iota_1(C') = 23$, $\iota_2(C) = \iota_2(C') = 58$, $\iota_3(C) = \iota_3(C') = 14$, $\iota_5(C) = \iota_5(C') = 5$, although $\iota_4(C) = 1$ and $\iota_4(C') = 2$.

- ι_3 is not a function of $(\iota_1, \iota_2, \iota_4, \iota_5)$: Choose

$$C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; C' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

  The third minors are given by $\iota_3(C) = 2$, $\iota_3(C') = 3$, hence $\iota_3(C) \neq \iota_3(C')$. On the contrary, $\iota_1(C) = \iota_1(C') = 5$, $\iota_2(C) = \iota_2(C') = 7$, $\iota_4(C) = \iota_4(C') = 2$, $\iota_5(C) = \iota_5(C') = 5$.

- ι_2 is not a function of $(\iota_1, \iota_3, \iota_4, \iota_5)$: Choose

$$C = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & \sqrt{\frac{2}{5}} \\ 1 & \sqrt{\frac{2}{5}} & 1 \end{pmatrix}, \; C' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & \frac{3}{4} & \sqrt{\frac{17}{80}} \\ 0 & \sqrt{\frac{17}{80}} & \frac{5}{4} \end{pmatrix}.$$

  One easily checks that $\iota_1(C) = \iota_1(C') = 4$, $\iota_3(C) = \iota_3(C') = \frac{1}{5}$, $\iota_4(C) = \iota_4(C') = 2$, $\iota_5(C) = \iota_5(C') = 5$. On the contrary, $\iota_2(C) = 3 + \frac{3}{5} \neq \iota_2(C') = 3 + \frac{29}{40}$.

- finally, ι_1 is not a function of $(\iota_2,\, \iota_3,\, \iota_4,\, \iota_5)$: Choose

$$C = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{5}{3} \end{pmatrix},\ C' = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, their traces are not equal. Nevertheless, $\iota_2(C) = \iota_2(C') = \frac{8}{3}$, $\iota_3(C) = \iota_3(C') = \frac{2}{3}$, $\iota_4(C) = \iota_4(C') = 1$, $\iota_5(C) = \iota_5(C') = \frac{4}{3}$. $\square$

2.2 Tranversely isotropic constitutive laws

As in Section 1.2, we assume that w can be extended in a function still denoted by w defined on $(\mathbb{R}^{+*})^5$ and differentiable with respect to ι^5. Then, from formula (19), we have

$$\forall F \in \mathbb{M}_3^+,\ \hat{\Sigma}(F) = 2\sum_{i=1}^{5} \frac{\partial w}{\partial \iota_i}(\iota^5(F^TF))\iota_i'(F^TF). \tag{26}$$

As ι_4 is linear, we have for any symmetric matrices C and D, $\iota_4'(C)(D) = \tau \cdot D\tau = \tau \otimes \tau : D$. We obtain ι_5' by writing that for any C and D

$$\iota_5'(C)(D) = \tau \cdot CD\tau + \tau \cdot DC\tau = (C\tau \otimes \tau + \tau \otimes C\tau) : D.$$

Therefore, writing for short ι^5 in place of $\iota^5(C)$, we have

$$\begin{aligned} \hat{\Sigma}(F) &= 2[\frac{\partial w}{\partial \iota_1}(\iota^5) + \iota_1(C)\frac{\partial w}{\partial \iota_2}(\iota^5)]Id - 2\frac{\partial w}{\partial \iota_2}(\iota^5)C + 2\iota_3(C)\frac{\partial w}{\partial \iota_3}(\iota^5)C^{-1} \\ &+ 2\frac{\partial w}{\partial \iota_4}(\iota^5)\tau \otimes \tau + 2\frac{\partial w}{\partial \iota_5}(\iota^5)(\tau \otimes C\tau + C\tau \otimes \tau). \end{aligned} \tag{27}$$

Formula (27) coincides with formula (71) in [6]. When dealing with the Cauchy stress tensor, and writing for short ι^5 in place of $\iota^5(B)$, we obtain

$$\begin{aligned} \hat{T}(F) = 2\iota_3(B)^{-1/2} \Big[&\iota_3(B)\frac{\partial w}{\partial \iota_3}(\iota^5)Id + [\frac{\partial w}{\partial \iota_1}(\iota^5) + \iota_1(B)\frac{\partial w}{\partial \iota_2}(\iota^5]B - \frac{\partial w}{\partial \iota_2}(\iota^5)B^2 \\ &+ \frac{\partial w}{\partial \iota_4}(\iota^5)F\tau \otimes F\tau + \frac{\partial w}{\partial \iota_5}(\iota^5)(F\tau \otimes BF\tau + BF\tau \otimes F\tau)\Big] \end{aligned} \tag{28}$$

that we write

$$\hat{T}(F) = \hat{T}_1(F) + \hat{T}_2(F) \tag{29}$$

where $\hat{T}_1(F)$ and $\hat{T}_2(F)$ assume the forms

$$\hat{T}_1(F) = \beta_0(\iota^5(B))Id + \beta_1(\iota^5(B))B + \beta_2(\iota^5(B))B^2, \tag{30}$$

$$\hat{T}_2(F) = \delta_0(\iota^5(B))F\tau \otimes F\tau + \delta_1(\iota^5(B))(F\tau \otimes BF\tau + BF\tau \otimes F\tau). \tag{31}$$

Formula (28) coincides with formula (73) in [6]. Let us investigate the contribution of the anisotropic part $\hat{T}_2(F)$. If it were reduced to its first term, then its action on any unit vector ν would result on a vector colinear to $F\tau$ and would be zero on vectors orthogonal to $F\tau$. But, the fact that the second term is present destroys this nice property. Indeed, stress vectors $\hat{T}_2(F)\nu$ belong to the two-dimensional vector space generated by $F\tau$ and by $BF\tau$, and only vectors ν that are both orthogonal to $F\tau$ and to $BF\tau$ give a zero value to $\hat{T}_2(F)\nu$. In all constitutive laws that we are aware of δ_1 is actually taken to be 0.

Equations (27) and (28) show that the symmetric matrices $\hat{\Sigma}(F)$ and $\hat{T}(F)$ decompose along a set of five matrices that vary with F. Coefficients in the decomposition depend on the five invariants defined till now.

3 Materials with two preferred directions. Orthotropy.

We recall that the variable x is dropped in most formulas.

In this section, we deal with materials which in a given point x of the reference configuration admit two fibers with unit tangent vectors τ and τ', $\tau' \neq \pm\tau$. Examples in engineeering are materials reinforced by two family of fibers and examples in physiology are artery walls. It is mechanically meaningful to separate the case when τ and τ' are orthogonal and the case when they are non orthogonal. In the particular case when the fibers are orthogonal, we will say that the materials are orthotropic with respect to τ and τ' at point x, see [Spencer] and [Holzapfel]. It is natural to require that performing a rotation with angle π around τ or τ' prior to applying a deformation does not change the energy. Indeed these deformations do not modify the fibers. For any vector ν, let R_ν denote the rotation with axis ν and angle π. Mathematically speaking, orthotropy reads as follows.

Definition 2 - *A material is said to be orthotropic at point x in Ω with respect to the orthogonal vectors τ and τ' if and only if*

$$\{R_\tau, R_{\tau'}\} \subset \mathcal{G}_x. \tag{32}$$

As $R_{\tau\wedge\tau'} = R_\tau \circ R_{\tau'}$, orthotropy implies that $R_\tau \wedge R_{\tau'}$ belongs to $\mathcal{G}_x$. If a material is isotropic transversely to τ, then it is orthotropic with respect to τ and to any τ' orthogonal to τ.

When τ and τ' are non orthogonal, $R_{\tau\wedge\tau'}$ is the only rotation Q, besides the identity, that preserves both directions τ and τ' ($Q\tau = \pm\tau, Q\tau' = \pm\tau'$). Therefore we state the following definition.

Definition 3 - *A material is said to admit at point x in Ω two non orthogonal preferred directions τ and τ' if and only if*

$$R_{\tau\wedge\tau'} \in \mathcal{G}_x. \tag{33}$$

It follows that, although they have no mechanical existence, any two non orthogonal unit vectors $\check{\tau}$ and $\check{\tau}'$ that generate the plane $< \tau, \tau' >$ are preferred directions as well.

3.1 Orthotropy.

In this subsection, we assume that τ and τ' are orthogonal. The endomorphism $\mathcal{C}$ whose matrix in the standard basis is C is represented in the basis $(\tau, \tau', \tau \wedge \tau')$ by

$$\begin{pmatrix} \iota_4(\mathcal{C}) & \iota_8(\mathcal{C}) & \iota_9(\mathcal{C}) \\ \iota_8(\mathcal{C}) & \iota_6(\mathcal{C}) & \iota_{10}(\mathcal{C}) \\ \iota_9(\mathcal{C}) & \iota_{10}(\mathcal{C}) & \iota_1(\mathcal{C}) - (\iota_4(\mathcal{C}) + \iota_6(\mathcal{C})) \end{pmatrix} \tag{34}$$

where

$$\iota_4(\mathcal{C}) = \tau \cdot \mathcal{C}\tau,\ \iota_6(\mathcal{C}) = \tau' \cdot \mathcal{C}\tau',\ \iota_8(\mathcal{C}) = \tau \cdot \mathcal{C}\tau', \tag{35}$$

$$\iota_9(\mathcal{C}) = \tau \cdot \mathcal{C}(\tau \wedge \tau'),\ \iota_{10}(\mathcal{C}) = \tau' \cdot \mathcal{C}(\tau \wedge \tau'). \tag{36}$$

As a consequence, for any energy $\tilde{W}$, $\tilde{W}(C)$ can be written under the form

$$\tilde{W}(C) = z(\iota_1(\mathcal{C}), \iota_4(\mathcal{C}), \iota_6(\mathcal{C}), \iota_8(\mathcal{C}), \iota_9(\mathcal{C}), \iota_{10}(\mathcal{C})), \tag{37}$$

where z is a function of six real numbers. The definition of $\iota_6(\mathcal{C})$ and of $\iota_8(\mathcal{C})$ is classical in the literature. We underline the fact that these so-called invariants are nothing but the coefficients of $\mathcal{C}$ in a basis attached to the preferred directions. The definition domain of z is the set O of the sextuplets $\iota_O = (\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10})$ in $\mathbb{R}^6$ such that

$$\begin{pmatrix} \iota_4 & \iota_8 & \iota_9 \\ \iota_8 & \iota_6 & \iota_{10} \\ \iota_9 & \iota_{10} & \iota_1 - (\iota_4 + \iota_6) \end{pmatrix}$$

is a symmetric positive definite matrix. By writing the explicit expressions of the principal minors, this gives

$$\begin{aligned} O = \quad & \{\iota_O = (\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10}) \in \mathbb{R}^6;\ \iota_4 > 0, \iota_4\iota_6 - \iota_8^2 > 0, \\ & (\iota_4\iota_6 - \iota_8^2)(\iota_1 - (\iota_4 + \iota_6)) - \iota_4\iota_{10}^2 - \iota_6\iota_9^2 + 2\iota_8\iota_9\iota_{10} > 0\}. \end{aligned}$$

For any $\iota_O \in O$, the assertions $\iota_1 > 0$ and $\iota_6 > 0$ hold true. On the contrary, ι_8, ι_9, ι_{10} need not be positive and O is a strict open subset of $(\mathbb{R}^{+*})^3 \times \mathbb{R}^3$. From now on, we will use the notation ι_O for $(\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10})$.

Proposition 5 - *A material is orthotropic at point x with respect to the orthogonal vectors τ and τ' if and only if the function z satisfies the following symmetry conditions*

$$\forall \iota_O \in O,\ z(\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10}) = z(\iota_1, \iota_4, \iota_6, -\iota_8, -\iota_9, \iota_{10}), \tag{38}$$

$$\forall \iota_O \in O,\ z(\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10}) = z(\iota_1, \iota_4, \iota_6, -\iota_8, \iota_9, -\iota_{10}). \tag{39}$$

Proof - First of all, if $(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})$ belongs to O, then $(\iota_1,\iota_4,\iota_6,-\iota_8,-\iota_9,\iota_{10})$ and $(\iota_1,\iota_4,\iota_6,-\iota_8,\iota_9,-\iota_{10})$ belong to O as well. Let r_τ (resp. $r_{\tau'}$) be the rotation with axis τ (resp. τ') and angle π. The material is orthotropic if and only if for any symmetric positive definite endomorphism $\mathcal{C}$, one has

$$\begin{aligned}
z(\iota_1(r_\tau^T\mathcal{C}r_\tau),\iota_4(r_\tau^T\mathcal{C}r_\tau),\iota_6(r_\tau^T\mathcal{C}r_\tau),\iota_8(r_\tau^T\mathcal{C}r_\tau),\iota_9(r_\tau^T\mathcal{C}r_\tau),\iota_{10}(r_\tau^T\mathcal{C}r_\tau)) = \\
z(\iota_1(\mathcal{C}),\iota_4(\mathcal{C}),\iota_6(\mathcal{C}),\iota_8(\mathcal{C}),\iota_9(\mathcal{C}),\iota_{10}(\mathcal{C})), \\
z(\iota_1(r_{\tau'}^T\mathcal{C}r_{\tau'}),\iota_4(r_{\tau'}^T\mathcal{C}r_{\tau'}),\iota_6(r_{\tau'}^T\mathcal{C}r_{\tau'}),\iota_8(r_{\tau'}^T\mathcal{C}r_{\tau'}),\iota_9(r_{\tau'}^T\mathcal{C}r_{\tau'}),\iota_{10}(r_{\tau'}^T\mathcal{C}r_{\tau'})) = \\
z(\iota_1(\mathcal{C}),\iota_4(\mathcal{C}),\iota_6(\mathcal{C}),\iota_8(\mathcal{C}),\iota_9(\mathcal{C}),\iota_{10}(\mathcal{C})).
\end{aligned}$$

The result follows from the facts that for $k=1,4,6$, $\iota_k(r_\tau^T\mathcal{C}r_\tau)=\iota_k(r_{\tau'}^T\mathcal{C}r_{\tau'})=\iota_k(\mathcal{C})$ and that

$$\iota_8(r_\tau^T\mathcal{C}r_\tau)=\iota_8(r_{\tau'}^T\mathcal{C}r_{\tau'})=-\iota_8(\mathcal{C}), \tag{40}$$

$$\iota_9(r_\tau^T\mathcal{C}r_\tau)=-\iota_9(\mathcal{C})=-\iota_9(r_{\tau'}^T\mathcal{C}r_{\tau'}),\ \iota_{10}(r_\tau^T\mathcal{C}r_\tau)=\iota_{10}(\mathcal{C})=-\iota_{10}(r_{\tau'}^T\mathcal{C}r_{\tau'}). \tag{41}$$

$\square$

Remark 5 - *It is easily derived from the above proposition that the identity*

$$\forall\iota_O=(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})\in O,\ z(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})=z(\iota_1,\iota_4,\iota_6,\iota_8,-\iota_9,-\iota_{10}) \tag{42}$$

is valid as well. This identity corresponds to the $r_{\tau\wedge\tau'}$ invariance.

The example of $z(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})=\iota_8\iota_9\iota_{10}$ shows that orthotropy does not mean that z can be written as

$$z(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})=\tilde{z}(\iota_1,\iota_4,\iota_6,|\iota_8|,|\iota_9|,|\iota_{10}|) \text{ on } O.$$

A correct formulation close to this one is given in the following proposition where $\mathrm{sgn}(t)=1$ if $t\geq 0$, $\mathrm{sgn}(t)=-1$ if $t<0$.

Proposition 6 - *Let*

$$\begin{aligned}
\bar{O} = \quad & \{(\iota_1,\iota_4,\iota_6,j_8,j_9,j_{10},t)\in(\mathbb{R}^{+*})^6\times\{-1,1\};\ \iota_4>0,\ \iota_4\iota_6-j_8^2>0, \\
& (\iota_4\iota_6-j_8^2)(\iota_1-(\iota_4+\iota_6))-\iota_4 j_{10}^2-\iota_6 j_9^2+2t\,j_8j_9j_{10}>0\} \\
& \cup\{(\iota_1,\iota_4,\iota_6,j_8,j_9,j_{10},t)\in(\mathbb{R}^{+})^6\times\{1\};\ j_8j_9j_{10}=0,\ \iota_4>0,\ \iota_4\iota_6-j_8^2>0, \\
& (\iota_4\iota_6-j_8^2)(\iota_1-(\iota_4+\iota_6))-\iota_4 j_{10}^2-\iota_6 j_9^2>0\}.
\end{aligned} \tag{43}$$

A material is orthotropic at point x with respect to the orthogonal vectors τ and τ' if and only there exists a function $\bar{z}$ defined on $\bar{O}$ such that, for any $\iota_O=(\iota_1,\iota_4,\iota_6,\iota_8,\iota_9,\iota_{10})$ in O,

$$z(\iota_O)=\bar{z}(|\iota_O|,\mathrm{sgn}(\iota_8\iota_9\iota_{10})) \tag{44}$$

where

$$|\iota_O|:=(\iota_1,\iota_4,\iota_6,|\iota_8|,|\iota_9|,|\iota_{10}|). \tag{45}$$

Proof - The set $\bar{O}$ defined in (43) is the image of O by the mapping $\iota_O \mapsto (|\iota_O|, \text{sgn}(\iota_8\iota_9\iota_{10}))$. Obviously, if z reads as in (44), then it satisfies conditions (38) and (39) and the material is orthotropic. For proving the converse assertion, it suffices to prove that if the material is orthotropic and if $(\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10})$ and $(\iota_1, \iota_4, \iota_6, \iota_8', \iota_9', \iota_{10}')$ in O satisfy $|\iota_8| = |\iota_8'|, |\iota_9| = |\iota_9'|, |\iota_{10}| = |\iota_{10}'|$ and $\text{sgn}(\iota_8\iota_9\iota_{10}) = \text{sgn}(\iota_8'\iota_9'\iota_{10}')$, then

$$z(\iota_1, \iota_4, \iota_6, \iota_8, \iota_9, \iota_{10}) = z(\iota_1, \iota_4, \iota_6, \iota_8', \iota_9', \iota_{10}').$$

Indeed, the following cases may arise:

- $\iota_8 = \iota_8', \iota_9 = \iota_9', \iota_{10} = \iota_{10}'$ and the result is obvious.
- $\iota_8 = \iota_8', \iota_9 = \iota_9', \iota_{10} = -\iota_{10}'$. In this case, either all values are non zero and the hypothesis on the signs is not fulfilled. Or one value is equal to 0: if $\iota_{10} = 0$, we are back to the previous case, if $\iota_9 = 0$, then $\iota_9 = -\iota_9'$ and we use (38), if $\iota_8 = 0$, then $\iota_8 = -\iota_8'$ and we use (42).
- $\iota_8 = \iota_8', \iota_9 = -\iota_9', \iota_{10} = \iota_{10}'$. This item is treated is the same way as the previous item.
- $\iota_8 = \iota_8', \iota_9 = -\iota_9', \iota_{10} = -\iota_{10}'$. We use (42).
- Cases when $\iota_8 = -\iota_8'$ are treated analogously to the previous four ones. $\square$

Remark 6 - *Formulations (38)-(39) and (44) underline the fact that $\tilde{W}$ can be expressed as a function z of six scalar variables that has some symmetry properties. Orthotropy is a weak restriction, since it requires the invariance of $\tilde{W}$ under the product by two matrices only. As can be expected, this results in a weak restriction on z: none of the variables in z disappears, the only restriction is that some parity has to be enforced.*

3.2 Comparison with classical results.

Equation (44) is a nice and simple formulation. In the sequel we try to link it to formulations that can be found in the literature.

Going back to existing works, we see that the set of pseudo-invariants that is commonly used is not the set $\iota_O = (\iota_k)_{k=1,4,6,8,9,10}$, but the set of seven elements $\iota^7 = (\iota_k)_{k=1}^7$ where $\iota_7(C) = \tau' \cdot C^2\tau'$. For instance, [3], [8] state that a material is orthotropic at point x in Ω with respect to τ and τ' if and only if there exists a mapping $w(x, \cdot)$ such that

$$\forall C \in \mathbb{S}_3^+, \, \tilde{W}(x, C) = w(x, \iota^7(C)). \tag{46}$$

Therefore, two questions arise: What is the relationship of results of Section 3.1 with these usual statements? Second, are seven variables necessary to express a function of six variables?

3.2.1 Focus on the six invariants $(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$.

In this section, we see that $\iota_2, \iota_5, \iota_7$ used by other authors might seem natural candidates to replace $\iota_8, \iota_9, \iota_{10}$ in the set of six coefficients ι_O used in Section 3.1. But, it turns out that they are only able to replace $\iota_8^2, \iota_9^2, \iota_{10}^2$, and that the sign of $\iota_8\iota_9\iota_{10}$ remains to be taken into account. Let

$$\begin{aligned} O_2 = \quad & \{(\iota_1, \iota_4, \iota_6, u, v, w, t) \in (\mathbb{R}^{+*})^6 \times \{-1, 1\};\ \iota_4 > 0,\ \iota_4\iota_6 - u > 0, \\ & (\iota_4\iota_6 - u)(\iota_1 - (\iota_4 + \iota_6)) - \iota_4 w - \iota_6 v + 2t\, u^{1/2} v^{1/2} w^{1/2} > 0\} \\ & \cup \{(\iota_1, \iota_4, \iota_6, u, v, w, t) \in (\mathbb{R}^{+})^6 \times \{1\};\ uvw = 0,\ \iota_4 > 0,\ \iota_4\iota_6 - u > 0, \\ & (\iota_4\iota_6 - u)(\iota_1 - (\iota_4 + \iota_6)) - \iota_4 w - \iota_6 v > 0\} \end{aligned}$$

and let $z_2 : O_2 \mapsto \mathbb{R}$ be defined by

$$z_2(\iota_1, \iota_4, \iota_6, u, v, w, t) = \bar{z}(\iota_1, \iota_4, \iota_6, u^{1/2}, v^{1/2}, w^{1/2}, t). \tag{47}$$

The subscript 2 has been chosen to recall that u, v and w play the role of squares. Then, we can reformulate (44) under the form

$$\tilde{W}(C) = z_2\big[\big(\iota_1, \iota_4, \iota_6, \iota_8^2, \iota_9^2, \iota_{10}^2, \operatorname{sgn}(\iota_8\iota_9\iota_{10})\big)(C)\big] = z_2\big[\big(\iota_{O,2}, \operatorname{sgn}(\iota_8\iota_9\iota_{10})\big)(C)\big] \tag{48}$$

where $\iota_{O,2} = (\iota_1, \iota_4, \iota_6, \iota_8^2, \iota_9^2, \iota_{10}^2)$. We remark that we can equivalently replace the triplet $(\iota_8^2, \iota_9^2, \iota_{10}^2)$ by $(\iota_2, \iota_5, \iota_7)$. Indeed, by computing ι_2, ι_5, ι_7 from (34), we have

$$\iota_2 = \iota_4\iota_6 + (\iota_4 + \iota_6)(\iota_1 - (\iota_4 + \iota_6)) - \iota_8^2 - \iota_9^2 - \iota_{10}^2, \tag{49}$$

$$\iota_5 = \iota_4^2 + \iota_8^2 + \iota_9^2, \tag{50}$$

$$\iota_7 = \iota_6^2 + \iota_8^2 + \iota_{10}^2. \tag{51}$$

Formulas (49)-(50)-(51) can readily be inverted under the form

$$\iota_8^2 = a(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7), \tag{52}$$

$$\iota_9^2 = b(\iota_1, \iota_2, \iota_4, \iota_6, \iota_7), \tag{53}$$

$$\iota_{10}^2 = c(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6). \tag{54}$$

where

$$a : \mathbb{R}^6 \mapsto \mathbb{R}, a(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7) = \iota_4\iota_6 - \iota_1(\iota_4 + \iota_6) + \iota_2 + \iota_5 + \iota_7, \tag{55}$$

$$b : \mathbb{R}^5 \mapsto \mathbb{R}, b(\iota_1, \iota_2, \iota_4, \iota_6, \iota_7) = (\iota_1 - \iota_4)(\iota_4 + \iota_6) - \iota_2 - \iota_7, \tag{56}$$

$$c : \mathbb{R}^5 \mapsto \mathbb{R}, c(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6) = (\iota_1 - \iota_6)(\iota_4 + \iota_6) - \iota_2 - \iota_5. \tag{57}$$

Therefore, letting

$$w(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7, t) = z_2(\iota_1, \iota_4, \iota_6, a, b, c, t), \tag{58}$$

$\tilde{W}(C)$ can be expressed for an orthotropic material as

$$\tilde{W}(C) = w[(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7, \operatorname{sgn}(\iota_8\iota_9\iota_{10}))(C)]. \tag{59}$$

But, there is no way of expressing $\mathrm{sgn}(\iota_8\iota_9\iota_{10})$ in terms of the first six invariants ι_1, ι_2, ι_4, ι_5, ι_6, ι_7 appearing in (59). Indeed, equations (49)-(50)-(51) show that for unchanged $\iota_1, \iota_4, \iota_6$, any element ι_8, ι_9 or ι_{10} can be changed in its opposite without modifying $\iota_2, \iota_5, \iota_7$. We can nevertheless make the product $\iota_8\iota_9\iota_{10}$ disappear, but at the cost of introducing ι_3. Indeed,

$$\iota_3 = (\iota_4\iota_6 - \iota_8^2)(\iota_1 - (\iota_4 + \iota_6)) + 2\,\iota_8\iota_9\iota_{10} - \iota_6\iota_9^2 - \iota_4\iota_{10}^2, \tag{60}$$

which, by use of (52)-(53)-(54), can be rewritten as

$$2\,\iota_8\iota_9\iota_{10} = \iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7) \tag{61}$$

where

$$d = (\iota_4 + \iota_6)(\iota_1^2 - 2\iota_1(\iota_4 + \iota_6) + 2\iota_4\iota_6 + 2\iota_2 + \iota_5 + \iota_7) + \iota_4\iota_5 + \iota_6\iota_7 - \iota_1(\iota_2 + \iota_5 + \iota_7). \tag{62}$$

Therefore, we have proved the following proposition.

Proposition 7 - *A material is orthotropic at point x with respect to the orthogonal vectors τ and τ' if and only if its energy can be written under the form*

$$\tilde{W}(C) = w\Big[\Big(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7,\ \mathrm{sgn}\big(\iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)\big)\Big)(C)\Big] \tag{63}$$

where w is defined on $\Big(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7,\ \mathrm{sgn}\big(\iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)\big)\Big)(\mathbb{S}_3^+)$.

3.2.2 An alternative proof.

In the previous section, we examined the following question: Are the values $(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$ evaluated in C sufficient to determine $\tilde{W}(C)$ when $\tilde{W}$ is orthotropic? Another way of solving this problem consists in identifying symmetric matrices that share the same invariants $(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$.

Proposition 8 - *There exists a symmetric matrix C such that $\iota_k(C) = \iota_k$ for any $k = 1, 2, 4, 5, 6, 7$ if and only if $a(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7) \geq 0$, $b(\iota_1, \iota_2, \iota_4, \iota_6, \iota_7) \geq 0$, $c(\iota_1, \iota_2, \iota_4, \iota_5, \iota_7) \geq 0$. When $a(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$, $b(\iota_1, \iota_2, \iota_4, \iota_6, \iota_7)$ and $c(\iota_1, \iota_2, \iota_4, \iota_5, \iota_7)$ are strictly positive, there are eight such matrices. They separate into two subsets: a subset of four matrices that can be written under the form $\{C_1, R_\tau^T C_1 R_\tau, R_{\tau'}^T C_1 R_{\tau'}, R_{\tau\wedge\tau'}^T C_1 R_{\tau\wedge\tau'}\}$, and another subset that can be written under the form $\{C_2, R_\tau^T C_2 R_\tau, R_{\tau'}^T C_2 R_{\tau'}, R_{\tau\wedge\tau'}^T C_2 R_{\tau\wedge\tau'}\}$.*

Proof - Computations are similar to what as already been written. Actually, we are looking for C that reads

$$C = \begin{pmatrix} \iota_4 & \alpha & \beta \\ \alpha & \iota_6 & \gamma \\ \beta & \gamma & \iota_1 - (\iota_4 + \iota_6) \end{pmatrix},$$

where α, β, γ must be such that

$$\iota_2 = \iota_4\iota_6 + (\iota_4 + \iota_6)(\iota_1 - (\iota_4 + \iota_6) - \alpha^2 - \beta^2 - \gamma^2, \tag{64}$$
$$\iota_5 = \iota_4^2 + \alpha^2 + \beta^2, \tag{65}$$
$$\iota_7 = \iota_6^2 + \alpha^2 + \gamma^2. \tag{66}$$

This is inverted into

$$\alpha^2 = \iota_4\iota_6 - \iota_1(\iota_4 + \iota_6) + \iota_2 + \iota_5 + \iota_7, \tag{67}$$
$$\beta^2 = (\iota_1 - \iota_4)(\iota_4 + \iota_6) - \iota_2 - \iota_7, \tag{68}$$
$$\gamma^2 = (\iota_1 - \iota_6)(\iota_4 + \iota_6) - \iota_2 - \iota_5. \tag{69}$$

As soon as a, b and c are nonnegative, the set of solutions of the above system is non empty and is given by $\alpha = \eta a^{\frac{1}{2}}, \beta = \varepsilon b^{\frac{1}{2}}, \gamma = \varepsilon' c^{\frac{1}{2}}$ where $\eta, \varepsilon, \varepsilon' = \pm 1$. Let

$$C_1 = \begin{pmatrix} \iota_4 & a^{\frac{1}{2}} & b^{\frac{1}{2}} \\ a^{\frac{1}{2}} & \iota_6 & c^{\frac{1}{2}} \\ b^{\frac{1}{2}} & c^{\frac{1}{2}} & \iota_1 - (\iota_4 + \iota_6) \end{pmatrix}, \; C_2 = \begin{pmatrix} \iota_4 & -a^{\frac{1}{2}} & b^{\frac{1}{2}} \\ -a^{\frac{1}{2}} & \iota_6 & c^{\frac{1}{2}} \\ b^{\frac{1}{2}} & c^{\frac{1}{2}} & \iota_1 - (\iota_4 + \iota_6) \end{pmatrix}.$$

Suppose $a > 0$, $b > 0$, $c > 0$. Then the four matrices such that $\eta\varepsilon\varepsilon' = 1$ are C_1, $R_\tau^T C_1 R_\tau$, $R_{\tau'}^T C_1 R_{\tau'}$, and $R_{\tau\wedge\tau'}^T C_1 R_{\tau\wedge\tau'}$ and the four matrices such that $\eta\varepsilon\varepsilon' = -1$ are C_2, $R_\tau^T C_2 R_\tau$, $R_{\tau'}^T C_2 R_{\tau'}$, and $R_{\tau\wedge\tau'}^T C_2 R_{\tau\wedge\tau'}$. □

Proposition 8 shows that it is not true that two matrices C and C' with the same six invariants $\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7$ satisfy either $C' = R_\tau^T C R_\tau$, or $C' = R_{\tau'}^T C R_{\tau'}$, or $C' = R_{\tau\wedge\tau'}^T C R_{\tau\wedge\tau'}$. Therefore, factorization of $\tilde{W}$ through $(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$ is not possible.

Let us compute the determinants of matrices C_1 and C_2 that have been introduced in the previous proof. Letting for short $a = a(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)$, $b = b(\iota_1, \iota_2, \iota_4, \iota_6, \iota_7)$, $c = c(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6)$, we obtain

$$\begin{aligned} \iota_3(C_1) &= (\iota_4\iota_6 - a)(\iota_1 - (\iota_4 + \iota_6)) - \iota_6 b - \iota_4 c + 2\,(abc)^{\frac{1}{2}}, \\ \iota_3(C_2) &= (\iota_4\iota_6 - a)(\iota_1 - (\iota_4 + \iota_6)) - \iota_6 b - \iota_4 c - 2\,(abc)^{\frac{1}{2}}, \end{aligned}$$

and, using d defined in (62),

$$\iota_3(C_1) - d = 2\,(abc)^{\frac{1}{2}},$$
$$\iota_3(C_2) - d = -2\,(abc)^{\frac{1}{2}}.$$

Therefore, knowledge of the sign of $\iota_3 - d$ selects one of the two subsets described in Proposition 8. This provides a new proof of Proposition 7.

3.2.3 Focus on the six invariants $(\iota_1, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7)$.

Another attempt in trying to reduce the classical set of seven invariants to a set of six invariants consists in trying to eliminate ι_2 while keeping ι_3. Indeed, for $\iota_1, \iota_4, \iota_6$ given, equations (61)-(50)-(51) constitute a set of three equations that provides ι_3, ι_5, ι_7 in terms of ι_8, ι_9, ι_{10}. Indeed,

$$\iota_3 = \bar{\kappa}_{1,4,6} - \kappa_{1,4,6}\,\iota_8^2 - \iota_6\,\iota_9^2 - \iota_4\,\iota_{10}^2 + 2\,\iota_8\iota_9\iota_{10}, \tag{70}$$
$$\iota_5 = \iota_4^2 + \iota_8^2 + \iota_9^2, \tag{71}$$
$$\iota_7 = \iota_6^2 + \iota_8^2 + \iota_{10}^2, \tag{72}$$

where we have let $\kappa_{1,4,6} = \iota_1 - (\iota_4 + \iota_6)$, $\bar{\kappa}_{1,4,6} = \iota_4\iota_6\,\kappa_{1,4,6}$. Let us examine the converse problem. We consider the system with unknowns $(\iota_8, \iota_9, \iota_{10})$ which reads

$$\kappa_{1,4,6}\,\iota_8^2 + \iota_6\,\iota_9^2 + \iota_4\,\iota_{10}^2 - 2\iota_8\iota_9\iota_{10} = \bar{\kappa}_{1,4,6} - \iota_3, \tag{73}$$
$$\iota_8^2 + \iota_9^2 = \iota_5 - \iota_4^2, \tag{74}$$
$$\iota_8^2 + \iota_{10}^2 = \iota_7 - \iota_6^2. \tag{75}$$

We assume that this system has at least one solution in O. It would be convenient that ι_8^2, ι_9^2, ι_{10}^2, and $\mathrm{sgn}(\iota_8\iota_9\iota_{10})$ are then uniquely determined. In such a case, we would be allowed to replace in (48) these quantities by functions of ι_1, ι_3, ι_4, ι_5, ι_6 and ι_7. But, this is not true. Indeed, we can construct symmetric definite positive matrices with identical $(\iota_1, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7)$ and non identical ι_8^2, ι_9^2, ι_{10}^2. Indeed, choose

$$C = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \end{pmatrix}, \; C' = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For both of them, $\iota_1 = 4, \iota_3 = 1, \iota_4 = 1, \iota_5 = \frac{3}{2}, \iota_6 = 1, \iota_7 = \frac{3}{2}$. But, for instance, $\iota_8(C) = 0$, $\iota_8(C') = \frac{1}{\sqrt{2}}$. Note that $\iota_2(C) = 4 \neq \iota_2(C') = \frac{5}{2}$.

Remark 7 - *We mentioned at the beginning of this section that if a material is isotropic transversely to τ, then it is orthotropic with respect to τ and to any τ' orthogonal to τ. This can be checked on the representation formulas. Actually, let $\tilde{W}$ be such that $\tilde{W}(C) = w(\iota^5(C))$. Then defining $\bar{z}$ by*

$$\bar{z}(\iota_1, \iota_4, \iota_6, j_8, j_9, j_{10}, t) = w(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5), \tag{76}$$

where $\iota_2 = \iota_4\iota_6 + (\iota_4 + \iota_6)(\iota_1 - (\iota_4 + \iota_6)) - j_8^2 - j_9^2 - j_{10}^2$, $\iota_3 = (\iota_4\iota_6 - j_8^2)(\iota_1 - (\iota_4 + \iota_6)) + 2\,t\,j_8 j_9 j_{10} - \iota_6 j_9^2 - \iota_4 j_{10}^2$, $\iota_5 = \iota_4^2 + j_8^2 + j_9^2$ – see (49), (60), (50) – we recover the expected expression (44).

3.2.4 Summary

The main conclusion of Section 3.1 is that an orthotropic energy fonction $\tilde{W}$ undertakes the reduced form

$$\tilde{W} \;=\; \bar{z}(\iota_1, \iota_4, \iota_6, |\iota_8|, |\iota_9|, |\iota_{10}|, \mathrm{sgn}(\iota_8\iota_9\iota_{10})), \tag{77}$$

which is an easy to read formulation. In Section 3.2, in order to compare our result with available formulas, we wrote

$$\tilde{W} = w(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7, \,\mathrm{sgn}(\iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)). \tag{78}$$

Let us look more closely at (78). Letting

$$y(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7) = w(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7, \,\mathrm{sgn}(\iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)), \tag{79}$$

we have

$$\tilde{W} = y(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7), \tag{80}$$

and conversely any $\tilde{W}$ of the form (80) is obviously orthotropic. Therefore, one might think that we have recovered the classical formulation. But, we are aware of the fact, left unclear in the literature up to our knowledge, that $(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7)$ are not independent functions of C. Actually, our previous computations show that

$$\iota_3 = d \pm 2\,(abc)^{1/2}. \tag{81}$$

In particular, considering the partial derivative of W with respect to ι_3 while keeping $\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7$ fixed is meaningless and should not be done when giving the expressions of constitutive laws.

3.3 Orthotropic constitutive laws

We recall that a hyperelastic energy is, by construction, differentiable with respect to C. As ι_4, ι_6, ι_8, ι_9, ι_{10} are coefficients of the matrix representing $\mathcal{C}$ in a given basis and as ι_1 can be expressed linearly in terms of the coefficients, it is clear that z defined in (37) is differentiable on O and that

$$\forall F \in \mathbb{M}_3^+,\ \hat{\Sigma}(F) = 2 \sum_{i=1,4,6,8,9,10} \frac{\partial z}{\partial \iota_i}(\iota_O(F^T F))\iota_i'(F^T F). \tag{82}$$

In other words, letting $\tau'' = \tau \wedge \tau'$,

$$\hat{\Sigma}(F) = 2\frac{\partial z}{\partial \iota_1} Id + 2\frac{\partial z}{\partial \iota_4}\tau\otimes\tau + 2\frac{\partial z}{\partial \iota_6}\tau'\otimes\tau' + \frac{\partial z}{\partial \iota_8}(\tau\otimes\tau')^s + \frac{\partial z}{\partial \iota_9}(\tau\otimes\tau'')^s + \frac{\partial z}{\partial \iota_{10}}(\tau'\otimes\tau'')^s$$

where all partial derivatives are taken in $\iota_O(F^T F)$ and where we have set $(\tau\otimes\nu)^s = \tau\otimes\nu + \nu\otimes\tau$ for any two vectors τ and ν. This identity does not carry any information – apart from the fact that the material is hyperelastic – as long as we do not use the symmetries of z. It is a decomposition of $\hat{\Sigma}(F)$ along a set of six independent symmetric matrices.

Let us now use the orthotropy and let us turn to $\bar{z}$ defined in (44). For any $(\bar{\iota}_1, \bar{\iota}_4, \bar{\iota}_6, \bar{j}_8, \bar{j}_9, \bar{j}_{10}, \bar{t})$ in $\bar{O}$, letting five of the first six variables and $\bar{t}$ fixed, there is an interval centered on the remaining variable (or a semi-interval if this variable is equal to 0) such that the product of the fixed variables and of this interval is included in $\bar{O}$. Therefore, considering the partial derivative of $\bar{z}$ with respect to any of its first six variables is meaningful. From the identities

$$\bar{z}(\iota_1,\iota_4,\iota_6,j_8,j_9,j_{10},1) = z(\iota_1,\iota_4,\iota_6,\eta j_8,\varepsilon j_9,\varepsilon' j_{10}) \quad \text{for} \quad \eta\varepsilon\varepsilon' = 1,$$

$$\bar{z}(\iota_1,\iota_4,\iota_6,j_8,j_9,j_{10},-1) = z(\iota_1,\iota_4,\iota_6,\eta j_8,\varepsilon j_9,\varepsilon' j_{10}) \quad \text{for} \quad \eta\varepsilon\varepsilon' = -1,$$

we compute the partial derivatives of $\bar{z}$ in terms of the partial derivatives of z. By using the reverse form of the obtained formulas, we obtain

$$\forall \iota_O \in O,\ \forall k = 1,4,6,\ \frac{\partial z}{\partial \iota_k}(\iota_O) = \frac{\partial \bar{z}}{\partial \iota_k}(|\iota_O|, \mathrm{sgn}(\iota_8\iota_9\iota_{10})),$$

$$\forall \iota_O \in O,\ \forall k = 8,9,10,\ \frac{\partial z}{\partial \iota_k}(\iota_O) = \mathrm{sgn}(\iota_k)\frac{\partial \bar{z}}{\partial j_k}(|\iota_O|, \mathrm{sgn}(\iota_8\iota_9\iota_{10})),$$

where we recall that $|\iota_O| = (\iota_1,\iota_4,\iota_6,|\iota_8|,|\iota_9|,|\iota_{10}|)$. This allows us to write

$$\begin{aligned}\hat{\Sigma} \quad &= 2\frac{\partial \bar{z}}{\partial \iota_1} Id + 2\frac{\partial \bar{z}}{\partial \iota_4}\tau\otimes\tau + 2\frac{\partial \bar{z}}{\partial \iota_6}\tau'\otimes\tau' \\ &+\mathrm{sgn}(\iota_8)\frac{\partial \bar{z}}{\partial j_8}(\tau\otimes\tau')^s + \mathrm{sgn}(\iota_9)\frac{\partial \bar{z}}{\partial j_9}(\tau\otimes\tau'')^s + \mathrm{sgn}(\iota_{10})\frac{\partial \bar{z}}{\partial j_{10}}(\tau'\otimes\tau'')^s \end{aligned} \tag{83}$$

where all partial derivatives are taken in $(|\iota_O|, \mathrm{sgn}(\iota_8\iota_9\iota_{10}))$. Equation (83) is the general form of an orthotropic constitutive law.

For isotropic (*resp.* transverse isotropic) laws, we have seen that $\hat{\Sigma}(F)$ which belongs to a 6-dimensional vector space can actually be decomposed along a set of three (*resp.* five) matrices. This is no longer true for orthotropic materials. Equation (83) is a decomposition along six matrices. The orthotropy properties are seen in the coefficients. Let us check for instance that $\hat{\Sigma}(FR_\tau) = R_\tau^T\,\hat{\Sigma}(F)\,R_\tau$, *i.e.* that $\tilde{\Sigma}(R_\tau^T C R_\tau) = R_\tau^T\,\hat{\Sigma}(C)\,R_\tau$. From (40) and (41), we know that $(\iota_8\iota_9\iota_{10})(R_\tau^T C R_\tau) = (\iota_8\iota_9\iota_{10})(C)$. Then all partial derivatives are taken at the same value $(|\iota_O|, \mathrm{sgn}(\iota_8\iota_9\iota_{10}))$ for both C and $R_\tau^T C R_\tau$. Moreover, we have $\tau\otimes\tau = R_\tau^T(\tau\otimes\tau)R_\tau$ because $R_\tau(\tau) = \tau$, and we have $\tau'\otimes\tau' = R_\tau^T(\tau'\otimes\tau')R_\tau$ because $R_\tau(\tau') = -\tau'$. This proves that the first line of (83) behaves as expected. Let us examine the first term of the second line: we have, on the one hand, $\mathrm{sgn}(\iota_8(R_\tau^T C R_\tau)) = -\mathrm{sgn}(\iota_8(C))$ and on the other hand $R_\tau^T(\tau\otimes\tau')^s R_\tau = -(\tau\otimes\tau')^s$ which gives as well the expected behavior. Remaining terms are treated in the same way.

Following the same path as above, we are able to get closer to classical formulations. We use first the mapping z_2 defined in (47) and, assuming that its partial derivatives exist – which when $uvw = 0$ is not a consequence of the hypotheses made till now – we obtain

$$\begin{aligned}\hat{\Sigma} = \quad & 2\,\frac{\partial z_2}{\partial \iota_1} Id + 2\,\frac{\partial z_2}{\partial \iota_4}\tau\otimes\tau + 2\,\frac{\partial z_2}{\partial \iota_6}\tau'\otimes\tau' \\ & +2\,\iota_8\frac{\partial z_2}{\partial u}(\tau\otimes\tau')^s + 2\,\iota_9\frac{\partial z_2}{\partial v}(\tau\otimes\tau'')^s + 2\,\iota_{10}\frac{\partial z_2}{\partial w}(\tau'\otimes\tau'')^s \end{aligned} \tag{84}$$

where all partial derivatives are taken in $\big(\iota_{O,2}, \mathrm{sgn}(\iota_8\iota_9\iota_{10})\big)$ with $\iota_{O,2} = (\iota_1,\iota_4,\iota_6,\iota_8^2,\iota_9^2,\iota_{10}^2)$. Then, the mapping w defined in (58) admits partial derivatives with respect to ι_1, ι_2, ι_4, ι_5,

ι_6, ι_7, and we obtain

$$\hat{\Sigma} = 2(\frac{\partial w}{\partial \iota_1}+\iota_1\frac{\partial w}{\partial \iota_2})Id-2\frac{\partial w}{\partial \iota_2}C+2\frac{\partial w}{\partial \iota_4}\tau\otimes\tau+2\frac{\partial w}{\partial \iota_5}(\tau\otimes C\tau)^s+2\frac{\partial w}{\partial \iota_6}\tau'\otimes\tau'+2\frac{\partial w}{\partial \iota_7}(\tau'\otimes C\tau')^s \quad (85)$$

where all partial derivatives are taken in $\big(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7, \mathrm{sgn}(\iota_3 - d(\iota_1, \iota_2, \iota_4, \iota_5, \iota_6, \iota_7)\big)$. Formula (85) may be more convenient than formula (83) for the experimental determination of the coefficients in the sense that it does not use the vector τ'' which is not a material line, *e.g.* a fiber, but only the vectors τ and τ' which represent material entities and along which one can measure the effect of a deformation. One can check that as the case must be the six matrices used in (83) can be expressed in terms of the six matrices used in (85) as soon as $\iota_8\iota_9\iota_{10} \neq 0$. Indeed, it suffices to write the expression of C, $(\tau\otimes C\tau)^s$ and $(\tau'\otimes C\tau')^s$ along $(Id,\ \tau\otimes\tau,\ \tau'\otimes\tau',\ (\tau\otimes\tau')^s,\ (\tau\otimes\tau'')^s,\ (\tau'\otimes\tau'')^s)$ and to invert a linear system with three unknowns in $\mathbb{S}_3$ to obtain

$$\begin{aligned}
(\tau\otimes\tau')^s = \quad & \frac{1}{\iota_8}[(\iota_1-(\iota_4+\iota_6))\,Id + (\iota_1-\iota_6)\,\tau\otimes\tau + (\iota_1-\iota_4)\,\tau'\otimes\tau' \\
& - C + (\tau\otimes C\tau)^s + (\tau'\otimes C\tau')^s], \\
(\tau\otimes\tau'')^s = \quad & \frac{1}{\iota_9}[(\iota_4+\iota_6-\iota_1)\,Id + (\iota_6-\iota_1-2\iota_4)\,\tau\otimes\tau + (\iota_4-\iota_1)\,\tau'\otimes\tau' \\
& + C - (\tau'\otimes C\tau')^s], \\
(\tau'\otimes\tau'')^s = \quad & \frac{1}{\iota_{10}}[(\iota_4+\iota_6-\iota_1)\,Id + (\iota_6-\iota_1)\,\tau\otimes\tau + (\iota_4-\iota_1-2\iota_6)\,\tau'\otimes\tau' \\
& + C - (\tau\otimes C\tau)^s].
\end{aligned}$$

Formula (85) is worth comparing with the classical formula, quoted for instance in formula (87) in [Ogden]. In the latter, $\hat{\Sigma}$ is decomposed along a set of eight matrices, namely the six matrices that appear in (85) plus C^{-1} and $(\tau\otimes\tau')^s$. Our reasoning shows that this is unnecessary.

4 Non orthogonal preferred directions

For non orthogonal preferred directions, $\tilde{W}$ is seen in some classical works as a function of $\iota_k, k = 1, \ldots, 8$, which makes it a function of eight variables. It will be seen in this section that ι_2 and ι_3 can take two real values at most when the other six variables are known.

In Definition 3, we stated that a material admits two non orthogonal preferred directions τ and τ' if and only if $R_{\tau''} \in \mathcal{G}_x$ where $\tau'' = \tau\wedge\tau'$. Therefore, as already pointed out, this notion only uses the direction orthogonal to the plane generated by τ and τ' and not the vectors themselves. Let $\check{\tau}$ and $\check{\tau}'$ be two orthogonal unit vectors that generate the same plane as τ and τ'. We can decompose $\mathcal{C}$ along the orthonormal basis $(\check{\tau}, \check{\tau}', \tau'')$. The obtained matrix reads with obvious notations

$$\begin{pmatrix} \check{\iota}_4(\mathcal{C}) & \check{\iota}_8(\mathcal{C}) & \check{\iota}_9(\mathcal{C}) \\ \check{\iota}_8(\mathcal{C}) & \check{\iota}_6(\mathcal{C}) & \check{\iota}_{10}(\mathcal{C}) \\ \check{\iota}_9(\mathcal{C}) & \check{\iota}_{10}(\mathcal{C}) & \check{\iota}_1(\mathcal{C}) - (\check{\iota}_4(\mathcal{C}) + \check{\iota}_6(\mathcal{C})) \end{pmatrix}.$$

Let us denote by $\check{z}$ the energy expressed in terms of $(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, \check{\iota}_9, \check{\iota}_{10})$. As seen in (42) the only condition that $\check{z}$ has to satisfy is

$$\check{z}(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, \check{\iota}_9, \check{\iota}_{10}) = \check{z}(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, -\check{\iota}_9, -\check{\iota}_{10}),$$

or, equivalently,

$$\check{z}(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, \check{\iota}_9, \check{\iota}_{10}) = \bar{\check{z}}(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, |\check{\iota}_9|, |\check{\iota}_{10}|, \operatorname{sgn}(\check{\iota}_9\check{\iota}_{10})),$$

where $\bar{\check{z}}$ is defined on

$$\begin{aligned}\bar{\check{O}} = \quad & \{(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, \check{j}_9, \check{j}_{10}, t) \in \mathbb{R}^4 \times (\mathbb{R}^{+*})^2 \times \{-1, 1\};\, \check{\iota}_4 > 0,\, \check{\iota}_4\check{\iota}_6 - \check{\iota}_8^2 > 0, \\ & (\check{\iota}_4\check{\iota}_6 - \check{\iota}_8^2)(\check{\iota}_1 - (\check{\iota}_4 + \check{\iota}_6)) - \check{\iota}_4\check{j}_{10}^2 - \check{\iota}_6\check{j}_9^2 + 2\,t\,\check{\iota}_8\check{j}_9\check{j}_{10} > 0\} \\ & \cup \{(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, \check{j}_9, \check{j}_{10}, t) \in \mathbb{R}^4 \times (\mathbb{R}^{+})^2 \times \{1\};\, \check{j}_9\check{j}_{10} = 0,\, \check{\iota}_4 > 0,\, \check{\iota}_4\check{\iota}_6 - \check{\iota}_8^2 > 0, \\ & (\check{\iota}_4\check{\iota}_6 - \check{\iota}_8^2)(\check{\iota}_1 - (\check{\iota}_4 + \check{\iota}_6)) - \check{\iota}_4\check{j}_{10}^2 - \check{\iota}_6\check{j}_9^2 > 0\}.\end{aligned} \tag{86}$$

We recall that any element in (86) necessarily satisfies $\check{\iota}_1 > 0$, $\check{\iota}_6 > 0$ in addition to $\check{\iota}_4 > 0$. On the contrary, the sign of $\check{\iota}_8$ is arbitrary. Therefore, the general form of a constitutive law admitting two non orthogonal preferred directions in the plane generated by $\check{\tau}$ and $\check{\tau}'$ is

$$\begin{aligned}\hat{\Sigma} \quad & = 2\frac{\partial \bar{\check{z}}}{\partial \check{\iota}_1} Id + 2\frac{\partial \bar{\check{z}}}{\partial \check{\iota}_4}\check{\tau}\otimes\check{\tau} + 2\frac{\partial \bar{\check{z}}}{\partial \check{\iota}_6}\check{\tau}'\otimes\check{\tau}' \\ & + \frac{\partial \bar{\check{z}}}{\partial \check{\iota}_8}(\check{\tau}\otimes\check{\tau}')^s + \operatorname{sgn}(\check{\iota}_9)\frac{\partial \bar{\check{z}}}{\partial \check{j}_9}(\check{\tau}\otimes\tau'')^s + \operatorname{sgn}(\check{\iota}_{10})\frac{\partial \bar{\check{z}}}{\partial \check{j}_{10}}(\check{\tau}'\otimes\tau'')^s\end{aligned} \tag{87}$$

where all partial derivatives are taken in $(\check{\iota}_1, \check{\iota}_4, \check{\iota}_6, \check{\iota}_8, |\check{\iota}_9|, |\check{\iota}_{10}|, \operatorname{sgn}(\check{\iota}_9\check{\iota}_{10}))$. As for $k = 1, 4, 6, 8$, $\check{\iota}_k(R^T_{\tau''}CR_{\tau''}) = \check{\iota}_k(C)$ and as for $k = 9, 10$, $\check{\iota}_k(R^T_{\tau''}CR_{\tau''}) = -\check{\iota}_k(C)$, one can check that $\hat{\Sigma}$ satisfies $\hat{\Sigma}(R^T_{\tau''}CR_{\tau''}) = R^T_{\tau''}\,\hat{\Sigma}(C)\,R_{\tau''}$ as it has to.

The disadvantage of this formulation is that it does not use the elements τ and τ' that have material existence. In order to make τ and τ' apparent in the final formulation, we choose to decompose the operators on the non orthonormal basis $(\tau, \tau', \tau\wedge\tau')$. In so doing we can prove the following proposition.

Proposition 9 - *i) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two symmetric operators such that ι_1, ι_4, ι_6, ι_8, $|\iota_9|$, $|\iota_{10}|$, $\operatorname{sgn}(\iota_9\iota_{10})$ take the same values on $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, either $\mathcal{C}_2 = \mathcal{C}_1$, or $\mathcal{C}_2 = r^T_{\tau''}\mathcal{C}_1 r_{\tau''}$.*

ii) There are generically four symmetric operators such that the six invariants ι_1, ι_4, ι_5, ι_6, ι_7, ι_8 take the same given values, when the set of these values belongs to $(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8)(\mathbb{S}_3^+)$. For those four operators ι_3 can take at most two distinct values. Choosing one of this values reduces the set of operators to two – possibly equal – operators that read $\mathcal{C}$ and $r^T_{\tau''}\mathcal{C}r_{\tau''}$.

Proof - *i)* Let $\mathcal{C}$ be a symmetric operator and let

$$E = \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{pmatrix} \tag{88}$$

be its matrix on the basis $(\tau, \tau', \tau \wedge \tau')$. By the symmetry hypothesis, $\mathcal{C}\tau \cdot \tau' = \mathcal{C}\tau' \cdot \tau$, $\mathcal{C}\tau' \cdot \tau'' = \mathcal{C}\tau'' \cdot \tau'$, and $\mathcal{C}\tau'' \cdot \tau = \mathcal{C}\tau \cdot \tau''$, which gives

$$\alpha' = \beta + q\,(\alpha - \beta'),\ \alpha'' = |\tau''|^2\, p\,(\gamma - q\gamma'),\ \beta'' = |\tau''|^2\, p\,(\gamma' - q\gamma)$$

where we have set $q = \tau \cdot \tau'$, $p = (1 - q^2)^{-1}$. Therefore, E is entirely known when the six coefficients α, β, γ, β', γ', γ'' are known. Let us now use the fact that ι_4, ι_6, ι_8 are given. From

$$\begin{aligned} \iota_4 &= \mathcal{C}\tau \cdot \tau = \alpha + q\,\beta, \quad & \iota_8 &= \mathcal{C}\tau \cdot \tau' = q\,\alpha + \beta, \\ \iota_6 &= \mathcal{C}\tau' \cdot \tau' = q\,\alpha' + \beta', \quad & \iota_8 &= \mathcal{C}\tau' \cdot \tau = \alpha' + q\,\beta', \end{aligned}$$

we derive the four values α, β, α', β'. Namely,

$$\alpha = p\,(\iota_4 - q\,\iota_8),\ \beta = p\,(\iota_8 - q\,\iota_4),\ \alpha' = p\,(\iota_8 - q\,\iota_6),\ \beta' = p\,(\iota_6 - q\,\iota_8). \tag{89}$$

The value γ'' is determined by assuming that ι_1 is given. Indeed,

$$\gamma'' = \iota_1 - (\alpha + \beta'). \tag{90}$$

Now,

$$\iota_9 = \mathcal{C}\tau \cdot \tau'' = \gamma\,|\tau''|^2,\ \iota_{10} = \mathcal{C}\tau' \cdot \tau'' = \gamma'\,|\tau''|^2.$$

Therefore, if $|\iota_9|$ and $|\iota_{10}|$ are given, then $|\gamma|$ and $|\gamma'|$ are determined. Finally, if in addition $\mathrm{sgn}(\iota_9\iota_{10})$ is given, then $\gamma = \varepsilon(|\iota_9|/|\tau''|^2)$, $\gamma' = \varepsilon'(|\iota_{10}/|\tau''|^2)$, with $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$ and $\varepsilon\varepsilon' = \mathrm{sgn}(\iota_9\iota_{10})$. And E reads

$$E = \begin{pmatrix} \alpha & \alpha' & p\,(\varepsilon|\iota_9| - q\,\varepsilon'|\iota_{10}|) \\ \beta & \beta' & p\,(\varepsilon''|\iota_{10}| - q\,\varepsilon|\iota_9|) \\ \varepsilon\dfrac{|\iota_9|}{|\tau''|^2} & \varepsilon'\dfrac{|\iota_{10}|}{|\tau''|^2} & \gamma'' \end{pmatrix} \tag{91}$$

with $\varepsilon\varepsilon' = \mathrm{sgn}(\iota_9\iota_{10})$. The conclusion follows.

ii) With a slight misuse of notation, let $(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8)$ be an element of $(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8)(\mathbb{S}_3^+)$. From point *i)*, we know that the matrix of a symmetric operator with the given values $(\iota_1, \iota_4, \iota_6, \iota_8)$ is of the form (88) with α, β, α', β' given by (89) and γ'' given by (90). It remains to determine γ and γ' from which α'' and β'' follow by the formulas $\alpha'' = |\tau''|^2\, p\,(\gamma - q\gamma')$, $\beta'' = |\tau''|^2\, p\,(\gamma' - q\gamma)$. Since

$$\begin{aligned} \iota_5 &= \alpha^2 + \beta^2 + 2\alpha\beta(\tau \cdot \tau') + \gamma^2|\tau''|^2, \\ \iota_7 &= \alpha'^2 + \beta'^2 + 2\alpha'\beta'(\tau \cdot \tau') + \gamma'^2|\tau''|^2, \end{aligned}$$

γ and γ' satisfy $\gamma = \varepsilon\Gamma$, $\gamma' = \varepsilon'\Gamma'$ where

$$\Gamma = \frac{[\iota_5 - (\alpha^2 + \beta^2 + 2\,q\,\alpha\beta)]^{1/2}}{|\tau''|} = \frac{[\iota_5 - p(\iota_4^2 + \iota_8^2 - 2\,q\,\iota_4\iota_8)]^{1/2}}{|\tau''|}$$

$$\Gamma' = \frac{[\iota_7 - (\alpha'^2 + \beta'^2 + 2\,q\,\alpha'\beta')]^{1/2}}{|\tau''|} = \frac{[\iota_7 - p(\iota_6^2 + \iota_8^2 - 2\,q\,\iota_6\iota_8)]^{1/2}}{|\tau''|} \tag{92}$$

and $\varepsilon = \pm 1$, $\varepsilon' = \pm 1$. When Γ and Γ' are non equal to 0, this gives exactly four symmetric opertors and all of them may be definite positive. Let us now compute the determinant of these operators. Their matrices read

$$E = \begin{pmatrix} \alpha & \alpha' & |\tau''|^2\,p\,(\varepsilon\Gamma - q\,\varepsilon'\Gamma') \\ \beta & \beta' & |\tau''|^2\,p\,(\varepsilon'\Gamma' - q\,\varepsilon\Gamma) \\ \varepsilon\Gamma & \varepsilon'\Gamma' & \gamma'' \end{pmatrix}. \tag{93}$$

We obtain

$$\begin{aligned} \iota_3 &= (\alpha\beta' - \beta\alpha')\gamma'' - p\,|\tau''|^2\big((q\,\alpha' + \beta')\Gamma^2 + (q\,\beta + \alpha)\Gamma'^2 - (\alpha' + \beta + q\,(\alpha + \beta'))\Gamma\Gamma'\varepsilon\varepsilon'\big) \\ &= (\alpha\beta' - \beta\alpha')\gamma'' - p\,|\tau''|^2(\iota_6\Gamma^2 + \iota_4\Gamma'^2) + 2p\,|\tau''|^2\iota_8\Gamma\Gamma'\varepsilon\varepsilon', \end{aligned} \tag{94}$$

where we recall that α, β', β, α', γ'', Γ, Γ' are known. As $\varepsilon\varepsilon'$ can only be equal to 1 or to -1, this shows that ι_3 can take at most two distinct values. $\square$

An immediate consequence of the first point of Proposition 9 is the characterization of the energies we aimed at.

Corollary 1 - *The energy of a material admitting two non orthogonal preferred directions τ and τ' can be written under the form*

$$\tilde{W} = \bar{z}(\iota_1, \iota_4, \iota_6, \iota_8, |\iota_9|, |\iota_{10}|, \mathrm{sgn}(\iota_9\iota_{10})). \tag{95}$$

Remark 8 - *The second point of Proposition 9 shows that ι_3 is not independent of ι_1, ι_4, ι_5, ι_6, ι_7, ι_8. These six values are not sufficient to determine a symmetric operator. But, once they are known, ι_3 can only take two values. By selecting one of those values, one reduces the set of operators to two operators that read $\mathcal{C}$ and $r_{\tau''}^T \mathcal{C} r_{\tau''}$. Therefore, introducing a partial derivative with respect to ι_3 in the expression of the constitutive law in addition to partial derivative with respect to ι_1, ι_4, ι_5, ι_6, ι_7, ι_8 is meaningless. In a analog way, it is immediate that for E of the form (93), $\iota_2(E)$ can only take two values. Therefore, ι_2 should not be considered either as a variable independent of ι_1, ι_4, ι_5, ι_6, ι_7, ι_8.*

The main conclusion of this section is that the energy fonction $\tilde{W}$ of a material that admits two non orthogonal preferred directions reads

$$\tilde{W} = \bar{z}(\iota_1, \iota_4, \iota_6, \iota_8, |\iota_9|, |\iota_{10}|, \mathrm{sgn}(\iota_9\iota_{10})). \tag{96}$$

By rewriting (94) under the form

$$\iota_3 = f(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8) + \varepsilon\varepsilon'\,g(\iota_4, \iota_5, \iota_6, \iota_7, \iota_8) \tag{97}$$

with

$$f(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8) = p\,(\iota_4\iota_6 - \iota_8^2)\iota_1 + 2p^2\,\iota_8^2(\iota_4 + \iota_6 - q\,\iota_8) - 2p^2\,q\,\iota_4\iota_6\iota_8 - p\,(\iota_5\iota_6 + \iota_4\iota_7), \quad (98)$$

$$g(\iota_4, \iota_5, \iota_6, \iota_7, \iota_8) = 2p\,\iota_8(\iota_5 - p(\iota_4^2 + \iota_8^2 - 2\,q\,\iota_4\iota_8))^{1/2}(\iota_7 - p(\iota_6^2 + \iota_8^2 - 2\,q\,\iota_6\iota_8))^{1/2}, \quad (99)$$

the energy reads as well

$$\tilde{W} = w(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8, \mathrm{sgn}\big((\iota_3 - f)/g\big). \quad (100)$$

From (96), we derive the general form of constitutive law admitting the non orthogonal preferred directions τ and τ'. It reads

$$\begin{aligned} \hat{\Sigma} \quad &= 2\frac{\partial \bar{z}}{\partial \iota_1} Id + 2\frac{\partial \bar{z}}{\partial \iota_4}\tau\otimes\tau + 2\frac{\partial \bar{z}}{\partial \iota_6}\tau'\otimes\tau' \\ &+\frac{\partial \bar{z}}{\partial \iota_8}(\tau\otimes\tau')^s + \mathrm{sgn}(\iota_9)\frac{\partial \bar{z}}{\partial j_9}(\tau\otimes\tau'')^s + \mathrm{sgn}(\iota_{10})\frac{\partial \bar{z}}{\partial j_{10}}(\tau'\otimes\tau'')^s \end{aligned} \quad (101)$$

where all partial derivatives are taken in $(\iota_1, \iota_4, \iota_6, \iota_8, |\iota_9|, |\iota_{10}|, \mathrm{sgn}(\iota_9\iota_{10}))$. This form is similar to (87) and, as in the first part of this section, one can check that $\hat{\Sigma}$ satisfies $\hat{\Sigma}(R_{\tau''}^T C R_{\tau''}) = R_{\tau''}^T \hat{\Sigma}(C) R_{\tau''}$ as it has to.

From (100), we derive another formulation closer to traditional ones. Actually,

$$\hat{\Sigma} = 2\frac{\partial w}{\partial \iota_1}Id + 2\frac{\partial w}{\partial \iota_4}\tau\otimes\tau + 2\frac{\partial w}{\partial \iota_5}(\tau\otimes C\tau)^s + 2\frac{\partial w}{\partial \iota_6}\tau'\otimes\tau' + 2\frac{\partial w}{\partial \iota_7}(\tau'\otimes C\tau')^s + \frac{\partial w}{\partial \iota_8}(\tau\otimes\tau')^s \quad (102)$$

where all partial derivatives are taken in $\big(\iota_1, \iota_4, \iota_5, \iota_6, \iota_7, \iota_8,\ \mathrm{sgn}(\iota_3 - f/g)\big)$.

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