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Bernard Roynette, Pierre Vallois, Marc Yor

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B. Roynette[(1)], P. Vallois[(2)], M. Yor[(3), (4)]

[(1)] Institut Élie Cartan, Nancy-Université, UMR 7502,CNRS, INRIA.

[(2)] Institut Élie Cartan, Nancy-Université, UMR 7502,CNRS, INRIA.

[(3)] Laboratoire de Probabilités et Modèles Aléatoires,
Université Paris VI et VII, 4 place Jussieu - Case 188
F - 75252 Paris Cedex 05

[(4)] Institut Universitaire de France

Abstract We develop a Brownian penalisation procedure related to weight processes (F_t) of the type : $F_t := f(I_t, S_t)$ where f is a bounded function with compact support and S_t (resp. I_t) is the one-sided maximum (resp. minimum) of the Brownian motion up to time t. Two main cases are treated : either F_t is the indicator function of $\{I_t \geq \alpha,\ S_t \leq \beta\}$ or F_t is null when $\{S_t - I_t > c\}$ for some $c > 0$. Then we apply these results to some kind of asymptotic Skorokhod embedding problem.

Key words : Skorokhod's problem, penalisation, one-sided maximum and minimum, Laplace's method.

Mathematics subject classification : 60G17, 60G40, 60G44, 60H10,60J25,60J60, 60J65.

1 Introduction

In a series of papers [18], [17], [16], [21], [23], [22], [24] (see also the surveys [19] and [20] and the monograph [25]) we have introduced a penalisation procedure of Brownian paths and applied it to many settings. To present the aim of this paper, we first introduce a few notations.
Let $\big(\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R}), (X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}\big)$ be the canonical space, where $(X_t)_{t\geq 0}$ denotes the coordinate maps : $X_t(\omega) = \omega(t)$, for any $t \geq 0$. Let $(P_x)_{x\in\mathbb{R}}$ be the family of Wiener probability measures on Ω : under P_x, $(X_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion started at x.
Next, we consider a stochastic process $(F_t)_{t\geq 0}$ which takes its values in $[0, \infty[$ and satisfies :

$$0 < E_0(F_t) < \infty \quad \forall t \geq 0. \tag{1.1}$$

We shall say that the penalisation procedure (associated with the weight process (F_t)) holds if :

$$Q_0^F(\Lambda_s) := \lim_{t\to\infty} \frac{E_0[1_{\Lambda_s} F_t]}{E_0[F_t]} \text{ exists for any } s \geq 0 \text{ and } \Lambda_s \in \mathcal{F}_s. \tag{1.2}$$

We briefly recall (see Theorem 3.6 in [17], [16] and [21]) that the penalisation procedure holds with $F_t = \varphi(S_t)$ where :

1. $\varphi : \ [0, \infty[\to [0, \infty[$ is a Borel function such that $\displaystyle\int_0^\infty \varphi(x)dx = 1$
2. $(S_t)_{t\geq 0}$ is the one-sided maximum process associated with $(X_t)_{t\geq 0}$:

$$S_t := \max_{u\leq t} X_u \ ; \quad t \geq 0. \tag{1.3}$$

According to Theorem 4.6 of [17], under Q_0^F, the r.v. S_∞ is finite and admits φ as a density function. Consequently, the procedure (1.2) forces Brownian motion to have a finite one-sided total maximum with the given probability density φ. This result presents some analogy with Skorokhod's embedding problem.

1.1 On Skorokhod's problem for linear Brownian motion

Let μ be a probability measure (p.m.) on $\mathbb{R}$, such that :

$$\int_{\mathbb{R}} |y| \, \mu(dy) < \infty \quad \text{and} \quad \int_{\mathbb{R}} y \, \mu(dy) = 0. \tag{1.4}$$

A number of constructions of (finite) stopping times T such that (i) T is standard, i.e. $(X_{s\wedge T} \ ; \ s \geq 0)$ is a uniformly integrable martingale, and (ii) the distribution of X_T, under P_0 is μ, have been made by many authors, see e.g. Obłój's thorough survey [11] of the subject. We briefly recall the particular construction given by Azéma and Yor ([2], [1]) : there exists a non-decreasing function $\phi_\mu : [0, +\infty[\to \mathbb{R}$ such that for :

$$T_\mu := \inf \big\{ t \geq 0, \quad X_t \leq \phi_\mu(S_t) \big\}, \tag{1.5}$$

T_μ is standard and $X_{T_\mu} \sim \mu$ (under P_0).
Precisely, ϕ_μ is the right-continuous inverse of ψ_μ where : $\displaystyle\psi_\mu(x) = \frac{1}{\mu(]x, \infty[)} \int_{]x,\infty[} y d\mu(y).$

1.2 An asymptotic resolution of Skorokhod's problem for diffusions

Similarly to Skorokhod embedding problem for Brownian motion, let us start with a target p.m. μ on $\mathbb{R}$. For simplicity we suppose that μ admits a density function μ_0 which is supposed to be positive and of class C_b^1 : $\mu(dx) = \mu_0(x)dx$.
Then, there exists a p.m. $\mathbb{Q}_0$ on $(\Omega, \mathcal{F}_\infty)$ such that :

$$X_t = B_t + \frac{1}{2} \int_0^t \frac{\mu_0'(X_s)}{\mu_0(X_s)} \, ds, \quad t \geq 0 \tag{1.6}$$

and $(B_t)_{t\geq 0}$ is a $\mathbb{Q}_0$-Brownian motion started at 0.
Moreover, under $\mathbb{Q}_0$, X_t converges in distribution to μ, as $t \to \infty$. In other words, we have introduced a diffusion process, whose limit distribution is the given p.m. μ, which may be considered as an asymptotic kind of resolution of Skorokhod's problem.

1.3 Solving Skorokhod's problem for (S_t, X_t)

The solution of Skorokhod's problem given by Azéma and Yor [2] suggests to consider the embedding problem for the two-dimensional process $(S_t, X_t)_{t\geq 0}$. A complete answer is given by Rogers [13] (see also [15]). Starting with a p.m. ν on $\mathbb{R}_+ \times \mathbb{R}$, assumed to satisfy :

$$\int_{\mathbb{R}_+\times\mathbb{R}} |y|\, \nu(dx, dy) < \infty, \quad \int_{\mathbb{R}_+\times\mathbb{R}} y\, \nu(dx, dy) = 0 \tag{1.7}$$

$$a\, \nu\big([a, \infty[\times\mathbb{R}\big) = \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{x\geq a\}}\, y\, \nu(dx, dy), \quad \forall a \geq 0, \tag{1.8}$$

it is shown that there exists a finite standard stopping time T such that $(S_T, X_T) \sim \nu$ (under P_0).

Note that (1.7) and (1.8) correspond to

$$E\big[|X_T|\big] < \infty, \quad E\big[X_T\big] = 0, \tag{1.9}$$

resp.

$$aP(S_T \geq a) = E\big[X_T | S_T \geq a\big], \quad \forall\, a \geq 0. \tag{1.10}$$

1.4 The (S_t, X_t) asymptotic resolution of Skorokhod's problem for diffusion processes

Comparing Subsections 1.2 and 1.3, we may ask the following question : for which class of p.m.'s ν on $\mathbb{R}_+ \times \mathbb{R}$, does there exist a p.m. $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_\infty)$ under which :

$$X_t = B_t + \int_0^t b(u, X.)du \tag{1.11}$$

where

$(B_t)_{t\geq 0}$ is a $\mathbb{Q}_1$-standard Brownian motion with $B_0 = 0$, (1.12)

$\big(b(t, X.)\big)_{t\geq 0}$ is an $(\mathcal{F}_t)$-adapted process, (1.13)

the couple (S_t, X_t) converges in distribution towards ν, as $t \to \infty$. (1.14)

Our approach is based on a Brownian penalisation procedure. This method is well fitted for our purpose since it permits to obtain Markov processes whose distributions are locally equivalent to that of Brownian motion and enjoy new path properties (for instance a finite total unilateral maximum, see the beginning of Introduction).

Unfortunately we have not been able to solve completely this question. In Section 6, we only give a class of p.m.'s ν verifying (1.11)-(1.14).

1.5 Organisation of the paper

Section 2 is a short survey of Brownian penalisations. To show that the penalisation procedure (see (1.2) or Section 2 for more details) associated with a weight process (F_t) holds, we need to be able to determine the rates of decay of $E_0[F_t]$ and $E[F_t|\mathcal{F}_s]$ as $t \to \infty$. In this paper, we consider penalisations with $F_t = f(I_t, S_t)$ where $f :]-\infty, 0] \times [0, \infty[\to]0, \infty[$ is a Borel bounded function with compact support and

$$I_t := \inf_{u \le t} X_u \qquad (t \ge 0).$$

To determine the rate of decay of $t \mapsto E\big[f(I_t, S_t)\big]$ as $t \to \infty$, we have been led to consider two classes of functions f. These developments are given in Section 3. With these estimates at hand, we shall show in Section 4 that the associated penalisation procedures hold. These schemes give rise to two families of p.m. on the canonical space $(\Omega, \mathcal{F}_\infty)$. In Section 5 we determine the law of the process (X_t) under each of these new p.m.'s. Finally, in Section 6 we apply our results to the question discussed above in subsection 1.4.

2 A survey of Brownian penalisations

We keep notations from the Introduction.

2.1 The goal

Penalisations provide a method to define $P_0(\cdot\ |A)$ for certain negligible events A in $\mathcal{F}_\infty := \bigvee_{t \ge 0} \mathcal{F}_t$, i.e. $P_0(A) = 0$.

This question arises naturally in probability theory and especially in the study of stochastic processes. Let us give a few explicit examples :

$$A_1 = \{X_t \ge 0\ ;\ \forall t \ge 0\} \tag{2.1}$$

$$A_2 = \{\sup_{t \ge 0} X_t \le a\}, \quad (a > 0) \tag{2.2}$$

$$A_3 = \{\inf_{t \ge 0} X_t \ge \alpha,\ \sup_{t \ge 0} X_t \le \beta\} \quad (\alpha < 0,\ \beta > 0). \tag{2.3}$$

Conditioning by A_1 may be treated by h-Doob's transforms (see for instance Section 4, Chap. V in [4]). As for (2.2), it is proved in [9], [10] that for any $0 < t_1 < \cdots < t_n$, the conditional distribution of the random vector $\big(X_{t_1}, \cdots, X_{t_n}\big)$ given $\{S_t \le a\}$ converges as $t \to \infty$.

The third case is the subject of our study. It is then demanded that $(X_t)_{t \ge 0}$ stays in the strip $[\alpha, \beta]$.

2.2 A solution via approximation

Given a decreasing family $(A_t)_{t \ge 0}$ of events in $\mathcal{F}_\infty$ such that $P_0(A_t) > 0$, $\forall\ t \ge 0$, we set $A = \bigcap_{t \ge 0} A_t$.

As an example, the set A_3 given by (2.3) satisfies : $A_3 = \bigcap_{t \ge 0} A_{3,t}$, with

$A_{3,t} := \{I_t \ge \alpha, S_t \le \beta\}$, where $(I_t)_{t \ge 0}$ is the one-sided minimum process, i.e.

$$I_t := \inf_{u \le t} X_u, \quad t \ge 0. \tag{2.4}$$

Going back to the general case of the family $(A_t)_{t \ge 0}$, we would like to define :

$$P_0^A(\Lambda) := \lim_{t \to \infty} P_0(\Lambda | A_t), \tag{2.5}$$

for $\Lambda \in \mathcal{F}_\infty$ such that the limit exists.
At this stage, three questions arise immediately :

$$\text{for which } \Lambda, \text{ does } P_0^A(\Lambda) \text{ exist ?} \tag{2.6}$$

$$\text{Can } P_0^A \text{ be extended to a p.m. on } (\Omega, \mathcal{F}_\infty) \text{ ?} \tag{2.7}$$

$$\text{How does } P_0^A \text{ depend on the family } (A_t)_{t\geq 0} \text{ ?} \tag{2.8}$$

2.3 A penalisation procedure

It is actually easier to generalize the previous approach by replacing $(A_t)_{t\geq 0}$ by a stochastic process $(F_t)_{t\geq 0}$ which takes its values in $[0,\infty[$ and satisfies (1.1).
Our penalisation procedure is the following : the assumptions of the next theorem have been shown to be satisfied for a large number of such weight processes $\big(F_t\big)_{t\geq 0}$; see [18], [17], [16], [21], [23] and [24].

Theorem 2.1 *Let $(F_t)_{t\geq 0}$ as above. Assume that :*

$$\frac{E_0[F_t|\mathcal{F}_s]}{E_0(F_t)} \underset{t\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} M_s^F \quad \forall s \geq 0 \tag{2.9}$$

and

$$E_0(M_s^F) = 1 \quad \forall s \geq 0. \tag{2.10}$$

Then :

1. *$(M_s^F \; ; \; s \geq 0)$ is a non-negative P_0-martingale.*
2. *For any $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E_0[1_{\Lambda_s} F_t]}{E_0[F_t]} = Q_0^F(\Lambda_s)$$

   and

$$Q_0^F(\Lambda_s) = E_0[1_{\Lambda_s} M_s^F].$$

3. *Q_0^F extends as a p.m. on $(\Omega, \mathcal{F}_\infty)$.*

Note that if we choose for the weight process $(F_t)_{t\geq 0} : F_t = 1_{A_t}$, where $(A_t)_{t\geq 0}$ is a decreasing family of events such that $P(A_t) > 0$, then assuming that (2.9) and (2.10) hold in this framework, we get :

$$Q_0^F(\Lambda_s) = \lim_{t\to\infty} P_0\big(\Lambda_s | A_t\big).$$

Consequently Q_0^F agrees with P_0^A, as tentatively defined by (2.5). Morever a solution to the questions (2.6) and (2.7) has been given.

3 Preliminary results

In subsection 3.1 below we shall study in a general framework the asymptotic behavior of $t \mapsto E[f(I_t, S_t)]$ as $t \to \infty$, where $f :]-\infty, 0] \times [0, \infty[\to [0, \infty[$ is a bounded function with compact support. However, in order to obtain an explicit rate of convergence, we will need to impose some restrictions on f, see subsection 3.2.

3.1 A general result

For any Borel function $f :]-\infty, 0] \times [0, \infty[\to [0, \infty[$ with compact support, let us define :

$$K_f = \sup\{\beta - \alpha \; ; \; f(\alpha, \beta) > 0\}. \tag{3.1}$$

This means that the support of f is included in the triangle with vertices $(-K_f, 0)$, $(0, 0)$ and $(0, K_f)$.
Let us state the main result of this subsection.

Proposition 3.1 *Let $f :]-\infty, 0] \times [0, \infty[\to [0, \infty[$, bounded with compact support, then*

$$E[f(I_t, S_t)] = \Delta_t(f a_0^{(t)}) + t\Delta_t(f a_1^{(t)}) + t^2 \Delta_t(f a_2) + R_t(f) \; ; \; t \geq 1 \tag{3.2}$$

where

1. *Δ_t is the linear operator acting on functions $g :]-\infty, 0] \times [0, \infty[\to [0, \infty[$:*

$$\Delta_t(g) := \int_{]-\infty,0]\times[0,\infty[} \frac{g(\alpha, \beta)}{(\beta - \alpha)^6} \exp\Big\{ - \frac{\pi^2 t}{2(\beta - \alpha)^2} \Big\} d\alpha d\beta \tag{3.3}$$

2. *$a_0^{(t)}$ and $a_1^{(t)}$ are two continuous functions defined on $]-\infty, 0] \times [0, \infty[$ satisfying :*

$$|a_i^{(t)}(\alpha, \beta)| \leq C(1 + K_f^4), \quad i = 0, 1 \tag{3.4}$$

and

$$a_2(\alpha, \beta) := 4\pi^3 \sin\left(\frac{\pi\beta}{\beta - \alpha}\right) \cdot \tag{3.5}$$

3. *$R_t(f)$ is a remainder term, which satisfies :*

$$|R_t(f)| \leq C(1 + K_f^4)\Big(\sup_{\alpha,\beta} f(\alpha, \beta)\Big) \exp\Big\{ - \frac{8\pi^2 t}{K_f^2} \Big\} \; ; \; t \geq 1. \tag{3.6}$$

To prove Proposition 3.1, three Lemmas are required.

Lemma 3.2 *For any $t > 0$, $\alpha < 0$ and $\beta > 0$, we have :*

$$P_0(I_t > \alpha, \, S_t < \beta) = \frac{4}{\pi} \sum_{k \geq 0} \frac{1}{2k+1} \sin\Big(\frac{(2k+1)\pi\beta}{\beta - \alpha}\Big) \exp\Big\{ - \frac{(2k+1)^2 \pi^2}{2(\beta - \alpha)^2} t \Big\} \tag{3.7}$$

Proof.
According to ([3] section 11 of chap 2 ; [12], ex. 3.15, chap. III) we have :

$$P_0(I_t > \alpha,\, S_t < \beta,\, X_t \in dx) = 1_{[\alpha,\beta]}(x)dx \\ \times \sum_{k\in\mathbb{Z}} p_t\big(x + 2k(\beta-\alpha)\big) - p_t\big(2\beta - x + 2k(\beta-\alpha)\big) \quad (3.8)$$

where $p_t(x)$ is the density function of the Gaussian distribution with mean 0 and variance t :

$$p_t(x) = \frac{1}{\sqrt{2\pi t}}\, \exp\Big\{-\frac{x^2}{2t}\Big\}. \quad (3.9)$$

Using the Poisson summation formula (see for instance [7], Chap. XIX, p. 630) we get :

$$\sum_{k\in\mathbb{Z}} p_t\big(x + 2k(\beta-\alpha)\big) - p_t\big(2\beta - x + 2k(\beta-\alpha)\big) \\ = \frac{1}{\beta-\alpha}\sum_{k\geq 1}\Big[\cos\Big(\frac{k\pi x}{\beta-\alpha}\Big) - \cos\Big(\frac{k\pi(2\beta-x)}{\beta-\alpha}\Big)\Big]\exp\Big\{-\frac{k^2\pi^2 t}{2(\beta-\alpha)^2}\Big\}$$

Since $\cos\, a - \cos\, b = -2\sin\Big(\frac{a+b}{2}\Big)\sin\Big(\frac{a-b}{2}\Big)$, we obtain :

$$P_0\big(I_t > \alpha,\, S_t < \beta,\, X_t \in dx\big) = 1_{[\alpha,\beta]}(x)dx \quad (3.10) \\ \times \frac{2}{\beta-\alpha}\sum_{k\geq 1}\sin\Big(\frac{k\pi\beta}{\beta-\alpha}\Big)\sin\Big(\frac{k\pi(\beta-x)}{\beta-\alpha}\Big)\exp\Big\{-\frac{k^2\pi^2 t}{2(\beta-\alpha)^2}\Big\}.$$

Integrating (3.10) over $[\alpha,\beta]$, we easily obtain the announced result. ■

Lemma 3.3 *Let* $h_1 :\ [0,\infty[\times]-\infty,0]\times]0,\infty[\to\mathbb{R}$ *be the function :*

$$h_1(t,\alpha,\beta) := \frac{4}{\pi}\sin\Big(\frac{\pi\beta}{\beta-\alpha}\Big)\exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\}. \quad (3.11)$$

Then

$$\frac{\partial^2 h_1}{\partial\alpha\partial\beta}(t,\alpha,\beta) = \Big(b_0(t,\alpha,\beta) + b_1(t,\alpha,\beta)t + b_2(\alpha,\beta)t^2\Big)\exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\} \quad (3.12)$$

where $b_0(t,\cdot)$ *and* $b_1(t,\cdot)$ *are of* C^∞ *class (in the variables* α *and* β*) except at 0, and :*

$$|b_i(t,\alpha,\beta)| \leq \frac{C}{(\beta-\alpha)^6}\big(1 + (\beta-\alpha)^4\big), \quad i = 0,1 \quad (3.13)$$

and

$$b_2(\alpha,\beta) := -\frac{4\pi^3}{(\beta-\alpha)^6}\sin\Big(\frac{\pi\beta}{\beta-\alpha}\Big) \quad (3.14)$$

Proof.
We first observe that $h_1(t,\alpha,\beta)$ is the term which is obtained by taking $k=0$ in the series (3.7).
We begin with the α-partial derivative of h_1 :

$$\frac{\partial h_1}{\partial \alpha}(t,\alpha,\beta) = \frac{4}{\pi}\left[\frac{\pi\beta}{(\beta-\alpha)^2}\cos\left(\frac{\pi\beta}{\beta-\alpha}\right) - \frac{\pi^2 t}{(\beta-\alpha)^3}\sin\left(\frac{\pi\beta}{\beta-\alpha}\right)\right]\exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\}.$$

Taking the β-partial derivative in the above expression, it is clear that (3.12) holds. Then (3.14) and (3.13) follow after straightforward calculations and estimates. ■

Lemma 3.4 *Let h_2 be the function :*

$$h_2(t,\alpha,\beta) := \frac{4}{\pi}\sum_{k\geq 2}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi\beta}{\beta-\alpha}\right)\exp\Big\{-\frac{(2k+1)^2\pi^2 t}{2(\beta-\alpha)^2}\Big\}. \tag{3.15}$$

Then

$$\left|\frac{\partial^2 h_2}{\partial\alpha\partial\beta}(t,\alpha,\beta)\right| \leq \frac{C\big(1+(\beta-\alpha)^4\big)}{t}\exp\Big\{-\frac{8\pi^2 t}{(\beta-\alpha)^2}\Big\}. \tag{3.16}$$

Proof.
We proceed as in the proof of Lemma 3.3 :

$$\left|\frac{\partial^2 h_2}{\partial\alpha\partial\beta}(t,\alpha,\beta)\right| \leq C\,t^2\big(1+(\beta-\alpha)^4\big)\left(\sum_{k\geq 2}\frac{(2k+1)^3}{(\beta-\alpha)^6}\exp\Big\{-\frac{(2k+1)^2\pi^2 t}{2(\beta-\alpha)^2}\Big\}\right)$$

Since $k\geq 2$, we have :

$$(2k+1)^2 = (2k-3+4)^2 \geq (2k-3)^2+16$$

As a result :

$$\left|\frac{\partial^2 h_2}{\partial\alpha\partial\beta}(t,\alpha,\beta)\right| \leq C\,t^2\big(1+(\beta-\alpha)^4\big)\left(\sum_{k\geq 2}\frac{(2k+1)^3}{(\beta-\alpha)^6}\exp\Big\{-\frac{(2k-3)^2\pi^2 t}{2(\beta-\alpha)^2}\Big\}\right)\times\exp\Big\{-\frac{8\pi^2 t}{(\beta-\alpha)^2}\Big\}.$$

Let $C := \sup\limits_{x\geq 0} x^3 e^{-x}$. Then :

$$x^3 e^{-ax} \leq \frac{C}{a^3}, \qquad \forall x\geq 0.$$

Taking $x = \dfrac{1}{(\beta-\alpha)^2}$ and $a = \dfrac{(2k-3)^2\pi^2 t}{2}$ in the above inequality, we get :

$$\frac{1}{(\beta-\alpha)^6}\exp\Big\{-\frac{(2k-3)^2\pi^2 t}{2(\beta-\alpha)^2}\Big\} \leq \frac{C}{(2k-3)^6 t^3}.$$

This implies (3.16). ■

Proof of Proposition 3.1

It is clear that the definition of $h_1(t,\alpha,\beta)$ $\big($resp. $h_2(t,\alpha,\beta)\big)$ given by (3.11) $\big($resp. (3.15)$\big)$ implies that

$$P(I_t > \alpha,\ S_t < \beta) = h_1(t,\alpha,\beta) + h_2(t,\alpha,\beta), \qquad \alpha < 0,\ \beta > 0.$$

Consequently :

$$E_0\big[f(I_t,S_t)\big] = -\int_{]-\infty,0]\times[0,\infty[} f(\alpha,\beta)\,\frac{\partial^2 h_1}{\partial\alpha\partial\beta}(t,\alpha,\beta)\,d\alpha d\beta + R_t(f)$$

where

$$R_t(f) := -\int_{]-\infty,0]\times[0,\infty[} f(\alpha,\beta)\,\frac{\partial^2 h_2}{\partial\alpha\partial\beta}(t,\alpha,\beta)\,d\alpha d\beta.$$

It is obvious that (3.16) implies (3.6).
From (3.13) and (3.14), we may deduce :

$$\begin{aligned} E_0\big[f(I_t,\,S_t)\big] &= -\sum_{i=0}^{1} t^i \int_{]-\infty,0]\times[0,\infty[} b_i(t,\alpha,\beta)(\beta-\alpha)^6\,\frac{f(\alpha,\beta)}{(\beta-\alpha)^6}\exp\Big\{-\frac{\pi^2 t}{(\beta-\alpha)^2}\Big\}d\alpha d\beta \\ &\quad + t^2\int_{]-\infty,0]\times[0,\infty[} b_2(\alpha,\beta)(\beta-\alpha)^6\,\frac{f(\alpha,\beta)}{(\beta-\alpha)^6}\,\exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\}d\alpha d\beta + R_t(f). \end{aligned}$$

Setting :

$$a_i^{(t)}(\alpha,\beta) = -(\beta-\alpha)^6\, b_i(t,\alpha,\beta) \qquad (i=0,1)$$

leads to (3.2).
Obviously, (3.4) is a consequence of (3.13). □

3.2 Applications

Let $f :]-\infty,0]\times[0,\infty[\to[0,\infty[$ be a bounded function with compact support. Recall that K_f has been defined by (3.1). This quantity and the set :

$$\mathcal{S}_f := \big\{\beta\in]0,\ K_f[;\ f(\beta-K_f,\,\beta)>0\big\}$$

will play an important role in our study.
One aim of our paper is to show that the penalisation procedure holds with the weight process $F_t := f(I_t,S_t)$. We briefly detail our approach. Formula (2.9) shows that it is natural to first investigate the asymptotic behavior of $t\mapsto E\big[f(I_t,S_t)\big]$, as $t\to\infty$. Roughly speaking (3.2) tells us that the dominant term is $\Delta_t(fa_2)$. We observe that :

$$\sup_{\alpha<0,\beta>0,f(\alpha,\beta)>0}\exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\} = \exp\Big\{-\frac{\pi^2 t}{2K_f^2}\Big\}, \tag{3.17}$$

and the above maximum is achieved at any point of the type $(\beta-K_f,\beta)$ where $\beta\in\ \mathcal{S}_f$.

In this general setting, it seems difficult to obtain an equivalent for $E\big[f(I_t,S_t)\big]$ as $t\to\infty$. This led us to consider two extreme cases : either $\mathcal{S}_f$ reduces to a single point or $\mathcal{S}_f=]0,\ K_f[$. The two corresponding prototypes of functions f are either :

$$f(\alpha,\beta)=1_{\{\alpha\geq\alpha_0,\,\beta\leq\beta_0\}}\qquad\text{where}\ \ \alpha_0<0,\beta_0>0 \tag{3.18}$$

or

$$f(\alpha,\beta)=1_{\{\beta-\alpha\leq c\}}\qquad\text{where}\ \ c>0. \tag{3.19}$$

More generally we have been able to deal with the two following cases.

Case 1.

$$f(\alpha,\beta)=\Phi(\alpha,\beta)\ 1_{\{\alpha\geq\alpha_0,\,\beta\leq\beta_0\}} \tag{3.20}$$

where $\alpha_0<0,\ \beta_0>0$ and $\Phi:\ [\alpha_0,0]\times[0,\beta_0]\to\mathbb{R}_+$ is continuous and

$$\Phi(\alpha_0,\ \beta_0)>0. \tag{3.21}$$

Note that then $K_f=\beta_0-\alpha_0$, $\mathcal{S}_f=\{\beta_0\}$, and (α_0,β_0) is the unique point which achieves the maximum in (3.17). It seems reasonable to believe that :

$$E\big[f(I_t,S_t)\big]\underset{t\to\infty}{\sim}C_{\alpha_0,\beta_0}\ \Phi(\alpha_0,\beta_0)\exp\Big\{-\frac{\pi^2t}{2(\beta_0-\alpha_0)^2}\Big\}. \tag{3.22}$$

Case 2.

$$f(\alpha,\beta)=\Phi(\alpha,\beta)\ 1_{\{\beta-\alpha\leq c\}} \tag{3.23}$$

where $c>0$, $\Phi:\ \big\{(\alpha,\beta);\ \alpha<0,\ \beta>0,\ \beta-\alpha\leq c\big\}\to\mathbb{R}_+$ is continuous and

$$\int_0^c\Phi(\beta-c,\beta)d\beta>0. \tag{3.24}$$

In this case $K_f=c$ and $\mathcal{S}_f=\big\{\beta\in]0,c[;\ \Phi(\beta-c,\beta)>0\big\}$. Therefore the maximum in (3.17) is achieved at any $(\beta-c,\beta)$, where β belongs to $]0,c[$. The expected behavior would be :

$$E\big[f(I_t,S_t)\big]\underset{t\to\infty}{\sim}C(\Phi,t)\exp\Big\{-\frac{\pi^2t}{2c^2}\Big\}, \tag{3.25}$$

where $t\mapsto C(\Phi,t)$ has a polynomial rate of decay in t and depends on all the values of Φ over the segment in $\mathbb{R}^2$ with endpoints $(-c,0)$ and $(0,c)$.

The heuristic arguments leading to (3.22) and (3.25) will be justified in the remainder of the section. Precisely we shall show :

Proposition 3.5 *1. In Case 1, we have :*

$$E\big[f(I_t,S_t)\big]\underset{t\to\infty}{\sim}\frac{8}{\pi}\ \Phi(\alpha_0,\beta_0)\sin\left(\frac{\pi\beta_0}{\beta_0-\alpha_0}\right)\exp\left\{-\frac{\pi^2t}{2(\beta_0-\alpha_0)^2}\right\} \tag{3.26}$$

2. In Case 2, we have :

$$E\big[f(I_t,S_t)\big] \underset{t\to\infty}{\sim} \frac{4\pi}{c^2}\left(\int_0^1 \Phi\big(c(r-1),cr\big)\sin(\pi r)dr\right) t\,\exp\left\{\frac{-\pi^2 t}{2c^2}\right\}. \tag{3.27}$$

Remark 3.6 *Suppose that $\varphi:\ [0,\infty[\to[0,\infty[$ is a Borel function such that $\int_0^\infty \varphi(y)dy<\infty$. Note that the rate of decay of $t\mapsto E\big[\varphi(S_t)\big]$ as $t\to\infty$ is very different from that of $E\big[f(I_t,S_t)\big]$. Indeed, (cf. Lemma 3.8 of [17]) it is easy to prove that :*

$$E\big[\varphi(S_t)\big] \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(y)dy\ \frac{1}{\sqrt{t}}.$$

Our proof of Proposition 3.5 requires 3 steps. First, we specify the $\Delta_t^i(f)$ introduced in Proposition 3.1, in Cases 1 and 2.

Lemma 3.7 *Let $a\ :]-\infty,0]\times[0,\infty[\to\mathbb{R}$ be a continuous function. Assume that f is a function which satisfies either (3.20) or (3.23). Then*

$$\Delta_t(fa)=\frac{1}{2}\int_{1/K_f^2}^\infty \Lambda(\Phi a)(x)\ x\exp\Big\{-\frac{\pi^2 t\,x}{2}\Big\}dx \tag{3.28}$$

where

$$\Lambda(g)(x)=\begin{cases}\displaystyle\int_{(1+\alpha_0\sqrt{x})_+}^{\inf\{1,\ \beta_0\sqrt{x}\}} g\Big(\frac{r-1}{\sqrt{x}},\ \frac{r}{\sqrt{x}}\Big) & \text{in Case 1}\\[2ex] \displaystyle\int_0^1 g\Big(\frac{r-1}{\sqrt{x}},\ \frac{r}{\sqrt{x}}\Big)dr & \text{in Case 2}\end{cases} \tag{3.29}$$

Proof.
We only consider Case 2. According to (3.3) we have

$$\Delta_t(fa)=\int_{]-\infty,0]\times[0,\infty[}\Phi(\alpha,\beta)\,\frac{a(\alpha,\beta)}{(\beta-\alpha)^6}\ \exp\Big\{-\frac{\pi^2 t}{2(\beta-\alpha)^2}\Big\}\,1_{\{\beta-\alpha\le c\}}d\alpha d\beta$$

Setting $x=\dfrac{1}{(\beta-\alpha)^2}$ (β fixed), we get :

$$\Delta_t(fa)=\frac{1}{2}\int_{1/c^2}^\infty x^{3/2}\exp\Big\{-\frac{\pi^2 t\,x}{2}\Big\}\left(\int_0^{1/\sqrt{x}}\Phi\big(\beta-\frac{1}{\sqrt{x}},\,\beta\big)a\big(\beta-\frac{1}{\sqrt{x}},\,\beta\big)d\beta\right)dx.$$

The change of variable $\beta=r/\sqrt{x}$ leads directly to (3.29). ■

The proof of (3.26) and (3.27) is based on Laplace's method, whose main result we briefly recall (see for instance [6], chap. IV). We consider :

$$I(t)=\int_0^\infty g(x)e^{t\,h(x)}dx,$$

where $g, h :]0, \infty[\to \mathbb{R}$ are continuous and satisfy the two following properties :

$$\int_0^\infty |g(x)| \; e^{t\, h(x)} dx < \infty, \quad \forall \, t > 0 \tag{3.30}$$

$$\exists \delta_0 > 0, \text{ such that } h(x) \leq h(\delta) \text{ for any } x \geq \delta \text{ and } 0 < \delta < \delta_0. \tag{3.31}$$

Proposition 3.8 *Suppose that the functions g and h satisfy :*

$$g(x) \underset{x \to 0_+}{\sim} g_0 x^\rho \tag{3.32}$$

$$h(x) = h_0 - h_1 \, x^\tau + o(x^\tau), \quad x \to 0 \tag{3.33}$$

for some :

$$g_0 \neq 0, \; \rho > -1, \; h_1 \geq 0, \; \tau > 0. \tag{3.34}$$

Then :

$$I(t) \underset{t \to \infty}{\sim} \frac{g_0}{\tau} \, \Gamma\Big(\frac{\rho + 1}{\tau}\Big) (h_1 t)^{-\frac{\rho+1}{\tau}} e^{h_0 t}. \tag{3.35}$$

As an application of the previous instance of Laplace's method, we obtain the following asymptotics.

Lemma 3.9 *Let $a :]-\infty, 0] \times [0, \infty[\to \mathbb{R}$ be a continuous function.*

1. *In Case 1, we have :*

$$\Delta_t(fa) \underset{t \to \infty}{\sim} \frac{2}{\pi^4} (\Phi a)(\alpha_0, \beta_0) \frac{1}{t^2} \, \exp\Big\{ -\frac{\pi^2 t}{2K_f^2} \Big\} \tag{3.36}$$

when $a(\alpha_0, \beta_0) \neq 0$.

2. *In Case 2, we have*

$$\Delta_t(fa) \underset{t \to \infty}{\sim} \frac{1}{\pi^2 K_f^2} \left(\int_0^1 (\Phi a)\big((r-1)K_f, \; rK_f\big) dr \right) \frac{1}{t} \, \exp\Big\{ -\frac{\pi^2 t}{2K_f^2} \Big\} \tag{3.37}$$

where it is assumed that $\int_0^1 (\Phi a)\big((r-1)K_f, \; rK_f\big) dr \neq 0.$

Proof.
According to (3.28), we have

$$\Delta_t(fa) = \frac{1}{2} \int_{1/K_f^2}^\infty \Lambda(\Phi a)(y) \; y \exp\Big\{ -\frac{\pi^2 t y}{2} \Big\} dy.$$

Setting $y = x + 1/K_f^2$ we get :

$$\Delta_t(fa) = \frac{1}{2} \left(\int_0^\infty \Lambda(\Phi a)\big(x + \frac{1}{K_f^2}\big) \Big(x + \frac{1}{K_f^2}\Big) \exp\Big\{ -\frac{\pi^2 t x}{2} \Big\} dx \right) \exp\Big\{ -\frac{\pi^2 t}{2K_f^2} \Big\}. \tag{3.38}$$

a) We begin with Case 2 which is easier.
From (3.29) we deduce :

$$\Lambda(\Phi a)\Big(x+\frac{1}{K_f^2}\Big)=\int_0^1(\Phi a)\left(\frac{r-1}{\sqrt{x+1/K_f^2}},\ \frac{r}{\sqrt{x+1/K_f^2}}\right)dr.$$

Since Φ and a are continuous, we obtain :

$$\lim_{x\to 0}\Lambda(\Phi a)\big(x+\frac{1}{K_f^2}\big)=\int_0^1(\Phi a)\big((r-1)K_f,rK_f\big)dr. \tag{3.39}$$

b) Next, we deal with Case 1. Due to (3.29) we have :

$$\Lambda(\Phi a)\big(x+\frac{1}{K_f^2}\big)=\int_{\tau_0(x)}^{\tau_1(x)}(\Phi a)\Big(\frac{r-1}{\sqrt{x+1/K_f^2}},\ \frac{r}{\sqrt{x+1/K_f^2}}\Big)dr$$

with

$$\tau_0(x)=\Big(1+\alpha_0\sqrt{x+1/K_f^2}\Big)_+,\quad \tau_1(x)=\inf\Big\{1,\ \beta_0\sqrt{x+1/K_f^2}\Big\}.$$

We observe that

$$\begin{aligned}\lim_{x\to 0_+}\tau_0(x)&=\Big(1+\frac{\alpha_0}{K_f}\Big)_+=\Big(1+\frac{\alpha_0}{\beta_0-\alpha_0}\Big)_+=\frac{\beta_0}{\beta_0-\alpha_0}\\ \lim_{x\to 0_+}\tau_1(x)&=\inf\Big\{1,\ \frac{\beta_0}{K_f}\Big\}=\inf\Big\{1,\ \frac{\beta_0}{\beta_0-\alpha_0}\Big\}=\frac{\beta_0}{\beta_0-\alpha_0}.\end{aligned}$$

This implies that :

$$\Lambda(\Phi a)\big(x+\frac{1}{K_f^2}\big)\underset{x\to 0_+}{\sim}\big(\tau_1(x)-\tau_0(x)\big)(\Phi a)(\alpha_0,\beta_0).$$

When x is small, we have

$$\begin{aligned}\tau_1(x)-\tau_0(x)&=\beta_0\sqrt{x+1/K_f^2}-1-\alpha_0\sqrt{x+1/K_f^2}\\&=\sqrt{xK_f^2+1}-1.\end{aligned}$$

As a result :

$$\Lambda(\Phi a)\big(x+\frac{1}{K_f^2}\big)\underset{x\to 0_+}{\sim}\frac{K_f^2}{2}(\Phi a)(\alpha_0,\beta_0)\ x. \tag{3.40}$$

c) We apply Laplace's method with

$$g(x)=\Big(x+\frac{1}{K_f^2}\Big)\Lambda(\Phi a)\big(x+\frac{1}{K_f^2}\big),\quad \text{and}\quad h(x)=-\frac{\pi^2 x}{2}\cdot$$

It is clear that (3.30), (3.31) and (3.33) hold with

$$h_0=0,\quad h_1=\frac{\pi^2}{2}\quad \text{and}\quad \tau=1.$$

In Case 2 (resp. Case 1), relation (3.39) $\big($resp. (3.40)$\big)$ implies that (3.32) is satisfied with $g_0 = \dfrac{1}{K_f^2}\displaystyle\int_0^1 (\Phi a)\big((r-1)K_f, rK_f\big)dr$ and $\rho = 0$ $\big($resp. $g_0 = \dfrac{1}{2}(\Phi a)(\alpha_0, \beta_0)$ and $\rho = 1\big)$.
Lemma 3.9 is a direct consequence of (3.35) and (3.38). ■

Proof of Proposition 3.5
1) We begin with Case 2. It is clear that (3.37) and (3.5) imply :

$$\Delta_t(fa_2) \underset{t\to\infty}{\sim} \frac{4\pi}{K_f^2}\left(\int_0^1 \Phi\big((r-1)K_f, rK_f\big)\sin(\pi r)dr\right)\frac{1}{t}\exp\Big\{-\frac{\pi^2 t}{2K_f^2}\Big\}. \tag{3.41}$$

Let $i = 0, 1$. From (3.4) we have :

$$|\Delta_t(fa_i^{(t)})| \le C(1 + K_f^4)\ \Delta_t(f). \tag{3.42}$$

Using (3.37), we get :

$$\Delta_t(f) \underset{t\to\infty}{\sim} \frac{\mu}{t}\exp\Big\{-\frac{\pi^2 t}{2K_f^2}\Big\} \quad \text{(for some } \ \mu > 0). \tag{3.43}$$

Applying (3.2), (3.6) and (3.41)-(3.43) shows (3.27).
2) Similarly to Case 2, in Case 1, the main term of $E\big[f(I_t, S_t)\big]$ is $\Delta_t(fa)$. As a result, (3.26) follows from (3.36).

4 Penalisation with the maximum and the minimum

4.1 Penalisation with $f(I_t, S_t)$, Case 1

In this subsection we suppose that f satisfies (3.20).

Theorem 4.1 *The Brownian penalisation procedure holds with the weight process* $F_t = f(I_t, S_t)$, *i.e.*

1. *Property (2.9) holds :*

$$\lim_{t\to\infty}\frac{E_0\big[f(I_t, S_t)|\mathcal{F}_u\big]}{E_0\big[f(I_t, S_t)\big]} = M_u^{\alpha_0,\beta_0}, \qquad \textit{for any } u \ge 0 \tag{4.1}$$

where

$$M_u^{\alpha_0,\beta_0} = N^{\alpha_0,\beta_0}(u \wedge T_{\alpha_0} \wedge T_{\beta_0})\ ; \quad u \ge 0 \tag{4.2}$$

$$N_u^{\alpha_0,\beta_0} = \frac{1}{\sin\big(\frac{\pi\beta_0}{\beta_0-\alpha_0}\big)}\ \sin\left(\frac{\pi(\beta_0 - X_u)}{\beta_0 - \alpha_0}\right)\exp\left\{\frac{\pi^2 u}{2(\beta_0-\alpha_0)^2}\right\} \quad u \ge 0. \tag{4.3}$$

$$T_x = \inf\{t \ge 0,\ X_t = x\}. \tag{4.4}$$

2. *Moreover (2.10) is satisfied :*

$$E_0[M_u^{\alpha_0,\beta_0}] = 1 \qquad \forall u \ge 0. \tag{4.5}$$

Remark 4.2 *1. According to Theorem 2.1 :*

(a) $(M_u^{\alpha_0,\beta_0}\ ;\ u \geq 0)$ *is a non-negative P_0-martingale, which converges to* 0.

(b) *For any $u \geq 0$ and $\Lambda_u \in \mathcal{F}_u$,*

$$Q_0^{\alpha_0,\beta_0}(\Lambda_u) := \lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_u}\Phi(I_t,S_t)1_{\{I_t\geq\alpha_0,S_t\leq\beta_0\}}\big]}{E_0\big[\Phi(I_t,S_t)1_{\{I_t\geq\alpha_0,S_t\leq\beta_0\}}\big]} \tag{4.6}$$

exists and $Q_0^{\alpha_0,\beta_0}$ is a p.m. on $(\Omega,\mathcal{F}_\infty)$ which satisfies :

$$Q_0^{\alpha_0,\beta_0}(\Lambda_u) = E_0[1_{\Lambda_u} M_u^{\alpha_0,\beta_0}] \qquad u \geq 0,\ \Lambda_u \in \mathcal{F}_u. \tag{4.7}$$

2. *In the particular case : $f = 1$ and $\beta_0 = -\alpha_0$, then*

$$F_t = 1_{\{I_t\geq-\beta_0,S_t\leq\beta_0\}} = 1_{\{X_t^*\leq\beta_0\}},$$

where $X_t^* = S_t \vee (-I_t) = \max_{u\leq t}|X_u|$. Moreover :

$$M_u^{-\beta_0,\beta_0} = \cos\Big(\frac{\pi X(u\wedge T_{\beta_0}^*)}{2\beta_0}\Big)\exp\Big\{\frac{\pi^2(u\wedge T_{\beta_0}^*)}{2\beta_0^2}\Big\},$$

where $T_{\beta_0}^* = \inf\{t \geq 0,\ |X_t| = \beta_0\}$.

3. *Note that in [17], a penalisation procedure has been considered with weight processes :*

$$F_t := \int_{]-\infty,0]\times[0,\infty]} 1_{\{I_t\geq\alpha,S_t\leq\beta\}}\exp\Big\{\frac{1}{2}\big(\frac{1}{\beta}-\frac{1}{\alpha}\big)L_t^0\Big\}\nu(d\alpha,d\beta)$$

where ν is a p.m. on $]-\infty,0]\times[0,\infty]$ and (L_t^0) is the local time process at 0 associated with (X_t).

It has been proved (cf Theorem 3.18 in [17]) that the martingales generated by this penalisation are functions of the 4-uple (I_t,S_t,L_t^0,X_t). Consequently, they are not of the form $\big(M_t^{-\beta_0,\beta_0}\big)$.

Proof of Theorem 4.1

To show that the penalisation procedure holds with $F_t = f(I_t,S_t)$ we need to prove (4.1) and (4.6). Observe that relation (3.26) in Proposition 3.5 gives the rate of decay of $t \to E_0\big[f(I_t,S_t)\big]$, $t\to\infty$. Next, we need to determine the asymptotic behavior of $t \to E\big[f(I_t,S_t)|\mathcal{F}_u\big]$. We will prove in step 1 below that the rate of decay of $E\big[f(I_t,S_t)|\mathcal{F}_u\big]$ may be deduced from (3.26). In step 2 we will prove (4.5).

1) Proof of (4.1)

Let $u > 0$ be fixed. We introduce $X'_t = X_{t+u} - X_u,\ t \geq 0$.

Under P_0, $(X'_t)_{t\geq0}$ is a Brownian motion started at 0 and independent from $\mathcal{F}_u$. Moreover :

$$S_{u+h} = S_u \vee (X_u + S'_h), \quad I_{u+h} = I_u \wedge (X_u + I'_h), \quad h > 0$$

where $S'_h = \max_{v\leq h} X'_v$ and $I'_h = \min_{v\leq h} X'_v$.

This implies that

$$E_0\Big[f(I_{u+v},S_{u+v})|\mathcal{F}_u\Big]=H(I_u,S_u,X_u,v)\ ;\ v\geq 0 \tag{4.8}$$

where

$$H(a,b,x,v):=E_0\Big[f\big(a\wedge(x+I'_v),b\vee(x+S'_v)\big)\Big]. \tag{4.9}$$

Suppose that a,b,x are fixed, $a\leq x,\ b\geq x$. We introduce :

$$\widetilde{f}(\alpha,\beta)=f\big(a\wedge(x+\alpha),b\vee(x+\beta)\big)\qquad \alpha\leq 0,\beta\geq 0. \tag{4.10}$$

Since :

$$a\wedge(x+\alpha)\geq\alpha_0\quad\Leftrightarrow\quad a\geq\alpha_0\text{ and }\alpha\geq\alpha_0-x$$

and

$$b\vee(x+\beta)\leq\beta_0\quad\Leftrightarrow\quad b\leq\beta_0\text{ and }\beta\leq\beta_0-x$$

then (3.20) implies that

$$H(a,b,x,v)=1_{\{a\geq\alpha_0,\beta\leq\beta_0\}}E_0\Big[\Phi\big(a\wedge(x+I_v),\ b\vee(x+S_v)\big)1_{\{I_v\geq\alpha_0-x,\ S_v\leq\beta_0-x\}}\Big].$$

We may apply (3.26) :

$$H(a,b,x,v)\underset{v\to\infty}{\sim}\frac{8}{\pi}\,\Phi(\alpha_0,\beta_0)\sin\left(\frac{\pi(\beta_0-x)}{\beta_0-\alpha_0}\right)\exp\Big\{-\frac{\pi^2v}{2(\beta_0-\alpha_0)^2}\Big\}1_{\{a\geq\alpha_0,\,b\leq\beta_0\}}.$$

Consequently :

$$E_0\big[f(I_{u+v},\ S_{u+v})|\mathcal{F}_u\big]\underset{v\to\infty}{\sim}\frac{8}{\pi}\,\Phi(\alpha_0,\beta_0)\sin\left(\frac{\pi(\beta_0-X_u)}{\beta_0-\alpha_0}\right)\exp\Big\{-\frac{\pi^2v}{2(\beta_0-\alpha_0)^2}\Big\}\times 1_{\{I_u\geq\alpha_0,\ S_u\leq\beta_0\}}. \tag{4.11}$$

Recall that

$$E_0\big[f(I_{u+v},\ S_{u+v})\big]\underset{v\to\infty}{\sim}\frac{8}{\pi}\,\Phi(\alpha_0,\beta_0)\sin\left(\frac{\pi\beta_0}{\beta_0-\alpha_0}\right)\exp\Big\{-\frac{\pi^2(u+v)}{2(\beta_0-\alpha_0)^2}\Big\}.$$

This proves (4.1) because

$$N^{\alpha_0,\beta_0}(T_{\alpha_0})=N^{\alpha_0,\beta_0}(T_{\beta_0})=0$$

and

$$\{I_u\geq\alpha_0\}=\{u\leq T_{\alpha_0}\},\quad\{S_u\leq\beta_0\}=\{u\leq T_{\beta_0}\}.$$

2) Proof of (4.5)

$(N_u^{\alpha_0,\beta_0},u\geq 0)$ is a continuous local martingale as combination of exponential martingales. Itô's formula confirms this :

$$dN_u^{\alpha_0,\beta_0}=-\frac{\pi}{\beta_0-\alpha_0}\,\frac{1}{\sin\big(\frac{\pi\beta_0}{\beta_0-\alpha_0}\big)}\cos\left(\frac{\pi(\beta_0-X_u)}{\beta_0-\alpha_0}\right)\exp\Big\{\frac{\pi^2u}{2(\beta_0-\alpha_0)^2}\Big\}dX_u. \tag{4.12}$$

It is clear that $N_u^{\alpha_0,\beta_0}$ is uniformly bounded on any interval $[0,T]$, T fixed. This shows that $(N_u^{\alpha_0,\beta_0},u\geq 0)$ is a martingale. Applying the Doob's optional stopping theorem we get :

$$E_0[M_u^{\alpha_0,\beta_0}]=E\big[N^{\alpha_0,\beta_0}(u\wedge T_{\alpha_0}\wedge T_{\beta_0})\big]=N^{\alpha_0,\beta_0}(0)=1. \qquad\blacksquare$$

4.2 Penalisation with $f(I_t, S_t)$, Case 2

We study penalisation in Case 2. Namely f satisfies (3.23) :

$$f(\alpha, \beta) = \Phi(\alpha, \beta) 1_{\{\beta - \alpha \le c\}}.$$

Theorem 4.3 *The Brownian penalisation holds with $F_t = f(I_t, S_t)$, f satisfying (3.23).*

1. *Property (2.9) holds :*

$$\lim_{t\to\infty} \frac{E_0\big[f(I_t, S_t)|\mathcal{F}_u\big]}{E_0\big[f(I_t, S_t)\big]} = M_u^{\Phi,c}, \qquad \forall u \ge 0 \tag{4.13}$$

where

$$M_u^{\Phi,c} = N^{\Phi,c}\big(u \wedge \theta(c)\big) \ ; \quad u \ge 0 \tag{4.14}$$

$$N^{\Phi,c}(u) = \frac{c + I_u - S_u}{c\rho(F)} \Big\{ \int_0^1 \Big[\Phi\big(S_u - c + r(c + I_u - S_u),\ S_u + r(c + I_u - S_u)\big) \tag{4.15}$$

$$\times \sin\Big(\frac{\pi}{c}\big(S_u - X_u + r(c + I_u - S_u)\big)\Big)\Big] dr\Big\} \exp\Big\{\frac{\pi^2 u}{2c^2}\Big\}$$

$$= \frac{1}{\rho(\Phi)} \left(\int_{(S_u - X_u)/c}^{(I_u - X_u + c)/c} \Phi(X_u + c(r-1),\ X_u + rc) \sin(\pi r) dr \right) \tag{4.16}$$

$$\times \exp\Big\{\frac{\pi^2 u}{2c^2}\Big\}$$

$$\rho(\Phi) = \int_0^1 \Phi\big(c(r-1),\ cr\big) \sin(\pi r) dr \tag{4.17}$$

$$\theta(c) = \inf\{t \ge 0;\ S_t - I_t = c\} \tag{4.18}$$

2. *Moreover (2.10) holds :*

$$E_0[M^{\Phi,c}(u)] = 1. \tag{4.19}$$

Remark 4.4 *1. Applying Theorem 2.1, we may deduce that :*

(a) $(M_u^{\Phi,c}\ ;\ u \ge 0)$ is a non-negative P_0-martingale, which converges to 0.

(b) For any $u \ge 0$ and $\Lambda_u \in \mathcal{F}_u$,

$$Q_0^{\Phi,c}(\Lambda_u) := \lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_u}\Phi(I_t, S_t) 1_{\{S_t - I_t \le c\}}\big]}{E_0\big[\Phi(I_t, S_t) 1_{\{S_t - I_t \le c\}}\big]}$$

exists.
Moreover $Q_0^{\Phi,c}$ is a p.m. on $(\Omega, \mathcal{F}_\infty)$ and

$$Q_0^{\Phi,c}(\Lambda_u) = E_0[1_{\Lambda_u} M_u^{\Phi,c}] \qquad u \ge 0,\ \Lambda_u \in \mathcal{F}_u. \tag{4.20}$$

2. When $\Phi(\alpha,\beta)=\Phi_0(\beta-\alpha)$ then $f(\alpha,\beta)=\Phi_0(\beta-\alpha)1_{\{\beta-\alpha\le c\}}$ and

$$N_u^{\Phi,c}=\frac{1}{2}\Big[\cos\Big(\frac{\pi(S_u-X_u)}{c}\Big)+\cos\Big(\frac{\pi(I_u-X_u)}{c}\Big)\Big]\exp\Big\{\frac{\pi^2 u}{2c^2}\Big\}. \tag{4.21}$$

Recall that $M_u^{\Phi,c}=N^{\Phi,c}(u\wedge\theta(c))$ $u\ge 0$.

Proof of Theorem 4.3

Our proof of Theorem 4.3 is close to that of Theorem 4.1. The details are left to the reader. However we would like to explain how the martingale $(M_t^{\Phi,c})_{t\ge 0}$ appears. We go back to the proof of Theorem 4.1. Obviously (4.8)-(4.10) are still valid. We have to make $\widetilde{f}$ explicit in Case 2. From (3.23) :

$$\widetilde{f}(\alpha,\beta)=\Phi\big(a\wedge(x+\alpha),\,b\vee(x+\beta)\big)1_{\{b\vee(x+\beta)-a\wedge(x+\alpha)\le c\}}$$

Since :

$$b\vee(x+\beta)-a\wedge(x+\alpha)\le c \quad\Leftrightarrow\quad b-a\le c,\ b-x-\alpha\le c,\ x+\beta-a\le c,\ \beta-\alpha\le c$$

then

$$\widetilde{f}(\alpha,\beta)=1_{\{b-a\le c\}}\,\widetilde{\Phi}(\alpha,\beta)\,1_{\{\beta-\alpha\le c\}}$$

where

$$\widetilde{\Phi}(\alpha,\beta)=\Phi\big(a\wedge(x+\alpha),\,b\vee(x+\beta)\big)\,1_{\{b-x-\alpha\le c,\,x+\beta-a\le c\}}.$$

Consequently, $K_{\widetilde{f}}=K_f=c$ and

$$\begin{aligned}\widetilde{\Phi}\big(c(r-1),cr\big) &= \Phi\big(a\wedge(x+cr-c),\,b\vee(x+cr)\big)1_{\{b-x\le cr\le c+a-x\}}\\ &= \Phi(x+cr-c,\,x+cr)\,1_{\{b-x\le cr\le c+a-x\}}.\end{aligned}$$

Applying (3.27) leads to (4.13) and (4.16). ■

5 The law of (X_t) under $Q_0^{\alpha_0,\beta_0}$ and $Q_0^{\Phi,c}$

We first consider the distribution of the canonical process (X_t) under $Q_0^{\alpha_0,\beta_0}$. Let $\alpha_0<0$ and $\beta_0>0$ be two fixed real numbers. Recall the definition of the p.m. $Q_0^{\alpha_0,\beta_0}$ on $(\Omega,\mathcal{F}_\infty)$:

$$Q_0^{\alpha_0,\beta_0}(\Lambda_u)=E_0[1_{\Lambda_u}M_u^{\alpha_0,\beta_0}] \qquad \Lambda_u\in\mathcal{F}_u \tag{5.1}$$

where $(M_u^{\alpha_0,\beta_0})_{u\ge 0}$ is the P_0-martingale defined by (4.2).

Theorem 5.1 *Under $Q_0^{\alpha_0,\beta_0}$:*

1. (X_t) is a diffusion process solving :

$$X_t=B_t-\frac{\pi}{\beta_0-\alpha_0}\int_0^t\cot\Big(\frac{\pi(\beta_0-X_u)}{\beta_0-\alpha_0}\Big)\,du\ ;\ t\ge 0, \tag{5.2}$$

where $(B_t)_{t\ge 0}$ is a $Q_0^{\alpha_0,\beta_0}$-Brownian motion started at 0.

2. (X_t) *has the following path properties :*

$$\alpha_0 < X_t < \beta_0 \qquad \forall t \geq 0 \tag{5.3}$$

$$S_\infty = \sup_{t\geq 0} X_t = \beta_0, \quad I_\infty = \inf_{t\geq 0} X_t = \alpha_0. \tag{5.4}$$

3. X_t *converges in distribution, as* $t \to \infty$ *to the p.m.* $p^{\alpha_0,\beta_0}(x)dx$ *on* $\mathbb{R}$*, with :*

$$p^{\alpha_0,\beta_0}(x) := \frac{2}{\beta_0 - \alpha_0} \sin^2\left(\frac{\pi(\beta_0 - x)}{\beta_0 - \alpha_0}\right) 1_{(\alpha_0,\beta_0)}(x). \tag{5.5}$$

Remark 5.2 *1. Property (5.3) follows intuitively from our penalisation procedure and the fact that the support of the p.m. on* $\mathcal{F}_u$ *:*

$$\Lambda_u \mapsto \frac{E_0\left[1_{\Lambda_u}\Phi(I_t, S_t)1_{\{I_t\geq\alpha_0, S_t\leq\beta_0\}}\right]}{E_0\left[\Phi(I_t, S_t)1_{\{I_t\geq\alpha_0, S_t\leq\beta_0\}}\right]} \quad (t > u)$$

is included in $\{I_u \geq \alpha_0, S_u \leq \beta_0\}$.

2. Note that $Q_0^{\alpha_0,\beta_0}$ *does not depend on* Φ*.*

Proof of Theorem 5.1
a) $(M_t^{\alpha_0,\beta_0}\ ;\ t \geq 0)$ is a non-negative $Q_0^{\alpha_0,\beta_0}$-martingale and can be written as :

$$M_t^{\alpha_0,\beta_0} = \mathcal{E}(J)_t := \exp\left\{\int_0^t J_u\, dX_u - \frac{1}{2}\int_0^t J_u^2 du\right\}$$

for any $t < T_{\alpha_0} \wedge T_{\beta_0}$. Indeed, from (4.12) and (4.3) we have :

$$J_u = -\frac{\pi}{\beta_0 - \alpha_0} \cot\left(\frac{\pi(\beta_0 - X_u)}{\beta_0 - \alpha_0}\right), \quad 0 \leq u < T_{\alpha_0} \wedge T_{\beta_0}.$$

Then Girsanov's theorem implies (5.2).
b) To investigate the path properties of (X_t), we define $Y_t = \beta_0 - X_t$, then

$$Y_t = \beta_0 - B_t + \frac{\pi}{\beta_0 - \alpha_0}\int_0^t \cot\left(\frac{\pi\, Y_u}{\beta_0 - \alpha_0}\right) du \tag{5.6}$$

Obviously $(Y_t)_{t\geq 0}$ is a one-dimensional diffusion. Let S $\big($resp. $m(dy)\big)$ denote its scale function (resp. speed measure), see for instance : (Section 1, chap. 4 of [8]), (Section 1, chap II of [4]). Using standard calculations (i.e. [14] (Section 52, chap V), [5] (Section 12, chap 16))) we easily get :

$$S(y) = -\cot\left(\frac{\pi y}{\beta_0 - \alpha_0}\right), \qquad y \in]0, \beta_0 - \alpha_0[$$

$$m\big([0, y]\big) = \frac{(\beta_0 - \alpha_0)}{\pi} y - \frac{(\beta_0 - \alpha_0)^2}{2\pi^2} \sin\left(\frac{2\pi y}{\beta_0 - \alpha_0}\right), \quad y \in]0, \beta_0 - \alpha_0[.$$

According to the classification of boundary points of a linear diffusion (cf Section 1, chap II of [4]), we have :

i) 0 is not an exit point since :

$$\int_{(0,x)} m\big([y,z]\big)\ S'(y)dy = \infty \qquad (z \in]0, \beta_0 - \alpha_0[). \tag{5.7}$$

ii) 0 is an entrance point since :

$$\int_{(0,z)} \big(S(z) - S(y)\big)m(dy) < \infty \qquad (z \in]0, \beta_0 - \alpha_0[). \tag{5.8}$$

Similarly $\beta_0 - \alpha_0$ is not an exit point and is an entrance one. This shows (5.3).
c) The diffusion $(Y_t)_{t\geq 0}$ which takes its values in $(0, \beta_0 - \alpha_0)$ is recurrent. This implies (5.4). Let p be the density of its invariant p.m. From the Fokker-Planck equation, p solves :

$$\frac{1}{2}\, p''(y) - \frac{\pi}{\beta_0 - \alpha_0} \left(\cot\Big(\frac{\pi, y}{\beta_0 - \alpha_0}\Big) p(y)\right)' = 0. \tag{5.9}$$

It is easy to verify that

$$p(y) = \frac{2}{\beta_0 - \alpha_0} \sin^2\left(\frac{\pi y}{\beta_0 - \alpha_0}\right) \qquad y \in (0, \beta_0 - \alpha_0)$$

is the unique density function solving (5.9). ■

To study the law of $(X_t)_{t\geq 0}$ under $Q_0^{\Phi,c}$ it is more convenient to express the p.m. $Q_0^{\Phi,c}$ via the family of p.m.'s $\big\{Q_0^{\beta-c,\beta}\ ;\ 0 < \beta < c\big\}$ (see Theorem 5.3 below). This result will allow to determine easily the distribution of (X_t) under $Q_0^{\Phi,c}$.

Theorem 5.3 *Let $c > 0$, and $\Phi : \big\{(\alpha, \beta)\ ;\ \alpha < 0, \beta > 0,\ \beta - \alpha \leq c\big\} \to \mathbb{R}_+$ be a continuous function satisfying (3.24). Then :*

$$Q_0^{\Phi,c}(\cdot) = \frac{1}{c\rho(\Phi)} \int_0^c \Phi(\beta - c, \beta) \sin\left(\frac{\pi\beta}{c}\right) Q_0^{\beta-c,\beta}(\cdot) d\beta. \tag{5.10}$$

Proof.
a) Assume for a while that the following holds :

$$E_0^{\Phi,c}\big[1_{\Lambda_u} h(S_u, S_\infty)\big] = \frac{1}{c\rho(\Phi)} \int_0^c \Phi(v - c, v) \sin\Big(\frac{\pi v}{c}\Big) E_0^{v-c,v}\big[1_{\Lambda_u} h(S_u, v)\big] dv \tag{5.11}$$

for any $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ Borel, $u > 0$, $\Lambda_u \in \mathcal{F}_u$, and that $E_0^{\Phi,c}$ (resp. $E_0^{\alpha_0,\beta_0}$) stands for the expectation under $Q_0^{\Phi,c}$ (resp. $Q_0^{\alpha_0,\beta_0}$).
Taking $h = 1$ in (5.1) implies that (5.10) holds on $\mathcal{F}_u$. Since the p.m.'s in (5.10) are defined on $\mathcal{F}_\infty$ they coincide on $\mathcal{F}_\infty$.
b) The remainder is devoted to the proof of (5.11).
i) We claim that :

$$Q_0^{\Phi,c}(S_\infty - I_\infty \leq c) = 1. \tag{5.12}$$

Indeed, from the definition of $Q_0^{\Phi,c}$ we have :

$$Q_0^{\Phi,c}(S_t - I_t > c) = E_0^{\Phi,c}[1_{\{S_t - I_t > c\}} M_t^{\Phi,c}] = E_0^{\Phi,c}[1_{\{t > \theta(c)\}} M_t^{\Phi,c}]$$

From Doob's optional stopping theorem and the fact that $M^{\Phi,c}_{\theta(c)} = 0$, we get :

$$Q_0^{\Phi,c}(S_t - I_t > c) = E_0^{\Phi,c}[1_{\{t>\theta(c)\}} M^{\Phi,c}_{\theta(c)}] = 0.$$

Taking $t \to \infty$, we obtain (5.12).

ii) Due to the monotone class theorem and (5.12), it is sufficient to show (5.11) with

$$h(x,y) := 1_{[0,\beta]}(x) 1_{]\beta',c]}(y) \qquad (0 < \beta < \beta' < c)$$

In this case, (5.11) reduces to :

$$A = \frac{1}{c\rho(\Phi)} \int_{\beta'}^{c} \Phi(v-c, v) \sin\left(\frac{\pi v}{c}\right) P_0^{v-c,v}\big(\Lambda_u \cap \{S_u \le \beta\}\big) dv, \tag{5.13}$$

where

$$A := P_0^{\Phi,c}\big(\Lambda_u \cap \{S_u \le \beta,\ S_\infty > \beta'\}\big). \tag{5.14}$$

It is clear that

$$A = \lim_{t\to\infty} A(t), \quad \text{ with } \ A(t) := P_0^{\Phi,c}\big(\Lambda_u \cap \{S_u \le \beta,\ S_t > \beta'\}\big).$$

Then, for $t > u$, we have :

$$\begin{aligned} A(t) &= E_0[1_{\{\Lambda_u \cap \{S_u \le \beta\}} \, 1_{\{S_t > \beta'\}} \, M_t^{\Phi,c}] \\ &= E_0[1_{\Lambda_u \cap \{S_u \le \beta\}} \, 1_{\{T_{\beta'} < t\}} \, M_t^{\Phi,c}] \end{aligned}$$

Due to Doob's optional stopping theorem we get :

$$A(t) = E_0\big[1_{\{\Lambda_u \cap \{S_u \le \beta\}} \, 1_{\{T_{\beta'} < t\}} \, M^{\Phi,c}(T_{\beta'})\big]$$

Taking the limit $t \to \infty$, we obtain :

$$A = E_0\big[1_{\Lambda_u \cap \{S_u \le \beta\}} \, M^{\Phi,c}(T_{\beta'})\big].$$

Using (4.14) and (4.16) we have :

$$\begin{aligned} A = \frac{1}{\rho(\Phi)} E_0 \Bigg[1_{\Lambda_u \cap \{S_u \le \beta\}} \Big(\int_0^1 1_{\{rc < I(T_{\beta'}) - \beta' + c\}} \sin(\pi r) \Phi\big(\beta' + c(r-1),\ \beta' + rc\big) dr \Big) \\ \times 1_{\{\beta' - I(T_{\beta'}) < c\}} \exp\left\{ \frac{\pi^2 T_{\beta'}}{2c^2} \right\} \Bigg]. \end{aligned}$$

Observe that $0 > I(T_{\beta'}) > rc + \beta' - c$ implies that $I(T_{\beta'}) > \beta' - c$ and $\beta' + rc < c$. Making the change of variable $v = \beta' + rc$ leads to

$$A = \frac{1}{c\rho(\Phi)} \int_{\beta'}^{c} \Phi(v-c, v) \sin\left(\frac{\pi(v-\beta')}{c}\right) A_1(v) dv \tag{5.15}$$

where

$$A_1(v) := E_0\Big[1_{\Lambda_u \cap \{S_u \le \beta\}} \, 1_{\{I(T_{\beta'}) > v - c\}} \exp\left\{\frac{\pi^2 T_{\beta'}}{2c^2}\right\}\Big].$$

Since $u \le T_\beta < T_{\beta'}$, we get, by using the Markov property at time u :

$$E_0\Big[1_{\{I(T_{\beta'})>v-c\}} \exp\Big\{\frac{\pi^2 T_{\beta'}}{2c^2}\Big\}\Big|\mathcal{F}_u\Big] = 1_{\{I_u>v-c\}} A_2(X_u) \exp\Big\{\frac{\pi^2 u}{2c^2}\Big\}$$

with :

$$A_2(x) := E\Big[1_{\{I(T_{\beta'-x})>v-c-x\}} \exp\Big\{\frac{\pi^2 T_{\beta'-x}}{2c^2}\Big\}\Big].$$

Let $\alpha_1 < 0 < \beta_1$ and assume that $\beta_1 - \alpha_1 < c$. Then, using the fact that

$$Z_t := \sin\Big(\frac{\pi(X_t - \alpha_1)}{c}\Big) \exp\Big\{\frac{\pi^2 t}{2c^2}\Big\}, \quad t \ge 0$$

is a martingale, and Doob's optional stopping theorem at $T_{\beta_1} \wedge T_{\alpha_1}$ lead to :

$$E_0\Big[1_{\{I(T_{\beta_1})>\alpha_1\}} \exp\Big\{\frac{\pi^2 T_{\beta_1}}{2c^2}\Big\}\Big] = -\frac{\sin\big(\pi\frac{\alpha_1}{c}\big)}{\sin\big(\frac{\pi}{c}(\beta_1-\alpha_1)\big)}. \tag{5.16}$$

Consequently, we obtain successively :

$$A_2(x) = \frac{\sin\big(\pi\frac{(v-x)}{c}\big)}{\sin\big(\frac{\pi(v-\beta')}{c}\big)}$$

$$A = \frac{1}{c\rho(\Phi)} \int_{\beta'}^{c} \Phi(v-c, v)\, A_3(v) dv,$$

with

$$A_3(v) := E_0\Big[1_{\Lambda_u \cap \{S_u \le \beta,\, I_u > v-c\}} \sin\Big(\frac{\pi(v - X_u)}{c}\Big) \exp\Big\{\frac{\pi^2 u}{2c^2}\Big\}\Big].$$

Note that $S_u \le \beta < \beta' < v$; then according to (4.2) and (4.3) we have :

$$A_2(v) = \sin\Big(\frac{\pi v}{c}\Big)\; E_0\Big[1_{\Lambda_u \cap \{S_u \le \beta\}}\, M_u^{v-c,v}\Big].$$

Finally

$$\begin{aligned} A &= \frac{1}{c\rho(\Phi)} \int_{\beta'}^{c} \Phi(v-c,v) \sin\Big(\frac{\pi v}{c}\Big) E_0\big[1_{\Lambda_u}\, M_u^{v-c,v} 1_{\{S_u \le \beta\}}\big] dv \\ &= \frac{1}{c\rho(\Phi)} \int_{\beta'}^{c} \Phi(v-c,v) \sin\Big(\frac{\pi v}{c}\Big) P_0^{v-c,v}\big(\Lambda_u \cap \{S_u \le \beta\}\big) dv. \end{aligned}$$

This shows (5.13). ■

We will deduce from Theorem 5.3 two main consequences (see Theorems 5.4 and 5.5 below). We first interpret the identity (5.10) in a more probabilistic way.

Theorem 5.4 *1. Conditionally on $S_\infty = v$, $(I_t, S_t, X_t)_{t\ge 0}$ is distributed under $Q_0^{\Phi,c}$ as the three dimensional process $(I_t, S_t, X_t)_{t\ge0}$ under $Q_0^{v-c,v}$.*

2. The density function of S_∞ under $Q_0^{\Phi,c}$ is

$$\frac{1}{c\rho(\Phi)}\, \Phi(v-c, v) \sin\Big(\frac{\pi v}{c}\Big)\, 1_{(0,c)}(v).$$

Proof.
Let $u > 0$ and $\Lambda_u \in \mathcal{F}_u$. Let us apply (5.11) with $h(x,y) = H(y)$, for $H : [0,\infty[\to [0,\infty[$:

$$E_0^{\Phi,c}\big[1_{\Lambda_u} H(S_\infty)\big] = \frac{1}{c\rho(\Phi)} \int_0^c \Phi(v-c,v) \sin\left(\frac{\pi v}{c}\right) H(v) Q_0^{v-c,v}(\Lambda_u) dv$$

If we take in particular $\Lambda_u = \Omega$, we get

$$E_0^{\Phi,c}\big[H(S_\infty)\big] = \frac{1}{c\rho(\Phi)} \int_0^c H(v)\Phi(v-c,v) \sin\left(\frac{\pi v}{c}\right) dv.$$

This shows *ii)*.
Then *i)* follows from

$$\begin{aligned} E_0^{\Phi,c}\big[1_{\Lambda_u} H(S_\infty)\big] &= E_0^{\Phi,c}\big[H(S_\infty) P_0^{\Phi,c}(\Lambda_u | S_\infty)\big] \\ &= \frac{1}{c\rho(\Phi)} \int_0^c H(v)\, P_0^{\Phi,c}(\Lambda_u | S_\infty = v) \Phi(v-c,v) \sin\left(\frac{\pi v}{c}\right) dv. \end{aligned}$$

■

We are now able to present a few path properties of (X_t) under $Q_0^{\Phi,c}$.

Theorem 5.5 *Under* $Q_0^{\Phi,c}$ *:*

1. *We have :*

$$I_t > I_\infty \quad and \quad S_t < S_\infty \quad for\ any\ t \geq 0 \tag{5.17}$$

$$S_\infty - I_\infty = c. \tag{5.18}$$

2. *When t goes to infinity, the couple (S_t, X_t) converges in distribution to the p.m. on $\mathbb{R}_+ \times \mathbb{R}$:*

$$\frac{2}{c^2\rho(\Phi)} \sin^2\left(\frac{\pi(x-y)}{c}\right) \sin\left(\frac{\pi x}{c}\right) \Phi(x-c,x) 1_{\{0<x<c,\, x-c<y<x\}} dxdy.$$

Remark 5.6 1. *Property (5.18) can be deduced intuitively from our penalisation procedure and the fact that the support of the p.m. on $\mathcal{F}_u$:*

$$\Lambda_u \mapsto \frac{E_0\big[1_{\Lambda_u}\Phi(I_t,S_t)1_{\{S_t - I_t \leq c\}}\big]}{E_0\big[\Phi(I_t,S_t)1_{\{S_t-I_t\leq c\}}\big]} \quad (t > u)$$

is included in $\big\{S_u - I_u \leq c\big\}$.

2. *Recall that the p.m. $Q_0^{\alpha_0,\beta_0}$ which arises from penalisation with $\Phi(I_t,S_t)1_{\{I_t\geq\alpha_0, S_t\leq\beta_0\}}$ does not depend on the values of Φ. Therefore penalisation associated with the process $\Phi(I_t,S_t)1_{\{S_t-I_t\leq c\}}$ is very different since the p.m. $Q_0^{\Phi,c}$ depends on the values of Φ over the segment $\big\{(\beta-c,\beta);\ \beta\in[0,c]\big\}$.*

Proof of Theorem 5.5.
It is easy to deduce (5.17) $\big($resp. (5.18)$\big)$ from (5.3) $\big($resp. (5.4)$\big)$ and Theorem 5.4. The details are left to the reader. Let $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. According to Theorem 5.4 we have :

$$E_0^{\Phi,c}\big[g(S_t, X_t)\big] = \frac{1}{c\rho(\Phi)} \int_0^c \Phi(v-c, v) \sin\Big(\frac{\pi v}{c}\Big) E_0^{v-c,v}\big[g(S_t, X_t)\big] dv.$$

Applying Theorem 5.1, we get :

$$\lim_{t\to\infty} E_0^{v-c,v}\big[g(S_t, X_t)\big] = \frac{2}{c} \int_{v-c}^{v} g(v, y) \sin^2\Big(\frac{\pi(v-y)}{c}\Big) dy.$$

Point *2)* of Theorem 5.5 is a direct consequence of the dominated convergence theorem. ■

We formulate differently item *2)* of Theorem 5.5.

Corollary 5.7 *The pair* $\Big(\dfrac{S_t + I_t}{2},\ X_t - \dfrac{S_t + I_t}{2}\Big)$ *converges in distribution as* $t \to \infty$ *to :*

$$\Big(\frac{1}{c\rho(\Phi)} \Phi\Big(x - \frac{c}{2}, x + \frac{c}{2}\Big) \cos\Big(\frac{\pi x}{c}\Big) 1_{\left[-\frac{c}{2}, \frac{c}{2}\right]}(x) dx\Big) \times \Big(\frac{2}{c} \cos^2\Big(\frac{\pi y}{c}\Big) 1_{\left[-\frac{c}{2}, \frac{c}{2}\right]}(y) dy\Big)$$

hence, its two components are asymptotically independent.

Proof.
This is a direct consequence of 2) of Theorem 5.4, (5.18) and simple changes of variables. ■

Remark 5.8 *Using Girsanov's theorem it may be proved that* (X_t) *solves :*

$$X_t = B_t + \int_0^t \frac{\frac{\partial \Gamma^{\Phi,c}}{\partial x}}{\Gamma^{\Phi,c}} (I_u, S_u, X_u) 1_{\{S_u - I_u < c\}} du \tag{5.19}$$

where $(B_t)_{t\geq 0}$ *is a* $Q_0^{\Phi,c}$ *Brownian motion, and*

$$\Gamma^{\Phi,c}(a, b, x) = \int_{(b-x)/c}^{(a-x+c)/c} \Phi(x + c(r-1), x + rc) \sin(\pi r) dr \tag{5.20}$$

$$\frac{\partial \Gamma^{\Phi,c}}{\partial x}(a, b, x) = -\frac{\pi}{c} \int_{(b-x)/c}^{(a-x+c)/c} \Phi(x + c(r-1), x + rc) \cos(\pi r) dr.$$

In the particular case $\Phi = 1$ *(i.e. penalisation with* $S_t - I_t \leq c$*) then :*

$$\Gamma^{1,c}(a, b, x) = \frac{1}{\pi} \Big(\cos\Big(\frac{\pi(b-x)}{c}\Big) + \cos\Big(\frac{\pi(a-x)}{c}\Big)\Big)$$

This implies that (X_t) *solves :*

$$X_t = B_t + \frac{\pi}{c} \int_0^t \tan\Big(\frac{\pi}{c}\Big(\frac{S_u + I_u}{2} - X_u\Big)\Big) 1_{\{S_u - I_u < c\}} du. \tag{5.21}$$

6 Application to diffusions

Our approach is based on Theorem 5.1. Let μ be a p.m. on $]-\infty,0]\times[0,\infty[$ which does not charge $(0,0)$. Let us consider the associated p.m. Q_0^μ on $(\Omega,\mathcal{F}_\infty)$:

$$Q_0^\mu(\cdot)=\int_{]-\infty,0]\times[0,\infty[}Q_0^{\alpha,\beta}(\cdot)\mu(d\alpha,d\beta) \tag{6.1}$$

where $Q_0^{\alpha,\beta}$ has been defined by (5.1). Note that from Theorem 5.3, the p.m. $Q_0^{\Phi,c}$ is equal to Q_0^μ where

$$\mu(d\alpha,d\beta)=\mu^{\Phi,c}(d\alpha,d\beta):=\frac{1}{c\rho(\Phi)}\,\Phi(\beta-c,\beta)\sin\left(\frac{\pi\beta}{c}\right)\delta_{\beta-c}(d\alpha)1_{[0,c]}(\beta)d\beta. \tag{6.2}$$

Proposition 6.1 *Under Q_0^μ :*

1. *(I_t,S_t) converges a.s. as $t\to\infty$ to (I_∞,S_∞) and the distribution of (I_∞,S_∞) is μ.*
2. *(I_t,S_t,X_t) converges in distribution, as $t\to\infty$ to the p.m. on $]-\infty,0]\times[0,\infty[\times\mathbb{R}$:*

   $$\lambda(d\alpha,d\beta,dx)=p^{\alpha,\beta}(x)\,\mu(d\alpha,d\beta)dx, \tag{6.3}$$

   where the density function $p^{\alpha,\beta}(x)$ has been defined by (5.5).

   In particular, (S_t,X_t) converges in law, as $t\to\infty$, to

   $$\nu(d\beta,dx):=\left(\int_{-\infty}^0 p^{\alpha,\beta}(x)\mu(d\alpha,d\beta)\right)dx. \tag{6.4}$$

Proof.
Proposition 6.1 is a direct consequence of Theorem 5.1.

■

Remark 6.2 1. *It seems difficult to characterize all possible p.m.'s ν obtained by this randomization procedure, i.e. to describe the set of p.m.'s ν which are defined by (6.4), μ varying in the set of p.m.'s on $]-\infty,0]\times[0,\infty[$.*

2. *It can be proved that if $\mu(d\alpha,d\beta)$ satisfies :*

   $$\int_{]-\infty,0]\times[0,\infty[}(\beta-\alpha)\mu(d\alpha,d\beta)<\infty, \tag{6.5}$$

   $$\int_{]-\infty,0]\times[0,\infty[}(\beta+\alpha)\mu(d\alpha,d\beta)=0, \tag{6.6}$$

   $$2\,a\,\mu\big(]-\infty,0]\times[a,\infty[\big)=\int_{]-\infty,0]\times[0,\infty[}(\alpha+\beta)\mu(d\alpha,d\beta). \tag{6.7}$$

   Then, the Rogers conditions (1.7) and (1.8) hold.

3. The p.m. Q_0^μ is locally absolutely continuous with respect to the Wiener measure P_0, namely :

$$Q_0^\mu(\Lambda_t) = E_0[1_{\Lambda_t} M_t^\mu], \qquad \Lambda_t \in \mathcal{F}_t \tag{6.8}$$

where

$$M_t^\mu = \int_{]-\infty,0]\times[0,\infty[} M_t^{\alpha,\beta} \mu(d\alpha, d\beta), \tag{6.9}$$

and $(M_t^{\alpha,\beta})_{t\geq 0}$ is the P_0-martingale introduced in Theorem 4.1. Moreover

$$M_t^\mu = \Gamma^\mu(I_t, S_t, X_t, t) \tag{6.10}$$

where

$$\Gamma^\mu(a,b,x,t) := \int_{]-\infty,0]\times[0,\infty[} \frac{1}{\sin\left(\frac{\pi\beta}{\beta-\alpha}\right)} \sin\left(\frac{\pi(\beta-x)}{\beta-\alpha}\right) \exp\left\{\frac{\pi^2 t}{2(\beta-\alpha)^2}\right\} \times 1_{\{\alpha\geq a,\, \beta\leq b\}}\, \mu(d\alpha, d\beta). \tag{6.11}$$

Due to Girsanov's theorem, it may be infered that X_t solves :

$$X_t = B_t + \int_0^t \frac{\frac{\partial}{\partial x}\Gamma^\mu}{\Gamma^\mu}(I_u, S_u, X_u, u)\, du \tag{6.12}$$

where $(B_t)_{t\geq 0}$ is a Q_0^μ-Brownian motion started at 0 and

$$\frac{\partial}{\partial x}\Gamma^\mu(a,b,x,t) = -\pi \int_{]-\infty,0]\times[0,\infty[} \frac{1}{(\beta-\alpha)\sin\left(\frac{\pi\beta}{\beta-\alpha}\right)} \cos\left(\frac{\pi(\beta-x)}{\beta-\alpha}\right) \tag{6.13}$$

$$\times \exp\left\{\frac{\pi^2 t}{2(\beta-\alpha)^2}\right\} 1_{\{\alpha\geq a,\, \beta\leq b\}}\, \mu(d\alpha, d\beta) \tag{6.14}$$

Consequently, (I_t, S_t, X_t) is a non-homogeneous Markov process.

4. There exist p.m.'s μ satisfying (6.5)-(6.7). It is easy to show that these conditions hold with

$$\mu(d\alpha, d\beta) = \left(\frac{2c^2}{(\beta+c)^3} 1_{\{\beta>0\}} d\beta\right) \mu_1(d\alpha) \tag{6.15}$$

where $\mu_1(d\alpha)$ is a p.m. on $]-\infty, 0]$, such that $-\int_{-\infty}^0 \alpha\, \mu_1(d\alpha) = c \in]0,\infty[$.

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