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REFINED CONVERGENCE FOR THE BOOLEAN MODEL

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Abstract

In a previous work [9] two of the authors proposed a new proof of a well known convergence result for the scaled elementary connected vacant component in the high intensity Boolean model towards the Crofton cell of the Poisson hyperplane process (see e.g. [4]). In this paper, we investigate the second-order term in this convergence when the two-dimensional Boolean model and the Poisson line process are coupled on the same probability space. We consider the particular case where the grains are discs with random radii. A precise coupling between the Boolean model and the Poisson line process is first established. A result of directional convergence in distribution for the difference of the two sets involved is then derived. Finally we show the convergence of this directional approximate defect process.

Keywords: Poisson point process; Crofton cell; Convergence; Stochastic geometry

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1. Introduction and notations

Since the first result of P. Hall [4, 5] and its generalizations in [12, 14], the scaled vacancy of the Boolean model is known to converge in some sense to its counterpart in the Poisson hyperplane process. In a previous paper [9] two of the authors gave another proof of this convergence result for the local occupation laws of a Boolean shell model in terms of Hausdorff distance. This convergence appears as a first order result, expressed in terms of weak convergence. Our aim in this paper is to give two generalisations of this result. We extend first the weak convergence to an almost sure convergence thanks to an adequate coupling between both models, and secondly we show a second order weak convergence for the difference of both sets, expressed as the convergence of a stochastic process in the Skorohod and L^1 senses.

We shall work in the plane $\mathbb{R}^2$, though some of our results might be stated in higher dimensions: let us consider a Boolean model based on a Poisson point process $\mathbf{X}_\lambda$ with intensity measure $\lambda^2\,\mathrm{d}x$ and generic shape an open disc centred at 0 of random radius R such that $\mathbf{E}[R] = 1$ and $\mathbf{E}[R^2] < +\infty$. The law of R will be denoted by μ, and we will assume that there exists $R_\star > 0$ such that $\mu(R_\star, +\infty) = 1$.
The choice of a random disc enables us to write simply the different couplings and computations presented below, generic convex smooth shapes could probably be treated in the same way, up to technical details.

The occupied phase of the Boolean model is denoted by

$$\mathscr{O}_\lambda = \bigcup_{x\in\mathbf{X}_\lambda} B(x, R_x),$$

where $B(x, r)$ denotes the disc centred at x and of radius r and where the radii R_x for each $x \in \mathbf{X}_\lambda$ are independent and identically distributed, independent of $\mathbf{X}_\lambda$ [13]. This process is supposed to leave the point 0 uncovered, which occurs with positive probability

$$\mathbf{P}(0 \notin \mathscr{O}_\lambda) = \exp(-\pi\lambda^2\mathbf{E}[R^2]).$$

From now on the Boolean model shall be conditioned by this event.
Let D_0^λ denote the (closed) connected component of $\mathbb{R}^2 \setminus \mathscr{O}_\lambda$ containing 0. The following

asymptotic result for this process (see [4, 12, 14, 9]) may be seen as a consequence of Steiner's formula [16]:

Theorem 1. *Let D^λ be the following compact set:*

- $D^\lambda = \lambda^2 D_0^\lambda$ *whenever this set is bounded,*
- $D^\lambda = K_0$ *a given fixed compact set otherwise.*

When λ tends to infinity, D^λ converges in law towards the Crofton cell $\mathscr{C}$ of a Poisson line process with intensity measure $\mathrm{d}\rho\,\mathrm{d}\theta$.

In [4] the convergence was stated for random discs and Hausdorff distance, whereas in [12] it was proved for generic shapes, using the hit or miss topology for random closed sets. The criterion developed in [14] gives the convergence for another general class of shapes, whereas in [9] the convergence is proved using a convergence result for random shells.

The Poisson line process with intensity measure $\mathrm{d}\rho\,\mathrm{d}\theta$ in $\mathbb{R}^2$ is defined as the set of (random) lines $D_{\rho,\theta} = \{(r,t) \in \mathbb{R}_+ \times [0, 2\pi) \;:\; r\cos(t-\theta) = \rho\}$, where (ρ, θ) are the points of a Poisson point process Φ in $\mathbb{R}_+ \times [0, 2\pi)$ with intensity measure $\mathrm{d}\rho\,\mathrm{d}\theta$. The Crofton cell $\mathscr{C}$ is defined as the polygon formed by those lines containing the point 0 (see [17] or [14] for a survey on Poisson line tessellations). Numerous distributional results on this model have been obtained notably by R. E. Miles [10, 11] and G. Matheron [8]. More recently, central limit theorems have been derived in [14, 15] for the two-dimensional case and in [6] for the general case. Besides, D. G. Kendall's conjecture on the shape of the Crofton cell when it is large has been proved in [7]. Additional distributional and asymptotic results at large inner radius have also been obtained in [3] and [2].

We shall first recall in section 2 the asymptotic properties of the outer radius of the Crofton cell and give some counterpart of those asymptotics for the rescaled outer radius of the Boolean empty connected component. Those results are useful for the next sections 3, 4 and 5, they also give some insight on the behaviour of the high intensity Boolean model with respect to the continuous percolation problem. It is a natural problem to try to estimate the error in theorem 1: one possible answer is to give a geometric description of the difference of those two sets. This description requires to couple the Boolean model with the Poisson line process. This coupling, which asserts as a consequence the almost sure convergence in Theorem 1, will be described in section 3, and its application to the second order convergence will be treated in section 4 for directional convergence, and in section 5 for the convergence of the *rescaled defect process* in the Skorohod and L^1 settings.

2. Estimates on the tail probability of the inner and outer radius

This section contains autonomous results about the inner and outer radii of both the Crofton cell and the the empty connected component $\lambda^2 D_0^\lambda$. Let us introduce some notations:

- *Crofton cell:* the inner radius is denoted by R_m, the outer radius by R_M;
- *rescaled Boolean model:* the inner radius is denoted by $R_m(\lambda)$, the outer radius by $R_M(\lambda)$;

and they are defined by

$$R_m = \sup\{r > 0 \ : \ B_2(0,r) \subset \mathscr{C}\}, \qquad R_M = \inf\{r > 0 \ : \ \mathscr{C} \subset B_2(0,r)\},$$
$$R_m(\lambda) = \sup\{r > 0 \ : \ B_2(0,r) \subset \lambda^2 D_0^\lambda\}, \qquad R_M(\lambda) = \inf\{r > 0 \ : \ \lambda^2 D_0^\lambda \subset B_2(0,r)\},$$

The laws of some of those quantities are well known and straightforward to obtain:

$$\forall r > 0, \qquad \mathbf{P}(R_m > r) = \exp(-2\pi r),$$
$$\mathbf{P}(R_m(\lambda) > r) = \exp(-(2\pi r + \pi r^2/\lambda^2)),$$

however for the outer radii we only have the following asymptotic result, proved in [2] in the context of a study of Kendall's conjecture on the shape of large Poisson polygons:

Theorem 2. (Theorem 8 in [2].) *We have for all $r > 0$*

$$\begin{aligned} & 2\pi r e^{-2r}\left(\cos 1 + \frac{e^{-(2\pi\cos 1 - 1)r}}{2\pi r}\right) \leq \mathbf{P}(R_M \geq r) \\ \leq\ & 2\pi r e^{-2r}\left(1 - (\pi-2) r e^{-2r} + \frac{2}{3}(\pi-3)^2 r^2 e^{-4r} + \frac{e^{-2(\pi-1)r}}{2\pi r}\right). \end{aligned}$$

Concerning the Boolean model, we prove below the following counterpart:

Theorem 3. *There exists a constant $S > 0$ and constants K and C depending only on $R_\star$ such that for all $r \geq 1$ and $\lambda^2 \geq Sr$ we have*

$$\mathbf{P}(R_M(\lambda) > r) \leq K \exp(-Cr).$$

Proof. The proof of this theorem relies on the following (non-optimal) reasoning : let N be a positive integer greater than 12, and define the angular sectors $S_{i,N}$ for $i \in \{1, \ldots, N\}$ as

$$S_{i,N} = \left\{(\rho,\theta)\ :\ \rho > 0, \theta + \frac{\pi}{N} \in \left[\frac{(2i-1)\pi}{N}, \frac{(2i+1)\pi}{N}\right)\right\},$$

for $i \in \{0, \ldots, N-1\}$. If $\lambda^2 D_0^\lambda$ is not included in $B(0,r)$, then there exists at least one of those sectors such that no disc of the rescaled Boolean model contains both points with polar coordinates $(r, (2i-1)\pi/N)$ and $(r, (2i+1)\pi/N)$ (this implies that λ must satisfy the condition $\lambda^2 R \geq r \sin \pi/N$).

Let us denote by $A_{\lambda,r,R}$ the set of the centres (ρ,θ) of discs of radius $\lambda^2 R$ such that this occurs for the sector $S_{0,N}$, we have by invariance under rotations

$$\mathbf{P}(R_M(\lambda) \geq r) \quad \leq \quad N \exp\left(-\lambda^{-2}\int_{\mathbb{R}_+\times[0,2\pi)\times\mathbb{R}_+} \mathbf{1}_{A_{\lambda,r,R}}(\rho,\theta)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\mu(R)\right), \quad (1)$$

and the aim of the computations is to bound the Lebesgue measure $a_{\lambda,r,R}$ of the set $A_{\lambda,r,R}$ from below. This set is

$$A_{\lambda,r,R} = \left[B\left((r,\pi/N), \lambda^2 R\right) \cap B\left((r,-\pi/N), \lambda^2 R\right)\right] \setminus B\left(0, \lambda^2 R\right).$$

The geometry of this set is quite easily described, let us indeed introduce the angle $\theta_0 = \operatorname{acos}(r/2\lambda^2 R) - \pi/N$, then for λ such that

$$\lambda^2 R \geq \max\left(\frac{r}{2\cos(\theta_0 - \pi/N)}, \frac{r}{2\cos(\pi/N)}\right), \quad (2)$$

one has

$$(\rho,\theta)\in A_{\lambda,r,R} \text{ if and only if } \begin{cases} -\theta_0\le\theta\le\theta_0, \\ \lambda^2R\le\rho\le\rho_e(\theta), \end{cases}$$

where $\rho_e(\theta)$ is given by

$$\rho_e(\theta) = r\cos(|\theta|+\pi/N) + \lambda^2R\sqrt{1-\left(\frac{r\sin(|\theta|+\pi/N)}{\lambda^2R}\right)^2}.$$

Let us now introduce $\theta_1=\pi/12$, this angle satisfies $\theta_1<\theta_0$ for λ large enough ($\lambda^2\ge Sr$ where the constant S is chosen greater than 2, and depends on N and $R_\star$). The computation of $a_{\lambda,r,R}$ becomes

$$\begin{aligned} a_{\lambda,r,R} &= \int_0^{\theta_0}\left[\rho_e(\theta)^2-(\lambda^2R)^2\right]\,d\theta, \\ &\ge \int_0^{\theta_1}\left[\rho_e(\theta)^2-(\lambda^2R)^2\right]\,d\theta, \\ &\ge \int_0^{\theta_1}\Big[\underbrace{r^2\cos(2\theta+2\pi/N)}_{\ge 0}+ \\ &\quad 2r\cos(\theta+\frac{\pi}{N})\underbrace{\sqrt{\lambda^4R^2-r^2\sin^2(\theta+\pi/N)}}_{\ge(\sqrt{3}/2)\lambda^2R\text{ from (2)}}\Big]\,d\theta, \\ &\ge \frac{3\pi}{24}\,\lambda^2Rr. \end{aligned}$$

Consequently we obtain that there exists a constant $C'>0$ such that

$$a_{\lambda,r,R}\ge C'\lambda^2Rr.$$

Inserting this estimate in inequality (1) completes the proof of theorem 3.

■

3. Coupling and almost sure convergence

Coupling the Boolean model with the Poisson line process is an easy task: indeed as λ tends to infinity the rescaled Boolean model *looks like* the Poisson line process as one can see from theorem 1 in [9]. The formal way to state this as a coupling result is to introduce a marked Poisson line process which couples both processes:

let $\mathbf{Z}$ be a Poisson point process with intensity measure $\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\mu$ on $\mathbb{R}_+\times[0,2\pi)\times\mathbb{R}_+$, and define the function

$$\psi_\lambda(\rho, R) = R\sqrt{1 + \frac{2\rho}{\lambda^2 R}}.$$

Define the following processes:

- $\mathbf{X} = \bigcup_{(\rho,\theta,R)\in\mathbf{Z}}\{(\rho,\theta)\}$,
- $\mathbf{X}_\lambda^M = \bigcup_{(\rho,\theta,R)\in\mathbf{Z}}\{(\psi_\lambda(\rho,R),\theta,R)\}$,

then one has:

Proposition 1. $\mathbf{X}$ *and* $\mathbf{X}_\lambda^M$ *are Poisson point processes with respective intensities* $\mathrm{d}\rho\,\mathrm{d}\theta$*, and* $\lambda^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\mu$ *on respectively* $\mathbb{R}_+\times[0,2\pi)$ *and* $\mathscr{B} := \{(r,t,R)\in\mathbb{R}_+\times[0,2\pi)\times\mathbb{R}_+ \;:\; r > R\}$*.*

We may then construct the polar lines at the points of $\mathbf{X}$, the Boolean model of discs associated to $\mathbf{X}_\lambda^M$: this Boolean model does not cover the origin, rescale this last Boolean model by an homothetic factor λ^2, and compare them, this procedure is illustrated in figure 1.

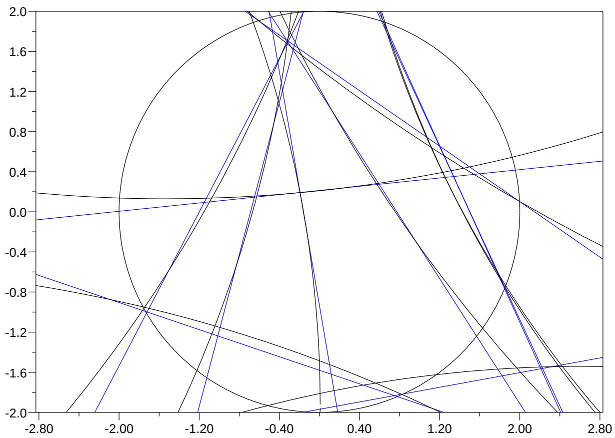

FIGURE 1: Coupling the Boolean model with the line process (the simulation is exact within the circle).

Remark 1. Conversely we could have introduced the coupling starting from the points (ρ,θ,R) of a marked Poisson point process $\mathbf{X}_\lambda^M$ with intensity $\lambda^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\mu$ on $\mathscr{B}$, yielding directly the Boolean model. In this setting the application

$$(\rho, \theta, R) \mapsto \left(\lambda^2(\rho - R) + \frac{1}{2R}\lambda^2(\rho - R)^2, \theta\right) \tag{3}$$

maps $\mathbf{X}_\lambda^M$ onto a Poisson point process $\mathbf{X}$ with intensity measure $\mathrm{d}\rho\,\mathrm{d}\theta$ on which we may construct the Poisson line process.

Proof. Proposition 1 is easily proved by the following:

- $\mathbf{X}$ is clearly a Poisson point process with the right intensity measure;
- and $\mathbf{X}_\lambda$ is also a Poisson point process, whose intensity measure is the image of the intensity measure of $\mathbf{Z}$ by the map

$$(\rho, \theta, R) \mapsto (\psi_\lambda(\rho, R), \theta, R),$$

  a straightforward computation shows the result.

■

From now on we shall use the coupling induced by $\mathbf{Z}$, thus the set D^λ will refer to the rescaled connected component in this Boolean model (when bounded, K_0 otherwise), and $\mathscr{C}$ will be the Crofton cell in this line process.

This coupling yields the following result on the local accuracy on the approximation of the rescaled Boolean model by the line process:

Proposition 2. *For all $M > 0$ and $\lambda \geq \lambda_0(M) = \sqrt{\frac{M}{R_\star}} > 0$ one has*

$$d_H^M(\lambda^2 D_0^\lambda, \mathscr{C}) \leq \frac{M'^2}{R_\star \lambda^2} \text{ a.s.},$$

where $M' = M + M^2/(\lambda^2 R_\star)$, and d_H^M denotes the M-Hausdorff distance in $B(0, M)$ defined by

$$\begin{aligned} d_H^M(F, G) = \inf\big\{\alpha > 0 \quad : \quad & (F \oplus B(0,\alpha)) \cap B(0,M) \supset G \cap B(0,M), \\ & (G \oplus B(0,\alpha)) \cap B(0,M) \supset F \cap B(0,M)\big\}, \end{aligned}$$

and for any two subsets A and B, the set $A \oplus B$ denotes their Minkowski sum: $A \oplus B = \{x + y,\ (x, y) \in A \times B\}$.

Proof. Let us denote by $(\rho_1, \alpha_1, R_1), \ldots, (\rho_K, \alpha_K, R_K)$ the points of $\mathbf{Z}$ such that the rescaled discs B_i associated to those points intersect $B(0, M)$. Since $\lambda \geq \lambda_0(M)$ no such disc can be included in $B(0, M)$. A straightforward computation with the help of formula (3) shows that the associated lines intersect the disc $B(0, M')$, where

$$M' = M + \frac{M^2}{\lambda^2 R_\star}.$$

More precisely, as is shown in figure 2, the M-Hausdorff distance between the intersection of the circle ∂B_i with $B(0, M)$ and the intersection of tangent line T_i with $B(0, M)$ is bounded by the distance between the points A and C, defined as respectively the intersection of T_i and $\partial B(0, M)$, and the point on ∂B_i aligned with A and the center of B_i. We have, if we define the distance from the origin to B_i as $u_i = \lambda^2 R_i(\sqrt{1 + 2\rho_i/(\lambda^2 R_i)} - 1) \leq M$:

$$\begin{aligned} AC &= \sqrt{\lambda^4 R_i^2 + M^2 - u_i^2} - \lambda^2 R_i, \\ &= \frac{M^2 - u_i^2}{\sqrt{\lambda^4 R_i^2 + M^2 - u_i^2} + \lambda^2 R_i}, \\ &\leq \frac{M^2}{2\lambda^2 R_i}. \end{aligned}$$

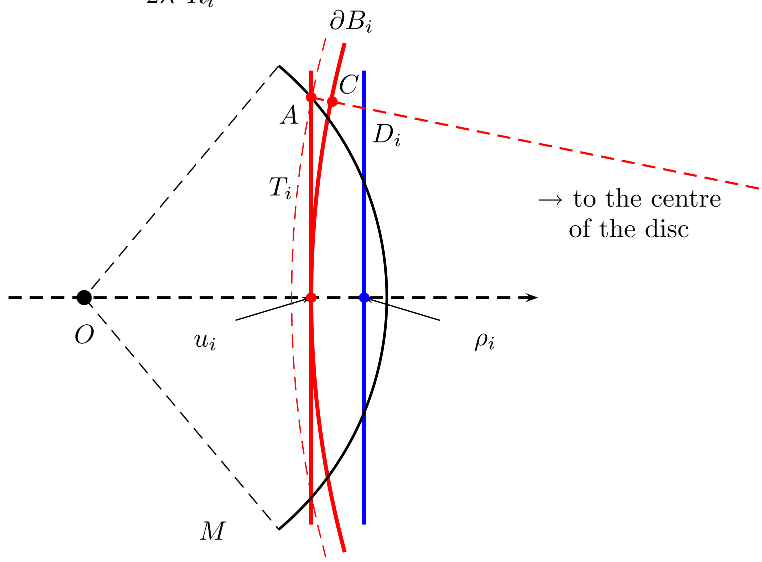

FIGURE 2: The circle ∂B_i and the tangent line T_i.

Consequently, for any disc B of radius $\lambda^2 R > \lambda^2 R_\star$ such that $0 \notin B$, and T the tangent to this disc at its closest point to 0 one has

$$d_H^M(\partial B, T) \leq \frac{M^2}{2\lambda^2 R_\star}. \tag{4}$$

On the other hand, as $u_i \leq M$, one has

$$\begin{aligned} |u_i - \rho_i| &= \rho_i \left| \frac{1 - \sqrt{1 + \frac{2\rho_i}{\lambda^2 R_i}}}{1 + \sqrt{1 + \frac{2\rho_i}{\lambda^2 R_i}}} \right| \\ &\leq \frac{\rho_i^2}{2\lambda^2 R_i}, \\ &\leq \frac{M'^2}{2\lambda^2 R_\star}, \end{aligned} \tag{5}$$

so that one obtains the result of proposition 2 by combining the two inequalities (4,5): if we denote by H_i (resp. H_i') the half plane with boundary D_i (resp. T_i) not containing the origin:

$$\begin{aligned} d_H^M(\lambda^2 D_0^\lambda, C_0^\lambda) &\leq \max_{i \in \{1,\dots,K\}} d_H^M(\complement B_i, \complement H_i), \\ &\leq \max_{i \in \{1,\dots,K\}} d_H^M(\complement B_i, \complement H_i') \\ &\quad + \max_{i \in \{1,\dots,K\}} d_H^M(\complement H_i, \complement H_i'), \\ &\leq \frac{M'^2}{\lambda^2 R_\star}, \end{aligned}$$

where $\complement G$ denotes the complementary set of G, and this concludes the proof of proposition 2.

■

From proposition 2 we may deduce the almost sure convergence in our coupled setting:

Theorem 4. *Almost surely D^λ converges in Hausdorff distance towards $\mathscr{C}$.*

Proof. Let us consider the subset Ω_r of those $\omega' s$ such that both R_M and $R_M(\lambda)$ are lesser than r, then the Hausdorff distance between $\mathscr{C}$ and $D^\lambda = \lambda^2 D_0^\lambda$ is lesser than $r'^2/(R_\star \lambda^2)$ where $r' = r + r^2/(\lambda^2 R_\star)$ for λ large enough thanks to proposition 2, and $\mathbf{P}(\Omega_r) \to 1$ as $r \to +\infty$ thanks to theorems 2 and 3, thus the conclusion.

■

Remark 2. The actual speed of convergence shown above could be stated in the following way: let $S_M(\lambda) = \max(R_M, R_M(\lambda))$, then

$$d_H(\mathscr{C}, D^\lambda) \leq \frac{S'_M(\lambda)^2}{\lambda^2 R_\star},$$

where $S'_M(\lambda) = S_M(\lambda) + S_M(\lambda)^2/(\lambda^2 R_\star)$.

4. Convergence of the second order-directional results

In order to prove a second order convergence result for the empty connected component towards the Crofton cell, we shall first give some notations and definitions, then we shall state the convergence results for one, then many directions in a second and third subsections.

4.1. Notations

Recall that for each $\omega \in \Omega$, we denote by $\mathscr{C}(\omega)$ the Crofton cell of the Poisson line process $\mathbf{X}$ induced by $\mathbf{Z}$, and by $D^\lambda(\omega)$ the empty connected component of the rescaled coupled Boolean model. We define the following quantities (almost surely they are all finite random variables):

- $N_e(\omega)$ the number of vertices of $\mathscr{C}(\omega)$, those points are denoted anti-clockwise by $V_1(\omega), \ldots, V_{N_e(\omega)}(\omega)$;
- $0 \leq \theta_1(\omega) < \ldots < \theta_{N_e(\omega)}(\omega) < 2\pi$ the polar angles of those vertices;
- we take the convention for the edge numbered i of $\mathscr{C}(\omega)$ to join vertices V_i (included) and $V_{(i \mod N_e(\omega))+1}$ (excluded);
- for each $i \in \{1, \ldots, N_e(\omega)\}$, set $(\Upsilon_i(\omega), \Theta_i(\omega), R_i(\omega))$ the polar coordinates (*angle* and *distance*) of the edges of $\mathscr{C}(\omega)$ marked with the associated radius of the disc in the coupled Boolean model (from now on we will write $i+1$ for $(i \mod N_e(\omega))+1$ and N_e for $N_e(\omega)$, for sake of simplicity).

For each $\omega \in \Omega$ and $t \in [0, 2\pi)$ we define Δ_t the half-line $\{(r,t) \ : \ r > 0\}$ and

- $\Theta(t, \omega)$ the polar angle of the edge intersecting the half-line Δ_t;
- $\Upsilon(t, \omega)$ the distance from the origin to this edge;

- $L(t,\omega) = \Upsilon(t,\omega)/\cos(\Theta(t,\omega)-t)$ the distance from the origin to the intersection of Δ_t with this edge;
- $R(t,\omega)$ the radius of the associated disc.

All quantities above are well defined on the same set of full probability for each t.

Definition 1. For each $\omega \in \Omega$ and $t \in [0, 2\pi)$ we define the *defect* at angle t by

$$d_\lambda(t,\omega) = \mathrm{dist}(0, \Delta_t \cap \partial D^\lambda(\omega)) - \mathrm{dist}(0, \Delta_t \cap \partial\mathscr{C}(\omega)),$$

where dist denotes the Euclidean distance, and the *approximate defect* at angle t by

$$\begin{aligned}\overline{d}_\lambda(t,\omega) \quad &= \quad \mathrm{dist}\left(0, \Delta_t \cap B\left(\lambda^2\psi_\lambda(\Upsilon(t,\omega),\Theta(t,\omega)), \lambda^2 R(t,\omega)\right)\right) - \\ &\quad\quad \mathrm{dist}(0, \Delta_t \cap \mathscr{C}(\omega)),\end{aligned}$$

when this quantity is well-defined (λ large enough).

We check easily that

$$\begin{aligned}\overline{d}_\lambda(t,\omega) \quad = \quad & \lambda^2 R(t,\omega)\cos(\Theta(t,\omega)-t)\sqrt{1+\frac{2\Upsilon(t,\omega)}{\lambda^2 R(t,\omega)}} \\ & -\lambda^2 R(t,\omega)\sqrt{1-\sin^2\left(\Theta(t,\omega)-t\right)\left(1+\frac{2\Upsilon(t,\omega)}{\lambda^2 R(t,\omega)}\right)} \\ & -\frac{\Upsilon(t,\omega)}{\cos(\Theta(t,\omega)-t)},\end{aligned} \tag{6}$$

for $\lambda \geq \sqrt{2\Upsilon(t,\omega)\tan^2(\Theta(t,\omega)-t)/R(t,\omega)}$, see figure 3.

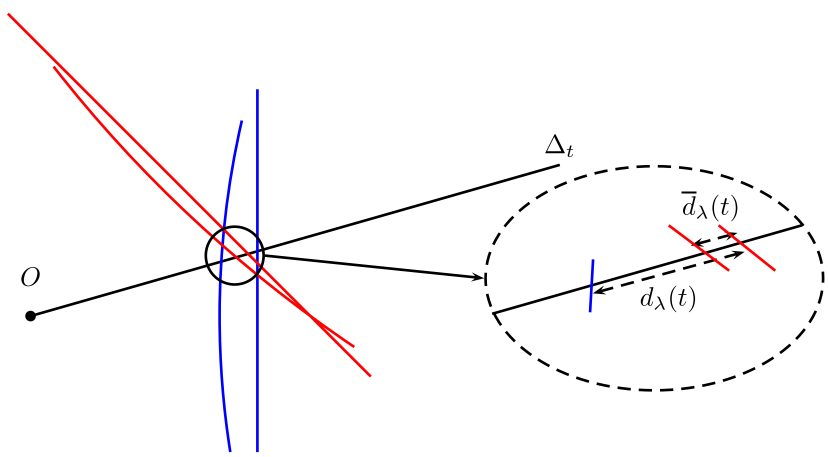

FIGURE 3: Defect and approximate defect

4.2. One directional convergence

The first result is an almost sure convergence of the defect function in one fixed direction:

Theorem 5. *For all $t \in [0, 2\pi)$ one has*

$$\lambda^2 d_\lambda(t, \cdot) \xrightarrow[\lambda \to +\infty]{a.s.} Z(t, \cdot),$$

where $Z(t, \cdot)$ is the random variable defined by

$$\forall \omega \in \Omega, \ Z(t, \omega) = -\frac{L(t,\omega)^2}{2} \frac{\cos 2(\Theta(t,\omega) - t)}{R(t,\omega)\cos(\Theta(t,\omega) - t)},$$

and the common law of $(L(t, \cdot), \Theta(t, \cdot), R(t, \cdot))$ is given by

$$\mathrm{d}(L(t,\cdot), \Theta(t,\cdot), R(t,\cdot))(P)(\ell, \alpha, r) = \pi \exp(-2\pi\ell) \cos(\alpha - t) \mathbf{1}_{\alpha \in (t-\pi/2, t+\pi/2)} \, \mathrm{d}\ell \, \mathrm{d}\alpha \, \mathrm{d}\mu(r).$$

Proof. The proof of this result proceeds in two steps:

- restrict the probability space to those events such that for λ large enough the defect is equal to the approximate defect;
- show that those events cover almost surely Ω.

Step 1: restricted events Let us consider $\delta > 0$, $\epsilon > 0$, $r > 0$ and $s > 0$, and consider the subset $\Omega_{\delta,\epsilon,r,s}$ of all $\omega \in \Omega$ such that

- $B(0, r) \subset \mathscr{C}(\omega) \subset B(0, s)$;
- for each $u \in (t - \delta, t + \delta)$, the intersection of $\mathscr{C}(\omega)$ with Δ_u is on the same edge of $\mathscr{C}(\omega)$;
- $\mathscr{C}(\omega) \oplus B(0, \epsilon)$ is not intersected by other lines of the Poisson line process than those on the boundary of $\mathscr{C}(\omega)$.

It is quite obvious for geometrical arguments that if λ is large enough, in direction t the defect will be exactly equal to the approximate defect, as in the disc of radius $s + \epsilon$ the Hausdorff distance between circles and lines gets smaller as λ increases, and thus in direction t the first intersecting line corresponds to the first intersecting disc. There remains to compute the exact asymptotics of the approximate defect, this is done in the following way, where we restrict ourselves thanks to invariance under rotations, to the angle $t = 0$.

One has on the one hand the following well-known classical result for the law of the first intersecting line:

Lemma 1. *Let L denote the distance from 0 to the first intersection on Δ_0 of the line process, and Θ the polar angle of this intersecting line, then the law of (L, Θ) is given by*

$$\mathrm{d}(L,\Theta)(P)(\ell,\theta) = e^{-2\ell}\,\cos\theta\,\mathbf{1}_{\theta\in(-\pi/2,\pi/2)}\,\mathrm{d}\ell\,\mathrm{d}\theta.$$

On the other hand, from formula 6 we get easily that

$$\begin{aligned} d_\lambda(0) &= \overline{d}_\lambda(0) \\ &= \lambda^2 R(0,\omega)\cos\Theta(0,\omega)\sqrt{1+\frac{2L(0,\omega)\cos\Theta(0,\omega)}{\lambda^2 R(0,\omega)}} \\ &\quad -\lambda^2 R(0,\omega)\sqrt{1-\sin^2\Theta(0,\omega)\left(1+\frac{2L(0,\omega)\cos\Theta(0,\omega)}{\lambda^2 R(0,\omega)}\right)} \\ &\quad -L(0,\omega) \end{aligned}$$

if the inner term of the square root is non negative (*i.e.* $\lambda \geq \sqrt{(2L\sin^2\Theta)/(R\cos\Theta)}$), $+\infty$ otherwise. The asymptotic expansion of those square roots gives easily

$$\overline{d}_\lambda(0) = -\frac{L(0,\omega)^2}{2\lambda^2}\frac{\cos 2\Theta(0,\omega)}{R(0,\omega)\cos\Theta(0,\omega)} + O(\lambda^{-4}).$$

Step 2: Almost sure covering We conclude the proof of theorem 5 by stating the following lemma:

Lemma 2. *As δ, r, ϵ tend to zero and s tends to $+\infty$, one has*

$$\mathbf{P}(\Omega_{\delta,\epsilon,r,s}) \to 1.$$

The proof of this lemma comes directly from the properties of the Poisson point process $\mathbf{X}$ and the asymptotic results on the law of the inner and outer radii of the Crofton cell stated in section 2.

■

Remark 3. The almost sure convergence above will not be used for the convergence of the defect process, only the convergence in law of the finite directional distributions is needed, however we shall state them almost surely.

4.3. Two and more directions

For more directions we may state similar results,

Theorem 6. *For all* $0 \leq t_1 < \cdots < t_n < 2\pi$*, the finite dimensional random vector* $\lambda^2(d_\lambda(t_1,\cdot),\ldots,d_\lambda(t_n,\cdot))$ *converges almost surely towards* $(Z(t_1,\cdot),\ldots,Z(t_n,\cdot))$*, where the law of this random vector may be fully explicited.*

The proof is essentially the same one as for one direction, only with more technical details.

This random vector depends only on the characteristics of the Crofton cell, let us for instance give the exact law of this vector for two directions (by invariance under rotations we choose directions 0 and $t \in (0, 2\pi)$): $\lambda^2(d_\lambda(0,\cdot), d_\lambda(t,\cdot))$ converges in law as λ goes to infinity towards

$$B\ (Z_0(\Upsilon,\Theta,R), Z_t(\Upsilon,\Theta,R)) + (1-B)\,(W_0(\Upsilon_1,\Theta_1,R_1), W_t(\Upsilon_2,\Theta_2,R_2))\,,$$

where

- B is Bernoulli random variable stating that the same line determines the intersections in directions 0 and t: this occurs with probability p,

$$\begin{aligned} p &= \mathbf{E}\left[\sum_{(\rho,\alpha,R)\in\mathbf{Z}} \mathbf{1}_{\text{The lines from } \mathbf{Z}\backslash\{\rho,\alpha,R\} \text{ do not intersect } \Delta_0 \text{ or } \Delta_t \text{ before } D_{(\rho,\alpha)}}\right], \\ &= \int_{\mathbb{R}_+\times[0,2\pi)\times[R_\star,+\infty)} \exp(-\mathfrak{p}(\Delta_{0,t}(\rho,\alpha)))\,\mathrm{d}\rho\,\mathrm{d}\alpha\,\mathrm{d}\mu(r), \end{aligned}$$

  where $\Delta_{0,t}(\rho,\alpha)$ is the triangle described by figure 4, $\mathfrak{p}$ denotes the perimeter function. This Bernoulli random variable is independent from the following ones,

- (Υ,Θ,R) has the following distribution:

$$\begin{aligned} \mathrm{d}(\Upsilon,\Theta)(P)(\rho,\alpha,r) &= p^{-1}\,\mathbf{1}_{\alpha\in(-\pi/2,\pi/2),\alpha-t\in(-\pi/2,\pi/2)} \\ &\quad \exp(-\mathfrak{p}(\Delta_{0,t}(\rho,\alpha)))\,\mathrm{d}\rho\,\mathrm{d}\alpha\,\mathrm{d}\mu(r), \end{aligned}$$

- $Z_0(\Upsilon,\Theta,R) = -\dfrac{\Upsilon^2}{2R\cos^2\Theta}\dfrac{\cos 2\Theta}{\cos\Theta}$,
- $Z_t(\Upsilon,\Theta,R) = -\dfrac{\Upsilon^2}{2R\cos^2(\Theta-\theta)}\dfrac{\cos 2(\Theta-t)}{\cos(\Theta-t)}$,

- $(\Upsilon_1, \Theta_1, R_1, \Upsilon_2, \Theta_2, R_2)$ has the following distribution,

$$\begin{aligned} & \mathrm{d}(\Upsilon_1, \Theta_1, R_1, \Upsilon_2, \Theta_2, R_2)(P)(\rho_1, \alpha_1, r_1, \rho_2, \alpha_2, r_2) \\ = \; & (1-p)^{-1}\, \mathbf{1}_{\alpha_1 \in (-\pi/2, \pi/2)} \mathbf{1}_{\alpha_2 - t \in (-\pi/2, \pi/2)} \mathbf{1}_{(\rho_1, \alpha_1) \notin B_t(\rho_2, \alpha_2)}\, \mathbf{1}_{(\rho_2, \alpha_2) \notin B_0(\rho_1, \alpha_1)} \\ & \exp(-\mathfrak{p}(\Delta'_{0,t}(\rho_1, \alpha_1, \rho_2, \alpha_2)))\, \mathrm{d}\rho_1\, \mathrm{d}\alpha_1\, \mathrm{d}\mu(r_1)\, \mathrm{d}\rho_2\, \mathrm{d}\alpha_2\, \mathrm{d}\mu(r_2), \end{aligned}$$

  where the sets B_0, B_t and $\Delta'_{0,t}$ are described by figure 4.
- $W_0(\Upsilon_1, \Theta_1, R_1) = -\dfrac{\Upsilon_1^2}{2R_1 \cos^2 \Theta_1} \dfrac{\cos 2\Theta_1}{\cos \Theta_1}$,
- $W_\theta(\Upsilon_2, \Theta_2, R_2) = -\dfrac{\Upsilon_2^2}{2R_2 \cos^2(\Theta_2 - t)} \dfrac{\cos 2(\Theta_2 - t)}{\cos(\Theta_2 - t)}$,

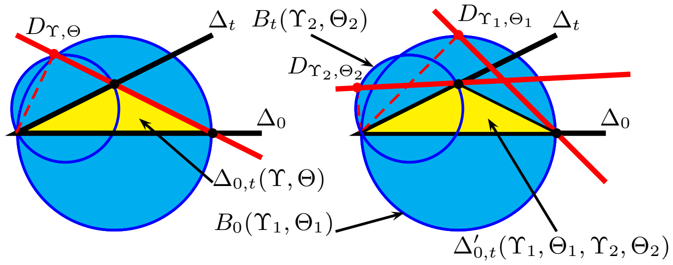

FIGURE 4: The sets $\Delta_{0,t}$, $\Delta'_{0,t}$, B_0 and B_t, and the corresponding lines.

5. Convergence of the stochastic process

In this section we consider the processes $(\lambda^2 d_\lambda(t))_{t \in [0, 2\pi)}$. Let us first remark the following: knowing the joint limit law of the couples $(\lambda^2 d_\lambda(0), \lambda^2 d_\lambda(t))$ gives some knowledge on this process, for instance by simulation we can obtain the covariogram $t \mapsto \mathrm{cov}(\lambda^2 d_\lambda(0), \lambda^2 d_\lambda(t))$, $t \in [0, \pi]$, in figure 5. One clearly observes the divergence as λ tends to infinity of the covariance for $t \to 0$, this is a consequence of the following elementary result coming from the explicit law of the defect:

Corollary 1. *The limit expected defect is an integrable random variable with*

$$\lim_{\lambda \to +\infty} \mathbf{E}[\lambda^2 d_\lambda(0)] = 0.$$

However, this limit expected defect is not square-integrable:

$$\lim_{\lambda \to +\infty} \mathbf{E}\left[\left(\lambda^2 d_\lambda(0) \right)^2 \right] = +\infty.$$

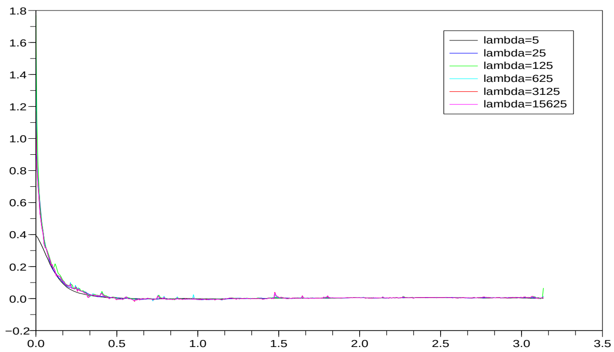

FIGURE 5: Covariograms, sample of size 250000

The matter of convergence of the whole process will be stated in the state $D([0, 2\pi])$. As a matter of fact for each λ the trajectory of the defect process is continous, however the limit process is not continuous: the choice of the space $D([0, 2\pi])$, even if there is no geometric justification in choosing right-continuity, seems to be quite natural. Let us thus consider such processes X_λ on $[0, 2\pi]$, according to theorem 15.4 in [1] the conditions for the convergence of processes $(X_\lambda)_{\lambda\geq 1}$ on $D([0, 2\pi])$ are:

- convergence in law of the finite-dimensional distributions, this is true thanks to theorem 6;
- tightness criterion, for instance the following one: $\forall \eta, \epsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{\lambda\to+\infty} \mathbf{P}\left(\sup_{t_1\leq t\leq t_2,\ t_2-t_1<\delta} \min(|X_\lambda(t) - X_\lambda(t_1)|, |X_\lambda(t_2) - X_\lambda(t)|) \geq \epsilon\right) \leq \eta.$$

In our case, unfortunately one can not use directly such a tightness criterion: indeed if we take $X_\lambda = \lambda^2 d_\lambda$ we see (figure 6) that the high slopes that appear near the angles corresponding to the vertices of the Crofton cell forbid us to use this kind of citerion, as well as all other classical criteria. Hence we shall first show the convergence of the approximate defect process $X_\lambda = \lambda^2 \overline{d}_\lambda$, as this process is the combination of a jump process and a smooth process, and then give an explicit estimate on the accuracy of this approximation in L^1 norm.

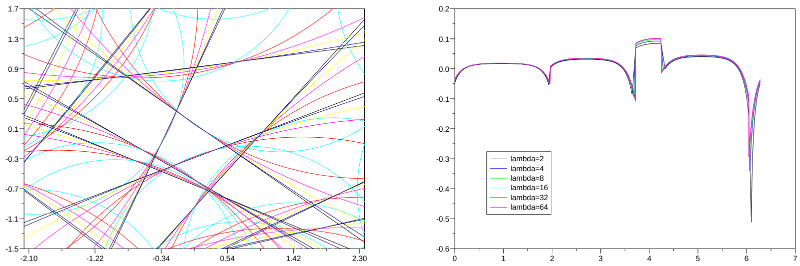

FIGURE 6: The Crofton cell and the related processes, one checks that there are four edges, and four singularities

5.1. Convergence of the approximate defect process

We prove the following theorem on the approximate defect:

Theorem 7. *The approximate defect process* $(\lambda^2 \overline{d}_\lambda(t))_{t\in[0,2\pi]}$ *converges in law in* $D([0,2\pi])$ *to the process* $(X_t)_{t\in[0,2\pi]}$ *defined for all* $i \in \{1,\ldots,N\}$ *and* $t \in [\theta_i, \theta_{i+1})$ *by*

$$\omega \mapsto X_t(\omega) = -\frac{\Upsilon_i(t,\omega)^2}{2R_i(t,\omega)} \frac{\cos 2(\Theta_i(t,\omega)-t)}{\cos^3(\Theta_i(t,\omega)-t)},$$

using the notations of definition 1.

Proof. Let us fix ϵ and η, both positive numbers, and define for $0 < r < s$ and $\delta_0 > 0$ the set Ω_{r,R,δ_0} of those $\omega \in \Omega$ such that the Crofton cell $\mathscr{C}(\omega)$ and D^λ satisfy:

- $B(0,r) \subset \mathscr{C}(\omega) \subset B(0,s)$, *i.e.* $R_m \geq r$ and $R_M \leq s$;
- $D^\lambda \subset B(0,s)$, *i.e.* $R_M(\lambda) \leq s$;
- the angular distance $\theta_{i+1}(\omega) - \theta_i(\omega)$ between any two consecutive vertices of $\mathscr{C}(\omega)$ is greater than δ_0.

We give without proof the following lemma, similar to lemma 2, stating that with high probability the Crofton cell is a 'gentle' polygon:

Lemma 3. *As* $r \to 0$, $s \to +\infty$ *and* $\delta_0 \to 0$ *one has* $\mathbf{P}(\Omega_{r,s,\delta_0}) \to 1$.

On this event Ω_{r,s,δ_0} we check easily from definition 1 that for $t \in [\theta_i, \theta_{i+1})$ the approximate defect $\lambda^2 \overline{d}_\lambda(t)$ is well defined for $\lambda \geq \sqrt{2s^3/(r^2 R_\star)}$ and is Lipschitz-continuous for $\lambda \geq 2\sqrt{s^3/(r^2 R_\star)}$ with Lipschitz constant lesser than $13s^6/(r^4 R_\star)$ on the interval $[\theta_i, \theta_{i+1})$.

It is then straightforward to check that if we choose $\delta < \delta_0$ and $\delta < C\epsilon r^4 R_\star / s^6$, we have for λ large enough (depending on r and s)

$$\begin{aligned} & \mathbf{P}\left(\sup_{t_1 \leq t \leq t_2,\ t_2 - t_1 < \delta} \min(|\lambda^2 \overline{d}_\lambda(t) - \lambda^2 \overline{d}_\lambda(t_1)|, |\lambda^2 \overline{d}_\lambda(t_2) - \lambda^2 \overline{d}_\lambda(t)|) \geq \epsilon \right) \\ \leq \quad & 1 - \mathbf{P}(\Omega_{r,s,\delta_0}). \end{aligned}$$

We may then conclude using lemma 3.

■

The asymptotic expansion of the approximate defect gives also a convergence in the spaces $L^p(0, 2\pi)$, $p \in [1, +\infty]$: indeed on the event Ω_{r,s,δ_0} one checks that $(\lambda^2 \overline{d}_\lambda(t) - X_t)_{t \in (0,2\pi)}$ is bounded uniformely on $(0, 2\pi)$ by $\lambda^{-2} s^5/(R_\star^2 r^2)$ times an explicit constant depending only on r, s and δ_0, hence

Proposition 3. *Almost surely one has for all* $p \in [1, +\infty]$

$$(\lambda^2 \overline{d}_\lambda(t))_{t \in (0,2\pi)} \underset{\lambda \to +\infty}{\xrightarrow{L^p}} (X_t)_{t \in (0,2\pi)}.$$

5.2. Estimate on the accuracy of the approximate defect process

Obviously the approximation of the rescaled defect process $(\lambda^2 d_\lambda(t))_{t \in [0,2\pi]}$ by the process $(\lambda^2 \overline{d}_\lambda(t))_{t \in [0,2\pi]}$ is not convergent to 0 in the space $L^\infty(0, 2\pi)$ because of the (common) jumps of both the rescaled and limit processes. We prove the following result:

Theorem 8. *Almost surely one has the following convergence:*

$$\lim_{\lambda \to +\infty} \lambda^2 \int_{(0,2\pi)} |d_\lambda(t) - \overline{d}_\lambda(t)|\, dt = 0.$$

Proof. The proof of this theorem will also be done in two steps:

- estimates on the widths of the 'almost jumps' of the defect process on almost-full probability events;
- L^∞ estimates on the difference of the two processes on those same events.

Step 1: widths of jumps

The quantities d_λ and $\overline{d}_\lambda$ are distinct only in the following case: the first intersecting line in direction t does not induce the first intersecting disc in direction t, this decomposes into three subcases:

- this first intersecting disc is associated to an other line of the line process that contains an edge of the Crofton cell, adjacent to the actual edge intersected by Δ_t,
- this disc is associated to a non-adjacent edge,
- this disc is associated to a line that does not induce any edge of the Crofton cell.

We shall show that the last two cases can be excluded on some event: let us first define for $\epsilon > 0$ the thick Crofton cell as $\mathscr{C}_\epsilon = \mathscr{C} \oplus B(0,\epsilon)$, we shall say that it is *equivalent* to $\mathscr{C}$ if it has the same edges and vertices as $\mathscr{C}$, more precisely if for each $i \in \{1,\ldots,N_e\}$ one has

$$(D_i \oplus B(0,\epsilon) \cap D_{i+1} \oplus B(0,\epsilon)) \cap (D_{i+1} \oplus B(0,\epsilon) \cap D_{i+2} \oplus B(0,\epsilon)) = \emptyset,$$

where the D_i, $i \in \{1,\ldots,N_e\}$ are the lines supporting the edges of $\mathscr{C}$. The intersections $D_i \oplus B(0,\epsilon) \cap D_{i+1} \oplus B(0,\epsilon)$ are lozenges, denoted by $C_{i,\epsilon}$ (crossings).

Remark 4. It is clear that by thickening the Poisson line process the thick Crofton cell is defined by at most the lines on the boundary of the Crofton cell. Our notion of equivalence is a little more demanding than just assuming that all those lines bound $\mathscr{C}_\epsilon$.

Lemma 4. *Let $r < s$ and ϵ be positive numbers and define $\overline{\Omega}_{r,s,\epsilon}$ the event such that for $\omega \in \overline{\Omega}_{r,s,\epsilon}$*

- *the Crofton cell $\mathscr{C}(\omega)$ is included in $B(0,s)$ and contains $B(0,r)$;*
- *the thick crofton cell $\mathscr{C}_\epsilon(\omega)$ is equivalent to the Crofton cell $\mathscr{C}$, and for each $(\rho,\theta,R) \in \mathbf{Z} \setminus \{(\Upsilon_i,\Theta_i,R_i),\ i \in \{1,\ldots,N_e\}\}$ one has*

  $$\mathscr{C}_\epsilon(\omega) \cap (D_{\rho,\theta} \oplus B(0,\epsilon)) = \emptyset,$$

  and for all $i \in \{1,\ldots,N_e\}$

  $$C_{i,\epsilon} \cap (D_{\rho,\theta} \oplus B(0,\epsilon)) = \emptyset,$$

Then one has

$$\lim_{r,\epsilon\to 0, s\to+\infty} \mathbf{P}(\overline{\Omega}_{r,s,\epsilon}) = 1.$$

Furthemore for λ large enough (depending on r, s and ϵ and (Υ_i, Θ_i), $i \in \{1, \ldots, N_e\}$) for all $t \in [0, 2\pi)$ the first intersecting line and disc are associated either to the same point $(\Upsilon_i, \Theta_i, R_i) \in \mathbf{Z}$ or to the two points $(\Upsilon_i, \Theta_i, R_i)$ and $(\Upsilon_{i\pm1}, \Theta_{i\pm1}, R_{i\pm1})$ where $i \in \{1, \ldots, N_e\}$ is the index of the corresponding edge of the Crofton cell.

The proof of this lemma follows classical lines, for instance for fixed r and s it is obvious that the conditional probability given that $R_M \leq s$ and $R_m \geq r$ that some line does violate the third or fourth hypothesis is of order ϵs. The last point is clearly illustrated in figure 7.

If λ is chosen large enough, depending only on r, s, ϵ, and $(\Upsilon_i, \Theta_i)_{i\in\{1,\ldots,N_e\}}$, then it is clear thanks to proposition 2 that the angles t for which the two edges are needed to determine the defect at t are at most those corresponding to the disjoint sub-lozenges $C_{i,\eta/\lambda^2}$ formed by the intersections of two polar thick lines $D_i \oplus B(0, \eta/\lambda^2)$ and $D_{i+1} \oplus B(0, \eta/\lambda^2)$ corresponding to edges i and $i+1$, where η depends on r, s, ϵ and $(\Upsilon_i, \Theta_i)_{i\in\{1,\ldots,N_e\}}$, with $\eta/\lambda^2 < \epsilon$.

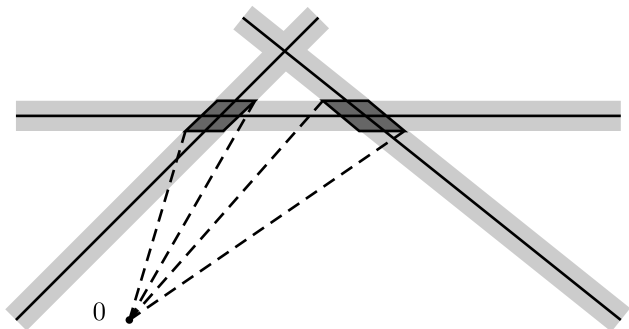

FIGURE 7: Thick Crofton cell and lozenges.

This implies that the total length of problematic angles t is lesser than $N \times \zeta/\lambda^2$, where ζ depends in quite a technical way on r, s, ϵ and $\mathscr{C}$ through the minimum of the differences $|\Theta_{i+1} - \Theta_i|$.

Step 2: L^∞ bounds From the properties of the Crofton cell on the set $\overline{\Omega}_{r,s,\epsilon}$, we may easily evaluate exactly the difference between both processes: indeed when this difference is non zero it is lesser than

$$|d_\lambda(t) - \overline{d}_\lambda(t)| \quad \leq \quad d_1(t) + d_2(t) + d_3(t),$$

where $d_1(t)$ (resp. $d_2(t)$) is the modulus of the distance between the intersections on Δ_t of D_i and the associated disc B_i (resp. $D_{i\pm1}$ and $B_{i\pm1}$), and $d_3(t)$ is the modulus of the distance between $D_i \cap \Delta_t$ and $D_{i\pm1} \cap \Delta_t$ (see figure 7 above).

Let us remark that one of $d_1(t)$ and $d_2(t)$ is exactly $\overline{d}_\lambda(t)$: thanks to proposition 2, those two terms are bounded from above by a constant for λ large enough:

$$d_j(t) \leq \frac{1}{\lambda^2 R_*}\frac{s}{r}\left(s + \frac{s^2}{\lambda^2 R_*}\right)^2, \quad j = 1, 2.$$

The third term is also bounded from above by a constant $M(r, s, \epsilon, \mathscr{C}) \times \eta/\lambda^2$. Hence we obtain

$$\lambda^2 |d_\lambda(t) - \overline{d}_\lambda(t)| \leq M_1(r, s, \epsilon, \mathscr{C}).$$

Hence we have the following estimate on the L^1 norm of the difference:

$$\lambda^2 \int_{(0,2\pi)} |d_\lambda(t) - \overline{d}_\lambda(t)|\, dt \quad \leq \quad \frac{1}{\lambda^2} M_2(r, s, \epsilon, \mathscr{C}) \times N_e,$$

this upper bound converges towards zero for each ω in the set $\overline{\Omega}_{r,s,\epsilon}$: this concludes the proof of theorem 8.

$\blacksquare$

As a consequence of this result and of proposition 3 we obtain eventually the following convergence

Theorem 9. *Almost surely one has*

$$(\lambda^2 d_\lambda(t))_{t\in(0,2\pi)} \underset{\lambda\to+\infty}{\overset{L^1(0,2\pi)}{\longrightarrow}} (X_t)_{t\in(0,2\pi)},$$

where the process $(X_t)_{t\in(0,2\pi)}$ *is defined in theorem 7.*

5.3. Tail probability for the supremum of the defect process

This short section is devoted to a uniform bound on the tail probabilities for the defect processes for large λ's:

Proposition 4. *One has the following estimate: for all $\beta < 1$*

$$\lim_{u \to +\infty} \left\{ s^{\beta} \limsup_{\lambda \to +\infty} \mathbf{P} \left(\sup_{\theta \in [0,2\pi)} \lambda^2 |d_\lambda(\theta)| \geq u \right) \right\} = 0.$$

Sketch. The proof uses the same tools as before, the estimates on the growth of both $\mathscr{C}$ and the empty component of the Boolean model around the origin. Indeed we may give an explicit upper bound of the defect on the set $\overline{\Omega}_{r,s,\epsilon}$ using the computations of section 5.2. This bound is roughly of order s^5/r^2. By using an adequate choice of $r \to 0$ and $s \to +\infty$ in terms of powers of u we may obtain the result. This tedious proof is left to the reader.

■

Remark 5. (*Splitting the defect process.*) The limit defect process may be decomposed in a continuous part and a pure jump part, such a splitting can be done for the defect process at fixed λ: let us indeed write

$$\lambda^2 d_\lambda(t) = \lambda^2 \overline{\overline{d}}_\lambda(t) + \left(\lambda^2 d_\lambda(t) - \lambda^2 \overline{\overline{d}}_\lambda(t) \right),$$

where $\overline{\overline{d}}_\lambda$ is the continuous part of the process $\overline{d}_\lambda$ (this is almost surely defined as being equal to $\overline{d}_\lambda$ at the angle $t = 0$, and the jumps are *deleted*). Then it can be shown that both terms above converge in law, the first one in the space $C(0, 2\pi)$, and the second one in a weak sense.

Remark 6. (*Directions for the general case.*) For more general shapes the coupling is more tricky to obtain, we may proceed in the following way:

- consider smooth shapes with no flat portion on the boundary: $\tau(F)$, where τ is a uniform rotation, and F a smooth random closed set;
- given τ and F, assign to a Poisson line the centre of the rotated rescaled shape tangent at the line, at the same distance from the origin than the line;
- compute the intensity of the point process of centres of shapes, and modify this intensity so that it becomes the Lebesgue measure multiplied by the parameter λ^2, this shall be done by a function

$$(\rho, \theta, \tau, F) \mapsto \Psi_\lambda(\rho, \theta, \tau, F) \in \mathbb{R}_+ \times [0, 2\pi);$$

Using this procedure, the computations might be done involving more technical details, the limiting process might be expressed with the curvature of F.

References

[1] Billingsley, P. (1999). *Convergence of probability measures* second ed. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York. A Wiley-Interscience Publication.

[2] Calka, P. (2002). The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *Adv. in Appl. Probab.* **34,** 702–717.

[3] Calka, P. (2003). Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process. *Adv. in Appl. Probab.* **35,** 551–562.

[4] Hall, P. (1985). Distribution of size, structure and number of vacant regions in a high-intensity mosaic. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **70,** 237–261.

[5] Hall, P. (1988). *Introduction to the Theory of Coverage Processes.* John Wiley & Sons, New York.

[6] Heinrich, L., Schmidt, H. and Schmidt, V. (2006). Central limit theorems for Poisson hyperplane tessellations. *Ann. Appl. Probab.* **16,** 919–950.

[7] Hug, D., Reitzner, M. and Schneider, R. (2004). The limit shape of the zero cell in a stationary Poisson hyperplane tessellation. *Ann. Probab.* **32,** 1140–1167.

[8] Matheron, G. (1975). *Random sets and integral geometry.* John Wiley & Sons, New York-London-Sydney. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[9] Michel, J. and Paroux, K. (2003). Local convergence of the Boolean shell model towards the thick Poisson hyperplane process in the Euclidean space. *Adv. in Appl. Probab.* **35,** 354–361.

[10] Miles, R. (1964). Random polygons determined by random lines in a plane I. *Proc. Nat. Acad. Sci. USA* **52,** 901–907.

[11] Miles, R. (1964). Random polygons determined by random lines in a plane II. *Proc. Nat. Acad. Sci. USA* **52,** 1157–1160.

[12] Molchanov, I. (1996). A limit theorem for scaled vacancies of the boolean model. *Stochastics and Stochastic Reports* **58,** 45–65.

[13] Molchanov, I. (2005). *Theory of random sets.* Probability and its Applications (New York). Springer-Verlag London Ltd., London.

[14] Paroux, K. (1997). Théorèmes centraux limites pour les processus poissoniens de droites dans le plan et questions de convergence pour le modèle booléen de l'espace euclidien. *PhD thesis.* Université Lyon 1.

[15] Paroux, K. (1998). Quelques théorèmes centraux limites pour les processus Poissoniens de droites dans le plan. *Adv. in Appl. Probab.* **30,** 640–656.

[16] Schneider, R. (1993). *Convex bodies: the Brunn-Minkowski theory.* Cambridge University Press.

[17] Stoyan, D., Kendall, W. S. and Mecke, J. (1987). *Stochastic geometry and its applications.* Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Ltd., Chichester. With a foreword by D. G. Kendall.