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RANK FOUR VECTOR BUNDLES WITHOUT THETA DIVISOR OVER A CURVE OF GENUS TWO

CHRISTIAN PAULY

Abstract. We show that the locus of stable rank four vector bundles without theta divisor over a smooth projective curve of genus two is in canonical bijection with the set of theta-characteristics. We give several descriptions of these bundles and compute the degree of the rational theta map.

1. Introduction

Let \( C \) be a complex smooth projective curve of genus 2 and let \( \mathcal{M}_r \) denote the coarse moduli space parametrizing semi-stable rank-\( r \) vector bundles with trivial determinant over the curve \( C \). Let \( C \cong \Theta \subset \text{Pic}^1(C) \) be the Riemann theta divisor in the degree 1 component of the Picard variety of \( C \). For any \( E \in \mathcal{M}_r \) we consider the locus

\[ \theta(E) = \left\{ L \in \text{Pic}^1(C) \mid h^0(C, L \otimes E) > 0 \right\}, \]

which is either a curve linearly equivalent to \( r\Theta \) or \( \theta(E) = \text{Pic}^1(C) \), in which case we say that \( E \) has no theta divisor. We obtain thus a rational map, the so-called theta map

\[ \theta : \mathcal{M}_r \dashrightarrow |r\Theta|, \]

between varieties having the same dimension \( r^2 - 1 \). We denote by \( \mathcal{B}_r \) the closed subvariety of \( \mathcal{M}_r \) parametrizing semi-stable bundles without theta divisor. It is known \cite{R} that \( \mathcal{B}_2 = \mathcal{B}_3 = \emptyset \) and that \( \mathcal{B}_r \neq \emptyset \) for \( r \geq 4 \).

It was recently shown that \( \theta \) is generically finite; see \cite{B1} Theorem A. Moreover the cases of low ranks \( r \) have been studied in the past: if \( r = 2 \) the theta map is an isomorphism \( \mathcal{M}_2 \cong \mathbb{P}^3 \) \cite{NR} and if \( r = 3 \) the theta map realizes \( \mathcal{M}_3 \) as a double covering of \( \mathbb{P}^8 \) ramified along a sextic hypersurface \cite{O}.

In this note we study the next case \( r = 4 \) and give a complete description of the locus \( \mathcal{B}_4 \). Our main result is the following

Theorem 1.1. Let \( C \) be a curve of genus 2.

1. The locus \( \mathcal{B}_4 \) is of dimension 0, reduced and of cardinality 16.
2. There exists a canonical bijection between \( \mathcal{B}_4 \) and the set of theta-characteristics of \( C \). Let \( E_\kappa \in \mathcal{B}_4 \) denote the stable vector bundle associated with the theta-characteristic \( \kappa \). Then

\[ \Lambda^2 E_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 E_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha, \]

where \( S(\kappa) \) is the set of 2-torsion line bundles \( \alpha \in J[2] \) such that \( \kappa \alpha \in \Theta \subset \text{Pic}^1(C) \).
3. If \( \kappa \) is odd, then \( E_\kappa \) is a symplectic bundle. If \( \kappa \) is even, then \( E_\kappa \) is an orthogonal bundle with non-trivial Stiefel-Whitney class.
4. The 16 vector bundles \( E_\kappa \) are invariant under the tensor product with the group \( J[2] \).

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The 16 vector bundles $E_\kappa$ already appeared in Raynaud’s paper \cite{R} as Fourier-Mukai transforms and were further studied in \cite{H} and \cite{P} — see section 2.2. We note that Theorem 1.1 completes the main result of \cite{H} which describes the restriction of $B_4$ to \textit{symplectic} rank-4 bundles. The method of this paper is different and is partially based on \cite{P}.

As an application of Theorem 1.1 we obtain the degree of the theta map for $r = 4$. We refer to \cite{BN} for a geometric interpretation of the general fiber of $\theta$ in terms of certain irreducible components of a Brill-Noether locus of the curve $\theta(E) \subset \text{Pic}^1(C)$.

**Corollary 1.2.** The degree of the rational theta map $\theta : M_4 \longrightarrow |4\Theta|$ equals 30.

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**Notations:** If $E$ is a vector bundle over $C$, we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of $E$ by $\mu(E) := \frac{\deg E}{\rk E}$, the canonical bundle over $C$ by $K$ and the degree $d$ component of the Picard variety of $C$ by $\text{Pic}^d(C)$. We denote by $J := \text{Pic}^0(C)$ the Jacobian of $C$ and by $J[n]$ its group of $n$-torsion points. The divisor $\Theta_\kappa \subset J$ is the translate of the Riemann theta divisor $C \cong \Theta \subset \text{Pic}^1(C)$ by a theta-characteristic $\kappa$. The line bundle $\mathcal{O}_J(2\Theta_\kappa)$ does not depend on $\kappa$ and will be denoted by $\mathcal{O}_J(2\Theta)$.

2. Proof of Theorem 1.1

2.1. The 16 vector bundles $E_\kappa$. We first show that the set-theoretical support of $B_4$ consists of 16 stable vector bundles $E_\kappa$, which are canonically labelled by the theta-characteristics of $C$.

We note that $B_4 \neq \emptyset$ by \cite{H}, see also \cite{H} Theorem 1.1. We consider a vector bundle $\mathcal{E} \in B_4$. Since $B_2 = B_3 = \emptyset$, we deduce that $\mathcal{E}$ is stable. We introduce $\mathcal{E}' = \mathcal{E}^* \otimes K$. Then $\mu(\mathcal{E}') = 2$ and since $\mathcal{E} \in B_4$, we obtain that $h^0(\mathcal{E}' \otimes \lambda^{-1}) = h^1(\mathcal{E} \otimes \lambda) = h^0(\mathcal{E} \otimes \lambda) > 0$ for any $\lambda \in \text{Pic}^1(C)$. In particular for any $x \in C$ we have $h^0(\mathcal{E}' \otimes \mathcal{O}_C(-x)) > 0$. On the other hand stability of $\mathcal{E}$ implies that $h^0(\mathcal{E}) = h^1(\mathcal{E}') = 0$. Hence $h^0(\mathcal{E}') = 4$ by Riemann-Roch. Thus we obtain that the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} \mathcal{E}'$$

is not of maximal rank. Let us denote by $I := \text{im} \ ev$ the subsheaf of $\mathcal{E}'$ given by the image of $ev$. Then clearly $h^0(I) = 4$. The cases $\rk I \leq 2$ are easily ruled out using stability of $\mathcal{E}'$. Hence we conclude that $\rk I = 3$. We then consider the natural exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} I \longrightarrow 0,$$

where $L$ is the line bundle such that $L^{-1} := \ker ev$.

**Proposition 2.1.** We have $h^0(I^*) = 0$.

**Proof.** Suppose on the contrary that there exists a non-zero map $I \rightarrow \mathcal{O}_C$. Its kernel $S \subset I$ is a rank-2 subsheaf of $\mathcal{E}'$ and by stability of $\mathcal{E}'$ we obtain $\mu(S) < \mu(\mathcal{E}') = 2$, hence $\deg S \leq 3$. Moreover $h^0(S) \geq h^0(I) - 1 = 3$.

Assume that $\deg S = 3$. Then $S$ is stable and $S$ can be written as an extension

$$0 \longrightarrow \mu \longrightarrow S \longrightarrow \nu \longrightarrow 0,$$

with $\deg \mu = 1$ and $\deg \nu = 2$. The condition $h^0(S) \geq 3$ then implies that $\mu = \mathcal{O}_C(x)$ for some $x \in C$, $\nu = K$ and that the extension has to be split, i.e., $S = K \oplus \mathcal{O}_C(x)$. This contradicts stability of $S$.

The assumption $\deg S \leq 2$ similarly leads to a contradiction. We leave the details to the reader. \qed
Now we take the cohomology of the dual of the exact sequence \([1]\) and we obtain — using \(h^0(I^*) = 0\) — an inclusion \(H^0(\mathcal{E}')^* \subset H^0(L)\). Hence \(h^0(L) \geq 4\), which implies \(\text{deg } L \geq 5\). On the other hand \(\text{deg } L = \text{deg } I\) and by stability of \(\mathcal{E}'\), we have \(\mu(I) < 2\), i.e., \(\text{deg } L \leq 5\). So we can conclude that \(\text{deg } L = 5\), that \(H^0(\mathcal{E}')^* = H^0(L)\) and that \(I = E_L\), where \(E_L\) is the evaluation bundle associated to \(L\) defined by the exact sequence
\[
(2) \quad 0 \to E_L^* \to H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \to 0.
\]

Moreover the subsheaf \(E_L \subset \mathcal{E}'\) is of maximal degree, hence \(E_L\) is a subbundle of \(\mathcal{E}'\) and we have an exact sequence
\[
(3) \quad 0 \to E_L \to \mathcal{E}' \to K^4L^{-1} \to 0,
\]
with extension class \(e \in \text{Ext}^1(K^4L^{-1}, E_L) = H^1(E_L \otimes K^{-4}L) = H^0(E_L^* \otimes K^5L^{-1})\). Using Riemann-Roch and stability of \(E_L\) (see e.g. \([5]\)) one shows that
\[
\begin{align*}
\text{deg } L & = 5, \\
h^0(E_L^* \otimes K^5L^{-1}) & = 7, \\
h^0(E_L^* \otimes K^5L^{-1}(-x)) & = 4, \\
h^0(E_L^* \otimes K^5L^{-1}(-x) - y)) & = 1
\end{align*}
\]
for general points \(x, y \in C\). In that case we denote by \(\mu_{x,y} \in \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})\) the point determined by the 1-dimensional subspace \(H^0(E_L^* \otimes K^5L^{-1}(-x - y))\). We also denote by \(S \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})\) the linear span of the points \(\mu_{x,y}\) when \(x\) and \(y\) vary in \(C\) and by \(H_e \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})\) the hyperplane determined by the non-zero class \(e\).

Tensoring the sequence \([3]\) with \(K^{-4}L(x + y)\) and taking cohomology one shows that \(\mu_{x,y} \in H_e\) if and only if \(h^0(\mathcal{E}' \otimes K^{-4}L(x + y)) > 0\). Since we assume \(\mathcal{E} \in \mathcal{B}_4\), we obtain
\[
S \subset H_e.
\]

We consider a general point \(x \in C\) such that \(h^0(E_L^* \otimes K^5L^{-1}(-x)) = 4\) and denote for simplicity \(A := E_L^* \otimes K^5L^{-1}(-x)\).

Then \(A\) is stable with \(\mu(A) = \frac{7}{3}\). We consider the evaluation map of global sections
\[
\text{ev}_A : \mathcal{O}_C \otimes H^0(A) \to A
\]
and consider the set \(S_A\) of points \(p \in C\) for which \((\text{ev}_A)_p\) is not surjective, i.e.
\[
S_A = \{ p \in C \mid h^0(A(-p)) \geq 2 \}.
\]

Then we have the following

**Lemma 2.2.** We assume that \(x\) is general.

(1) If \(L^2 \neq K^5\), then the set \(S_A\) consists of the 2 distinct points \(p_1, p_2\) determined by the relation \(\mathcal{O}_C(p_1 + p_2) = K^4L^{-1}(-x)\).

(2) If \(L^2 = K^5\), then the set \(S_A\) consists of the 2 distinct points \(p_1, p_2\) introduced in (1) and the conjugate \(\sigma(x)\) of \(x\) under the hyperelliptic involution \(\sigma\).

**Proof.** Given a point \(p \in C\), we tensorize the exact sequence \([3]\) with \(K^5L^{-1}(-x - p)\) and take cohomology:
\[
0 \to H^0(A(-p)) \to H^0(L) \otimes H^0(K^5L^{-1}(-x - p)) \to H^0(K^5(-x - p)) \to \cdots
\]
We note that \(h^0(K^5L^{-1}(-x - p)) = 2\). We distinguish two cases.

(a) The pencil \(|K^5L^{-1}(-x - p)|\) has a base-point, i.e. there exists a point \(q \in C\) such that \(K^5L^{-1}(-x - p) = K(q)\), or equivalently \(K^4L^{-1}(-x) = \mathcal{O}_C(p + q)\). Since \(x\) is general, we have \(h^0(K^4L^{-1}(-x)) = 1\), which determines \(p\) and \(q\), i.e., \(\{p, q\} = \{p_1, p_2\}\). In this case \(|K^5L^{-1}(-x - p)| = |K(q)| = |K|\) and \(h^0(A(-p)) = h^0(K^{-1}L) = 2\). This shows that \(p_1, p_2 \in S_A\).
(b) The pencil $|K^5L^{-1}(-x-p)|$ is base-point-free. By the base-point-free-pencil-trick, we have $H^0(A(-p)) \cong H^0(L^2K^{-5}(x+p))$. Since $\deg L^2K^{-5}(x+p) = 2$, we have $h^0(L^2K^{-5}(x+p)) = 2$ if and only if $L^2K^{-5}(x+p) = K$, or equivalently $O_C(p) = K^6L^{-2}(-x)$. If $K^6L^{-2} \neq K$, then for general $x \in C$ the line bundle $K^6L^{-2}(-x)$ is not of the form $O_C(p)$. If $K^6L^{-2} = K$, then for any $x \in C$, $K^6L^{-2}(-x) = O_C(x)$, which implies that $\sigma(x) \in S_A$.

This shows the lemma. \hfill \Box

Proposition 2.3. If $L^2 \neq K^5$, then $S = \mathbb{P}H^0(E^*_L \otimes K^5L^{-1})$.

Proof. We consider a general point $x \in C$ and the rank-3 bundle $A$. Let $B \subset A$ denote the subsheaf given by the image of $ev_A$. By Lemma 2.2 (1) we have $\deg B = \deg A - 2 = 5$. Moreover $H^0(B) = H^0(A)$ and there is an exact sequence

$$0 \to M^{-1} \to O_C \otimes H^0(B) \xrightarrow{\varphi} E_B \to 0,$$

where $M \in \text{Pic}^5(C)$. It follows that the rational map

$$\phi_x : C \dashrightarrow \mathbb{P}^5 = \mathbb{P}H^0(A) = \mathbb{P}^3,$$

factorizes through

$$C \xrightarrow{\varphi_M} |M|^* \to \mathbb{P}H^0(B),$$

where $\varphi_M$ is the morphism given by the linear system $|M|$ and the second map is linear and identifies with the projectivization of the dual of $\delta$, which is given by the long exact sequence obtained from (4) by dualizing and taking cohomology:

$$0 \to H^0(B^*) \to H^0(B) \xrightarrow{\delta} H^0(M) \to H^1(B^*) \to \cdots$$

We obtain that the linear span of im $\phi_x$ is non-degenerate if and only if $H^0(B^*) = 0$.

We now show that $h^0(B^*) = 0$. Suppose on the contrary that there exists a non-zero map $B \to O_C$. Its kernel $S \subset B$ is a rank-2 subsheaf of $A$ with $\deg S \geq \deg B = 5$, hence $\mu(S) \geq \frac{5}{2}$, which contradicts stability of $A$ — recall that $\mu(A) = \frac{5}{2}$.

This shows that im $\phi_x$ spans $\mathbb{P}H^0(A) \subset \mathbb{P}H^0(E^*_L \otimes K^5L^{-1})$ for general $x \in C$. We now take 2 general points $x, x' \in C$ and deduce from $\dim H^0(A) \cap H^0(A') = \dim H^0(E^*_L \otimes K^5L^{-1}(-x-x')) = 1$ that the linear span of the union $\mathbb{P}H^0(A) \cap \mathbb{P}H^0(A')$ equals the full space $\mathbb{P}H^0(E^*_L \otimes K^5L^{-1})$. This shows the proposition. \hfill \Box

We deduce from the proposition that the line bundle $L$ satisfies the relation $L^2 = K^5$, i.e.

$$L = K^2 \kappa$$

for some theta-characteristic $\kappa$ of $C$. In that case we note that $H^0(E^*_L \otimes K^5L^{-1})$ equals $H^0(E^*_L \otimes L)$ and we can consider the exact sequence

$$0 \to H^0(E^*_L \otimes L) \to H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \to 0,$$

obtained from (2) by tensoring with $L$ and taking cohomology. We also note that there is a natural inclusion $\Lambda^2 H^0(L) \subset H^0(E^*_L \otimes L)$, see e.g. [P] section 2.1. More precisely we can show

**Proposition 2.4.** The linear span $S$ equals

$$S = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E^*_L \otimes L).$$

Proof. Using the standard exact sequences and the base-point-free-pencil-trick, one easily works out that for general points $x, y \in C$

$$\mu(x,y) = \mathbb{P}\Lambda^2 H^0(L(-x-y)) \subset \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E^*_L \otimes L).$$

This implies that $S \subset \mathbb{P}\Lambda^2 H^0(L)$. In order to show equality one chooses 4 general points $x_i \in C$ such that their images $C \to |L|^* = \mathbb{P}^3$ linearly span the $\mathbb{P}^3$. We denote by $s_i \in H^0(L)$ the global
section vanishing on the points \(x_j\) for \(j \neq i\) and not vanishing on \(x_i\). Then one checks that for any choice of the indices \(i,j,k,l\) such that \(\{i,j,k,l\} = \{1,2,3,4\}\) one has \(s_i \wedge s_j = \mu_{x_i,x_j}\). Since the 6 tensors \(s_i \wedge s_j\) are a basis of \(\Lambda^2 H^0(L)\), we obtain equality. \(\square\)

The hyperplane \(S = \mathbb{P} \Lambda^2 H^0(L) \subset \mathbb{P} H^0(E^*_{\xi} \otimes L)\) determines a unique (up to a scalar) non-zero extension class \(e \in H^0(E^*_{\xi} \otimes L)^*\) by \(S = H_e\), which in turn determines a unique stable vector bundle \(E \in \mathcal{B}_4\), which we will denote by \(E_e\).

This shows that \(\mathcal{B}_4\) is of dimension 0 and of cardinality 16.

### 2.2. The Raynaud bundles

In this subsection we recall the construction of the Raynaud bundles introduced in [R] as Fourier-Mukai transforms. We refer to [H] section 9.2 for the details and the proofs.

The rank-4 vector bundle \(\mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*\) over \(J\) admits a canonical \(J[2]\)-linearization and descends therefore under the duplication map \([2] : J \rightarrow J\), i.e., there exists a rank-4 vector bundle \(M\) over \(J\) such that

\[
[2]^* M \cong \mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*.
\]

**Proposition 2.5.** For any theta-characteristic \(\kappa\) of \(C\) there exists an isomorphism

\[
\xi_\kappa : M \xrightarrow{\sim} M^* \otimes \mathcal{O}_J(\Theta_\kappa).
\]

Moreover if \(\kappa\) is even (resp. odd), then \(\xi_\kappa\) is symmetric (resp. skew-symmetric).

Let \(\gamma_\kappa : C \rightarrow J\) be the Abel-Jacobi map defined by \(\gamma_\kappa(p) = \kappa^{-1}(p)\). We define the Raynaud bundle

\[
R_\kappa := \gamma_\kappa^* M \otimes \kappa^{-1}.
\]

Then by [R] the bundle \(R_\kappa \in \mathcal{B}_4\). Since \(\gamma_\kappa^* \mathcal{O}_J(\Theta_\kappa) = K\) we see that the isomorphism \(\xi_\kappa\) induces an orthogonal (resp. symplectic) structure on the bundle \(R_\kappa\), if \(\kappa\) is even (resp. odd). In particular the bundle \(R_\kappa\) is self-dual, i.e., \(R_\kappa = R_\kappa^*\). The pull-back \(\gamma_\kappa^*(\xi'_\kappa)\) for a theta-characteristic \(\kappa' = \kappa \alpha\) with \(\alpha \in J[2]\) gives an isomorphism

\[
R_\kappa \xrightarrow{\sim} R_\kappa^* \otimes \alpha,
\]

hence a non-zero section in \(H^0(\Lambda^2 R_\kappa \otimes \alpha)\) (resp. \(H^0(\text{Sym}^2 R_\kappa \otimes \alpha)\)) if \(h^0(\kappa \alpha) = 1\) (resp. \(h^0(\kappa \alpha) = 0\)). We deduce that there are isomorphisms

\[
(5) \quad \Lambda^2 R_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 R_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha.
\]

In particular the 16 bundles \(R_\kappa\) are non-isomorphic. Each \(R_\kappa\) is invariant under tensor product with \(J[2]\). The isomorphisms (5) can be used to prove the relation

\[
(6) \quad R_\kappa \otimes \beta = R_{\kappa \beta^2}, \quad \forall \beta \in J[4].
\]

### 2.3. Symplectic and orthogonal bundles

In this subsection we give a third construction of the bundles in \(\mathcal{B}_4\) as symplectic and orthogonal extension bundles. Let \(\kappa\) be a theta-characteristic.

If \(\kappa\) is odd, then \(\kappa = \mathcal{O}_C(w)\) for some Weierstrass point \(w \in C\). The construction outlined in [P] section 2.2 gives a unique symplectic bundle \(E_e \in \mathcal{B}_4\) with \(e \in H^1(\text{Sym}^2 G)_+\). We denote this bundle by \(V_\kappa\).

If \(\kappa\) is even, there is an analogue construction, which we briefly outline for the convenience of the reader. The proofs are similar to those given in [H]. Using the Atiyah-Bott-fixed-point formula one observes that among all non-trivial extensions

\[
0 \rightarrow \kappa^{-1} \rightarrow G \rightarrow \mathcal{O}_C \rightarrow 0,
\]
there are 2 extensions (up to scalar), which are $e$-invariant. We take one of them. Then any non-zero class $e \in H^1(\Lambda^2 G) = H^1(\kappa^{-1})$ determines an orthogonal bundle $\mathcal{E}_e$, which fits in the exact sequence

$$0 \rightarrow G \rightarrow \mathcal{E}_e \rightarrow G^* \rightarrow 0.$$  

The composite map

$$D_\mathcal{E} : \mathbb{P}H^1(\Lambda^2 G) \rightarrow \mathcal{M}_4 \rightarrow |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_\mathcal{E} : H^1(\Lambda^2 G) \rightarrow H^0(\operatorname{Pic}^1(C), 4\Theta).$$

Moreover $\operatorname{im} \widetilde{D}_\mathcal{E} \subset H^0(\operatorname{Pic}^1(C), 4\Theta)_-$, which can be seen as follows. By [Se] Thm 2 the second Stiefel-Whitney class $w_2(\mathcal{E}_e)$ of an orthogonal bundle $\mathcal{E}_e$ is given by the parity of $h^0(\mathcal{E}_e \otimes \kappa')$ for any theta-characteristic $\kappa'$. This parity can be computed by taking the cohomology of the exact sequence $[\mathbb{P}]$ tensorized with $\kappa'$ and taking into account that the coboundary map is skew-symmetric. One obtains that $w_2(\mathcal{E}_e) \neq 0$ and one can conclude the above-mentioned inclusion by [Sp] Lemma 1.4.

We now observe that by the Atiyah-Bott-fixed-point-formula $h^1(\Lambda^2 G)_+ = h^1(\Lambda^2 G)_- = 1$. By the argument given in [Sp] section 2.2 we conclude that one of the two eigenspaces $H^1(\Lambda^2 G)_ \pm$ is contained in the kernel $\ker \widetilde{D}_\mathcal{E}$. We denote the corresponding bundle $\mathcal{E}_e$ by $V_\kappa \subset B_4$.

**2.4. Three descriptions of the same bundle.**

**Proposition 2.6.** For any theta-characteristic $\kappa$ the three bundles $E_\kappa, R_\kappa$ and $V_\kappa$ coincide.

**Proof.** If $\kappa$ is odd, this was worked out in detail in [SpI] section 8 and Theorem 29. If $\kappa$ is even, the proofs are similar. \hfill \square

This proposition shows all assertions of Theorem 1.1 except reducedness of $B_4$.

I am grateful to Olivier Serman for giving me the following fourth description of the bundle $E_\kappa$ for an even theta-characteristic $\kappa$. We recall that an even theta-characteristic $\kappa$ corresponds to a partition of the set of six Weierstrass points of $C$ into two subsets of three points, which we denote by $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. With this notation we have

**Proposition 2.7.** Let $\kappa$ be an even theta-characteristic. We denote by $A_\kappa$ (resp. $B_\kappa$) the unique stable rank-2 bundle with determinant $\kappa$ and which contains the four 2-torsion line bundles $\mathcal{O}_C, \mathcal{O}_C(w_1 - w_2), \mathcal{O}_C(w_1 - w_3)$ and $\mathcal{O}_C(w_2 - w_3)$ (resp. $\mathcal{O}_C, \mathcal{O}_C(w_4 - w_5), \mathcal{O}_C(w_4 - w_6)$ and $\mathcal{O}_C(w_5 - w_6)$). Then the orthogonal rank-4 vector bundle $E_\kappa$ is isomorphic to

$$\operatorname{Hom}(A_\kappa, B_\kappa)$$

equipped with the quadratic form given by the determinant.

We refer to [Sp] section 5.5 for the proof.

**2.5. Reducedness of $B_4$.** We denote by $\mathcal{L}$ the determinant line bundle over the moduli space $\mathcal{M}_4$ and recall that the set $B_4$ can be identified with the base locus of the linear system $|\mathcal{L}|$. This endows the set $B_4$ with a natural scheme-structure.

We start with a description of the space of global sections $H^0(\mathcal{M}_4, \mathcal{L})$.

**Proposition 2.8.** For any theta-characteristic $\kappa$ there is a section $s_\kappa \in H^0(\mathcal{M}_4, \mathcal{L})$ with zero divisor

$$\Delta_\kappa := \operatorname{Zero}(s_\kappa) = \{ E \in \mathcal{M}_4 \mid h^0(\Lambda^2 E \otimes \kappa) > 0 \}.$$

The 16 sections $s_\kappa$ form a basis of $H^0(\mathcal{M}_4, \mathcal{L})$. 

Proof. The Dynkin index of the second fundamental representation $\rho : \mathfrak{sl}_4(\mathbb{C}) \to \text{End}(\Lambda^2 \mathbb{C}^4)$ equals 2 (see e.g. [LS] Proposition 2.6). Moreover the bundle $\Lambda^2 E \otimes \kappa$ admits a $K$-valued non-degenerate quadratic form, which allows to construct the Pfaffian divisor $s_\kappa$, which is a section of $\mathcal{L}$ (see [LS]). The space $H^0(\mathcal{M}_4, \mathcal{L})$ is a representation of level 2 of the Heisenberg group $\text{Heis}(2)$, which is a central extension of $J[2]$ by $\mathbb{C}^*$. One can work out that the sections $s_\kappa$ generate the 16 one-dimensional character spaces for the $\text{Heis}(2)$-action on $H^0(\mathcal{M}_4, \mathcal{L})$. This shows that the sections $s_\kappa$ are linearly independent. 

Since $E_\kappa \in \mathcal{B}_4$, we have $E_\kappa \in \Delta_{\kappa'}$ for any theta-characteristic $\kappa'$. By the deformation theory of determinant and Pfaffian divisors (see e.g. [L], [LS]) the point $E_\kappa \in \mathcal{M}_4$ is a smooth point of the divisor $\Delta_{\kappa'} \subset \mathcal{M}_4$ if and only if the following two conditions hold

1. $h^0(\Lambda^2 E_\kappa \otimes \kappa') = 2$,
2. the natural linear form

$$\Phi_{\kappa'} : T_{E_\kappa} \mathcal{M}_4 = H^1(\text{End}_0(E_\kappa)) \longrightarrow \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa')^*$$

is non-zero.

Moreover if these two conditions holds, then $T_{E_\kappa} \Delta_{\kappa'} = \ker \Phi_{\kappa'}$. The map $\Phi_{\kappa'}$ is built up as follows: the exceptional isomorphism of Lie algebras $\mathfrak{sl}_4 \cong \mathfrak{so}_6$ induces a natural vector bundle isomorphism

\begin{equation}
\text{End}_0(E_\kappa) \xrightarrow{\sim} \Lambda^2(\Lambda^2 E_\kappa).
\end{equation}

Then $\Phi_{\kappa'}$ is the dual of the linear map given by the wedge product of global sections

$$\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \longrightarrow H^0(\Lambda^2(\Lambda^2 E_\kappa) \otimes K) = H^0(\text{End}_0(E_\kappa) \otimes K).$$

Proposition 2.9. The 0-dimensional scheme $\mathcal{B}_4$ is reduced.

Proof. Since $E_\kappa$ is a smooth point of $\mathcal{M}_4$ and $\dim T_{E_\kappa} \mathcal{M}_4 = 15$, it is sufficient to show that for any theta-characteristic $\kappa' \neq \kappa$ the divisor $\Delta_{\kappa'}$ is smooth at $E_\kappa$ and that the 15 hyperplanes $\ker \Phi_{\kappa'} \subset T_{E_\kappa} \mathcal{M}_4$ are linearly independent: using the isomorphism (3) we obtain that for $\kappa' \neq \kappa$

$$h^0(\Lambda^2 E_\kappa \otimes \kappa') = \sharp S(\kappa) \cap S(\kappa') = 2$$

and using the isomorphism (8) we obtain that

$$\text{End}_0(E_\kappa) = \bigoplus_{\alpha \in J[2] \setminus \{0\}} \alpha.$$

On the other hand one easily sees that if $\gamma, \delta \in J[2]$ are the two 2-torsion points in the intersection $S(\kappa) \cap S(\kappa')$, then $\kappa' = \kappa \gamma \delta$, hence $\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \cong H^0(K \gamma \delta)$. This implies that the linear form

$$\Phi_{\kappa'} : \bigoplus_{\alpha \in J[2] \setminus \{0\}} H^1(\alpha) \longrightarrow H^0(K \gamma \delta)^* = H^1(\beta)$$

is projection onto the direct summand $H^1(\beta)$, where $\beta = \kappa^{-1} \kappa' \in J[2]$. This description of the linear forms $\Phi_{\kappa'}$ clearly shows that they are non-zero and linearly independent. \qed

This completes the proof of Theorem 1.1.
3. Proof of Corollary 1.2

Since by Theorem 1.1 $\mathcal{B}_3$ is a reduced 0-dimensional scheme of length 16, the degree of the theta map $\theta$ is given by the formula

$$\deg \theta + 16 = c_{15},$$

where $\frac{c_{15}}{15!}$ is the leading coefficient of the Hilbert polynomial

$$P(n) = \chi(\mathcal{M}_4, \mathcal{L}^n) = \frac{c_{15}}{15!} n^{15} + \text{lower degree terms}. $$

In order to compute the polynomial $P$ we write

$$P(X) = \sum_{k=0}^{15} \alpha_k Q_k(X),$$

with $Q_k(X) = \frac{1}{k!} (X+7)(X+6) \cdots (X+8-k)$ and $Q_0(X) = 1$. Note that $\deg Q_k = k$ and that $c_{15} = \alpha_{15}$. The canonical bundle of $\mathcal{M}_4$ equals $\mathcal{L}^{-8}$. By the Grauert-Riemenschneider vanishing theorem we obtain that $h^i(\mathcal{M}_4, \mathcal{L}^n) = 0$ for any $i \geq 1$ and $n \geq -7$. Hence $P(n) = h^0(\mathcal{M}_4, \mathcal{L}^n)$ for $n \geq -7$. Moreover $P(n) = 0$ for $n = -7, -6, \ldots, -1$ and $P(0) = 1$. The values $P(n)$ for $n = 1, 2, \ldots, 8$ can be computed by the Verlinde formula and with the use of MAPLE. They are given in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(n)$</td>
<td>16</td>
<td>140</td>
<td>896</td>
<td>4680</td>
<td>21024</td>
<td>83628</td>
<td>300080</td>
<td>984539</td>
</tr>
</tbody>
</table>

Using the expression (9) of $P$ one straightforwardly deduces the coefficients $\alpha_k$ by increasing induction on $k$: $\alpha_k = 0$ for $k = 0, 1, \ldots, 6$ and the values $\alpha_k$ for $k = 7, \ldots, 15$ are given in the following table.

<table>
<thead>
<tr>
<th>$k$</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k$</td>
<td>1</td>
<td>8</td>
<td>32</td>
<td>96</td>
<td>214</td>
<td>328</td>
<td>324</td>
<td>184</td>
<td>46</td>
</tr>
</tbody>
</table>

Hence $\deg \theta = \alpha_{15} - 16 = 30$.

References


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