Preference modelling on totally ordered sets by the Sugeno integral

Agnès Rico, Michel Grabisch, Christophe Labreuche, Alain Chateauneuf

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A. Rico [a,c] M. Grabisch [a,b] Ch. Labreuche [a] A. Chateauneuf [c]

[a] *THALES R & T, Domaine de Corbeville 91404 Orsay Cedex, France*

[b] *LIP6, UPMC, 4 Place Jussieu, 75252 Paris, France*

[c] *CERMSEM Maison des sciences économiques 106-112, Bld de l'Hopital, 75013 Paris, France*

Abstract

We present in this paper necessary and sufficient conditions for the representation of preferences in a decision making problem, by the Sugeno integral, in a purely ordinal framework. We distinguish between strong representation (exact) and weak representation (no contradiction on strict preferences).

Key words: Preference representation, Sugeno integral, ordinal information.

1 Introduction

The main purpose of decision making theory is to find numerical representations of a given preference relation on a set of objects X. Usually, a preference relation on X is a binary relation denoted $\succeq$, which is complete, reflexive and transitive. Depending on the structure of X, there are many results (see e.g. [KLST71]) giving necessary and sufficient conditions on $\succeq$ in order to have a numerical *representation* of $\succeq$, i.e. a mapping $V : X \longrightarrow \mathbb{R}$ such that $\forall a, b \in X, a \succeq b \Leftrightarrow V(a) \geq V(b)$. A large class of decision making problems is concerned with (or can be turned into) the case where the objects are points in some n-dimensional space E^n, where E is a totally ordered set, typically $E = \mathbb{R} \cup \{-\infty, \infty\}$ or $[0, 1]$. In this case, denoting $a = (a_1, \ldots a_n)$ an object in E^n, we call a_i the *score* of object a on the ith dimension, $V(a)$ is the *global score* of a.

Email address: agnes.rico@free.fr (A. Rico).

In real situations, where scores have to be directly assessed by the decision maker, it is often the case that E is a finite totally ordered set such as $\{bad, medium, good\}$. In this case, two problems arise, one is due to the fact that V cannot use arithmetical operations and the second is the finiteness of E.

A complete treatment of this case is beyond the scope of a single paper, and we will focus only on a part of it. First, we will discard any problem coming from the (possible) finiteness of E, and will suppose that we always "have enough points" in E (this will be detailed in Section 2). The reader is referred to [Gra01] for a detailed analysis of this question. Second, we concentrate on a particular class of functions V, called Sugeno integrals with respect to a capacity [Sug74]. The reason is that the Sugeno integral w.r.t. a capacity coincides with the class of Boolean polynomials (i.e. expressions $P(a_1, \ldots, a_n)$ involving n variables and coefficients valued in E, a totally ordered set with a least element 0 and a greatest element 1, linked by minimum ($\wedge$) or maximum ($\vee$) in an arbitrary combination of parentheses, e.g. $((\alpha \wedge a_1) \vee (a_2 \wedge (\beta \vee a_3))) \wedge a_4)$, such that $P(0, 0, \ldots, 0) = 0$, $P(1, 1, \ldots, 1) = 1$, and P is non decreasing w.r.t. each variable [Mar01]. These three conditions are very natural in the context of score aggregation, since they mean that an object having the least (resp. the greatest) score on each dimension should receive as global score the least (resp. the greatest) one, and that improving a score on one dimension cannot decrease the global score. Thus, the Sugeno integral captures a large class of interest (however, see Section 6 for a discussion on limitations).

Suppose E is fixed, and a preference relation on a subset of E^n is given. Our aim is to know if this preference relation is representable by a Sugeno integral, and in the affirmative, by which capacities.

The paper is organized as follows. Section 2 presents the basic material for the sequel, and defines exactly the representation problem we address, introducing the notion of strong representation and weak representation. Section 3 solves the strong representation problem, while Section 4 solves the weak one. To conclude section 5 gives an example and section 6 presents a discussion about the Sugeno integral.

2 Framework and notations

2.1 The preference representation problem

Let $(E, \leq)$ be a totally ordered set with a least element 0 and a greatest element 1. We consider O a finite subset of E^n, containing objects of interest, on which the decision maker has a preference, expressed under the form of a complete, reflexive, and transitive binary relation $\succeq$. We denote by $\succ$ and $\sim$ the asymmetric and symmetric part of $\succeq$ respectively. The binary relation $\succ$ is called the *strict preference*, while $\sim$ is the *indifference* relation. Clearly, $\sim$ is an equivalence relation, and we denote by $[a]$ the equivalence class of $a \in O$. Since O is finite, so is the number of equivalence classes, which we call p. For the sake of convenience, we choose in each equivalence class a representative a^i, which we number so that $a^1 \prec a^2 \prec \cdots \prec a^p$.

We distinguish two levels of representation of the preference. The *strong representation* consists in finding a function $V : O \longrightarrow E$ such that

$$\forall a, b \in O, a \succeq b \Leftrightarrow V(a) \geq V(b). \tag{1}$$

It is well known and easy to prove (see [KLST71]) that when E is $\mathbb{R} \cup \{-\infty, \infty\}$, such a representation always exists when O is finite [1]. By contrast with the strong representation, the *weak representation* merely forbids to map strict preference of a over b to $b > a$. Hence, function V is such that:

$$\forall a, b \in O, a \succ b \Rightarrow \neg(V(b) > V(a)), \tag{2}$$

where $\neg$ denotes negation.

Note that if $a \sim b$, there is no restriction on $V(a)$ and $V(b)$. Clearly, the set of weak representations includes the set of strong ones.

2.2 Capacities and the Sugeno integral on finite sets

We call $C = \{1, \ldots, n\}$ the index set of dimensions used to score the objects.

Definition 1 *A capacity on C [Sug74] is an isotone mapping from the Boolean lattice 2^C to E preserving top and bottom, i.e. $\mu(\emptyset) = 0$, $\mu(C) = 1$, and $A \subset B$ implies $\mu(A) \leq \mu(B)$.*

We denote by $\mathcal{M}(C)$ the set of all capacities defined on C. On this set we introduce the pointwise order, i.e. $\mu \leq \mu'$ if and only if $\forall A \in 2^C, \quad \mu(A) \leq \mu'(A)$; and the capacities $\bigvee_{i \in I} \mu_i$ and $\bigwedge_{i \in I} \mu_i$ are defined pointwise.

Particular types of capacity useful in the sequel are called *maxitive capacity* and *minitive capacity*, which we denote Π and N respectively. Maxitive capacities are sup-preserving capacities, also called possibility measures [DuPrSa01]: $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$, for any $A, B \in 2^C$. The associated *possibility distribution* π is defined by $\pi(i) = \Pi(\{i\})$, for any $i \in C$. Minitive capacities are inf-preserving mappings, i.e $N(A \cap B) = N(A) \wedge N(B)$, for any $A, B \in 2^C$.

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[1] It suffices to assign a number to each equivalence class so that the ordering reflects the preference. So this remains possible if $|E| \geq p$.

We introduce now the Sugeno integral [Sug74] on a finite set. For any function $a : C \longrightarrow E$, we denote $a(i)$ by a_i, thus identifying E^C with E^n.

Definition 2 *Let $a \in E^n$, and μ be a capacity on C. The* Sugeno integral *of a with respect to μ is defined by: $S_\mu(a) := \bigvee_{i=1}^n [a_{(i)} \wedge \mu(A_{(i)})]$, where $(\cdot)$ indicates a permutation on C such that $a_{(1)} \leq \cdots \leq a_{(n)}$, and $A_{(i)} := \{(i), \ldots, (n)\}$.*

Note that the permutation $(\cdot)$ depends on a.

Property 1 *For any capacity $\mu \in \mathcal{M}(C)$ and any $a \in E^n$, we have*

(i) $\bigwedge_{i=1}^n a_i \leq S_\mu(a) \leq \bigvee_{i=1}^n a_i$.
(ii) *if $a \leq a'$ (pointwise order), then $S_\mu(a) \leq S_\mu(a')$.*
(iii) $S_\mu(a) = \bigwedge_{i=1}^n [a_{(i)} \vee \mu(A_{(i+1)})]$, *with* $A_{(n+1)} = \emptyset$.

The first two properties are elementary, the third one can be found in [Mar98].

The Sugeno integral w.r.t maxitive capacities Π with associated possibility distribution π reduces to, for any $a \in E^n$:

$$S_\Pi(a) = \bigvee_{i=1}^n [\pi(i) \wedge a_i]. \tag{3}$$

Similarly the Sugeno integral w.r.t minitive capacities N is

$$S_N(a) = \bigwedge_{i=1}^n [n(i) \vee a_i], \ \text{ with } n(i) = N\left(C \backslash \{i\}\right). \tag{4}$$

(for a proof, see [DuPr86]).

2.3 Representation of preference by the Sugeno integral

We restate the representation problem under the assumption that the function V we are looking for is a Sugeno integral. Hence, V will be entirely determined if μ is known. The problem amounts to finding if there exists a capacity μ such that (1) or (2) is satisfied, and in the case it exists, what is the set of all solutions. Solving this problem in the general case is difficult, hence our approach is to split it in two pieces. Let us consider first the strong representation problem. It amounts to find p "numbers" $\alpha_1 < \alpha_2 < \cdots < \alpha_p$ in E such that there exists a capacity μ satisfying

$$S_\mu(a) = \alpha_i, \quad \forall a \in [a^i], \quad \forall i = 1, \ldots, p, \tag{5}$$

according to notations of Section 2. For the weak representation problem, it suffices to find $p-1$ numbers $0 =: \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p := 1$ in E such that there exists a capacity μ satisfying

$$\alpha_{i-1} \leq S_\mu(a) \leq \alpha_i, \quad \forall a \in [a^i], \quad \forall i = 1, \dots, p. \tag{6}$$

When E is finite with $|E| \geq p$, it is possible to build an efficient enumerative algorithm, taking into account properties of the Sugeno integral, which generates p-uples $(\alpha_1, \dots, \alpha_p)$, in order to test conditions (5) or (6). If we note $\mathcal{S}_{\alpha_1,\dots,\alpha_p}$, the set of capacities which are solutions for a p-uple $(\alpha_1, \dots, \alpha_p)$, the solution set we are looking for is $\cup_{\alpha_1,\dots,\alpha_p}\mathcal{S}_{\alpha_1,\dots,\alpha_p}$. Hence, we limit ourself to the problem of finding the set of all capacities satisfying either conditions (5), or conditions (6) for a given p-uple.

3 Strong representation

In this section, we solve the strong representation problem, i.e. supposing to have p numbers $\alpha_1 < \cdots < \alpha_p$ in E, find the set of capacities satisfying all conditions (5). Let us denote this set by $\mathcal{S}$, avoiding subscripts $\alpha_1, \dots, \alpha_p$ unless necessary. We solve the problem first for a single equivalence class, say $[a^i]$. Let us call $\mathcal{S}_i$ the set of solutions. In order to find $\mathcal{S}_i$, we first build the set $\mathcal{S}_i^{\leq}(a)$ of capacities such that $S_\mu(a) \leq \alpha_i$, for some $a \in [a^i]$, and the set $\mathcal{S}_i^{\geq}(a)$ with the reversed inequality. Then clearly, $\mathcal{S}_i = \bigcap_{a \in [a^i]}(\mathcal{S}_i^{\leq}(a) \cap \mathcal{S}_i^{\geq}(a))$, and $\mathcal{S} = \bigcap_{i=1}^{p} \mathcal{S}_i$.

3.1 Construction of $\mathcal{S}_i$

We are looking for all capacities μ such that $S_\mu(a) = \alpha_i$, for a given a in $[a^i]$. For the sake of simplicity, we drop index i for α_i in all this section. To build our set of solutions we need the following steps:

Step 1: Construction of $\mathcal{S}_i^{\leq}(a)$

Let a be in $[a^i]$, commonly we use the notations: $a_{(0)} = 0$ and $a_{(n+1)} = 1$.

Definition 3 *Let $i^{\geq}_{a,\alpha}$ be the index such that $a_{\left(i^{\geq}_{a,\alpha}-1\right)} < \alpha \leq a_{\left(i^{\geq}_{a,\alpha}\right)}$ and $i^{>}_{a,\alpha}$ be the one such that $a_{\left(i^{>}_{a,\alpha}-1\right)} \leq \alpha < a_{\left(i^{>}_{a,\alpha}\right)}$.*

Note that in this definition $a_{(j)}$ means the j^{th} largest a_i (see definition 2). We illustrate the above definition by Fig. 1.

Definition 4 *Let $a \in O$ and $\alpha \in E$. We define the set function $\hat{\mu}^{a,\alpha}$ by:*

$$\forall A \in \mathcal{P}(C) \setminus \{\emptyset, C\}, \quad \hat{\mu}^{a,\alpha}(A) = \begin{cases} \alpha \text{ if } & A \subseteq A_{\left(i^{\geq}_{a,\alpha}\right)} \\ 1 \text{ otherwise} \end{cases}$$

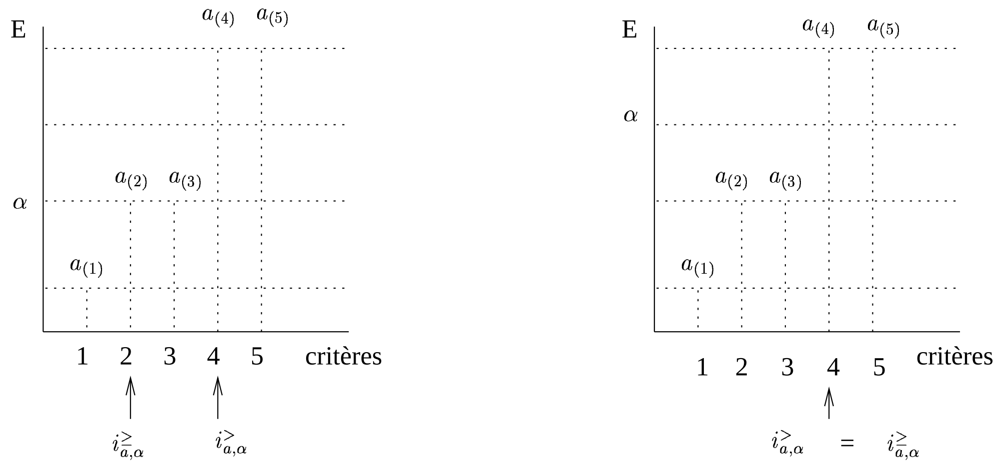

Fig. 1. $i^{\geq}_{a,\alpha}$ and $i^{>}_{a,\alpha}$

and $\hat{\mu}^{a,\alpha}(\emptyset) = 0,\ \hat{\mu}^{a,\alpha}(C) = 1.$

It is easy to check that $\hat{\mu}^{a,\alpha} \in \mathcal{M}(C)$.

Property 2 *If* $i^{>}_{a,\alpha} \neq 1$, $\hat{\mu}^{a,\alpha}$ *is a maxitive capacity with the possibility distribution:* $\pi(1) = \cdots = \pi\left(i^{>}_{a,\alpha} - 1\right) = 1$ *and* $\pi\left(i^{>}_{a,\alpha}\right) = \cdots = \pi(n) = \alpha$.

Proof : If $i^{>}_{a,\alpha} = 1$, $\hat{\mu}^{a,\alpha}$ is not a maxitive capacity since $1 = \hat{\mu}^{a,\alpha}(C) > \vee_{i \in C} \hat{\mu}^{a,\alpha}(\{i\}) = \alpha$. If $i^{>}_{a,\alpha} \neq 1$, we name Π the maxitive capacity with the possibility distribution $\pi(1) = \cdots = \pi\left(i^{>}_{a,\alpha} - 1\right) = 1$ and $\pi\left(i^{>}_{a,\alpha}\right) = \cdots = \pi(n) = \alpha$. $\forall A \subseteq C$, $\Pi(A) = \bigvee_{i \in A} \pi(i)$, which is clearly equal to $\hat{\mu}^{a,\alpha}$. ■

Property 3 *If* $i^{>}_{a,\alpha} = 1$, *then* $S_{\hat{\mu}^{a,\alpha}}(a) > \alpha$. *Otherwise* $S_{\hat{\mu}^{a,\alpha}}(a) \leq \alpha$.

Proof : Assume $i^{>}_{a,\alpha} \neq 1$, then using (3), we obtain $S_{\hat{\mu}^{a,\alpha}}(a) = \vee_{i=1}^{i^{>}_{a,\alpha}-1} a_{(i)} \vee \vee_{i \geq i^{>}_{a,\alpha}} \alpha$. Since $\vee_{i=1}^{i^{>}_{a,\alpha}-1} a_{(i)} \leq \alpha$, we get the desired result.

Now if $i^{>}_{a,\alpha} = 1$, clearly $S_{\hat{\mu}^{a,\alpha}}(a) \geq a_{(1)} > \alpha$. ■

From Property 3 we deduce immediately:

Corollary 1 $S_{\hat{\mu}^{a,\alpha}}(a) \leq \alpha$ *if and only if* $a_{(1)} \leq \alpha$.

Lemma 1 *Given* $a \in O$ *and* $\alpha \in E$,

$$\{\mu \in \mathcal{M}(C) \,|\, S_\mu(a) \leq \alpha\} = \begin{cases} \emptyset & \textit{if} \quad a_{(1)} > \alpha \\ \{\mu \in \mathcal{M}(C) \,|\, \quad \mu \leq \hat{\mu}^{a,\alpha}\} & \textit{otherwise.} \end{cases}$$

Proof : Let μ be a capacity such that for a subset A, $\mu(A) > \hat{\mu}^{a,\alpha}(A)$. Clearly

A is neither the set C nor the empty set. The case $A \nsubseteq A_{\left(i^{\geq}_{a,\alpha}\right)}$ cannot happen because it implies $\hat{\mu}^{a,\alpha}(A) = 1$; so we have $A \subseteq A_{\left(i^{\geq}_{a,\alpha}\right)}$. Then $\mu(A) > \alpha$, and the monotonicity of the capacity permits us to write $a_{(i^{\geq}_{a,\alpha})} \wedge \mu\left(A_{(i^{\geq}_{a,\alpha})}\right) > \alpha$ which implies $S_{\mu}(a) > \alpha$. ■

In substance, the result says that the upper envelope of the set of solutions, whenever nonempty, is a maxitive capacity (in possibility theory, these are the least informative capacities).

The next result gives a characterization of the capacities satisfying $\mu \leq \hat{\mu}^{a,\alpha}$.

Property 4 *Let μ be in $\mathcal{M}(C)$, $\mu \leq \hat{\mu}^{a,\alpha}$ if and only if $\mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \leq \alpha$.*

Proof : If we have $\mu \leq \hat{\mu}^{a,\alpha}$ then $\mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \leq \hat{\mu}^{a,\alpha}\left(A_{(i^{\geq}_{a,\alpha})}\right) = \alpha$. If $\mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \leq \alpha$ there are two possible cases. Either $A \subseteq A_{(i^{\geq}_{a,\alpha})}$ and we get $\mu(A) \leq \mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \leq \alpha = \hat{\mu}^{a,\alpha}(A)$. Or $A \nsubseteq A_{(i^{\geq}_{a,\alpha})}$ hence we have $\hat{\mu}^{a,\alpha}(A) = 1$ and so $\mu(A) \leq \hat{\mu}^{a,\alpha}(A)$. ■

In summary, the set $\{\mu \in \mathcal{M}(C) \,|S_{\mu}(a) \leq \alpha\}$ is empty if $a_{(1)} > \alpha$ and is the set $\left\{\mu \in \mathcal{M}(C) \,|\mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \leq \alpha\right\}$ otherwise.

Step 2: Construction of $\mathcal{S}_i^{\geq}(a)$

Definition 5 *Let $a \in O$ and $\alpha \in E$ be given, we define:*

$$\forall A \in \mathcal{P}(C) \setminus \{\emptyset, C\} \quad \check{\mu}^{a,\alpha}(A) = \begin{cases} \alpha \textit{ if } & A_{(i^{\geq}_{a,\alpha})} \subseteq A \\ 0 \textit{ otherwise} \end{cases}$$

and $\check{\mu}^{a,\alpha}(\emptyset) = 0$, $\check{\mu}^{a,\alpha}(C) = 1$.

It is easy to check that $\check{\mu}^{a,\alpha} \in \mathcal{M}(C)$.

Property 5 *If $i^{\geq}_{a,\alpha} \neq n+1$, $\check{\mu}^{a,\alpha}$ is a minitive capacity.*

Proof : The proof is quite similar to the proof of Property 2 ■

Property 6 *$S_{\check{\mu}^{a,\alpha}}(a) \geq \alpha$ if and only if $a_{(n)} \geq \alpha$.*

Proof : If $a_{(n)} \geq \alpha$, $S_{\check{\mu}^{a,\alpha}}(a) = \bigvee_{i=1}^{n} [a_{(i)} \wedge \check{\mu}^{a,\alpha}(A_{(i)})] \geq a_{(i^{\geq}_{a,\alpha})} \wedge \check{\mu}^{a,\alpha}\left(A_{(i^{\geq}_{a,\alpha})}\right) \geq \alpha$.
If $S_{\check{\mu}^{a,\alpha}}(a) \geq \alpha$ then we get $\forall i \quad a_{(i)} \vee \check{\mu}^{a,\alpha}\left(A_{(i+1)}\right) \geq \alpha$. So for $(i) = n$ we have $a_{(n)} \geq a_{(n)} \vee \check{\mu}^{a,\alpha}(A_{(n+1)}) \geq \alpha$. ■

Lemma 2 *Let $a \in O$ and $\alpha \in E$ be given.*

$$\{\mu \in \mathcal{M}(C) \mid \quad S_\mu(a) \geq \alpha\} = \begin{cases} \emptyset & \text{if} \quad a_{(n)} < \alpha \\ \{\mu \in \mathcal{M}(C) \mid \quad \mu \geq \check{\mu}^{a,\alpha}\} & \text{otherwise.} \end{cases}$$

Proof : The proof is similar as the proof of Lemma 1. ■

Property 7 *Let μ be a capacity, $\mu \geq \check{\mu}^{a,\alpha}$ if and only if $\mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \geq \alpha$.*

Proof : The proof is the dual of the proof of Lemma 2. ■

In conclusion, the set $\{\mu \in \mathcal{M}(C) \,|S_\mu(a) \geq \alpha\}$ is empty if $a_{(n)} < \alpha$ and is $\left\{\mu | \mu\left(A_{(i^{\geq}_{a,\alpha})}\right) \geq \alpha\right\}$ otherwise.

Step 3: Construction of $\mathcal{S}_i^{\leq}(a) \cap \mathcal{S}_i^{\geq}(a)$

Let a be in $[a^i]$, the association of Lemma 1 and 2 implies the following result.

Theorem 1

$$\{\mu \in \mathcal{M}(C) \,|S_\mu(a) = \alpha\} = \begin{cases} \emptyset & \text{if } a_{(n)} < \alpha \text{ or } a_{(1)} > \alpha \\ \{\mu \in \mathcal{M}(C) \,|\check{\mu}^{a,\alpha} \leq \mu \leq \hat{\mu}^{a,\alpha}\} & \text{otherwise.} \end{cases}$$

Step 4: Construction of $\mathcal{S}_i$

In this section we are interested of representing an equivalence class.

Theorem 2 *The set of capacities μ such that $S_\mu(a) = \alpha \quad \forall a \in [a^i]$ is*

- $\emptyset$ *if* $\exists a \in [a^i]$ *such as* $a_{(1)} > \alpha$ *or* $a_{(n)} < \alpha$,
- $\left\{\mu \in \mathcal{M}(C) \,| \bigvee_{a\in[a^i]} \check{\mu}^{a,\alpha} \leq \mu \leq \bigwedge_{a\in[a^i]} \hat{\mu}^{a,\alpha}\right\}$ *otherwise* .

Note that the capacity $\bigvee_{a\in[a^i]} \check{\mu}^{a,\alpha}$ is no longer a minitive capacity, and that the capacity $\bigwedge_{a\in[a^i]} \hat{\mu}^{a,\alpha}$ is no longer a maxitive capacity.

Proof : According to theorem 1, we can find a solution if α is such that $a_{(1)} \leq \alpha \leq a_{(n)} \,\forall a \in [a^i]$ that is to say if and only if $\bigvee_{a\in[a^i]} a_{(1)} \leq \alpha \leq \bigwedge_{a\in[a^i]} a_{(n)}$. When we have $\bigvee_{a\in[a^i]} a_{(1)} \leq \bigwedge_{a\in[a^i]} a_{(n)}$, a capacity μ is a solution if and only if $\check{\mu}^{a,\alpha} \leq \mu \leq \hat{\mu}^{a,\alpha} \,\forall a \in [a^i]$ in other words if and only if $\bigvee_{a\in[a^i]} \check{\mu}^{a,\alpha} \leq \mu \leq \bigwedge_{a\in[a^i]} \hat{\mu}^{a,\alpha}$. We know that the $\check{\mu}^{a,\alpha}$ take the values 0 or α and the capacities

$\hat{\mu}^{a,\alpha}$ the values 1 or α. Henceforth, $\bigvee_{a\in[a^i]} \check{\mu}^{a,\alpha} \leq \bigwedge_{a\in[a^i]} \hat{\mu}^{a,\alpha}$. ■

3.2 Construction of $\mathcal{S}$

In this section, the goal is to find one or several common capacities for representing simultaneously several equivalence classes. Hence the set of solutions is the intersection of the set of solutions for each class.

We define $\check{\mu}^i := \bigvee_{a\in[a^i]} \check{\mu}^{a,\alpha_i}$, and $\hat{\mu}^i := \bigwedge_{a\in[a^i]} \hat{\mu}^{a,\alpha_i}$.

With these new notations, for a given equivalence class $[a^i]$, the solutions are such that $\check{\mu}^i \leq \mu \leq \hat{\mu}^i$. Consequently our solution set is the set of capacities such that $\bigvee_{i=1}^{p} \check{\mu}^i \leq \mu \leq \bigwedge_{i=1}^{p} \hat{\mu}^i$. Hence we must find a necessary and sufficient condition for this double inequality to be true.

Theorem 3 *There exists a common capacity for the different equivalent classes if and only if* $\forall i,j,\ \alpha_i < \alpha_j \Rightarrow A_{\left(i^{\geq}_{b,\alpha_j}\right)} \not\subseteq A_{\left(i^{>}_{a,\alpha_i}\right)},\ \forall a \in [a^i],\ \forall b \in [a^j]$.

Proof : For $i = 1,\dots,p$, the capacities $\check{\mu}^i$ can take the values α_i or 0 and the capacities $\hat{\mu}^i$ the values α_i or 1. We have a solution if and only if $\vee_{i=1}^{p}\check{\mu}^i \leq \wedge_{i=1}^{p}\hat{\mu}^i$. Consequently, if $\check{\mu}^j$ takes the value α_j for a given index j, $\hat{\mu}^i$ cannot reach the values $\alpha_i < \alpha_j$. Let $\alpha_i < \alpha_j \in E$. The definition of the measures $\check{\mu}^j$ associated to α_j implies $A_{\left(i^{\geq}_{b,\alpha_j}\right)} \subseteq A, \forall b \in [a^j]$. For such sets A, we must have $A \not\subseteq A_{\left(i^{>}_{a,\alpha_i}\right)} \forall a \in [a^i]$. Writing this property for the set $A_{\left(i^{\geq}_{b,\alpha_j}\right)}$, we obtain $\check{\mu}^i \leq \hat{\mu}^i \Rightarrow A_{\left(i^{\geq}_{b,\alpha_j}\right)} \not\subseteq A_{\left(i^{>}_{a,\alpha_i}\right)}$ for all $\alpha_i < \alpha_j,\ \forall a \in [a^i],\ \forall b \in [a^j]$.

On the other hand, if for all $\alpha_i < \alpha_j$, $\forall a \in [a^i]$, $\forall b \in [a^j]$, we get $A_{\left(i_{b,\alpha_j}\geq\right)} \not\subseteq A_{\left(i_{a,\alpha_i}>\right)}$, then when α_j is the value of $\check{\mu}^j$, $\hat{\mu}^i$ cannot take the value α_i. It completes the proof of the equivalence. ■

4 Weak representation

We address now the weak representation problem. We suppose to have $p-1$ numbers $0 =: \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p := 1$ in E, and we try to find the set of capacities such that all conditions (6) are satisfied. Let us call $\mathcal{W}$ this set of solutions, avoiding as before the subscripts $\alpha_1,\dots,\alpha_{p-1}$.

4.1 Construction of $\{\mu \in \mathcal{M}(C)\,|S_\mu(a) \leq \alpha \leq S_\mu(b)\}$.

If $a_{(1)} \leq \alpha$, the set of the capacities μ such that $S_\mu(a) \leq \alpha$ has a greatest element $\hat{\mu}^{a,\alpha}$ and if $b_{(n)} \geq \alpha$, the set of the capacities μ such that $S_\mu(b) \geq \alpha$ has a least element $\check{\mu}^{b,\alpha}$. In other words we obtain the following result:

Property 8 *Let $a, b \in O$ and $\alpha \in E$ be given, the set of capacities μ such that $S_\mu(a) \leq \alpha \leq S_\mu(b)$ is equal to*

- $\emptyset$ *if $a_{(1)} > \alpha$ or $b_{(n)} < \alpha$,*
- $\left\{\mu \in \mathcal{M}(C) \mid \quad \check{\mu}^{b,\alpha} \leq \mu \leq \hat{\mu}^{a,\alpha}\right\}$ *otherwise.*

4.2 Construction of $\mathcal{W}$

First we introduce the following capacities:

Definition 6 *Let $\check{\mu}$ and $\hat{\mu}$ be two capacities defined by:*

$$\forall A \in \mathcal{P}(\mathcal{C}), \quad \check{\mu}(A) = \bigvee_{i=1}^{p-1} \bigvee_{a \in [a^{i+1}]} \check{\mu}^{a,\alpha_i}(A), \quad \hat{\mu}(A) = \bigwedge_{i=1}^{p-1} \bigwedge_{a \in [a^i]} \hat{\mu}^{a,\alpha_i}(A)$$

Theorem 4 *The set of capacities μ such that $S_\mu(a) \leq \alpha_i \leq S_\mu(b), \forall a \in [a^i], \forall b \in [a^{i+1}], \forall i = 1, \ldots, p-1$ is*

- $\emptyset$ *if $\exists i$ such that $a_{(1)} > \alpha_i$ for some $a \in [a^i]$ or $\exists i$ such that $b_{(n)} < \alpha_i$ for some $b \in [a^{i+1}]$,*
- $\{\mu$ *such that* $\check{\mu} \leq \mu \leq \hat{\mu}\}$ *otherwise.*

Proof : If there exists i such that for $a \in [a^i]$, $a_{(1)} > \alpha_i$ or such that for $b \in [a^{i+1}]$, $b_{(n)} < \alpha_i$; then $\{\mu$ such that $S_\mu(a) \leq \alpha_i \leq S_\mu(b)\}$ is empty. So the solutions set is empty. Otherwise, if μ is such that $S_\mu(a) \leq \alpha_i \leq S_\mu(b), \forall a \in [a^i]$, $b \in [a^{i+1}]$, $\forall i = 1, \ldots, p-1$, then for all $a \in [a^i]$ and for all $b \in [a^{i+1}]$, we have $\check{\mu}^{b,\alpha_i}(A) \leq \mu(A) \leq \hat{\mu}^{a,\alpha_i}(A), \ \forall i \in 1, \ldots, p-1$, which implies $\check{\mu} \leq \mu \leq \hat{\mu}$.

Let μ be a capacity such that $\check{\mu} \leq \mu \leq \hat{\mu}$, hence $\hat{\mu} \leq \hat{\mu}^{a,\alpha_i}$, $\check{\mu} \geq \check{\mu}^{b,\alpha_i}$, $\forall i \in 1, \ldots, p-1$, $\forall a \in [a^i]$ and $\forall b \in [a^{i+1}]$. So $\check{\mu}^{b,\alpha_i} \leq \mu \leq \hat{\mu}^{a,\alpha_i} \quad \forall i \in 1, \cdots, p-1$, $\forall a \in [a^i]$ $\forall b \in [a^{i+1}]$ and the property 8 implies μ is a solution. ∎

The solution set is not empty if and only if $\check{\mu} \leq \hat{\mu}$.

Theorem 5 *$\check{\mu} \leq \hat{\mu}$ if and only if $\forall i, j$ such that $\alpha_i > \alpha_j$, $A_{(i^{\geq}_{b,\alpha_i})} \not\subseteq A_{(i^{>}_{a,\alpha_j})}$, $\forall a \in [a^i], \forall b \in [a^j]$.*

Proof : The proof is similar as the proof of Theorem 3. ∎

5 Example

We consider three students a, b, c who are evaluated according to two criteria $1, 2$.
The scores are given in the evaluation scale $E = \{bad, medium, good, excellent\}$.
Moreover, the decision maker gives the following preferences: $a \sim b \prec c$. There are two equivalence classes. In this paper we focus on the solution where a pair (α_1, α_2) is given to represent a class. We consider the following data:

students	score according to 1	score according to 2	α_i
a	bad	good	medium
b	good	bad	medium
c	good	excellent	good

We try to build strong and weak representations.

Strong representation

$A_{(i^{\geq}_{c,good})} = \{1, 2\}$, $A_{(i^{>}_{a,medium})} = \{2\}$ and $A_{(i^{>}_{b,medium})} = \{1\}$ which entail $A_{(i^{\geq}_{c,good})} \not\subseteq A_{(i^{>}_{a,medium})}$ and $A_{(i^{\geq}_{c,good})} \not\subseteq A_{(i^{>}_{b,medium})}$. Consequently Theorem 3 entails the existence of solutions.

So now we look for capacities such that $S_\mu(a) = S_\mu(b) = medium$. We obtain the following result:

	$\emptyset$	1	2	$\{1, 2\}$
$\check{\mu}^{a,medium} \vee \check{\mu}^{b,medium}$	bad	medium	medium	excellent
$\hat{\mu}^{a,medium} \wedge \hat{\mu}^{b,medium}$	bad	medium	medium	excellent

So there is one solution: $\mu(\emptyset) = bad$, $\mu(1) = medium$, $\mu(2) = medium$, $\mu(1, 2) = excellent$.

To conclude, we verify that the capacity μ can represent the second equivalence class.

	$\emptyset$	1	2	$\{1, 2\}$
$\check{\mu}^{c,good}$	bad	bad	bad	excellent
$\hat{\mu}^{c,good}$	bad	excellent	good	excellent

We have $\check{\mu}^{c,good} \leq \mu \leq \hat{\mu}^{c,good}$, so μ can represent the second equivalence class.

Weak representation

We look for the capacities which satisfy $S_\mu(a) \leq \alpha \leq S_\mu(c)$ and $S_\mu(b) \leq \beta \leq S_\mu(c)$ where $\alpha, \beta \in E$. In this paper, we focus on the solution for a given ordered pair (α, β). We fix for this example $\alpha = \beta = medium$. It is easy to check that these two equations have solutions. Now we are going to compute the capacities which define the set of solutions.

	$\emptyset$	1	2	$\{1,2\}$
$\check{\mu}^{c,medium}$	bad	bad	bad	excellent
$\hat{\mu}^{a,medium}$	bad	excellent	medium	excellent
$\hat{\mu}^{b,medium}$	bad	medium	excellent	excellent

μ satisfies $\check{\mu}^{c,medium} \leq \mu \leq \hat{\mu}^{a,medium}$ and $\check{\mu}^{c,medium} \leq \mu \leq \hat{\mu}^{b,medium}$, so we have $\check{\mu}^{c,medium} \leq \mu \leq \hat{\mu}^{a,medium} \wedge \hat{\mu}^{b,medium}$. In conclusion, the capacities which are solutions satisfy $bad \leq \mu(1) \leq medium$ and $bad \leq \mu(2) \leq medium$.

6 Discussion and related results

We have presented general results on preference representation by a Sugeno integral, illustrated by a detailed example. As one can guess, there is a high probability that the preference cannot be represented by a Sugeno integral in the strong sense as soon as the set O of objects becomes large. The weak representation has however, less drastic conditions. In case of a large set O, we think that only an approximate representation can be obtained. The exact way of doing this approximation is still a topic of research.

Despite the fact that the Sugeno integral covers almost all the class of "suitable" functions built with $\vee, \wedge$ as explained in the introduction, the Sugeno integral has several drawbacks and curious properties for preference representation. Due to space limitations, we do not detail them and refer the reader to a survey of the topic in [DMPRS01]. However, we think that all these limitations have a common origin, which is related to Pareto conditions. We summarize below these facts, see [Mur01] for a detailed study of them. Let us take $E = [0,1]$, and $a, b \in [0,1]^n$. We say that $a \leq b$ if $a_i \leq b_i$ for all $i \in C$, and that $a < b$ if $a \leq b$ and $a_i < b_i$ for some $i \in C$. Lastly, we write $a \ll b$ if $a_i < b_i$ for all $i \in C$. We consider a preference relation $\preceq$ on $[0,1]^n$, and define the following conditions:

- Monotonicity: $a \leq b$ implies $a \preceq b$.
- Strong Pareto condition: $a < b$ implies $a \prec b$.
- Weak Pareto condition: $a \ll b$ implies $a \prec b$.

Monotonicity is a fundamental condition for any preference representation, but the weak Pareto condition is also desirable, otherwise the model could be said to be "blind" or insensitive in certain situations. It can be shown that the Sugeno integral always satisfies monotonicity, but it can never satisfy the strong Pareto condition. More surprisingly, it satisfies the weak Pareto condition if and only if the capacity is valued on $\{0, 1\}$. This last property shows clearly the weakness of the Sugeno integral. A possible way to escape this is to consider a lexicographic use of the Sugeno integral, as shown in [Mur01].

Lastly, we mention the fact that the Sugeno integral can be represented under the form of decision rules, as shown by Greco *et al.* [GMS01].

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