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Block thresholding for a density estimation problem with a change-point

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Abstract

We consider a density estimation problem with a change-point. We develop an adaptive wavelet estimator constructed from a block thresholding rule. Adopting the minimax point of view under the $\mathbb{L}^p$ risk (with $p \geq 1$) over Besov balls, we prove that it is near optimal.

Key words: Density estimation, Wavelets, Block thresholding.

1991 MSC: 62G07

1 MOTIVATIONS

The standard density estimation problem can be formulated as follows. Let $n \in \mathbb{N}^*$ and $(X_i)_{i \in \{1, \dots, n\}}$ be an i.i.d. sample from a distribution with density function f . The goal is to estimate the density function f based on the sample. In the statistical literature, various estimation techniques have been studied. We refer to the books of Devroye and Györfi [5], Silverman [13], Efromovich [6], Härdle et al. [9] and Tsybakov [14].

In this paper, we consider a different density estimation problem inspired from models for control charts (type Shewhart). Let $n \in \mathbb{N}^*$ and $T_n \in \mathbb{N}^*$. Let $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, T_n\}}$ be independent random variables. For each $r \in \{1, \dots, T_n\}$, $(X_{i,r})_{i \in \{1, \dots, n\}}$ is an i.i.d. sample from a distribution with density function f_r . We suppose that there exists $\eta_n \in \{1, \dots, T_n - 1\}$ such that

- for any $r \in \{1, \dots, \eta_n\}$, we have $f_r = f$,

- for any $r \in \{\eta_n + 1, \dots, T_n\}$, we have $f_r = h \neq f$ and $\mathbb{E}(X_{1,\eta_n}) \neq \mathbb{E}(X_{1,\eta_n+1})$.

The integer η_n and the function f are unknown. The goal is to estimate f from $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, T_n\}}$.

We make the two following assumptions.

(H1) We assume that $\lim_{n \rightarrow \infty} n^{-1} \log T_n = 0$. Thus T_n and, a fortiori, η_n , can be really greater than n .

(H2) We assume that, for any $r \in \{1, \dots, T_n\}$, $X_{1,r}(\Omega) = [0, 1]$ and that there exists a known constant $K > 0$ such that $\sup_{x \in [0, 1]} |f(x)| \leq K < \infty$.

To estimate f , one can only use the variables $(X_{i,1})_{i \in \{1, \dots, n\}}$ and take a standard density estimator (kernel, wavelets, ...). However, if we consider all the variables $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, \eta_n\}}$, we gain informations on f . Its estimation can be significantly improved. This motivates the construction of a plug-in estimator of f described as follows. Firstly, we estimate η_n via $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, T_n\}}$. Let $\hat{\eta}_n$ be the corresponding estimator. Then, we estimate f by a density estimator $\tilde{f}_n$ constructed from $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, \hat{\eta}_n\}}$. In this study, we adopt the wavelet methodology. The considered estimator uses a $\mathbb{L}^p$ version of the local block thresholding rule known under the name of BlockShrink. It has been initially developed for the standard density estimation under $\mathbb{L}^2$ risk by Hall et al. [8, 7] and recently improved by Cai and Chicken [1]. The $\mathbb{L}^p$ version of this thresholding rule, more general, has been introduced by Picard and Tribouley [12].

To measure the performances of $\tilde{f}_n$, we consider the minimax approach under the $\mathbb{L}^p$ risk with $p \geq 1$ (not only $p = 2$) over wide range of smoothness spaces: the Besov balls. We aim to evaluate the smallest bound w_n such that

$$\sup_{f \in B_{\pi,r}^s(M)} \mathbb{E} \left(\|\tilde{f}_n - f\|_p^p \right) \leq w_n,$$

where, for any $m \in \mathbb{L}^p([0, 1])$, $\|m\|_p^p = \int_0^1 |m(x)|^p dx$ and $B_{\pi,r}^s(M)$ is the Besov ball (to be defined in Section 2). In this study, we prove that w_n is of the form

$$w_n = C(n\eta_n)^{-\alpha} (\log(n\eta_n))^\beta,$$

where C is a constant independent of f and n , $\alpha \in]0, 1[$, $\beta \in [0, 1[$ and α, β only depend on s, π, r, s and p . It is near optimal in the minimax sense. The proof is based on several auxiliary results including one proved by Chesneau [3]. The originality of w_n resides in the presence of η_n in its expression: more η_n is large, more w_n is small. This illustrates the fact that our estimator takes into account all the pertinent observations for the estimation of f .

The rest of the paper is organized as follows. In Section 2, we present wavelets

and Besov balls. The estimators are defined in Section 3. Section 4 is devoted to the main result. The proofs are postponed in Section 5.

2 WAVELETS AND BESOV BALLS

We consider an orthonormal wavelet basis generated by dilations and translations of a compactly supported "father" wavelet ϕ and a compactly supported "mother" wavelet ψ . For the purposes of this paper, we use the periodized wavelet bases on the unit interval. For any $x \in [0, 1]$, any integer j and any $k \in \{0, \dots, 2^j - 1\}$, let $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ be the elements of the wavelet basis and

$$\phi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x - l), \quad \psi_{j,k}^{per}(x) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x - l),$$

their periodized version. There exists an integer τ such that the collection ζ defined by $\zeta = \{\phi_{\tau,k}^{per}, k = 0, \dots, 2^\tau - 1; \psi_{j,k}^{per}, j = \tau, \dots, \infty, k = 0, \dots, 2^j - 1\}$ constitutes an orthonormal basis of $\mathbb{L}^2([0, 1])$. In what follows, the superscript "per" will be suppressed from the notations for convenience. For any integer $l \geq \tau$, a function $f \in \mathbb{L}^2([0, 1])$ can be expanded into a wavelet series as

$$f(x) = \sum_{k=0}^{2^l-1} \alpha_{l,k} \phi_{l,k}(x) + \sum_{j=l}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where $\alpha_{l,k} = \int_0^1 f(t) \phi_{l,k}(t) dt$ and $\beta_{j,k} = \int_0^1 f(t) \psi_{j,k}(t) dt$. For further details about wavelet bases on the unit interval, we refer to Cohen et al. [4].

Let us now define the Besov balls. Let $M \in (0, \infty)$, $s \in (0, \infty)$, $\pi \in [1, \infty]$ and $r \in [1, \infty]$. Let us set $\beta_{\tau-1,k} = \alpha_{\tau,k}$. We say that a function f belongs to the Besov balls $B_{\pi,r}^s(M)$ if and only if there exists a constant $M^* > 0$ such that the associated wavelet coefficients satisfy

$$\left(\sum_{j=\tau-1}^{\infty} \left(2^{j(s+1/2-1/\pi)} \left(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^\pi \right)^{1/\pi} \right)^r \right)^{1/r} \leq M^*.$$

For a particular choice of parameters s , π and r , these sets contain the Hölder and Sobolev balls. See Meyer [10].

3 ESTIMATOR

For any $\kappa > 0$, set

$$\mathcal{A}_n(\kappa) = \left\{ r \in \{1, \dots, T_n\}; \left| \sum_{i=1}^n (X_{i,r} - X_{i,r+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right\}. \quad (3.1)$$

We estimate η_n by the random integer

$$\hat{\eta}_n(\kappa) = \min \mathcal{A}_n(\kappa). \quad (3.2)$$

This estimator satisfies $\mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)}$, where $h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa)$. See Proposition 4.1 below. A suitable value for κ will be specified later.

We are now in the position to describe the considered estimator of f . As mentioned in Section 1, it can be viewed as a generalization of the $\mathbb{L}^p$ version of the "BlockShrink estimator" initially developed under $\mathbb{L}^2$ risk by Hall et al. [8, 7] and Cai and Chicken [1]. For its $\mathbb{L}^p$ form, see Picard and Tribouley [12].

Let $p \geq 1$ and $u \in \{1, \dots, T_n\}$. Let $j_1(u)$ and $j_2(u)$ be the integers defined by $j_1(u) = \lfloor ((p \vee 2)/2) \log_2(\log(nu)) \rfloor$ and $j_2(u) = \lfloor \log_2(nu/\log(nu)) \rfloor$. Here, $p \vee 2 = \max(p, 2)$ and the quantity $\lfloor a \rfloor$ denotes the whole number part of a . For any $j \in \{j_1(u), \dots, j_2(u)\}$, set $L(u) = \lfloor (\log(nu))^{(p \vee 2)/2} \rfloor$ and $A_j(u) = \{1, \dots, \lfloor 2^j L(u)^{-1} \rfloor\}$. For any $K \in A_j(u)$, we consider the set

$$U_{j,K}(u) = \{k \in \{0, \dots, 2^j - 1\}; (K-1)L(u) \leq k \leq KL(u) - 1\}.$$

For any $u \in \{1, \dots, T_n\}$, we define $\hat{f}_n(x; u)$, $x \in [0, 1]$, by

$$\begin{aligned} \hat{f}_n(x; u) = & \sum_{k=0}^{2^{j_1(u)}-1} \hat{\alpha}_{j_1(u),k}(u) \phi_{j_1(u),k}(x) \\ & + \sum_{j=j_1(u)}^{j_2(u)} \sum_{K \in A_j(u)} \sum_{k \in U_{j,K}(u)} \hat{\beta}_{j,k}(u) 1_{\{\hat{b}_{j,K}(u) \geq dn^{-1/2}\}} \psi_{j,k}(x), \end{aligned} \quad (3.3)$$

where d is a constant independent of f and n , $\hat{b}_{j,K}(u) = (L^{-1} \sum_{k \in U_{j,K}(u)} |\hat{\beta}_{j,k}(u)|^p)^{1/p}$,

$$\hat{\alpha}_{j_1(u),k}(u) = (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u \phi_{j_1(u),k}(X_{i,r}), \quad \hat{\beta}_{j,k}(u) = (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u \psi_{j,k}(X_{i,r}). \quad (3.4)$$

We finally consider the estimator

$$\tilde{f}_n(x) = \hat{f}_n(x; \hat{\eta}_n(\kappa_*)), \quad (3.5)$$

where $\hat{\eta}_n(\kappa_*)$ is defined by (3.2), $\kappa_* = t^{-1}(2p+1)$ and t is the function defined by $t(x) = x^2/(8 + 3^{-1}4x)$, $x \in [0, \infty[$.

Let us mention that $\tilde{f}_n$ does not require any a priori knowledge on f and η_n in its construction. It is adaptive.

Remark. For any $u \in \{1, \dots, T_n\}$, the sets $A_j(u)$ and $U_{j,K}(u)$ are chosen such that $\cup_{K \in A_j(u)} U_{j,K}(u) = \{0, \dots, 2^j - 1\}$, $U_{j,K}(u) \cap U_{j,K'}(u) = \emptyset$ for any $K \neq K'$ with $K, K' \in A_j(u)$, and $|U_{j,K}(u)| = L(u)$.

4 RESULTS

4.1 MAIN RESULT

Theorem 4.1 below determines the rates of convergence achieved by the estimator $\tilde{f}_n$ under the $\mathbb{L}^p$ risk (with $p \geq 1$) over Besov balls.

Theorem 4.1 *Consider the density model described in Section 1 and the estimator $\tilde{f}_n$ defined by (3.5). Then there exists a constant $C > 0$ such that, for any $\pi \in [1, \infty]$, $r \in [1, \infty]$, $s \in (1/\pi, \infty)$, and d and n large enough, we have*

$$\sup_{f \in B_{\pi,r}^s(M)} \mathbb{E} \left(\|\tilde{f}_n - f\|_p^p \right) \leq C \omega_n,$$

where

$$\omega_n = \begin{cases} (n\eta_n)^{-\alpha_1 p} (\log(n\eta_n))^{\alpha_1 p 1_{\{p > \pi\}}}, & \text{when } \epsilon > 0, \\ ((n\eta_n)^{-1} \log(n\eta_n))^{\alpha_2 p} (\log(n\eta_n))^{(p-\pi/r)+1_{\{\epsilon=0\}}}, & \text{when } \epsilon \leq 0, \end{cases}$$

with $\alpha_1 = s/(2s+1)$, $\alpha_2 = (s - 1/\pi + 1/p)/(2(s - 1/\pi) + 1)$ and $\epsilon = \pi s + 2^{-1}(\pi - p)$.

Now, let us briefly discuss the optimal nature of ω_n . Using standard lower bound techniques, we can prove that there exists a constant $c > 0$ such that

$$\inf_{\tilde{f}} \sup_{f \in B_{\pi,r}^s(M)} \mathbb{E} \left(\|\tilde{f} - f\|_p^p \right) \geq c v_n,$$

where $\inf_{\tilde{f}}$ denotes the infimum over all the possible estimators $\tilde{f}$ of f and $v_n = (n\eta_n)^{-\alpha_1 p}$ when $\epsilon > 0$ and $v_n = ((n\eta_n)^{-1} \log(n\eta_n))^{\alpha_2 p}$ when $\epsilon \leq 0$. The proof is similar to the proof of the lower bound for the standard density estimation problem with $n\eta_n$ i.i.d. variables. Further details can be found in the book of Härdle et al. [9] (Section 10.4). Since ω_n is equal to v_n up to logarithmic terms, it is near optimal in the minimax sense. Moreover, one can show that it is better than those achieved by the conventional term-by-term thresholding estimators (hard, soft,...). The main difference is for the case $\{\pi \geq p\}$ where there is no extra logarithmic term.

4.2 AUXILIARY RESULTS

The proof of Theorem 4.1 is based on several auxiliary results.

Statistical properties satisfied by the set $\mathcal{A}_n(\kappa)$ and the estimator $\hat{\eta}_n(\kappa)$ are presented in Proposition 4.1 below.

Proposition 4.1 *Let $\kappa > 0$, $\mathcal{A}_n(\kappa)$ be defined by (3.1) and $\hat{\eta}_n(\kappa)$ be defined by (3.2). Then*

- we have

$$\mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) \geq 1 - 2(nT_n)^{-h(\kappa)},$$

where $h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa)$.

- for any $m \in \{1, \dots, T_n\} - \{\eta_n\}$, we have

$$\mathbb{P}(m \in \mathcal{A}_n(\kappa)) \leq 2(nT_n)^{-t(\kappa)},$$

where $t(\kappa) = \kappa^2/(8 + 3^{-1}4\kappa)$.

- we have

$$\mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)},$$

where $h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa)$.

Proposition 4.2 below investigates the performances of the non adaptive estimator $\hat{f}_n(x; \eta_n)$ via the minimax approach under the $\mathbb{L}^p$ risk (with $p \geq 1$) over Besov balls.

Proposition 4.2 *Consider the density model described in Section 1 and the estimator $\hat{f}_n(x; \eta_n)$ defined by (3.3). Then there exists a constant $C > 0$ such that, for any $\pi \in [1, \infty]$, $r \in [1, \infty]$, $s \in (1/\pi, \infty)$, and d and n large enough, we have*

$$\sup_{f \in B_{\pi, r}^s(M)} \mathbb{E} \left(\|\hat{f}_n(\cdot; \eta_n) - f\|_p^p \right) \leq C\omega_n,$$

where ω_n is defined as in Theorem 4.1.

By definition of η_n , the variables $(X_{i,r})_{(i,r) \in \{1, \dots, n\} \times \{1, \dots, \eta_n\}}$ are i.i.d. with probability density function f . By (H2), we have $\sup_{x \in [0,1]} |f(x)| \leq K < \infty$. Therefore, the proof of Proposition 4.2 is similar to those of Chesneau [2, Theorem 4.1] with $n\eta_n$ i.i.d. variables (and the weight function $w(x) = 1$). It is a consequence of a general result proved by Chesneau [3, Theorem 4.2]. The crucial points are some statistical properties satisfied by the estimators $\hat{\alpha}_{j,k}(u)$ and $\hat{\beta}_{j,k}(u)$ defined by (3.4) (moment inequality and concentration inequality). For the particular case $p = 2$, we refer to Cai and Chicken [1, Theorem 1].

5 PROOFS

In this section, C denotes a positive constant which can take different values for each mathematical term. It is independent of f and n .

Proof of Theorem 4.1. We have

$$\mathbb{E} \left(\|\tilde{f}_n - f\|_p^p \right) = \sum_{u=1}^{T_n} \mathbb{E} \left(\|\hat{f}_n(\cdot; u) - f\|_p^p 1_{\{\hat{\eta}_n(\kappa_*)=u\}} \right) = A + B, \quad (5.1)$$

where

$$A = \mathbb{E} \left(\|\hat{f}_n(\cdot; \eta_n) - f\|_p^p 1_{\{\hat{\eta}_n(\kappa_*)=\eta_n\}} \right)$$

and

$$B = \sum_{u \in \{1, \dots, T_n\} - \{\eta_n\}} \mathbb{E} \left(\|\hat{f}_n(\cdot; u) - f\|_p^p 1_{\{\hat{\eta}_n(\kappa_*)=u\}} \right).$$

The upper bound for A. It follows from Proposition 4.2 that, if $f \in B_{\pi, r}^s(M)$,

$$A \leq \mathbb{E} \left(\|\hat{f}_n(\cdot; \eta_n) - f\|_p^p \right) \leq C\omega_n, \quad (5.2)$$

where ω_n is the desired rate of convergence.

The upper bound for B. By (H2), we have $\sup_{x \in [0,1]} |f(x)| \leq K < \infty$. Using the elementary inequalities : $\|\hat{f}_n(\cdot; u) - f\|_p^p \leq (\|\hat{f}_n(\cdot; u)\|_p + \|f\|_p)^p \leq 2^{p-1}(\|\hat{f}_n(\cdot; u)\|_p^p + \|f\|_p^p) \leq 2^{p-1}(\|\hat{f}_n(\cdot; u)\|_p^p + K^p)$, we obtain

$$B \leq 2^{p-1} \sum_{u \in \{1, \dots, T_n\} - \{\eta_n\}} \mathbb{E} \left(\left(\|\hat{f}_n(\cdot; u)\|_p^p + K^p \right) 1_{\{\hat{\eta}_n(\kappa_*)=u\}} \right). \quad (5.3)$$

Since $\|\phi_{j,k}\|_p = \|\phi\|_p 2^{j(1/2-1/p)}$ and $\|\psi_{j,k}\|_p = \|\psi\|_p 2^{j(1/2-1/p)}$, the Minkowski inequality yields

$$\begin{aligned} \|\hat{f}_n(\cdot; u)\|_p^p &\leq \left(\sum_{k=0}^{2^{j_1(u)}-1} \left| \hat{\alpha}_{j_1(u),k}(u) \right| \|\phi_{j_1(u),k}\|_p \right. \\ &\quad \left. + \sum_{j=j_1(u)}^{j_2(u)} \sum_{K \in A_j(u)} \sum_{k \in U_{j,K}(u)} \left| \hat{\beta}_{j,k}(u) \right| 1_{\{\hat{b}_{j,K}(u) \geq dn^{-1/2}\}} \|\psi_{j,k}\|_p \right)^p \\ &\leq C \left(\sum_{k=0}^{2^{j_1(T_n)}-1} \left| \hat{\alpha}_{j_1(u),k}(u) \right| 2^{j_1(T_n)(1/2-1/p)} + \sum_{j=j_1(1)}^{j_2(T_n)} \sum_{k=0}^{2^j-1} \left| \hat{\beta}_{j,k}(u) \right| 2^{j(1/2-1/p)} \right)^p. \end{aligned}$$

Since the wavelet basis is compactly supported, we have $\sup_{x \in [0,1]} |\phi_{j,k}(x)| \leq$

$\|\phi\|_\infty 2^{j/2}$ and $\sup_{x \in [0,1]} |\psi_{j,k}(x)| \leq \|\psi\|_\infty 2^{j/2}$. It follows that $|\hat{\alpha}_{j_1(u),k}(u)| \leq (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u |\phi_{j_1(u),k}(X_{i,r})| \leq \sup_{x \in [0,1]} |\phi_{j_1(u),k}(x)| \leq C 2^{j_1(u)/2} \leq C 2^{j_1(T_n)/2}$ and $|\hat{\beta}_{j,k}(u)| \leq (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u |\psi_{j,k}(X_{i,r})| \leq \sup_{x \in [0,1]} |\psi_{j,k}(x)| \leq C 2^{j/2}$. This yields

$$\begin{aligned} \|\hat{f}_n(\cdot; u)\|_p^p &\leq C \left(2^{j_1(T_n)(2-1/p)} + \sum_{j=j_1(1)}^{j_2(T_n)} 2^{j(2-1/p)} \right)^p \leq C 2^{j_2(T_n)(2p-1)} \\ &\leq C (nT_n)^{2p-1}. \end{aligned} \quad (5.4)$$

Combining (5.3) and (5.4), and using the fact that, for any $u \in \{1, \dots, T_n\}$, we have $\{\hat{\eta}_n(\kappa_*) = u\} \subseteq \{u \in \mathcal{A}_n(\kappa_*)\}$, we obtain

$$\begin{aligned} B &\leq C (nT_n)^{2p-1} \sum_{u \in \{1, \dots, T_n\} - \{\eta_n\}} \mathbb{P}(\hat{\eta}_n(\kappa_*) = u) \\ &\leq C (nT_n)^{2p} \sup_{u \in \{1, \dots, T_n\} - \{\eta_n\}} \mathbb{P}(u \in \mathcal{A}_n(\kappa_*)). \end{aligned}$$

The second point of Proposition 4.1 and the definition of κ_* give

$$B \leq C (nT_n)^{2p} (nT_n)^{-t(\kappa_*)} \leq C (nT_n)^{2p-t(\kappa_*)} \leq C (nT_n)^{-1} \leq C \omega_n, \quad (5.5)$$

where ω_n is the desired rate of convergence. Putting the inequalities (5.1), (5.2) and (5.5) together, this completes the proof of Theorem 4.1. $\square$

Proof of Proposition 4.1. First of all, let us present the Bernstein inequality. See, for instance, Petrov [11].

Lemma 5.1 (Bernstein's inequality) *Let $(Y_i)_{i \in \mathbb{N}^*}$ be independent random variables such that, for any $n \in \mathbb{N}^*$ and any $i \in \{1, \dots, n\}$, we have $\mathbb{E}(Y_i) = 0$ and $|Y_i| \leq M < \infty$. Then, for any $\lambda > 0$, and any $n \in \mathbb{N}^*$, we have*

$$\mathbb{P} \left(\left| \sum_{i=1}^n Y_i \right| \geq \lambda \right) \leq 2 \exp \left(- \frac{\lambda^2}{2(d_n^2 + \frac{M\lambda}{3})} \right),$$

where $d_n^2 = \sum_{i=1}^n \mathbb{E}(Y_i^2)$.

• *Proof of the first point.* We have

$$\mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) = \mathbb{P} \left(\left| \sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right).$$

Set $\mu_n = \mathbb{E}(X_{1,\eta_n}) - \mathbb{E}(X_{1,\eta_n+1})$. By definition of η_n , we have $\mathbb{E}(X_{1,\eta_n}) \neq \mathbb{E}(X_{1,\eta_n+1})$. By (H1), we have $\lim_{n \rightarrow \infty} n^{-1} \log T_n = 0$. Therefore, for a large enough n , we have $|\mu_n| \geq 2\kappa\sqrt{n^{-1} \log(nT_n)}$. Using the triangular inequality, we obtain the inclusion $\left\{ \left| \sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \right| \leq \kappa\sqrt{n \log(nT_n)} \right\} \subseteq \left\{ \left| \sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1}) \right| \geq \kappa\sqrt{n \log(nT_n)} \right\}$. It follows that

$$\begin{aligned} \mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) &\geq \mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)\right| \leq \kappa\sqrt{n \log(nT_n)}\right) \\ &= 1 - \mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)\right| \geq \kappa\sqrt{n \log(nT_n)}\right). \end{aligned} \quad (5.6)$$

By (H2), we have, for any $i \in \{1, \dots, n\}$, $X_{i,\eta_n}(\Omega) = X_{i,\eta_n+1}(\Omega) = [0, 1]$. Therefore, for any $i \in \{1, \dots, n\}$, $|X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n| \leq |X_{i,\eta_n}| + |X_{i,\eta_n+1}| + \mathbb{E}(|X_{1,\eta_n}|) + \mathbb{E}(|X_{1,\eta_n+1}|) \leq 4$. Hence $\sum_{i=1}^n \mathbb{E}((X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)^2) \leq 16n$. The Bernstein inequality applied to the independent, uniformly bounded and centered random variables $(X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)_{i \in \{1, \dots, n\}}$ implies that

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)\right| \geq \kappa\sqrt{n \log(nT_n)}\right) \\ &\leq 2 \exp\left(-\frac{\left(\kappa\sqrt{n \log(nT_n)}\right)^2}{2\left(16n + \frac{4\kappa\sqrt{n \log(nT_n)}}{3}\right)}\right) \\ &= 2 \exp\left(-\frac{\kappa^2 \log(nT_n)}{2\left(16 + \frac{4\kappa}{3}\sqrt{n^{-1} \log(nT_n)}\right)}\right). \end{aligned}$$

Since, by (H1), for n large enough, $n^{-1} \log(nT_n) \leq 1$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)\right| \geq \kappa\sqrt{n \log(nT_n)}\right) \leq 2(nT_n)^{-h(\kappa)}, \quad (5.7)$$

where $h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa)$.

Putting (5.6) and (5.7) together, we obtain

$$\mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) \geq 1 - 2(nT_n)^{-h(\kappa)}.$$

This proved the first point of Proposition 4.1.

- *Proof of the second point.* We have, for any $m \in \{1, \dots, T_n\} - \{\eta_n\}$,

$$\mathbb{P}(m \in \mathcal{A}_n(\kappa)) = \mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,m} - X_{i,m+1})\right| \geq \kappa \sqrt{n \log(nT_n)}\right). \quad (5.8)$$

By (H2), we have, for any $i \in \{1, \dots, n\}$ and any $m \in \{1, \dots, T_n\}$, $X_{i,m}(\Omega) = X_{i,m+1}(\Omega) = [0, 1]$. Therefore, for any $i \in \{1, \dots, n\}$ and any $m \in \{1, \dots, T_n\} - \{\eta_n\}$, $|X_{i,m} - X_{i,m+1}| \leq |X_{i,m}| + |X_{i,m+1}| \leq 2$. Hence $\sum_{i=1}^n \mathbb{E}((X_{i,m} - X_{i,m+1})^2) \leq 4n$. The Bernstein inequality applied to the independent, uniformly bounded and centered random variables $(X_{i,m} - X_{i,m+1})_{i \in \{1, \dots, n\}}$ implies that

$$\begin{aligned} & \mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,m} - X_{i,m+1})\right| \geq \kappa \sqrt{n \log(nT_n)}\right) \\ & \leq 2 \exp\left(-\frac{(\kappa \sqrt{n \log(nT_n)})^2}{2\left(4n + \frac{2\kappa \sqrt{n \log(nT_n)}}{3}\right)}\right) \\ & = 2 \exp\left(-\frac{\kappa^2 \log(nT_n)}{2\left(4 + \frac{2\kappa}{3} \sqrt{n^{-1} \log(nT_n)}\right)}\right). \end{aligned}$$

Since, by (H1), for n large enough, $n^{-1} \log(nT_n) \leq 1$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n (X_{i,m} - X_{i,m+1})\right| \geq \kappa \sqrt{n \log(nT_n)}\right) \leq 2(nT_n)^{-t(\kappa)}, \quad (5.9)$$

where $t(\kappa) = \kappa^2 / (8 + 3^{-1} 4\kappa)$.

It follows from (5.8) and (5.9) that, for any $m \in \{1, \dots, T_n\} - \{\eta_n\}$,

$$\mathbb{P}(m \in \mathcal{A}_n(\kappa)) \leq 2(nT_n)^{-t(\kappa)}.$$

- *Proof of the third point.* It follows from the Bonferroni inequality that

$$\begin{aligned} \mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) &= \mathbb{P}\left(\cap_{m=1}^{\eta_n-1} \{m \notin \mathcal{A}_n(\kappa)\} \cap \{\eta_n \in \mathcal{A}_n(\kappa)\}\right) \\ &\geq \sum_{m=1}^{\eta_n-1} \mathbb{P}(m \notin \mathcal{A}_n(\kappa)) + \mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) - (\eta_n - 1). \end{aligned}$$

The two first points give

$$\begin{aligned}
& \mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \\
& \geq (\eta_n - 1)(1 - 2(nT_n)^{-t(\kappa)}) + 1 - 2(nT_n)^{-t(\kappa)} - (\eta_n - 1) \\
& = 1 - 2(\eta_n - 1)(nT_n)^{-t(\kappa)} - 2(nT_n)^{-h(\kappa)} \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)}.
\end{aligned}$$

This completes the proof of Proposition 4.1.

□

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