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Block thresholding for a density estimation problem with a change-point

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Abstract

We consider a density estimation problem with a change-point. We develop an adaptive wavelet estimator constructed from a block thresholding rule. Adopting the minimax point of view under the $L^p$ risk (with $p \geq 1$) over Besov balls, we prove that it is near optimal.

Key words: Density estimation, Wavelets, Block thresholding.

1991 MSC: 62G07

1 MOTIVATIONS

The standard density estimation problem can be formulated as follows. Let $n \in \mathbb{N}^*$ and $(X_i)_{i \in \{1, \ldots, n\}}$ be an i.i.d. sample from a distribution with density function $f$. The goal is to estimate the density function $f$ based on the sample. In the statistical literature, various estimation techniques have been studied. We refer to the books of Devroye and Györfi [5], Silverman [13], Efromovich [6], Härdele et al. [9] and Tsybakov [14].

In this paper, we consider a different density estimation problem inspired from models for control charts (type Shewhart). Let $n \in \mathbb{N}^*$ and $T_n \in \mathbb{N}^*$. Let $(X_{i,r})_{(i,r) \in \{1, \ldots, n\} \times \{1, \ldots, T_n\}}$ be independent random variables. For each $r \in \{1, \ldots, T_n\}$, $(X_{i,r})_{i \in \{1, \ldots, n\}}$ is an i.i.d. sample from a distribution with density function $f_r$. We suppose that there exists $\eta_n \in \{1, \ldots, T_n - 1\}$ such that

- for any $r \in \{1, \ldots, \eta_n\}$, we have $f_r = f$,
• for any \( r \in \{ \eta_n + 1, \ldots, T_n \} \), we have \( f_r = h \neq f \) and \( \mathbb{E}(X_{1, \eta_n}) \neq \mathbb{E}(X_{1, \eta_n + 1}) \).

The integer \( \eta_n \) and the function \( f \) are unknown. The goal is to estimate \( f \) from \( (X_{i,r})_{(i,r) \in \{1,\ldots,n\} \times \{1,\ldots,T_n\}} \).

We make the two following assumptions.

\((H1)\) We assume that \( \lim_{n \to \infty} n^{-1} \log T_n = 0 \). Thus \( T_n \) and, a fortiori, \( \eta_n \), can be really greater than \( n \).

\((H2)\) We assume that, for any \( r \in \{1, \ldots, T_n\} \), \( X_{1,r}(\Omega) = [0,1] \) and that there exists a known constant \( K > 0 \) such that \( \sup_{x \in [0,1]} |f(x)| \leq K < \infty \).

To estimate \( f \), one can only use the variables \( (X_{i,1})_{i \in \{1,\ldots,n\}} \) and take a standard density estimator (kernel, wavelets, ...). However, if we consider all the variables \( (X_{i,r})_{(i,r) \in \{1,\ldots,n\} \times \{1,\ldots,T_n\}} \), we gain informations on \( f \). Its estimation can be significantly improved. This motivates the construction of a plug-in estimator described as follows. Firstly, we estimate \( \eta_n \) via \( (X_{i,r})_{(i,r) \in \{1,\ldots,n\} \times \{1,\ldots,T_n\}} \). Let \( \hat{\eta}_n \) be the corresponding estimator. Then, we estimate \( f \) by a density estimator \( \hat{f}_n \) constructed from \( (X_{i,r})_{(i,r) \in \{1,\ldots,n\} \times \{1,\ldots,\hat{\eta}_n\}} \). In this study, we adopt the wavelet methodology. The considered estimator uses a \( \mathbb{L}^p \) version of the local block thresholding rule known under the name of BlockShrink. It has been initially developed for the standard density estimation under \( \mathbb{L}^2 \) risk by Hall et al. [8, 7] and recently improved by Cai and Chicken [1]. The \( \mathbb{L}^p \) version of this thresholding rule, more general, has been introduced by Picard and Tribouley [12].

To measure the performances of \( \hat{f}_n \), we consider the minimax approach under the \( \mathbb{L}^p \) risk with \( p \geq 1 \) (not only \( p = 2 \)) over wide range of smoothness spaces: the Besov balls. We aim to evaluate the smallest bound \( w_n \) such that

\[
\sup_{f \in B^s_{r,p}(M)} \mathbb{E} \left( \| \hat{f}_n - f \|_p^p \right) \leq w_n,
\]

where, for any \( m \in \mathbb{L}^p([0,1]) \), \( \|m\|_p = \int_0^1 |m(x)|^p dx \) and \( B^s_{r,p}(M) \) is the Besov ball (to be defined in Section 2). In this study, we prove that \( w_n \) is of the form

\[
w_n = C(n\eta_n)^{-\alpha}(\log(n\eta_n))^{\beta},
\]

where \( C \) is a constant independent of \( f \) and \( n \), \( \alpha \in [0,1[ \), \( \beta \in [0,1[ \) and \( \alpha, \beta \) only depend on \( s, \pi, r, s \) and \( p \). It is near optimal in the minimax sense. The proof is based on several auxiliary results including one proved by Chesneau [3]. The originality of \( w_n \) resides in the presence of \( \eta_n \) in its expression: more \( \eta_n \) is large, more \( w_n \) is small. This illustrates the fact that our estimator takes into account all the pertinent observations for the estimation of \( f \).

The rest of the paper is organized as follows. In Section 2, we present wavelets
and Besov balls. The estimators are defined in Section 3. Section 4 is devoted to the main result. The proofs are postponed in Section 5.

\section{WAVELETS AND BESOV BALLS}

We consider an orthonormal wavelet basis generated by dilations and translations of a compactly supported "father" wavelet \( \phi \) and a compactly supported "mother" wavelet \( \psi \). For the purposes of this paper, we use the periodized wavelet bases on the unit interval. For any \( x \in [0, 1] \), any integer \( j \) and any \( k \in \{0, \ldots, 2^j - 1\} \), let \( \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \) and \( \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \) be the elements of the wavelet basis and

\[
\phi_{j,k}^{\text{per}}(x) = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x - l), \quad \psi_{j,k}^{\text{per}}(x) = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x - l),
\]

their periodized version. There exists an integer \( \tau \) such that the collection \( \zeta = \{\phi_{\tau,k}, k = 0, \ldots, 2^{\tau} - 1; \psi_{j,k}^{\text{per}}, j = \tau, \ldots, \infty, k = 0, \ldots, 2^j - 1\} \) constitutes an orthonormal basis of \( L^2([0, 1]) \). In what follows, the superscript "\( \text{per} \)" will be suppressed from the notations for convenience. For any integer \( l \geq \tau \), a function \( f \in L^2([0, 1]) \) can be expanded into a wavelet series as

\[
f(x) = \sum_{k=0}^{2^l-1} \alpha_{l,k}\phi_{l,k}(x) + \sum_{j=\tau}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \quad x \in [0, 1],
\]

where \( \alpha_{l,k} = \int_0^1 f(t)\phi_{l,k}(t)dt \) and \( \beta_{j,k} = \int_0^1 f(t)\psi_{j,k}(t)dt \). For further details about wavelet bases on the unit interval, we refer to Cohen et al. [4].

Let us now define the Besov balls. Let \( M \in (0, \infty) \), \( s \in (0, \infty) \), \( \pi \in [1, \infty] \) and \( r \in [1, \infty] \). Let us set \( \beta_{\tau-1,k} = \alpha_{\tau,k} \). We say that a function \( f \) belongs to the Besov balls \( B_{\pi,r}^s(M) \) if and only if there exists a constant \( M^* > 0 \) such that the associated wavelet coefficients satisfy

\[
\left( \sum_{j=\tau}^{\infty} \left( 2^{j(s+1/2-1/\pi)} \left( \sum_{k=0}^{2^j-1} |\beta_{j,k}|^\pi \right)^{1/\pi} \right)^r \right)^{1/r} \leq M^*.
\]

For a particular choice of parameters \( s, \pi \) and \( r \), these sets contain the Hölder and Sobolev balls. See Meyer [10].

\section{ESTIMATOR}

For any \( \kappa > 0 \), set
\[ \mathcal{A}_n(\kappa) = \left\{ r \in \{1, \ldots, T_n\}; \quad \sum_{i=1}^n (X_{i,r} - X_{i,r+1}) \geq \kappa \sqrt{n \log(nT_n)} \right\}. \] (3.1)

We estimate \( \eta_n \) by the random integer
\[ \hat{\eta}_n(\kappa) = \min \mathcal{A}_n(\kappa). \] (3.2)

This estimator satisfies \( \mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)} \), where \( h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa) \). See Proposition 4.1 below. A suitable value for \( \kappa \) will be specified later.

We are now in the position to describe the considered estimator of \( f \). As mentioned in Section 1, it can be viewed as a generalization of the \( \mathbb{L}^p \) version of the "BlockShrink estimator" initially developed under \( \mathbb{L}^2 \) risk by Hall et al. [8, 7] and Cai and Chicken [1]. For its \( \mathbb{L}^p \) form, see Picard and Tribouley [12].

Let \( p \geq 1 \) and \( u \in \{1, \ldots, T_n\} \). Let \( j_1(u) \) and \( j_2(u) \) be the integers defined by \( j_1(u) = \lfloor (p \vee 2)/2 \rfloor \log_2(\log(nu)) \) and \( j_2(u) = \lceil \log_2(nu/\log(nu)) \rceil \). Here, \( p \vee 2 = \max(p, 2) \) and the quantity \( \lfloor a \rfloor \) denotes the whole number part of \( a \). For any \( j \in \{j_1(u), \ldots, j_2(u)\} \), set \( L(u) = \lceil \log(nu)/(p/2) \rceil \) and \( A_j(u) = \{1, \ldots, 2^jL(u)^{-1}\} \). For any \( K \in A_j(u) \), we consider the set
\[ U_{j,K}(u) = \{k \in \{0, \ldots, 2^j - 1\}; (K-1)L(u) \leq k \leq KL(u) - 1\}. \]

For any \( u \in \{1, \ldots, T_n\} \), we define \( \hat{f}_n(x; u), x \in [0, 1] \), by

\[ \hat{f}_n(x; u) = \sum_{k=0}^{2^{j_1(u)}-1} \hat{a}_{j_1(u),k}(u) \phi_{j_1(u),k}(x) + \sum_{j=j_1(u)}^{j_2(u)} \sum_{K \in A_j(u)} \sum_{k \in U_{j,K}(u)} \hat{\beta}_{j,k}(u) 1\{\hat{\eta}_{j,K}(u) \geq d_{n^{-1/2}}\} \psi_{j,k}(x), \] (3.3)

where \( d \) is a constant independent of \( f \) and \( n \), \( \hat{\beta}_{j,K}(u) = (L^{-1} \sum_{k \in U_{j,K}(u)} |\hat{\beta}_{j,k}(u)|^p)^{1/p} \),
\[ \hat{a}_{j_1(u),k}(u) = (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u \phi_{j_1(u),k}(X_{i,r}), \quad \hat{\beta}_{j,k}(u) = (nu)^{-1} \sum_{i=1}^n \sum_{r=1}^u \psi_{j,k}(X_{i,r}). \] (3.4)

We finally consider the estimator
\[ \tilde{f}_n(x) = \tilde{f}_n(x; \hat{\eta}_n(\kappa_*)), \] (3.5)

where \( \tilde{\eta}_n(\kappa_*) \) is defined by (3.2), \( \kappa_* = t^{-1}(2p + 1) \) and \( t \) is the function defined by \( t(x) = x^2/(8 + 3^{-1}4x) \), \( x \in [0, \infty[. \)
Let us mention that \( \tilde{f}_n \) does not require any a priori knowledge on \( f \) and \( \eta_n \) in its construction. It is adaptive.

Remark. For any \( u \in \{1, \ldots, T_n\} \), the sets \( A_j(u) \) and \( U_{j,K}(u) \) are chosen such that \( \cup_{K \in A_j(u)} U_{j,K}(u) = \{0, \ldots, 2^j - 1\} \), \( U_{j,K}(u) \cap U_{j,K'}(u) = \emptyset \) for any \( K \neq K' \) with \( K, K' \in A_j(u) \), and \( |U_{j,K}(u)| = L(u) \).

4 RESULTS

4.1 MAIN RESULT

Theorem 4.1 below determines the rates of convergence achieved by the estimator \( \tilde{f}_n \) under the \( L^p \) risk (with \( p \geq 1 \)) over Besov balls.

**Theorem 4.1** Consider the density model described in Section 1 and the estimator \( \tilde{f}_n \) defined by (3.5). Then there exists a constant \( C > 0 \) such that, for any \( \pi \in [1, \infty] \), \( r \in [1, \infty] \), \( s \in \left(\frac{1}{\pi}, \infty\right) \), and \( d \) and \( n \) large enough, we have

\[
\sup_{f \in B_{s\pi,r}(M)} E \left( \|\tilde{f}_n - f\|_p^p \right) \leq C \omega_n,
\]

where

\[
\omega_n = \begin{cases} 
(n \eta_n)^{-\alpha_1} (\log(n \eta_n))^{\alpha_1 + 1} (r > \pi), & \text{when } \epsilon > 0, \\
((n \eta_n)^{-\alpha_2} (\log(n \eta_n))^{\alpha_2 - \pi/r} + 1^{(\epsilon = 0)}), & \text{when } \epsilon \leq 0,
\end{cases}
\]

with \( \alpha_1 = s/(2s + 1) \), \( \alpha_2 = (s - 1/\pi + 1/p)/(2s - 1/\pi + 1) \) and \( \epsilon = \pi s + 2^{-1}(\pi - p) \).

Now, let us briefly discuss the optimal nature of \( \omega_n \). Using standard lower bound techniques, we can prove that there exists a constant \( c > 0 \) such that

\[
\inf_{\tilde{f}} \sup_{f \in B_{s\pi,r}(M)} E \left( \|\tilde{f} - f\|_p^p \right) \geq c v_n,
\]

where \( \inf_{\tilde{f}} \) denotes the infimum over all the possible estimators \( \tilde{f} \) of \( f \) and \( v_n = (n \eta_n)^{-\alpha_1} \) when \( \epsilon > 0 \) and \( v_n = ((n \eta_n)^{-1} \log(n \eta_n))^{\alpha_2} \) when \( \epsilon \leq 0 \).

The proof is similar to the proof of the lower bound for the standard density estimation problem with \( n \eta_n \) i.i.d. variables. Further details can be found in the book of Härdle et al. [9] (Section 10.4). Since \( \omega_n \) is equal to \( v_n \) up to logarithmic terms, it is near optimal in the minimax sense. Moreover, one can show that it is better than those achieved by the conventional term-by-term thresholding estimators (hard, soft,...). The main difference is for the case \( \{ \pi \geq p \} \) where there is no extra logarithmic term.
4.2 AUXILIARY RESULTS

The proof of Theorem 4.1 is based on several auxiliary results.

Statistical properties satisfied by the set $\mathcal{A}_n(\kappa)$ and the estimator $\hat{\eta}_n(\kappa)$ are presented in Proposition 4.1 below.

**Proposition 4.1** Let $\kappa > 0$, $\mathcal{A}_n(\kappa)$ be defined by (3.1) and $\hat{\eta}_n(\kappa)$ be defined by (3.2). Then

- we have
  $$\mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) \geq 1 - 2(nT_n)^{-h(\kappa)},$$
  where $h(\kappa) = \kappa^2/(32 + 3^{-18}\kappa)$.

- for any $m \in \{1, ..., T_n\} \setminus \{\eta_n\}$, we have
  $$\mathbb{P}(m \in \mathcal{A}_n(\kappa)) \leq 2(nT_n)^{-t(\kappa)},$$
  where $t(\kappa) = \kappa^2/(8 + 3^{-14}\kappa)$.

- we have
  $$\mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)},$$
  where $h(\kappa) = \kappa^2/(32 + 3^{-18}\kappa)$.

Proposition 4.2 below investigates the performances of the non adaptive estimator $\hat{f}_n(x; \eta_n)$ via the minimax approach under the $\mathbb{L}^p$ risk (with $p \geq 1$) over Besov balls.

**Proposition 4.2** Consider the density model described in Section 1 and the estimator $f_n(x; \eta_n)$ defined by (3.3). Then there exists a constant $C > 0$ such that, for any $\pi \in [1, \infty]$, $r \in [1, \infty]$, $s \in (1/\pi, \infty)$, and $d$ and $n$ large enough, we have

$$\sup_{f \in B_{s,\pi}(M)} \mathbb{E}\left(\|\hat{f}_n(:, \eta_n) - f\|_p^p\right) \leq C\omega_n,$$

where $\omega_n$ is defined as in Theorem 4.1.

By definition of $\eta_n$, the variables $(X_{i,r})_{(i,r) \in \{1, ..., n\} \times \{1, ..., \eta_n\}}$ are i.i.d. with probability density function $f$. By $(H2)$, we have $\sup_{x \in [0,1]} |f(x)| \leq K < \infty$. Therefore, the proof of Proposition 4.2 is similar to those of Chesneau [2, Theorem 4.1] with $n\eta_n$ i.i.d. variables (and the weight function $w(x) = 1$). It is a consequence of a general result proved by Chesneau [3, Theorem 4.2]. The crucial points are some statistical properties satisfied by the estimators $\widehat{\alpha}_{j,k}(u)$ and $\widehat{\beta}_{j,k}(u)$ defined by (3.4) (moment inequality and concentration inequality). For the particular case $p = 2$, we refer to Cai and Chicken [1, Theorem 1].
5 PROOFS

In this section, $C$ denotes a positive constant which can take different values for each mathematical term. It is independent of $f$ and $n$.

Proof of Theorem 4.1. We have

$$
\mathbb{E} \left( \| \hat{f}_n - f \|_p^p \right) = \sum_{u=1}^{T_n} \mathbb{E} \left( \| \hat{f}_n ; u \|_p^p \{ \bar{\eta}_n = u \} \right) = A + B, \tag{5.1}
$$

where

$$
A = \mathbb{E} \left( \| \hat{f}_n ; \eta_n \|_p^p \{ \bar{\eta}_n = \eta_n \} \right)
$$

and

$$
B = \sum_{u \in \{1, \ldots, T_n\} - \{\eta_n\}} \mathbb{E} \left( \| \hat{f}_n ; u \|_p^p \{ \bar{\eta}_n = u \} \right).
$$

The upper bound for $A$. It follows from Proposition 4.2 that, if $f \in B_{\pi,r}^s (M)$,

$$
A \leq \mathbb{E} \left( \| \hat{f}_n ; \eta_n \|_p^p \right) \leq C \omega_n, \tag{5.2}
$$

where $\omega_n$ is the desired rate of convergence.

The upper bound for $B$. By $(H2)$, we have $\sup_{x \in [0,1]} |f(x)| \leq K < \infty$. Using the elementary inequalities: $\| f \|_p \leq \| f \|_p^p \leq \| f \|_p \leq 2^{p-1} (\| f \|_p + \| f \|_p)^p \leq 2^{p-1} (\| f \|_p + \| f \|_p)^p \leq 2^{p-1} (\| f \|_p + K)$, we obtain

$$
B \leq 2^{p-1} \sum_{u \in \{1, \ldots, T_n\} - \{\eta_n\}} \mathbb{E} \left( \left[ \| \hat{f}_n ; u \|_p^p + K^p \right] 1 \{ \bar{\eta}_n = u \} \right). \tag{5.3}
$$

Since $\| \phi_{j,k} \|_p = \| \phi \|_p 2^{j/(2-1/p)}$ and $\| \psi_{j,k} \|_p = \| \psi \|_p 2^{j/(2-1/p)}$, the Minkowski inequality yields

$$
\| \hat{f}_n ; u \|_p \leq \left( \sum_{k=0}^{2^{j_1(u)}-1} | \hat{\alpha}_{j_1(u),k}(u) | \| \phi_{j_1(u),k} \|_p \right) + \sum_{j=j_1(u)}^{j_2(u)} K \sum_{K \in A_j(u)} K \sum_{K \in U_{j,K}(u)} \left[ \| \beta_{j,K}(u) \| \right] \left( \sum_{k=0}^{2^{j_2(u)}-1} \| \psi_{j,K}(u) \|_p \right)^p
\leq C \left( \sum_{k=0}^{2^{j_1(T_n)}-1} | \hat{\alpha}_{j_1(u),k}(u) | \left( \sum_{j=j_1(u)}^{j_2(T_n)} 2^{j_1(T_n)(1/(2-1/p))} \right)^p \right).
$$

Since the wavelet basis is compactly supported, we have $\sup_{x \in [0,1]} | \phi_{j,k}(x) | \leq \cdots$
Lemma 5.1 (Bernstein’s inequality) Let \((Y_i)_{i \in \mathbb{N}^*}\) be independent random variables such that, for any \(n \in \mathbb{N}^*\) and any \(i \in \{1, \ldots, n\}\), we have \(\mathbb{E}(Y_i) = 0\) and \(|Y_i| \leq M < \infty\). Then, for any \(\lambda > 0\), and any \(n \in \mathbb{N}^*\), we have

\[
\mathbb{P} \left( \left| \sum_{i=1}^{n} Y_i \right| \geq \lambda \right) \leq 2 \exp \left( -\frac{\lambda^2}{2(d_n^2 + M\lambda/3)} \right),
\]

where \(d_n^2 = \sum_{i=1}^{n} \mathbb{E}(Y_i^2)\).

Proof of the first point. We have

\[
\mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) = \mathbb{P} \left( \left| \sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right).
\]
Set $\mu_n = \mathbb{E}(X_{i,\eta_n}) - \mathbb{E}(X_{i,\eta_n+1})$. By definition of $\eta_n$, we have $\mathbb{E}(X_{i,\eta_n}) \neq \mathbb{E}(X_{i,\eta_n+1})$. By (H1), we have $\lim_{n \to \infty} n^{-1} \log T_n = 0$. Therefore, for a large enough $n$, we have $|\mu_n| \geq 2\kappa\sqrt{n\log(nT_n)}$. Using the triangular inequality, we obtain the inclusion $\{\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \leq \kappa\sqrt{n\log(nT_n)}\} \subseteq \{\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1}) \geq \kappa\sqrt{n\log(nT_n)}\}$. It follows that

$$
\mathbb{P}(\eta_n \in A_n(\kappa)) \geq \mathbb{P}\left(\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \leq \kappa\sqrt{n\log(nT_n)}\right) = 1 - \mathbb{P}\left(\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \geq \kappa\sqrt{n\log(nT_n)}\right).
$$

(5.6)

By (H2), we have, for any $i \in \{1, ..., n\}$, $X_{i,\eta_n}(\Omega) = X_{i,\eta_n+1}(\Omega) = [0, 1]$. Therefore, for any $i \in \{1, ..., n\}$, $|X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n| \leq |X_{i,\eta_n}| + |X_{i,\eta_n+1}| + \mathbb{E}(|X_{i,\eta_n}|) + \mathbb{E}(|X_{i,\eta_n+1}|) \leq 4$. Hence $\sum_{i=1}^{n} \mathbb{E}((X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)^2) \leq 16n$. The Bernstein inequality applied to the independent, uniformly bounded and centered random variables $(X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n)_{i \in \{1, ..., n\}}$ implies that

$$
\mathbb{P}\left(\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \geq \kappa\sqrt{n\log(nT_n)}\right) \leq 2\exp\left(-\frac{(\kappa\sqrt{n\log(nT_n)})^2}{2(16n + 4\kappa\sqrt{n\log(nT_n)})}\right)
$$

$$
= 2\exp\left(-\frac{\kappa^2\log(nT_n)}{2(16 + 8\kappa\sqrt{n^{-1}\log(nT_n)})}\right).
$$

Since, by (H1), for $n$ large enough, $n^{-1}\log(nT_n) \leq 1$, we have

$$
\mathbb{P}\left(\sum_{i=1}^{n} (X_{i,\eta_n} - X_{i,\eta_n+1} - \mu_n) \geq \kappa\sqrt{n\log(nT_n)}\right) \leq 2(nT_n)^{-h(\kappa)},
$$

(5.7)

where $h(\kappa) = \kappa^2/(32 + 3^{-1}8\kappa)$.

Putting (5.6) and (5.7) together, we obtain

$$
\mathbb{P}(\eta_n \in A_n(\kappa)) \geq 1 - 2(nT_n)^{-h(\kappa)}.
$$

This proved the first point of Proposition 4.1.
\[ \mathbb{P}(m \in \mathcal{A}_n(\kappa)) = \mathbb{P}\left( \left| \sum_{i=1}^{n} (X_{i,m} - X_{i,m+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right). \]  

(5.8)

By (H2), we have, for any \( i \in \{1, \ldots, n\} \) and any \( m \in \{1, \ldots, T_n\} \), \( X_{i,m}(\Omega) = X_{i,m+1}(\Omega) = [0, 1] \). Therefore, for any \( i \in \{1, \ldots, n\} \) and any \( m \in \{1, \ldots, T_n\} \) − \( \{\eta_n\} \), \( |X_{i,m} - X_{i,m+1}| \leq |X_{i,m}| + |X_{i,m+1}| \leq 2 \). Hence \( \sum_{i=1}^{n} \mathbb{E}((X_{i,m} - X_{i,m+1})^2) \leq 4n \). The Bernstein inequality applied to the independent, uniformly bounded and centered random variables \( (X_{i,m} - X_{i,m+1})_{i \in \{1, \ldots, n\}} \) implies that

\[
\mathbb{P}\left( \left| \sum_{i=1}^{n} (X_{i,m} - X_{i,m+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right) \leq 2 \exp\left( -\frac{\kappa^2 \log(nT_n)}{2 \left( 4 + \frac{2\kappa}{3} \sqrt{n^{-1} \log(nT_n)} \right)} \right).
\]

(5.9)

Since, by (H1), for \( n \) large enough, \( n^{-1} \log(nT_n) \leq 1 \), we have

\[
\mathbb{P}\left( \left| \sum_{i=1}^{n} (X_{i,m} - X_{i,m+1}) \right| \geq \kappa \sqrt{n \log(nT_n)} \right) \leq 2(nT_n)^{-t(\kappa)},
\]

(5.9)

where \( t(\kappa) = \kappa^2/(8 + 3^{-1}4\kappa) \).

It follows from (5.8) and (5.9) that, for any \( m \in \{1, \ldots, T_n\} \) − \( \{\eta_n\} \),

\[ \mathbb{P}(m \in \mathcal{A}_n(\kappa)) \leq 2(nT_n)^{-t(\kappa)}.
\]

• Proof of the third point. It follows from the Bonferroni inequality that

\[
\mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) = \mathbb{P}\left( \bigcap_{m=1}^{\eta_n-1} \{m \notin \mathcal{A}_n(\kappa)\} \cap \{\eta_n \in \mathcal{A}_n(\kappa)\} \right) \\
\geq \sum_{m=1}^{\eta_n-1} \mathbb{P}(m \notin \mathcal{A}_n(\kappa)) + \mathbb{P}(\eta_n \in \mathcal{A}_n(\kappa)) - (\eta_n - 1).
\]

The two first points give
\[
\mathbb{P}(\hat{\eta}_n(\kappa) = \eta_n) \\
\geq (\eta_n - 1)(1 - 2(nT_n)^{-t(\kappa)}) + 1 - 2(nT_n)^{-t(\kappa)} - (\eta_n - 1) \\
= 1 - 2(\eta_n - 1)(nT_n)^{-t(\kappa)} - 2(nT_n)^{-h(\kappa)} \geq 1 - 2\eta_n(nT_n)^{-h(\kappa)}.
\]

This completes the proof of Proposition 4.1.

\[\square\]

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**References**


