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To cite this version:

HAL Id: hal-00267367
https://hal.archives-ouvertes.fr/hal-00267367v3
Submitted on 11 Apr 2008

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GROTHENDIECK GROUP AND GENERALIZED MUTATION RULE FOR 2-CALABI–YAU TRIANGULATED CATEGORIES

YANN PALU

Abstract. We compute the Grothendieck group of certain 2-Calabi–Yau triangulated categories appearing naturally in the study of the link between quiver representations and Fomin–Zelevinsky’s cluster algebras. In this setup, we also prove a generalization of Fomin–Zelevinsky’s mutation rule.

Introduction

In their study of the connections between cluster algebras (see [22]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well–defined on the Grothendieck group of a cluster–tilted algebra associated with a finite–dimensional hereditary algebra. The conjecture was proved in [19] in the more general context of Hom–finite 2-Calabi–Yau triangulated categories. It was used in order to study the existence of a cluster character on such a category \( \mathcal{C} \), by using a formula proposed by Caldero–Keller.

In the present paper, we restrict to the case where \( \mathcal{C} \) is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster–tilting subcategories in \( \mathcal{C} \). When the cluster–tilting subcategories are related by a single mutation, this shows, via the method of [9], that their quivers are related by the Fomin–Zelevinsky mutation rule. This special case was already proved in [3], without assuming \( \mathcal{C} \) to be algebraic.

We also compute the Grothendieck group of the triangulated category \( \mathcal{C} \). In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category \( \mathcal{C}_A \) with the group \( K_0(\mathcal{C}_A) \). The latter group was defined in [1] by only considering the triangles in \( \mathcal{C}_A \) which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite dimensional hereditary algebra, with its triangulated structure defined by B. Keller in [16].

This paper is organized as follows: The first section is dedicated to notation and necessary background from [5], [8], [17], [19]. In section 2, we compute the Grothendieck group of the triangulated category \( \mathcal{C} \). In section 3, we prove a generalized mutation rule for quivers of cluster–tilting subcategories in \( \mathcal{C} \). In particular, this yields a new proof of the Fomin–Zelevinsky mutation rule, under the restriction that \( \mathcal{C} \) is algebraic. We finally show that \( K_0(\mathcal{C}_A) = K_0(\mathcal{C}_A) \) for any finite dimensional hereditary algebra \( A \).

Acknowledgements

This article is part of my PhD thesis, under the supervision of Professor B. Keller. I would like to thank him deeply for introducing me to the subject and for his infinite patience.
1. Notations and background

Let $E$ be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field $k$. By a result of Happel [10], its stable category $C = E$ is triangulated. We assume moreover, that $C$ is Hom-finite, 2-Calabi–Yau and has a cluster–tilting subcategory (see section 1.2), and we denote by $\Sigma$ its suspension functor. Note that we do not assume that $E$ is Hom-finite.

We write $X(\cdot, \cdot)$, or $\text{Hom}_X(\cdot, \cdot)$, for the morphisms in a category $X$ and $\text{Hom}_X(\cdot, \cdot)$ for the morphisms in the category of $X$-modules. We also denote by $X^*$ the projective $X$-module represented by $X$: $X^* = X(?, X)$.

1.1. Fomin–Zelevinsky mutation for matrices. Let $B = (b_{ij})_{i,j \in I}$ be a finite or infinite matrix, and let $k$ be in $I$. The Fomin and Zelevinsky mutation of $B$ (see [8]) in direction $k$ is the matrix $\mu_k(B) = (b'_{ij})$ defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + |b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that $\mu_k(\mu_k(B)) = B$ and that if $B$ is skew-symmetric, then so is $\mu_k(B)$.

We recall two lemmas of [9], stated for infinite matrices, which will be useful in section 3. Note that lemma 7.2 is a restatement of [2, (3.2)]. Let $S = (s_{ij})$ be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

**Lemma 7.1** ([9, Geiss–Leclerc–Schröer]): Assume that $B$ is skew-symmetric. Then, $S^2 = 1$ and the $(i,j)$-entry of the transpose of the matrix $S$ is given by

$$s^t_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix $S$ yields a convenient way to describe the mutation of $B$ in the direction $k$: 

1.2. Cluster–tilting subcategories

1.3. The antisymmetric bilinear form
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Lemma 7.2 ([9, Geiss–Leclerc–Schröer], [2, Berenstein–Fomin–Zelevinsky]): Assume that $B$ is skew-symmetric. Then we have:

$$\mu_k(B) = S^kBS.$$  

Note that the product is well-defined since the matrix $S$ has a finite number of non vanishing entries in each column.

1.2. Cluster–tilting subcategories. A cluster–tilting subcategory (see [17]) of $C$ is a full subcategory $T$ such that

a) $T$ is a linear subcategory;

b) for any object $X$ in $C$, the contravariant functor $C(?, X)|_T$ is finitely generated;

c) for any object $X$ in $C$, we have $C(X, \Sigma T) = 0$ for all $T$ in $T$ if and only if $X$ belongs to $T$.

We now recall some results from [17], which we will use in the sequel. Let $T$ be a cluster–tilting subcategory of $C$, and denote by $M$ its preimage in $E$. In particular $M$ contains the full subcategory $P$ of $E$ formed by the projective-injective objects, and we have $M = T$.

The following proposition will be used implicitly, extensively in this paper.

Proposition [17, Keller–Reiten]:

a) The category $\text{mod } M$ of finitely presented $M$-modules is abelian.

b) For each object $X \in C$, there is a triangle

$$\Sigma^{-1}X \rightarrow T_1^X \rightarrow T_0^X \rightarrow X$$

of $C$, with $T_0^X$ and $T_1^X$ in $T$.

Recall that the perfect derived category $\text{per } M$ is the full triangulated subcategory of the derived category of $\text{DMod } M$ generated by the finitely generated projective $M$-modules.

Proposition [17, Keller–Reiten]:

a) For each $X \in E$, there are conflations

$$0 \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow X \rightarrow M^0 \rightarrow M^1 \rightarrow 0$$

in $E$, with $M_0$, $M_1$, $M^0$ and $M^1$ in $M$.

b) Let $Z$ be in $\text{mod } M$. Then $Z$ considered as an $M$-module lies in the perfect derived category $\text{per } M$ and we have canonical isomorphisms

$$D(\text{per } M)(Z, ?) \cong (\text{per } M)(?, Z[3]).$$

1.3. The antisymmetric bilinear form. In section 3, we will use the existence of the antisymmetric bilinear form $\langle \cdot, \cdot \rangle_a$ on $K_0(\text{mod } M)$. We thus recall its definition from [3].

Let $\langle \cdot, \cdot \rangle$ be a truncated Euler form on $\text{mod } M$ defined by

$$\langle M, N \rangle = \dim \text{Hom}_M(M, N) - \dim \text{Ext}^1_M(M, N)$$

for any $M, N \in \text{mod } M$. Define $\langle \cdot, \cdot \rangle_a$ to be the antisymmetrization of this form:

$$\langle M, N \rangle_a = \langle M, N \rangle - \langle N, M \rangle.$$

This bilinear form descends to the Grothendieck group $K_0(\text{mod } M)$:

Lemma [3, section 3]: The antisymmetric bilinear form

$$\langle M, N \rangle_a : K_0(\text{mod } M) \times K_0(\text{mod } M) \rightarrow \mathbb{Z}$$

is well-defined.
2. **Grothendieck groups of algebraic 2-CY categories with a cluster-tilting subcategory**

We fix a cluster-tilting subcategory $T$ of $C$, and we denote by $M$ its preimage in $E$. In particular $M$ contains the full subcategory $P$ of $E$ formed by the projective-injective objects, and we have $M = T$.

We denote by $\mathcal{H}^b(E)$ and $D^b(E)$ respectively the bounded homotopy category and the bounded derived category of $E$. We also denote by $\mathcal{H}_{E-\text{ac}}^b(E)$, $\mathcal{H}^b(P)$, $\mathcal{H}^b(M)$ and $\mathcal{H}_{E-\text{ac}}^b(M)$ the full subcategories of $\mathcal{H}^b(E)$ whose objects are the $E$-acyclic complexes, the complexes of projective objects in $E$, the complexes of objects of $M$ and the $E$-acyclic complexes of objects of $M$, respectively.

2.1. **A short exact sequence of triangulated categories.**

**Lemma 1.** Let $A_1$ and $A_2$ be thick, full triangulated subcategories of a triangulated category $A$ and let $B$ be $A_1 \cap A_2$. Assume that for any object $X$ in $A$ there is a triangle $X_1 \to X \to X_2 \to \Sigma X_1$ in $A$, with $X_1$ in $A_1$ and $X_2$ in $A_2$. Then the induced functor $A_1/B \to A_2$ is a triangle equivalence.

**Proof.** Under these assumptions, denote by $F$ the induced triangle functor from $A_1/B$ to $A_2$. We are going to show that the functor $F$ is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful, $F$ will then be an equivalence of categories.

We first show that $F$ is full. Let $X_1$ and $X'_1$ be two objects in $A_1$. Let $f$ be a morphism from $X_1$ to $X'_1$ in $A/A_2$ and let

\[
\begin{array}{ccc}
X_1 & \overset{w}{\to} & Y \\
\downarrow & & \downarrow \\
X'_1 & \overset{w}{\to} & Y
\end{array}
\]

be a left fraction which represents $f$. The morphism $w$ is in the multiplicative system associated with $A_2$ and thus yields a triangle $\Sigma^{-1}A_2 \to Y \to X'_1 \to A_2$ where $A_2$ lies in the subcategory $A_2$. Moreover, by assumption, there exists a triangle $Y_1 \to Y \to Y_2 \to \Sigma Y_1$ with $Y_1$ in $A_1$. Applying the octahedral axiom to the composition $Y_1 \to Y \to X'_1$ yields a commutative diagram whose two middle rows and columns are triangles in $A$

\[
\begin{array}{cccc}
\Sigma^{-1}A_2 & \to & \Sigma^{-1}A_2 \\
\downarrow & & \downarrow \\
Y_1 & \to & Y_2 \to \Sigma Y_1 \\
\downarrow & & \downarrow \\
Y_1 & \to & X'_1 \to Z \to \Sigma Y_1 \\
\downarrow & & \downarrow \\
A_2 & = & A_2
\end{array}
\]

Since $Y_2$ and $A_2$ belong to $A_2$, so does $Z$. And since $X'_1$ and $Y_1$ belong to $A_1$, so does $Z$. This implies, that $Z$ belongs to $B$. The morphism $Y_1 \to X'_1$ is in the multiplicative system of $A_1$ associated with $B$ and the diagram

\[
\begin{array}{ccc}
X_1 & \overset{f}{\to} & X'_1 \\
\downarrow & & \downarrow \\
Y_1 & \to & Y
\end{array}
\]
is a left fraction which represents $f$. This implies that $f$ is the image of a morphism in $A_1/B$. Therefore the functor $F$ is full.

We now show that $F$ is conservative. Let $X_1 \xrightarrow{f} Y_1 \rightarrow Z_1 \rightarrow \Sigma X_1$ be a triangle in $A_1$. Assume that $Ff$ is an isomorphism in $A/A_2$, which implies that $Z_1$ is an object of $A_2$. Therefore, $Z_1$ is an object of $B$ and $f$ is an isomorphism in $A_1/B$.

We finally show that $F$ is dense. Let $X$ be an object of the category $A/A_2$, and let $X_1 \rightarrow X \rightarrow X_2 \rightarrow \Sigma X_1$ be a triangle in $A$ with $X_i$ in $A_i$. Since $X_2$ belongs to $A_2$, the image of the morphism $X_1 \rightarrow X$ in $A/A_2$ is an isomorphism. Thus $X$ is isomorphic to the image by $F$ of an object in $A_1/B$. □

As a corollary, we have the following:

**Lemma 2.** The following sequence of triangulated categories is short exact:

$$0 \rightarrow \mathcal{H}^b_{\pi} (\mathcal{M}) \rightarrow \mathcal{H}^b (\mathcal{M}) \rightarrow \mathcal{D}^b (\mathcal{E}) \rightarrow 0.$$

Remark: This lemma remains true if $\mathcal{C}$ is $d$-Calabi–Yau and $\mathcal{M}$ is $(d-1)$-cluster-tilting, using section 5.4 of [17].

**Proof.** For any object $X$ in $\mathcal{H}^b (\mathcal{E})$, the existence of an object $M$ in $\mathcal{H}^b (\mathcal{M})$ and of a quasi-isomorphism $w$ from $M$ to $X$ is obtained using the approximation conflations of Keller–Reiten (see section 1.2). Since the cone of the morphism $w$ belongs to $\mathcal{H}^b_{\pi} (\mathcal{E})$, lemma 1 applies to the subcategories $\mathcal{H}^b_{\pi} (\mathcal{M})$, $\mathcal{H}^b (\mathcal{M})$ and $\mathcal{H}^b_{\pi} (\mathcal{E})$ of $\mathcal{H}^b (\mathcal{E})$.

**Proposition 3.** The following diagram is commutative with exact rows and columns:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{H}^b_{\pi} (\mathcal{M}) & \xrightarrow{i_{\mathcal{M}}} & \mathcal{H}^b (\mathcal{M}) / \mathcal{H}^b (\mathcal{P}) & \rightarrow & \mathcal{E} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{H}^b_{\pi} (\mathcal{M}) & \rightarrow & \mathcal{H}^b (\mathcal{M}) & \rightarrow & \mathcal{D}^b (\mathcal{E}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}^b (\mathcal{P}) & \rightarrow & \mathcal{H}^b (\mathcal{P}) & \rightarrow & 0 & & & & \\
\end{array}
\]

\[(D)\]

**Proof.** The column on the right side has been shown to be exact in [18] and [20]. The second row is exact by lemma 1. The subcategories $\mathcal{H}^b_{\pi} (\mathcal{M})$ and $\mathcal{H}^b (\mathcal{P})$ of $\mathcal{H}^b (\mathcal{M})$ are left and right orthogonal to each other. This implies that the induced functors $i_{\mathcal{M}}$ and $i_{\mathcal{P}}$ are fully faithful and that taking the quotient of $\mathcal{H}^b (\mathcal{M})$ by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact. □

2.2. **Invariance under mutation.** A natural question is then to which extent the diagram $(D)$ depends on the choice of a particular cluster-tilting subcategory. Let thus $T'$ be another cluster-tilting subcategory of $\mathcal{C}$, and let $M'$ be its preimage in $\mathcal{E}$. Let $\text{mod} \mathcal{M}$ (resp. $\text{mod} \mathcal{M}'$) be the category of $\mathcal{M}$-modules (resp. $\mathcal{M}'$-modules), i.e. of $k$-linear contravariant functors from $\mathcal{M}$ (resp. $\mathcal{M}'$) to the category of $k$-vector spaces.

Let $X$ be the $\mathcal{M}$-$\mathcal{M}'$-bimodule which sends the pair of objects $(M, M')$ to the $k$-vector space $\mathcal{E}(M, M')$. The bimodule $X$ induces a functor $F = \otimes_{\mathcal{M}'} X : \text{mod} \mathcal{M} \rightarrow \text{mod} \mathcal{M}$ denoted by $T_X$ in [15] section 6.1.
Recall that the perfect derived category per \( \mathcal{M} \) is the full triangulated subcategory of the derived category \( \mathcal{D} \text{Mod} \mathcal{M} \) generated by the finitely generated projective \( \mathcal{M} \)-modules.

**Proposition 4.** The left derived functor

\[
\mathbb{L}F : \mathcal{D} \text{Mod} \mathcal{M}' \longrightarrow \mathcal{D} \text{Mod} \mathcal{M}
\]

is an equivalence of categories.

**Proof.** Recall that if \( X \) is an object in a category \( \mathcal{X} \), we denote by \( X \) the functor \( \mathcal{X}(?, X) \) represented by \( X \). By [13, 6.1], it is enough to check the following three properties:

1. For all objects \( M', M'' \) of \( \mathcal{M} \), the group \( \text{Hom}_{\mathcal{D} \text{Mod} \mathcal{M}}(\mathbb{L}FM'', \mathbb{L}FM''[n]) \) vanishes for \( n \neq 0 \) and identifies with \( \text{Hom}_{\mathcal{M}'}(M', M'') \) for \( n = 0 \);
2. for any object \( M' \) of \( \mathcal{M}' \), the complex \( \mathbb{L}FM'' \) belongs to \( \mathbb{P} \mathcal{M} \):
3. the set \( \{\mathbb{L}FM'', M' \in \mathcal{M}'\} \) generates \( \mathcal{D} \text{Mod} \mathcal{M} \) as a triangulated category with infinite sums.

Let \( M' \) be an object of \( \mathcal{M}' \), and let \( M_1 \overset{i}{\longrightarrow} M_0 \longrightarrow M' \) be a conflation in \( \mathcal{E} \), with \( M_0 \) and \( M_1 \) in \( \mathcal{M} \), and whose deflation is a right \( \mathcal{M} \)-approximation (c.f. section 4 of [17]). The surjectivity of the map \( M_0 \longrightarrow \mathcal{E}(?, M')|_{\mathcal{M}} \) implies that the complex \( \hat{P} = (\cdots \rightarrow M_1^- \rightarrow M_0^- \rightarrow 0 \rightarrow \cdots) \) is quasi-isomorphic to \( \mathbb{L}FM'' = \mathcal{E}(?, M'')|_{\mathcal{M}} \). Therefore \( \mathbb{L}FM'' \) belongs to the subcategory per \( \mathcal{M} \) of \( \mathcal{D} \text{Mod} \mathcal{M} \). Moreover, we have, for any \( n \in \mathbb{Z} \) and any \( M'' \in \mathcal{M}' \), the equality

\[
\text{Hom}_{\mathcal{D} \text{Mod} \mathcal{M}}(\mathbb{L}FM'', \mathbb{L}FM''[n]) = \text{Hom}_{\mathcal{D} \text{Mod} \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}[n])
\]

where the right-hand side vanishes for \( n \neq 0, 1 \). In case \( n = 1 \) it also vanishes, since \( \text{Ext}^1_{\mathcal{M}}(M', M'') \) vanishes. Now,

\[
\text{Hom}_{\mathcal{D} \text{Mod} \mathcal{M}}(P, \mathcal{E}(?, M'')|_{\mathcal{M}}) \simeq \text{Ker}(\mathcal{E}(M_0, M'') \rightarrow \mathcal{E}(M_1, M'')) \\
\simeq \mathcal{E}(M', M'').
\]

It only remains to be shown that the set \( R = \{\mathbb{L}FM'', M' \in \mathcal{M}'\} \) generates \( \mathcal{D} \text{Mod} \mathcal{M} \). Denote by \( \mathcal{R} \) the full triangulated subcategory with infinite sums of \( \mathcal{D} \text{Mod} \mathcal{M} \) generated by the set \( R \). The set \( \{M, M \in \mathcal{M}\} \) generates \( \mathcal{D} \text{Mod} \mathcal{M} \) as a triangulated category with infinite sums. Thus it is enough to show that, for any object \( M \) of \( \mathcal{M} \), the complex \( \mathcal{M}' \) concentrated in degree 0 belongs to the subcategory \( \mathcal{R} \). Let \( M \) be an object of \( \mathcal{M} \), and let \( M \overset{i}{\longrightarrow} M_0 \longrightarrow M_1 \) be a conflation of \( \mathcal{E} \) with \( M_0 \) and \( M_1 \) in \( \mathcal{M}' \). Since \( \text{Ext}^1_{\mathcal{M}}(?, M)|_{\mathcal{M}} \) vanishes, we have a short exact sequence of \( \mathcal{M} \)-modules

\[
0 \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M_0)|_{\mathcal{M}} \longrightarrow \mathcal{E}(?, M_1)|_{\mathcal{M}} \longrightarrow 0,
\]

which yields the triangle

\[
M' \longrightarrow \mathbb{L}FM_0'' \longrightarrow \mathbb{L}FM_1'' \longrightarrow \Sigma M'.
\]

As a corollary of proposition [1], up to equivalence the diagram (D) does not depend on the choice of a cluster–tilting subcategory. To be more precise: Let \( G \) be the functor which sends an object \( X \) in the category \( \mathcal{H}^b(\mathcal{M}') \) to a representative of \( (\mathbb{L}F)X \) in \( \mathcal{H}^b(\mathcal{M}) \), and a morphism in \( \mathcal{H}^b(\mathcal{M}') \) to the induced one in \( \mathcal{H}^b(\mathcal{M}) \).
Corollary 5. The following diagram is commutative

and the functor $G$ is an equivalence of categories.

We denote by $\text{per}_M$ the full subcategory of $\text{per} \mathcal{M}$ whose objects are the complexes with homologies in $\text{mod} \mathcal{M}$. The following lemma will allow us to compute the Grothendieck group of $\text{per}_M$ in section 2.3:

Lemma 6. The canonical $t$-structure on $D \text{Mod} \mathcal{M}$ restricts to a $t$-structure on $\text{per}_M$, whose heart is $\text{mod} \mathcal{M}$.

Proof. By [13], it is enough to show that for any object $M^*$ of $\text{per}_M$, its truncation $\tau_{\leq 0} M^*$ in $D \text{Mod} \mathcal{M}$ belongs to $\text{per}_M$. Since $M^*$ is in $\text{per}_M$, $\tau_{\leq 0} M^*$ is bounded, and is thus formed from the complexes $H^i(M^*)$ concentrated in one degree by taking iterated extensions. But, for any $i$, the $\mathcal{M}$-module $H^i(M^*)$ actually is an $\mathcal{M}$-module. Therefore, by [17] (see section 1.2), it is perfect as an $\mathcal{M}$-module and it lies in $\text{per}_M$. □

The next lemma already appears in [21]. For the convenience of the reader, we include a proof.

Lemma 7. The Yoneda equivalence of triangulated categories $H^b(M) \rightarrow \text{per} \mathcal{M}$ induces a triangle equivalence $H^b_{\mathcal{E}}(\mathcal{M}) \rightarrow \text{per}_M$.

Proof. We first show that the cohomology groups of an $\mathcal{E}$-acyclic bounded complex $M$ vanish on $\mathcal{P}$. Let $P$ be a projective object in $\mathcal{E}$ and let $E$ be a kernel in $\mathcal{E}$ of the map $M^n \rightarrow M^{n+1}$. Since $M$ is $\mathcal{E}$-acyclic, such an object exists, and moreover, it is an image of the map $M^{n-1} \rightarrow M^n$. Any map from $P$ to $M^n$ whose composition with $M^n \rightarrow M^{n+1}$ vanishes factors through the kernel $E \rightarrow M^n$. Since $P$ is projective, this factorization factors through the deflation $M^{n-1} \rightarrow E$.

Therefore, we have $H^n(M)(P) = 0$ for all projective objects $P$, and $H^n(M)$ belongs to $\text{mod} \mathcal{M}$. Thus the Yoneda functor induces a fully faithful functor from $H^b_{\mathcal{E}-\text{ac}}(\mathcal{M})$ to $\text{per}_M$. To prove that it is dense, it is enough to prove that any object of the heart $\text{mod} \mathcal{M}$ of the $t$-structure on $\text{per}_M$ is in its essential image.

But this was proved in [17] section 4] (see section 1.2). □
Proposition 8. There is a triangle equivalence of categories
\[ \text{per}_M M \cong \text{per}_{M'} M' \]

Proof. Since the categories \( \mathcal{H}^b(P) \) and \( \mathcal{H}_{-ac}(M) \) are left-right orthogonal in \( \mathcal{H}^b(M) \), this is immediate from corollary \( 3 \) and lemma \( 4 \). \( \square \)

2.3. Grothendieck groups. For a triangulated (resp. additive, resp. abelian) category \( \mathcal{A} \), we denote by \( K_{\text{tri}}(\mathcal{A}) \) or simply \( K_0(\mathcal{A}) \) (resp. \( K_{\text{add}}(\mathcal{A}) \), resp. \( K_{\text{ab}}(\mathcal{A}) \)) its Grothendieck group (with respect to the mentioned structure of the category). For an object \( A \) in \( \mathcal{A} \), we also denote by \([A]\) its class in the Grothendieck group of \( \mathcal{A} \).

The short exact sequence of triangulated categories
\[ 0 \rightarrow \mathcal{H}_{-ac}^b(M) \rightarrow \mathcal{H}^b(M) / \mathcal{H}^b(P) \rightarrow \mathcal{L} \rightarrow 0 \]
given by proposition \( 3 \) induces an exact sequence in the Grothendieck groups
\[ (\ast) \quad K_0(\mathcal{H}_{-ac}^b(M)) \rightarrow K_0(\mathcal{H}^b(M) / \mathcal{H}^b(P)) \rightarrow K_0(\mathcal{L}) \rightarrow 0. \]

Lemma 9. The exact sequence (\( \ast \)) is isomorphic to an exact sequence
\[ (\ast\ast) \quad K_0^b(\text{ mod } M) \xrightarrow{\varphi} K_0^{\text{add }}(M) \rightarrow K_0^{\text{tri }}(\mathcal{L}) \rightarrow 0. \]

Proof. First, note that, by \( 2 \), see also lemma \( 3 \), we have an isomorphism between the Grothendieck groups \( K_0(\mathcal{H}_{-ac}^b(M)) \) and \( K_0(\text{per}_M M) \). The t-structure on \( \text{per}_M M \) whose heart is \( \text{mod } M \), see lemma \( 1 \), in turn yields an isomorphism between the Grothendieck groups \( K_0(\text{per}_M M) \) and \( K_0^{\text{add }}(M) \). Next, we show that the canonical additive functor \( M \xrightarrow{\alpha} \mathcal{H}^b(M) / \mathcal{H}^b(P) \) induces an isomorphism between the Grothendieck groups \( K_0^{\text{add }}(M) \) and \( K_0^{\text{tri }}(\mathcal{H}^b(M) / \mathcal{H}^b(P)) \). For this, let us consider the canonical additive functor \( M \xrightarrow{\beta} \mathcal{H}^b(M) \) and the triangle functor \( \mathcal{H}^b(M) \xrightarrow{\gamma} \mathcal{H}^b(P) \).

The following diagram describes the situation:
\[ \begin{array}{ccc}
\mathcal{H}^b(M) & \xrightarrow{\gamma} & \mathcal{H}^b(P) \\
\mathcal{M} & \xrightarrow{\alpha} & \mathcal{H}^b(M) / \mathcal{H}^b(P) \\
\end{array} \]

The functor \( \gamma \) vanishes on the full subcategory \( \mathcal{H}^b(P) \), thus inducing a triangle functor, still denoted by \( \gamma \), from \( \mathcal{H}^b(M) / \mathcal{H}^b(P) \) to \( \mathcal{H}^b(M) \). Furthermore, the functor \( \beta \) induces an isomorphism at the level of Grothendieck groups, whose inverse \( K_0(\beta)^{-1} \) is given by
\[ K_0^{\text{tri }}(\mathcal{H}^b(M)) \rightarrow K_0^{\text{add }}(M) \]
\[ [M] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [M^i]. \]

As the group \( K_0^{\text{tri }}(\mathcal{H}^b(M) / \mathcal{H}^b(P)) \) is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms \( K_0(\alpha) \) and \( K_0(\beta)^{-1} K_0(\gamma) \) are inverse to each other. \( \square \)

As a consequence of the exact sequence (\( \ast\ast \)), we have an isomorphism between \( K_0^{\text{tri }}(\mathcal{L}) \) and \( K_0^{\text{add }}(M) / \text{Im } \varphi \). In order to compute \( K_0^{\text{tri }}(\mathcal{L}) \), the map \( \varphi \) has to be made explicit. We first recall some results from Iyama–Yoshino \( 4 \) which generalize results from \( 1 \): For any indecomposable \( M \) of \( \mathcal{M} \) not in \( P \), there exists \( M^* \) unique up to isomorphism such that \( (M, M^*) \) is an exchange pair. This means that \( M \) and \( M^* \) are not isomorphic and that the full additive subcategory of \( \mathcal{C} \) generated
by all the indecomposable objects of \( \mathcal{M} \) but those isomorphic to \( M \), and by the indecomposable objects isomorphic to \( M^* \) is again a cluster–tilting subcategory. Moreover, \( \dim \mathcal{E}(M, \Sigma M^*) = 1 \). We can thus fix two (non-split) exchange triangles \( M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^* \) and \( M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M \).

We may now state the following:

**Theorem 10.** The Grothendieck group of the triangulated category \( \mathcal{E} \) is the quotient of that of the additive subcategory \( \mathcal{M} \) by all relations \([B_{M^*}] - [B_M] \):

\[
K_0^{\text{tri}}(\mathcal{E}) \simeq K_0^{\text{add}}(\mathcal{M})/([B_{M^*}] - [B_M])_M.
\]

**Proof.** We denote by \( S_M \) the simple \( \mathcal{M} \)-module associated to the indecomposable object \( M \). This means that \( S_M(M') \) vanishes for all indecomposable objects \( M' \) in \( \mathcal{M} \) not isomorphic to \( M \) and that \( S_M(M) \) is isomorphic to \( k \). The abelian group \( K_0^{\text{ab}}(\mathcal{M}) \) is generated by all classes \([S_M]\). In view of lemma [3], it is sufficient to prove that the image of the class \([S_M]\) under \( \varphi \) is \([B_M]\). First note that the \( \mathcal{M} \)-module \( \text{Ext}_1^\mathcal{E}(\cdot, M^*)|_\mathcal{M} \) vanishes on the projectives; \( \mathcal{M} \) can thus be viewed as a \( \mathcal{M} \)-module, and as such, is isomorphic to \( S_M \). After replacing \( B_M \) and \( B_{M^*} \) by isomorphic objects of \( \mathcal{E} \), we can assume that the exchange triangles \( M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^* \) and \( M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M \) come from conflations \( M^* \rightarrow B_M \rightarrow M \rightarrow \Sigma M^* \) and \( M \rightarrow B_{M^*} \rightarrow M^* \rightarrow \Sigma M \). The spliced complex

\[
\cdots \rightarrow M \rightarrow B_M \rightarrow B_M \rightarrow M \rightarrow 0 \rightarrow \cdots
\]

denoted by \( C^* \), is then an \( E \)-acyclic complex, and it is the image of \( S_M \) under the functor \( \text{Ext}^1_{\mathcal{E}}(\cdot, B_M)|_\mathcal{M} \). Indeed, we have two long exact sequences induced by the conflations above:

\[
0 \rightarrow \mathcal{M}(?, M) \rightarrow \mathcal{M}(?, B_{M^*}) \rightarrow \mathcal{E}(?, M^*)|_\mathcal{M} \rightarrow \text{Ext}_1^\mathcal{E}(?, M)|_\mathcal{M} = 0 \text{ and }
\]

\[
0 \rightarrow \mathcal{E}(?, M^*)|_\mathcal{M} \rightarrow \mathcal{M}(?, B_M) \rightarrow \mathcal{M}(?, M) \rightarrow \text{Ext}_1^\mathcal{E}(?, M^*)|_\mathcal{M} \rightarrow \text{Ext}_1^\mathcal{E}(?, B_M)|_\mathcal{M}.
\]

Since \( B_M \) belongs to \( \mathcal{M} \), the functor \( \text{Ext}_1^\mathcal{E}(\cdot, B_M) \) vanishes on \( \mathcal{M} \), and the complex:

\[
(C) : \quad \cdots \rightarrow 0 \rightarrow M \rightarrow (B_M) \rightarrow (B_M) \rightarrow M \rightarrow 0 \rightarrow \cdots
\]

is quasi-isomorphic to \( S_M \).

Now, in the notations of the proof of lemma [3], \( \varphi[S_M] \) is the image of the class of the \( E \)-acyclic complex \( C^* \) under the morphism \( K_0(\beta^{-1})K_0(\gamma) \). This is \([M] - [B_M] + [B_{M^*}] - [M]\) which equals \([B_{M^*}] - [B_M]\) as claimed. \( \square \)

### 3. The generalized mutation rule

Let \( T \) and \( T' \) be two cluster–tilting subcategories of \( \mathcal{C} \). Let \( Q \) and \( Q' \) be the quivers obtained from their Auslander–Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating \( Q' \) to \( Q \), and to prove that it generalizes the Fomin–Zelevinsky mutation rule.

**Remark:**

1. Assume that \( \mathcal{C} \) has cluster–tilting objects. Then it is proved in [5, Theorem 1.1.6], without assuming that \( \mathcal{C} \) is algebraic, that the Auslander–Reiten quivers of two cluster–tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin–Zelevinsky mutation rule.

2. To prove that the generalized mutation rule actually generalizes the Fomin–Zelevinsky mutation rule, we use the ideas of section 7 of [5].
3.1. The rule. As in section 3, we fix a cluster–tilting subcategory \( T \) of \( \mathcal{C} \), and write \( \mathcal{M} \) for its preimage in \( \mathcal{E} \), so that \( T = \mathcal{M} \). Define \( Q \) to be the quiver obtained from the Auslander–Reiten quiver of \( \mathcal{M} \) by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object \( L \) will also be labeled by \( L \). We denote by \( a_{LN} \) the number of arrows from vertex \( L \) to vertex \( N \) in the quiver \( Q \). Let \( B_M \) be the matrix whose entries are given by \( b_{LN} = a_{LN} - a_{NL} \).

Let \( R_{AM} \) be the matrix of \( (\ , \) \) : \( K_0(\mod \mathcal{M}) \times K_0(\mod \mathcal{M}) \to \mathbb{Z} \) in the basis given by the classes of the simple modules.

**Lemma 11.** The matrices \( R_{AM} \) and \( B_M \) are equal: \( R_{AM} = B_M \).

**Proof.** Let \( L \) and \( N \) be two non-projective indecomposable objects in \( \mathcal{M} \). Then \( \dim \text{Hom}(S_L, S_N) - \dim \text{Hom}(S_N, S_L) = 0 \) and we have:

\[
([S_L], [S_N])_a = \dim \text{Ext}^1(S_N, S_L) - \dim \text{Ext}^1(S_L, S_N) = b_{LN}.
\]

\[\square\]

Let \( T' \) be another cluster–tilting subcategory of \( \mathcal{C} \), and let \( \mathcal{M}' \) be its preimage in the Frobenius category \( \mathcal{E} \). Let \( (M'_i)_{i \in I} \) (resp. \( (M_j)_{j \in J} \)) be representatives for the isoclasses of non-projective indecomposable objects in \( \mathcal{M}' \) (resp. \( \mathcal{M} \)). The equivalence of categories \( \text{per} \mathcal{M} \to \text{per} \mathcal{M}' \) of proposition 3 induces an isomorphism between the Grothendieck groups \( K_0(\mod \mathcal{M}) \) and \( K_0(\mod \mathcal{M}') \) whose matrix, in the bases given by the classes of the simple modules, is denoted by \( S \). The equivalence of categories \( \mathcal{D} \text{Mod} \mathcal{M} \to \mathcal{D} \text{Mod} \mathcal{M}' \) restricts to the identity on \( \mathcal{H}^b(\mathcal{P}) \), so that it induces an equivalence \( \text{per} \mathcal{M} / \text{per} \mathcal{P} \to \text{per} \mathcal{M}' / \text{per} \mathcal{P} \). Let \( T \) be the matrix of the induced isomorphism from \( K_0(\text{proj} \mathcal{M}) / K_0(\text{proj} \mathcal{P}) \) to \( K_0(\text{proj} \mathcal{M}') / K_0(\text{proj} \mathcal{P}) \), in the bases given by the classes \([M_i(M_j)],[j \in J, i \in I] \). The matrix \( T \) is much easier to compute than the matrix \( S \). Its entries \( t_{ij} \) are given by the approximation triangles of Keller and Reiten in the following way: For all \( j \), there exists a triangle of the form

\[
\Sigma^{-1}M_j \to \bigoplus_i \alpha_{ij}M'_i \to \bigoplus_i \beta_{ij}M'_i \to M_j.
\]

Then, we have:

**Theorem 12.**

a) (Generalized mutation rule) The following equalities hold:

\[
t_{ij} = \alpha_{ij} - \beta_{ij}
\]

and

\[
B_{AM'} = TB_MT^t.
\]

b) The category \( \mathcal{C} \) has a cluster–tilting object if and only if all its cluster–tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.

c) All cluster–tilting objects of \( \mathcal{C} \) have the same number of indecomposable direct summands (up to isomorphism).

Note that point c) was shown in [1, 5.3.3(1)] (see also [3, 1.1.8]) and, in a more general context, in [2]. Note also that, for the generalized mutation rule to hold, the cluster–tilting subcategories do not need to be related by a sequence of mutation.

**Proof.** Assertions b) and c) are consequences of the existence of an isomorphism between the Grothendieck groups \( K_0(\mod \mathcal{M}) \) and \( K_0(\mod \mathcal{M}') \). Let us prove the equalities a). Recall from [3, section 3.3], that the antisymmetric bilinear form
\((\cdot,\cdot)_a\) on \(\text{mod} M\) is induced by the usual Euler form \((\cdot,\cdot)_E\) on \(\text{mod} M\). The following commutative diagram
\[
\begin{array}{ccc}
\text{per} M \times \text{per} M & \xrightarrow{\sim} & \text{per} M' \times \text{per} M' \\
(\cdot,\cdot)_E & \downarrow & (\cdot,\cdot)_E \\
\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}
\end{array}
\]
thus induces a commutative diagram
\[
\begin{array}{ccc}
\text{K}_0(\text{mod} M) \times \text{K}_0(\text{mod} M) & \xrightarrow{S \times S} & \text{K}_0(\text{mod} M') \times \text{K}_0(\text{mod} M') \\
(\cdot,\cdot)_a & \downarrow & (\cdot,\cdot)_a \\
\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}
\end{array}
\]
This proves the equality \(R_M = S^t R_M S\), or, by lemma \([1]\),
\[
(1) \quad B_M = S^t B_M' S.
\]
Any object of \(\text{per} M\) becomes an object of \(\text{per} M/\text{per} P\) through the composition \(\text{per} M \hookrightarrow \text{per} M \twoheadrightarrow \text{per} M/\text{per} P\). Let \(M\) and \(N\) be two non-projective indecomposable objects in \(M\). Since \(S_N\) vanishes on \(P\), we have
\[
\text{Hom}_{\text{per} M/\text{per} P} (M(?,M), S_N) = \text{Hom}_{\text{per} M} (M(?,M), S_N) = \text{Hom}_{\text{Mod} M} (M(?,M), S_N) = S_N(M).
\]
Thus \(\dim \text{Hom}_{\text{per} M/\text{per} P} (M(?,M), S_N) = \delta_{MN}\), and the commutative diagram
\[
\begin{array}{ccc}
\text{per} M/\text{per} P \times \text{per} M/\text{per} P & \xrightarrow{\sim} & \text{per} M'/\text{per} P \times \text{per} M'/\text{per} P \\
R \text{Hom} & \downarrow & R \text{Hom} \\
\text{per} k & \xrightarrow{\sim} & \text{per} k
\end{array}
\]
induces a commutative diagram
\[
\begin{array}{ccc}
\text{K}_0(\text{proj} M)/\text{K}_0(\text{proj} P) \times \text{K}_0(\text{mod} M) & \xrightarrow{T \times S} & \text{K}_0(\text{proj} M')/\text{K}_0(\text{proj} P) \times \text{K}_0(\text{mod} M') \\
\text{Id} & \downarrow & \text{Id} \\
\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}
\end{array}
\]
In other words, the matrix \(S\) is the inverse of the transpose of \(T\):
\[
(2) \quad S = T^{-t}
\]
Equalities (1) and (2) imply what was claimed, that is
\[
B_{M'} = T B_M T^t.
\]
Let us compute the matrix \(T\): Let \(M\) be indecomposable non-projective in \(M\), and let
\[
\Sigma^{-1} M \longrightarrow M_1' \longrightarrow M_0' \longrightarrow M
\]
be a Keller–Reiten approximation triangle of \(M\) with respect to \(M'\), which we may assume to come from a conflation in \(E\). This conflation yields a projective resolution
\[
0 \longrightarrow (M_1')^- \longrightarrow (M_0')^- \longrightarrow E(?,M)|_{M'} \longrightarrow \text{Ext}_{E}^1(?,M)|_{M'} = 0.
\]
so that \(T\) sends the class of \(M'\) to \([[(M_0')^-]] - [[(M_1')^-]]\). Therefore, \(t_{ij}\) equals \(\alpha_{ij} - \beta_{ij}\). \(\square\)
3.2. Examples.

3.2.1. As a first example, let $C$ be the cluster category associated with the quiver of type $A_4$: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. Its Auslander–Reiten quiver is the Moebius strip:

Let $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where the indecomposable $M_i$ corresponds to the vertex labelled by $i$ in the picture. Let also $M' = M'_1 \oplus M'_2 \oplus M'_3 \oplus M'_4$, where $M'_1 = M_1$, and where the indecomposable $M'_i$ corresponds to the vertex labelled by $i'$ if $i \neq 1$. One easily computes the following Keller–Reiten approximation triangles:

$\Sigma^{-1}M_1 \rightarrow 0 \rightarrow M'_1 \rightarrow M_1$,
$\Sigma^{-1}M_2 \rightarrow M'_2 \rightarrow M'_1 \rightarrow M_2$,
$\Sigma^{-1}M_3 \rightarrow M'_3 \rightarrow 0 \rightarrow M_4$ and
$\Sigma^{-1}M_4 \rightarrow M'_4 \rightarrow M'_3 \rightarrow M_4$;

so that the matrix $T$ is given by:

$$
T = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1
\end{pmatrix}.
$$

We also have

$$
B_{M'} = \begin{pmatrix}
0 & -1 & 1 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$

Let maple compute

$$
T^{-t}B_{M'}T^{-t} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 1 \\
0 & 1 & 0 & -1 \\
0 & -1 & 1 & 0
\end{pmatrix},
$$

which is $B_M$.

3.2.2. Let us look at a more interesting example, where one cannot easily read the quiver of $M'$ from the Auslander–Reiten quiver of $C$. Let $C$ be the cluster category associated with the quiver $Q$:

For $i = 0, 1, 2$, let $M_i$ be (the image in $C$ of) the projective indecomposable (right) $kQ$-module associated with vertex $i$. Their dimension vectors are respectively $[1, 0, 0], [2, 1, 0]$ and $[2, 0, 1]$. Let $M$ be the direct sum $M_0 \oplus M_1 \oplus M_2$. Let $M'$ be the direct sum $M'_0 \oplus M'_1 \oplus M'_2$, where $M'_0, M'_1$ and $M'_2$ are (the images in $C$ of) the indecomposable regular $kQ$-modules with dimension vectors $[1, 2, 0], [0, 1, 0]$.
and \([2, 4, 1]\) respectively. As one can check, using \([1]\), \(M\) and \(M'\) are two cluster-tilting objects of \(\mathcal{C}\). To compute Keller–Reiten’s approximation triangles, amounts to computing projective resolutions in \(\text{mod } kQ\), viewed as \(\text{mod } \text{End}_\mathcal{C}(M)\). One easily computes these projective resolutions, by considering dimension vectors:
\[
0 \to 8M_0 \to 2M_1 \oplus 4M_2 \to M'_2 \to 0,
\]
\[
0 \to 2M_0 \to M_1 \to M'_1 \to 0\quad \text{and}
\]
\[
0 \to 3M_0 \to 2M_1 \to M'_0 \to 0.
\]
By applying the generalized mutation rule, one gets the following quiver
\[
\begin{array}{c}
(6) \\
\downarrow \downarrow \downarrow \\
(4) \\
\downarrow \\
(2) \\
\downarrow \\
2,
\end{array}
\]
which is therefore the quiver of \(\text{End}_\mathcal{C}(M')\) since by \([5]\), there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster-tilting object in a cluster category.

3.3. Back to the mutation rule. We assume in this section that the Auslander–Reiten quiver of \(\mathcal{T}\) has no loops nor 2-cycles. Under the notations of section 3.1, let \(k\) be in \(I\) and let \((M_k, M'_k)\) be an exchange pair (see section 2.3). We choose \(M'\) to be the cluster-tilting subcategory of \(\mathcal{C}\) obtained from \(M\) by replacing \(M_k\) by \(M'_k\), so that \(M'_i = M_i\) for all \(i \neq k\). Recall that \(T\) is the matrix of the isomorphism \(K_0(\text{proj } M)/K_0(\text{proj } P) \to K_0(\text{proj } M')/K_0(\text{proj } P)\).

Lemma 13. Then, the \((i, j)\)-entry of the matrix \(T\) is given by
\[
t_{ij} = \begin{cases} 
-\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k \\
\delta_{ij} & \text{else.}
\end{cases}
\]

Proof. Let us apply theorem \([12]\) to compute the matrix \(T\). For all \(j \neq k\), the triangle \(\Sigma^{-1}M_j \to 0 \to M'_j = M_j\) is a Keller–Reiten approximation triangle of \(M_j\) with respect to \(\mathcal{M}'\). We thus have \(t_{ij} = \delta_{ij}\) for all \(j \neq k\). There is a triangle unique up to isomorphism
\[
M'_k \to B_{M_k} \to M_k \to \Sigma M'_k
\]
where \(B_{M_k} \to M_k\) is a right \(T \cap T'\)-approximation. Since the Auslander–Reiten quiver of \(\mathcal{T}\) has no loops and no 2-cycles, \(B_{M_k}\) is isomorphic to the direct sum: \(\bigoplus_{i \in I} (M'_i)^{a_{ik}}\). We thus have \(t_{ik} = -\delta_{ik} + a_{ik}\), which equals \(\frac{|b_{ik}| + b_{ik}}{2}\). Remark that, by lemma 7.1 of \([1]\), as stated in section 1.1, we have \(T^2 = Id\), so that \(S = T^t\) and
\[
s_{ij} = \begin{cases} 
-\delta_{ij} + \frac{|b_{ij}| + b_{ij}}{2} & \text{if } i = k \\
\delta_{ij} & \text{else.}
\end{cases}
\]

□

Theorem 14. The matrix \(B_{\mathcal{M}'}\) is obtained from the matrix \(B_{\mathcal{M}}\) by the Fomin–Zelevinski mutation rule in the direction \(M\).

Proof. By \([2]\) (see section 1.1), and by lemma \([13]\), we know that the mutation of the matrix \(B_{\mathcal{M}}\) in direction \(M\) is given by \(T B_{\mathcal{M}} T^t\), which is \(B_{\mathcal{M}}\), by the generalized mutation rule (theorem \([12]\)). □
3.4. Cluster categories. In [1], the authors study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated to an algebra $A$ which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category $\mathcal{C}_A$ is called admissible in [1] if the projection functor from the bounded derived category $D^b(\text{mod } A)$ to $\mathcal{C}_A$ is exact (triangulated). They define a Grothendieck group $K_0(\mathcal{C}_A)$ with respect to the triangles induced by those of $D^b(\text{mod } A)$, and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

**Theorem 15.** [Barot–Kussin–Lenzing] We have $K_0(\mathcal{C}_A) = K_0(\mathcal{C}_A)$ in each of the following three cases:

(i) $A$ is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.

(ii) $A$ is tubular.

(iii) $A$ is hereditary of finite representation type.

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of theorem 15:

**Theorem 16.** Let $A$ be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in [16]. Then we have $K_0(\mathcal{C}_A) = K_0(\mathcal{C}_A)$.

**Proof.** By lemma 3.2 in [1], this theorem is a corollary of the following lemma. □

**Lemma 17.** Under the assumptions of section 3.1, and if moreover $\mathcal{M}$ has a finite number $n$ of non-isomorphic indecomposable objects, then we have an isomorphism $K_0(\mathcal{C}) \approx \mathbb{Z}^n/\text{Im } B_M$.

**Proof.** This is a restatement of theorem 10. □

**References**


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