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Mean Square Error Approximation for Wavelet-Based Semiregular Mesh Compression

Frédéric Payan and Marc Antonini

Abstract—The objective of this paper is to propose an efficient model-based bit allocation process optimizing the performances of a wavelet coder for semiregular meshes. More precisely, this process should compute the best quantizers for the wavelet coefficient subbands that minimize the reconstructed mean square error for one specific target bitrate. In order to design a fast and low complex allocation process, we propose an approximation of the reconstructed mean square error relative to the coding of semiregular mesh geometry. This error is expressed directly from the quantization errors of each coefficient subband. For that purpose, we have to take into account the influence of the wavelet filters on the quantized coefficients. Furthermore, we propose a specific approximation for wavelet transforms based on lifting schemes. Experimentally, we show that, in comparison with a "naïve" approximation (depending on the subband levels), using the proposed approximation as distortion criterion during the model-based allocation process improves the performances of a wavelet-based coder for any model, any bitrate, and any lifting scheme.

Index Terms—Weighted mean square error (MSE), biorthogonal wavelet, lifting scheme, butterfly scheme, bit allocation, geometry coding, semiregular meshes.

1 INTRODUCTION

Wavelets are now frequently exploited to perform efficient compression. Based on multiresolution analysis, not only do wavelet coders achieve better compression rates [1], [2], [3], [4] than methods based on signal quantization, but they also make the progressive transmission, the adaptive displaying, or the level of details control easier.

Compression algorithms always attempt to optimize the trade-off between rate and distortion. Because of the multiresolution representation of the transformed data, a low frequency signal, and several levels of details (see Fig. 1), the rate-distortion optimization problem for a wavelet coder amounts to dispatch pertinently the bits across the different subbands, in order to obtain the highest quality of the reconstructed signal for a specific bitrate. One relevant method to solve this problem is to include a bit allocation process in the compression algorithm. This process should compute the best quantizers for all the coefficient subbands that minimize the reconstructed mean square error (MSE) due to data quantization for one specific target bitrate.

1.1 Goal and Contributions

The main objective of our current work is to design such an allocation process for a wavelet-based geometry coder of semiregular meshes [5], [6]. In particular, we aim to design a fast and low complex algorithm. For this purpose, some previous works have proposed to express the MSE relative to a quantized signal directly from the MSE of each subband of wavelet coefficients by taking into account the influence of the wavelet filters on the quantized coefficients [7], [8], [9], [10], [11]. Unfortunately, these approximations have been developed and exploited only on canonical sampling grids (in image or video processing for instance). Moreover, the proposed formulations are always adapted for "classical" wavelet transforms, i.e., defined by low pass and high pass filters.

Therefore, the contribution of this paper is the development of an approximation of the MSE relative to the reconstructed mesh geometry (that is, a signal sampled on a triangular edge lattice) across a wavelet coder. We show precisely that this specific MSE can be expressed as a weighted sum of the MSE of each coefficient subband. We also observe that, like in the previous works on the canonical sampling grids, the weighting on a triangular edge lattice depends on the wavelet transform used. Furthermore, as our work is only focused on wavelet transforms based on lifting schemes, we also show that the weighting can be expressed only in a function of the prediction and update operators of lifting schemes.

We would like to draw the readers’ attention to the fact that we already used this MSE in a previous paper [6]. However, in [6], its computation was not developed and only focused the unlifted butterfly transform [4]. So, this paper has three specific objectives. First, this manuscript complements other works by the authors. In particular, it fills in details about the computation of the weighting and provides the relative numerical values used in [6]. Moreover, this paper shows that this weighting can be applied easily to any lifting scheme and used for any kind of (semi)regular meshes. Finally, we experimentally show the interest of using this weighting in a wavelet-based geometry coder by evaluating the coding gains relative to its use during the bit allocation.

The remainder of this paper is organized as follows: Section 2 presents the background and the problem...
statement of our work. Section 3 develops the MSE approximation of a triangular mesh geometry. Then, Section 4 provides the numerical values of the weighting for the Butterfly-based lifting schemes. Finally, we show experimentally the interest of using the MSE approximation during a model-based bit allocation process in Section 5, and conclude in Section 6.

2 PRELIMINARIES

As introduced previously, the final objective of our work is to design an efficient wavelet-based geometry coder for semiregular triangular meshes. For this purpose, we aim to develop a bit allocation process to optimize the trade off between the reconstructed MSE relative to the geometry quantization and the global bitrate.

2.1 Overall Coding Scheme

The overall scheme of the wavelet coder used in this paper is presented in Fig. 2. The source semiregular mesh $M$ is transformed by a Discrete Wavelet Transform $DWT$ in several subbands of wavelet coefficients. The coordinates of the wavelet coefficients are then encoded with scalar quantizers $SQ$ (depending on the quantization steps computed during the allocation process). An entropy coding is finally applied on the quantized coefficients to obtain the bitstream. In parallel, the coarse mesh connectivity can be encoded with any connectivity coder. In this paper, we use the coder of [12].

2.2 The Allocation Process

The objective of the allocation process is to determine the set of the quantization steps $\{q\}$ (used to quantize the subbands) that minimizes the distortion defined by the MSE $\sigma^2$ relative to the reconstructed mesh geometry, at one given specific bitrate $R_{target}$. This can be modeled by the following problem $\mathcal{P}$:

\[
\begin{align*}
\begin{cases}
\text{minimize} & \sigma^2(\{q\}) \\
\text{with constraint} & R_T(\{q\}) = R_{target},
\end{cases}
\end{align*}
\]

where $R_T$ represents the total bitrate, $R_{target}$ a user-given bitrate, and $\{q\}$ the set of quantization steps.

This constrained allocation problem can be formulated by a Lagrangian criterion:

\[
J_\lambda(\{q\}) = \sigma^2(\{q\}) + \lambda (R_T(\{q\}) - R_{target}),
\]

with $\lambda$ the lagrangian operator. Hence, the solutions of the allocation problem $\mathcal{P}$, i.e., the optimal quantization steps $\{q^*\}$ are obtained by minimizing this lagrangian criterion. So, we have to solve the following system [6]:

\[
\begin{align*}
\frac{\partial J_\lambda(\{q\})}{\partial q} &= 0 \\
\frac{\partial J_\lambda(\{q\})}{\partial \lambda} &= 0.
\end{align*}
\]

3 MSE APPROXIMATION ON A TRIANGULAR EDGE LATTICE

Let us first introduce some global notations useful for the understanding of the rest of the paper.
corresponding to the sublattice filters, defined by

\[ x_i = \frac{1}{\sqrt{2}} x_{i-1} + \frac{1}{\sqrt{2}} x_{i+1} \]

where \( \mathcal{K} = \mathbb{Z}^d \) with \( \mathbb{Z} \) an invertible \( d \times d \) matrix permitting to obtain data sampled on lattices other than the canonical lattice \( \mathbb{Z}^d \), for instance, the triangular edge lattice used in the rest of the paper.

A sublattice of \( \mathcal{K} \) can be obtained by \( D \mathbb{Z}^d \), where \( D \) is a dilation matrix \( d \times d \). The determinant of \( D \) is an integer \( m \in \mathbb{Z} \). Then, the lattice \( \mathbb{Z}^d \) can be written as a sum of sublattices

\[ \mathbb{Z}^d = \bigcup_{j=0}^{m-1} (D \mathbb{Z}^d + t_j), \]

with \( t_j \in \mathbb{Z}^d \) the shift related to the \( j \)th coset. Hence, we can define a coset \( s_i \) as the set of elements of the signal \( s \) corresponding to the sublattice \( \mathcal{L} = D \mathbb{Z}^d + t_i \), and given by

\[ s_i(\mathcal{L}) = \{ s(Dk + t_i) \mid k \in \mathbb{Z}^d \}. \]

Note that \( s_i(\mathcal{L}) \) is a sequence of real-valued numbers indexed by \( \mathbb{Z}^d \) and not by \( \mathcal{L} \) [13].

According to the definition of a sublattice, an \( m \)-channel filter bank \( \{ g_i \} \) on a lattice \( \mathcal{K} \) can be formulated according to the polyphase notation as:

\[ G_i(z) = \sum_{j=0}^{m-1} z^{-t_j} G_{i,j}(z^d) \text{ for } i \in \{0, \ldots, m-1\}, \]

with \( G_{i,j}(z) \) the \( i, j \)th polyphase component of the synthesis filters, defined by

\[ G_{i,j}(z) = \sum_{k \in \mathbb{Z}^d} g_i(Dk + t_j) z^{-k}, \]

\[ z^{-t_j} \text{ the shift related to the } j \text{th coset given by} \]

\[ z^{-t_j} = \prod_{n=1}^{d} z_n^{-t_j(n)}, \]

and

\[ z^d = \{ z_1^d, z_2^d, \ldots, z_m^d \}. \]

The vector \( d_j \) is the \( j \)th column vector of the matrix \( D \), and \( z^d_i \) is defined by:

\[ \varepsilon_i = \frac{1}{N_s} [r_c(0)], \]

where \( r_c(t) \) is the autocorrelation function of \( \varepsilon \), and \( 0 \) is the null vector of dimension 2, and \( N_s \) is the number of samples of the input signal. The energy of the signal \( \varepsilon \), denoted \( \sigma^2 \), can be developed by using the deterministic correlation function

\[ \sigma^2 = \frac{1}{N_s} [r_c(0)], \]

1. We prefer using the “additive noise” model because some assumptions relative to the “gain-plus-additive noise” model proposed by Jayant [14] and used by Park and Haddad have often been controversial [15].

3.2 Challenge

A semiregular mesh is based on a triangular edge lattice [13] (see Fig. 3). A wavelet transform for meshes corresponds consequently to a 4-channel filter bank. Hence, the geometry of a semiregular mesh \( M \) is transformed into four cosets \( \{ s_i, i = 0, 1, 2, 3 \} \) on account of an analysis filter bank \( \{ h_i, i = 0, 1, 2, 3 \} \) and a downsampling \( \downarrow D \) (see Fig. 4). The cosets are then quantized. Assuming that quantization error is an additive noise [16], [15], the quantization error \( \varepsilon_i \) between the \( i \)th coset \( s_i \) and its quantized value \( \hat{s}_i \) is given by:

\[ \varepsilon_i = (s_i - \hat{s}_i). \]

An upsampling \( \uparrow D \) followed by a synthesis wavelet transform \( g_i \) provides the geometry of the reconstructed mesh \( \hat{M} \). The challenge is thus to obtain the MSE relative to the reconstructed mesh geometry according to the quantization error of each subband and the knowledge of the synthesis filters.

3.3 MSE of the Quantized Mesh Geometry

In order to simplify the derivation, we propose to follow a deterministic approach, unlike the previous works of [7], [8], [9], [10], [11] which follow a statistical approach. So, let us consider the geometry of the source mesh \( M \) as a realization of a stationary and ergodic random process [16]. The total quantization error \( \varepsilon \) can thus be considered as a deterministic quantity defined by

\[ \sigma^2 = \frac{1}{N_s} [r_c(0)], \]

where \( r_c(t) \) is the autocorrelation function of \( \varepsilon \), \( 0 \) is the null vector of dimension 2, and \( N_s \) is the number of samples of the input signal. The energy of the signal \( \varepsilon \), denoted \( \sigma^2 \), can be developed by using the deterministic correlation function

\[ \sigma^2 = \frac{1}{N_s} [r_c(0)], \]

1. We prefer using the “additive noise” model because some assumptions relative to the “gain-plus-additive noise” model proposed by Jayant [14] and used by Park and Haddad have often been controversial [15].
\[
R_e(z) = \mathcal{E}(z) \mathcal{E}(z^{-1}),
\]
with \(\mathcal{E}(z)\) the \(z\)-transform of the reconstruction error \(e\), and \(z = (z_1, z_2)\). According to Fig. 4, \(\mathcal{E}(z)\) can be formulated in a function of the error of each coset \(x_i\) [17]:

\[
\mathcal{E}(z) = \sum_{i=0}^{3} G_i(z) \mathcal{E}_i(z^D),
\]
with \(G_i(z)\) and \(G_i(z)\), respectively, the \(z\)-transform of the reconstruction error \(e_{i}\) related to the coset \(x_i\) and of the synthesis filter \(g_i\). \(D\) is the dilation matrix defined in Section 3.1. By assuming there is no cross-correlation between errors \(e_{i}(k)\) and \(e_{i}(k')\) (for all \(k \neq k'\)) [16], [15], we can write

\[
E_i(z^D) \mathcal{E}_i(z^D) = \delta_{i,j} R_{\mathcal{E}_i}(z^D)
\]

with \(R_{\mathcal{E}_i}(z)\) the \(z\)-transform of the autocorrelation function of the reconstruction error \(e_{i}\), and \(\delta_{i,j}\) the Kronecker symbol defined by

\[
\delta_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Hence, (14) and (15) provide:

\[
R_{\mathcal{E}}(z) = \sum_{i=0}^{3} R_{G_i}(z) R_{\mathcal{E}_i}(z^D).
\]

Applying the inverse \(z\)-transform on (16) yields the formulation of the autocorrelation function of the reconstruction error:

\[
r_{\mathcal{E}}(t) = \sum_{i=0}^{3} \sum_{\tau} r_{g_i}(\tau) r_{\mathcal{E}_i}(Dt - \tau).
\]

The energy \(r_{\mathcal{E}}(0)\) of the signal \(\mathcal{E}\) is then given by:

\[
r_{\mathcal{E}}(0) = \sum_{i=0}^{3} \sum_{\tau} r_{g_i}(\tau) r_{\mathcal{E}_i}(-\tau).
\]

By assuming that the quantization error samplings are uncorrelated [16], \(r_{\mathcal{E}_i}(-\tau) = 0\) if \(\tau \neq 0\) and, consequently,

\[
r_{\mathcal{E}}(0) = \sum_{i=0}^{3} r_{g_i}(0) r_{\mathcal{E}_i}(0).
\]

Now, the problem is to deal with \(r_{g_i}(0)\) and \(r_{\mathcal{E}_i}(0)\). In the Appendix, we show that the energy \(r_{g_i}(0)\) of the synthesis filter can be developed in

\[
r_{g_i}(0) = \sum_{k \in \mathbb{Z}^2} g_i(Dk + t_j)^2,
\]

with \(g_i(Dk + t_j) = g_{i,j}(k)\) the coefficient \(k\) of the \(j\)th polyphase component of the synthesis filter \(i\). In parallel, by assuming that the quantization error samplings are uncorrelated [16], the energy \(r_{\mathcal{E}_i}(0)\) of the quantization error is:

\[
r_{\mathcal{E}_i}(0) = \sum_{k \in \mathbb{Z}^2} \epsilon_{i}(k)^2 = N_{s_i} \sigma_{e_i}^2,
\]

where \(\sigma_{e_i}^2\) stands for the MSE of the coset \(x_i\) and \(N_{s_i}\) the number of samples of \(x_i\).

### 3.4 Solution for a One-Level Decomposition

Finally, by merging (20), (21), and (19) in (13), we obtain an expression of the MSE relative to the reconstructed mesh geometry:

\[
\sigma_{e}^2 = \sum_{j=0}^{3} N_{s_j} w_j \sigma_{e_i}^2 \quad \text{with} \quad w_i = \sum_{j=0}^{3} g_{i,j}(k)^2,
\]

where \(g_{i,j}(k)\) represents the coefficient \(k\) of the \(j\)th polyphase component of the synthesis filter \(i\). We stress that the weights \(w_i\) depend only on the polyphase matrix components of the synthesis filters. Now, in case of lifting schemes, the polyphase components depend only on the predictive and update operators (see the next section). We finally obtain an approximation specific to the lifting schemes, as expected.

### 3.5 Solution for an \(N\)-Level Decomposition

Wavelet coders generally exploit several levels of decomposition by applying several times the wavelet transform on the coset of lowest frequency. For example, the \(z\)-transform of the reconstruction error \(\mathcal{E}\) according to a two-level decomposition (see Fig. 5) can be written as:

\[
\mathcal{E}(z) = G_0(z) \sum_{l=0}^{M-1} G_l(z^D) \mathcal{E}_{i,j}(z^D)
\]

\[
+ \sum_{l=1}^{M-1} G_l(z) \mathcal{E}_{i,j}(z^D),
\]

where \(\mathcal{E}_{i,j}(z)\) stands for the \(z\)-transform of the quantization error \(e_{i,j}\) related to the coset \((i, j)\), with \(i\) the level of decomposition and \(j\) the channel index. By the same way as for the one-level decomposition, the MSE on a triangular edge lattice across a two-level wavelet coder can be simplified in:

\[
\sigma_{e}^2 = \frac{N_{s_{i,j}}}{N_s} w_{0} \sum_{j=0}^{3} \left[ \frac{N_{s_{i,j}}}{N_{s_{0,0}}} w_{0} \sigma_{e_{i,j}}^2 \right]
\]

\[
+ \sum_{j=1}^{3} \left[ \frac{N_{s_{i,j}}}{N_s} w_{0} \sigma_{e_{i,j}}^2 \right],
\]

where \(N_{s_{i,j}}\) is the number of samples of the \(i, j\)th coset. Thus, it is easy to generalize (24) to an \(N\)-level decomposition:

\[
\sigma_{e}^2 = \frac{N_{s_{i-1,j}}}{N_s} W_{N-1,0} \sigma_{e_{i-1,j}}^2
\]

\[
+ \sum_{j=1}^{N-1} \sum_{j=1}^{3} \frac{N_{s_{i,j}}}{N_s} W_{i,j} \sigma_{e_{i,j}}^2,
\]

where \(W_{i,j}\) represent the weights relative to the coset \((i, j)\), with \(i\) the level of decomposition and \(j\) the channel index.\(^2\)

\(^2\) \((N-1, 0)\) corresponds to the lowest decomposition level. Hence, the index \((N-1, 0)\) is relative to the low frequency subband of the mesh geometry.
corresponding weights \( w_i \) and using the formulation (22) allows to compute the one-level decomposition. They are given by the update operator

\[
G = \begin{pmatrix}
1 & p_1 & p_2 & p_3 \\
-1 & -w_1 p_1 & -u_1 p_2 & -u_1 p_3 \\
-1 & -w_2 p_1 & -u_2 p_2 & -w_2 p_3 \\
-1 & -w_3 p_1 & -u_3 p_2 & -w_3 p_3
\end{pmatrix},
\]

with \( p_i \) and \( u_i \) the prediction and update operators associated to the \( i \)th coset. Hence, identifying this matrix with the operators \( p_i \) and \( u_i \) of any 4-channel lifting scheme and using the formulation (22) allows to compute the corresponding weights \( w_i \).

The weights for the lifted Butterfly-based transform, computed by substituting the \( z \)-transform of the prediction and update operators in each component of the polyphase matrix given by (27), are:

\[
\begin{align*}
w_0 &= \frac{169}{256} \approx 0.66015625 \\
w_1 &= \frac{1727}{2048} \approx 0.843261715 \\
w_2 &= \frac{1727}{2048} \approx 0.843261715 \\
w_3 &= \frac{1727}{2048} \approx 0.843261715.
\end{align*}
\]

Similarly, the weights for the unlifted Butterfly-based transform are:

\[
\begin{align*}
w_0 &= \frac{169}{256} \approx 0.66015625 \\
w_1 &= 1 \\
w_2 &= 1 \\
w_3 &= 1.
\end{align*}
\]

4 Computation of Weights for the Butterfly-Based Lifting Scheme

In this paper, we only focus on the Butterfly-based lifting schemes [13] since our coding algorithm uses these filters. However, the proposed weighting can obviously be applied to any lifting scheme for semiregular meshes.

There generally exist two different versions for the Butterfly-based lifting scheme: the lifted version (a prediction step and an update step) and the unlifted version (only a prediction step). The prediction and update operator filters of the lifted Butterfly scheme are presented in Fig. 6 [13]. The description of this lifting scheme can be found in [20].

As stated before, only the polyphase components are needed to compute the weights. We consider the polyphase matrix of a 4-channel lifting scheme [13], given by

\[
G = \begin{pmatrix}
1 & p_1 & p_2 & p_3 \\
-1 & -w_1 p_1 & -u_1 p_2 & -u_1 p_3 \\
-1 & -w_2 p_1 & -u_2 p_2 & -w_2 p_3 \\
-1 & -w_3 p_1 & -u_3 p_2 & -w_3 p_3
\end{pmatrix},
\]

with \( p_i \) and \( u_i \) the prediction and update operators associated to the \( i \)th coset. Hence, identifying this matrix with the operators \( p_i \) and \( u_i \) of any 4-channel lifting scheme and using the formulation (22) allows to compute the corresponding weights \( w_i \).

The weights for the lifted Butterfly-based transform, computed by substituting the \( z \)-transform of the prediction and update operators in each component of the polyphase matrix given by (27), are:

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w_0 &= \frac{169}{256} \approx 0.66015625 \\
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w_2 &= \frac{1727}{2048} \approx 0.843261715 \\
w_3 &= \frac{1727}{2048} \approx 0.843261715.
\end{align*}
\]

Similarly, the weights for the unlifted Butterfly-based transform are:

\[
\begin{align*}
w_0 &= \frac{169}{256} \approx 0.66015625 \\
w_1 &= 1 \\
w_2 &= 1 \\
w_3 &= 1.
\end{align*}
\]

4.1 Complexity Reduction

As said previously, using this MSE approximation leads to faster distortion computations during the bit allocation since we do not have to apply the synthesis filters before. In the specific case of the Butterfly-based lifting scheme, the complexity is the following:

- Eight multiplications and eight additions per sample coordinate during the prediction stage, i.e., for the high frequency subbands \( x_1, x_2, \) and \( x_3 \). In other words, 48 arithmetic operations (a.o.) per wavelet coefficient.
- Six multiplications and six additions per sample coordinate during the update stage, i.e., for the low frequency coset \( x_0 \). In other words, 36 a.o. per low frequency coefficient.

Thus, the complexity \( C_b(N) \) for a \( N \)-level decomposition of the lifted version of the Butterfly-based scheme is

\[
\sum_{i=0}^{N-1} \left[ 36 \times N_{s,i} + 36 \times \sum_{i=0}^{N-1} N_{s,i} + 48 \times \sum_{i=1}^{3} N_{s,i} \right] \text{ a.o.}
\]

In parallel, the complexity \( C_{ab}(N) \) for a \( N \)-level decomposition of the unlifted version of the butterfly-based scheme (only a prediction step) is

\[
\sum_{i=0}^{N-1} \left[ 48 \times \sum_{i=1}^{3} N_{s,i} \right] \text{ a.o.}
\]

As we use an iterative algorithm to solve the optimization problem (like in [6], for instance), the distortion is computed several times during the allocation process. The convergence of our algorithm is generally reached in approximately five iterations (the maximal number of iterations we observe is 10). Thus, given \( I \) the number of iterations, the whole complexity reduction relative to the use of the weighted MSE is given by \( I \times C_b(N) \) for the lifted version of the butterfly-based scheme, and by \( I \times C_{ab}(N) \) for the unlifted one. For instance, the synthesis for the lifted version takes about 0.5 second on a Pentium III 1GHz with 512 Mbytes, involving a significant average time cost reduction of 2.5 seconds for the allocation process (for five iterations). Furthermore, as explained in [6], exploiting this MSE approximation allows us to develop a model-based algorithm for the allocation process as well. Combining the MSE approximation with a model-based algorithm finally leads to a more significant time cost reduction. As an example, the allocation is finally processed in less than 0.4 second for instance on a Pentium III 1GHz with 512 Mbytes RAM.

5 Simulation Results

In order to evaluate the coding gain when using the weighted MSE as distortion criterion during the bit allocation, we experiment the compression algorithm presented in Section 2 (see Fig. 2) according to two cases:
the distortion criterion is the MSE approximation
with the weighting obtained in Section 3 and
the distortion criterion is a so-called “naïve”
weighted MSE. In that case, the weights depend
only on the resolution levels and, thus, are defined
by inverse powers of 2.
Moreover, to show that the weighting can be exploited for
any kind of semiregular meshes (independently of the
remeshing method) and for any scheme, we propose two
versions of our coder:
- the first version, named the MAPS coder, deals with
  meshes issued from the remesher MAPS [21] and
  includes the lifted Butterfly scheme, and
- the second version, named the NORMAL coder, deals
  with Normal Meshes [22] and includes the unlifted
  Butterfly scheme.

Then, we compare the quality of the meshes obtained after
decompression, quantized at the same given bitrate. Figs. 7a,
7b, 7c, and 7d show, respectively, the corresponding curve
PSNR/bitrate for the models BUNNY, VENUS, RABBIT, and
HORSE encoded with the proposed MAPS coder. As
numerous papers about wavelet geometry coders [3], [4],
[23], the PSNR is given by

\[
\text{PSNR} = 20 \log_{10} \left( \frac{\text{peak}}{d_S} \right),
\]

where \( \text{peak} \) is the bounding box diagonal of the original
object, and \( d_S \) is the Root MSE between the original irregular
mesh and the reconstructed semiregular one (computed with
MESH [24]). The bitrate is reported with respect to the
number of vertices in the original irregular mesh (\( b/v \)). In
addition, Figs. 8a, 8b, 8c, and 8c show, respectively, the curves
PSNR/bitrate for the models SKULL, HORSE, RABBIT, and
VENUS, encoded with the proposed NORMAL coder.

Theoretically the quantization error assumptions used
during derivation in Section 3 are only valid for high
bitrates [16], and we could expect that the coding gain is
limited for low bitrates. However, compared to the “naïve”
proposed coder, we globally observe that using the
weighted MSE significantly improves the coding perfor-
mances for any model, any version of the lifting scheme,
and at any bitrate. Note that the gain reaches more than
3.4 dB on the Normal mesh SKULL and 2.7 dB on the MAPS
mesh VENUS at low bitrates.

Also, Fig. 9 provides the visual benefits by showing the
distribution of the local reconstruction error on several
models. The colour corresponds to the magnitude of the
distance point-surface (see Fig. 9a) between the original
irregular mesh and the reconstructed semiregular one
(computed with MESH [24]). We globally observe that, for
one specific bitrate, the quality of the reconstructed meshes
is always higher when the weighted MSE is exploited
during the allocation process.

In order to evaluate the performances of our algorithm
when using the proposed MSE approximation as distortion
criterion of the bit allocation, we also compare our algorithms
with some state-of-the-art coders. To be coherent, we
obviously compare our MAPS coder with the zerotree coder
for MAPS meshes named PGC [3], exploiting the lifted
version of the Butterfly-based transform. On the other hand,
we compare our Normal coder with the zerotree coder named
for Normal meshes NMC [4] and with EQMC [23], both
exploiting the unlifted version of the Butterfly-based trans-
form. The curves PSNR/bitrate (see Fig. 7 and Fig. 8) show

that, for MAPS meshes, the proposed coder provides better
results (up to 3.5 dB) or, in the worst case, similar results. In
parallel, our Normal coder always outperforms the state-of-
the-art coders NMC and EQMC (up to more than 3.5 dB.

Fig. 7. PSNR curves for meshes encoded with the MAPS coder (lifted
Butterfly-based transform). (a) BUNNY, (b) VENUS, (c) RABBIT, and
(d) HORSE.
compared to NMC). Notice that without the proposed weighting, our coders generally provide worse results than the state-of-the-art coders, showing definitively the interest of using the weighted MSE.

**Fig. 8.** PSNR curves for meshes encoded with the NORMAL coder (unlifted Butterfly-based transform). (a) SKULL, (b) HORSE, (c) RABBIT, and (d) VENUS.

**Fig. 9.** Distribution of the reconstruction error on different models (at its finest resolution) according to the distortion criterion. (a) Error scale. The warmer colors correspond to the higher errors. (b) Naive MSE: 2.0 b/iv, PSNR = 69.58 dB. (c) Weighted MSE: 2.0 b/iv, PSNR = 70.92 dB. (d) Naive MSE: 2.0 b/iv, PSNR = 69.15 dB. (e) Weighted MSE: 2.2 b/iv, PSNR = 70.65 dB. (f) Naive MSE: 2.2 b/iv, PSNR = 54.9 dB. (g) Weighted MSE: 1.69 b/iv, PSNR = 57.54 dB.

### 6 Conclusions

In this paper, we proposed a weighted MSE relative to the triangular mesh geometry. The final objective was to propose an efficient model-based bit allocation process optimizing the performances of a wavelet-based geometry coder for semi-regular meshes. We have particularly shown that the weights can depend only on the polyphase components of the synthesis filters, which is very useful in case of lifting schemes. Experimentally, we observe that using this MSE approximation as distortion criterion of a bit allocation process significantly improves the coding performances of a wavelet coder (for any kind of semiregular meshes, any bitrate, and any lifting scheme), so that the proposed algorithms outperform the relative state-of-the-art coders.
APPENDIX
Energy of the Synthesis Filter
The energy of the synthesis filter $r_g(0)$ is given by:

$$r_g(0) = \frac{1}{2\pi} \int_{\mathbb{T}} G_i(z) G_i(z^{-1}) z^{-1} dz.$$  \hspace{1cm} (32)

According to (7), a synthesis filter bank $\{g_i\}$ on a triangular edge lattice can be formulated according to the polyphase notation:

$$G_i(z) = \sum_{j=0}^{3} z^{-j} G_{i,j}(z^0) \quad \text{for} \quad i \in \{0, \ldots, 3\},$$  \hspace{1cm} (33)

with $G_{i,j}(z)$ the $i, j$th polyphase component of the synthesis filters, defined by

$$G_{i,j}(z) = \sum_{k \in \mathbb{Z}^2} g_i(Dk + t_j) z^{-k},$$  \hspace{1cm} (34)

and $z^{-t}$ the shift relative to the $j$th coset. By exploiting (33) and (34), (32) can be developed in:

$$r_g(0) = \frac{1}{2\pi} \sum_{i=0}^{3} \sum_{k \in \mathbb{Z}^2} \sum_{v \in \mathbb{Z}^2} g_i(Dk + t_v) \int_{\mathbb{T}} z^{-Dk - Dk' - t_v + t_v} dz.$$  \hspace{1cm} (35)

Using the Cauchy theorem, that is,

$$\frac{1}{2\pi} \int_{\mathbb{T}} z^{-1} dz = \begin{cases} 1 & \text{if} \quad l = 0, \\ 0 & \text{else} \end{cases},$$

the integral operator of (35) is equal to 1 if

$$-Dk + Dk' - t_v + t_v = 0$$  \hspace{1cm} (36)

is satisfied. The dilation matrix $D$ being invertible, this condition becomes $(-k + k') - (D^{-1} t_v - D^{-1} t_v) = 0$. From [13], we know that $D^{-1} t_v$ is restricted to the unit hypercube, that is, $[0, 1]^2$. On the other hand, $k \in \mathbb{Z}^2$. These two remarks yield that

$$\{ (-k + k') \in \mathbb{Z}^2 \mid (D^{-1} t_v - D^{-1} t_v) \in (-1, 1)^2 \}.$$  \hspace{1cm} (37)

As $\mathbb{Z}^2 \cap (-1, 1)^2 = 0$, we have to solve separately $(-k + k') = 0$, and $(D^{-1} t_v - D^{-1} t_v) = 0$ to satisfy (36). Consequently, the set of solutions is $k = k'$ and $u = v$. Finally, the energy of the synthesis filter $r_g(0)$ is given by

$$r_g(0) = \sum_{j=0}^{3} \sum_{k \in \mathbb{Z}^2} g_i(Dk + t_j)^2,$$  \hspace{1cm} (38)

with $g_i(Dk + t_j) = g_{i,j}(k)$ the coefficient $k$ of the $j$th polyphase component of the synthesis filter $i$.

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