Optimal control of unilateral obstacle problem with a source term

Radouen Ghanem

Optimal control of unilateral obstacle problem with a source term

Radouen Ghanem
UMR 6628-MAPMO, Fdration Denis Poisson, Universit d'Orlans
BP. 6759, F-45067 Orlans Cedex 2
radouen.ghanem@univ-orleans.fr

Abstract

We consider an optimal control problem for the obstacle problem with an elliptic variational inequality. The obstacle function which is the control function is assumed in H^2. We use an approximate technique to introduce a family of problems governed by variational equations. We prove optimal solutions existence and give necessary optimality conditions.

Key words: Optimal control, obstacle problem, variational inequality.
AMS Classification (2000): 35R35, 49J40, 49J20.

1 Introduction

The study of variational inequalities and free boundary problems finds application in a variety of disciplines including physics, engineering, and economies as well as potential theory and geometry. In the past years, the optimal control of variational inequalities has been studied by many authors with different formulations. For example optimal control problems for obstacle problems (where the obstacle is a given (fixed) function) were considered with the control variables in the variational inequality. Roughly speaking, the control is different from the obstacle, see for example works by [4], [8], [15], [16] and the references therein.

Here we deal with the obstacle as the control function. This kind of problem appears in shape optimization for example. It may concern a dam optimal shape. The obstacle gives the form to be designed such that the pressure of the fluid inside the dam is close to a desired value. This is equivalent in some sense to controlling the free boundary [10].

The main difficulty of this type of problem comes from the fact that the mapping between the control and the state (control-to-state operator) is not differentiable but only Lipschitz-continuous and so it is not easy to get first order optimality conditions.

These problems have been considered from the theoretical and/or numerical points of view by many authors (see for example Adams and Lenhart [3], Ito and Kunisch [12]). They have used either an approximation of the variational inequality by penalization-regularization or a complementarity constraint formulation. Adams et Lenhart [3] consider optimal control problem governed by a linear elliptic variational inequality without source terms. The main result is that any optimal pair must satisfy "state = obstacle". Adams and Lenhart [2] treat control of H^1- obstacle, and convergence results in [2], [3] are given under implicit monotonicity assumptions.

Ito and Kunisch [12] consider the optimal control problem to minimize a functional involving the H^1 norm of the obstacle, subject to a variational inequality of the type $y \in \operatorname{argmin}\{a(z) - \langle f, z\rangle | z \leq \psi\}$ in a Hilbert lattice H. Under appropriate conditions, they show that the variational inequality can be expressed by the system $Ay + \lambda = f, \lambda := \max(0, \lambda + c(y - \psi))$. Smoothing the max-operation, this system is approximated by a semilinear elliptic equation containing only smooth expressions. Passing to

the limit, the optimality system of the associated differentiable optimal control problem is used to derive an optimality system of the original nonsmooth control problem with only H^1-regularity for the obstacle. Bergounioux and Lenhart [6], [7] have studied obstacle optimal control for semilinear and bilateral obstacle problem, where the admissible controls (obstacles) are H^2-bounded and the convergence results are given with a compactness assumption. Yuquan and Chen [17], consider an obstacle control problem in a elliptic variational inequality without source terms. We can see also quote the paper of Lou [14] for more generalized regularity results. In this paper we consider an optimal control problem: we seek an optimal pair of optimal solution (state, control), when the state is close to a desired target profile and satisfies an unilateral variational inequality with a source term, and the control function is the lower obstacle. Convergence results are proved with compactness techniques.
The new feature in this paper is the regularity on the control function (obstacle) and an optimality conditions system more complete than the one given in [17].
Let us give the outline of the paper. Next section, is devoted to the formulation of the optimal control problem, we give assumptions for the state equation, and give preliminaries results. In section 3 we study the variational inequality, give control-to-state operator properties and assert an existence result for optimal solution. The last section is devoted to optimality condition system.

2 Optimal control problem

Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \leq 3)$, with lipschitz boundary $\partial\Omega$. We adopt the standard notation $H^m(\Omega)$ for the Sobolev space of order m in Ω with norm $\|\cdot\|_{H^m(\Omega)}$, where

$$H^m(\Omega) := \left\{ v \mid v \in L^2(\Omega), \partial^q v \in L^2(\Omega)\ \forall q, |q| \leqslant m \right\},$$

and

$$H_0^m(\Omega) := \left\{ v \mid v \in H^m(\Omega), \left.\frac{\partial^k v}{\partial \eta^k}\right|_{\partial\Omega} = 0,\ 0 \leq k \leq m-1 \right\},$$

defined as the closure of $\mathscr{D}(\Omega)$ in the space $H^m(\Omega)$, where $\mathscr{D}(\Omega)$, the space of $\mathcal{C}^\infty(\Omega)$-functions, with compact support in Ω (see for example [1]). We shall denote by $\|\cdot\|_V$, the Banach space V norm, and $\|\cdot\|_{L^p(\Omega)}$ the $p-$summable functions $u : \Omega \to \mathbb{R}$ endowed with the norm $\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}$ for $1 \leq p < \infty$ and $\|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$ for $p = \infty$. In the same way, $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and $(\cdot,\cdot)$ the $L^p(\Omega)$ inner product. It is well known that $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ with compact and dense injection. We consider the bilinear form $a(\cdot,\cdot)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$a(u,v) := \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\ dx + \sum_{i=1}^{n} \int_\Omega a_i \frac{\partial u}{\partial x_i} v\ dx + \int_\Omega a_0 u\, v\ dx, \tag{2.1}$$

where

$$\begin{cases} a_0, a_i, a_{ij} \in L^\infty(\Omega), \\ \sum\limits_{i,j=1}^{n} a_{ij}\theta_i\theta_j \geq m \sum\limits_{i=0}^{n} \theta_i^2, \quad m > 0, \quad \text{a.e. in } \Omega, \quad \forall \theta \in \mathbb{R}^n. \end{cases} \tag{\textbf{H}}$$

Moreover, we suppose that $a_{ij} \in \mathcal{C}^{0,1}(\bar{\Omega})$ (the space of lipschitz continuous functions in Ω, where $\bar{\Omega}$ is the closure of Ω) and that a_0 is nonnegative to ensure a good regularity of the solution (see for example [13]). We suppose that the bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$

$$\exists M > 0, \forall \varphi, \psi \in H_0^1(\Omega), |a(\varphi,\psi)| \leq M \|\varphi\|_{H_0^1(\Omega)} \|\psi\|_{H_0^1(\Omega)}, \tag{2.2}$$

and coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$

$$\exists\, m > 0, \forall \varphi \in H_0^1(\Omega), a(\varphi, \varphi) \geq m \left\|\varphi\right\|_{H_0^1(\Omega)}^2. \tag{2.3}$$

We call $A \in \mathcal{L}(H_o^1(\Omega), H^{-1}(\Omega))$ the linear (elliptic) operator associated to $a(\cdot,\cdot)$ such that $\langle Au, v\rangle := a(u,v)$. We note that the coercivity assumption (2.3) on a implies that

$$\forall\ \varphi \in H_0^1(\Omega),\ \ \langle A\varphi, \varphi\rangle \geq m \left\|\varphi\right\|_{H_0^1(\Omega)}^2.$$

For any $\varphi \in H_0^1(\Omega)$, we define

$$\mathcal{K}(\varphi) := \left\{ y \in H_0^1(\Omega) \mid y \geq \varphi \text{ a.e. in } \Omega \right\},$$

and consider the following variational inequality

$$a\left(y,\, v - y\right) \geq \left(f,\, v - y\right), \quad \forall v \in \mathcal{K}(\varphi), \tag{2.4}$$

where f belongs to $L^2(\Omega)$ as a source term. In addition c or C denotes a general positive constant independent of δ.

Theorem 2.1. *Under the hypothesis* (2.2) *and* (2.3)*, for any* $f \in L^2(\Omega)$ *and* $\varphi \in H_0^1(\Omega)$*, the variational inequality* (2.4)*, has a unique solution* y *in* $\mathcal{K}(\varphi)$*. In addition if* φ *belongs to* $H^2(\Omega)$*, the solution* y *belongs to* $H^2(\Omega) \cap H_0^1(\Omega)$*.*

Proof. See [9]. □

From now we define the operator $\mathcal{T}$ (control-to-state) from $H^2(\Omega) \cap H_0^1(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$, such that $y := \mathcal{T}(\varphi)$ is the unique solution to the variational inequality (2.4).
Now, we consider the optimal control problem $(\mathcal{P})$, defined as follows

$$\min \left\{ J(\varphi) := \tfrac{1}{2} \int_\Omega \left(\mathcal{T}(\varphi) - z\right)^2 dx + \tfrac{\nu}{2} \left(\int_\Omega (\Delta\varphi)^2\, dx \right),\ \varphi \in \mathcal{U}_{ad} \right\}, \tag{$\mathcal{P}$}$$

where ν is a given positive constant, $z \in L^2(\Omega)$ and $\mathcal{U}_{ad}$ (the set of admissible control) is a closed convex subset of $H^2(\Omega) \cap H_0^1(\Omega)$: we seek an obstacle (optimal control) φ^* in $\mathcal{U}_{ad}$, such that the corresponding state is close to a target profile z. In the sequel we set $\mathcal{U} := H^2(\Omega) \cap H_0^1(\Omega)$.

3 Approximation of problem $(\mathcal{P})$

3.1 Approximation of operator $\mathcal{T}$

The obstacle problem (2.4) can be equivalently written as follows

$$Ay + \partial I_{\mathcal{K}(\varphi)}(y) \ni f \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where

$$\partial I_{\mathcal{K}(\varphi)}(y) = \partial I_{\mathcal{K}^+(y-\varphi)}(y) := \left\{ v \in L^2(\Omega) \mid v \in \beta_o(y - \varphi),\ \text{a.e. in } \Omega \right\},$$

and

$$\mathcal{K}^+ := \left\{ y \in H_0^1(\Omega) \mid y \geq 0,\ \text{a.e. in } \Omega \right\},$$

and $\beta_o : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is the maximal monotone (multivalued) graph,

$$\beta_o(r) := \begin{cases} 0 & \text{if } r \geq 0 \\ \mathbb{R}^- & \text{if } r = 0 \\ \emptyset & \text{if } r < 0. \end{cases}$$

Equation (3.1), can be approximated by the following smooth semilinear equation

$$Ay + \beta_\delta (y - \varphi) = f \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega, \tag{3.2}$$

where β_δ is an approximation of β_o. One possible approximation of β_o si given as follow

$$\beta_\delta(r) := \frac{1}{\delta} \begin{cases} 0 & \text{if } r \geq 0 \\ -r^2 & \text{if } r \in \left[-\frac{1}{2}, 0\right] \\ r + \frac{1}{4} & \text{if } r \leq -\frac{1}{2}, \end{cases}$$

Where $\delta > 0$ and we note that $r_\delta^- := \frac{1}{\delta} \min\{0, r\} \leq \beta_\delta(r) \leq 0$ and $\beta \in \mathscr{C}^\infty(\mathbb{R})$ and β_δ' is given by

$$\beta_\delta'(r) := \frac{1}{\delta} \begin{cases} 0 & \text{if } r \geq 0 \\ -2r & \text{if } r \in \left[-\frac{1}{2}, 0\right] \\ 1 & \text{if } r \leq -\frac{1}{2}. \end{cases}$$

As $\beta_\delta(\cdot - \varphi)$ is nondecreasing, it is well known (see for example [11]), that the boundary value problem (3.2) admits a unique solution y^δ in $H^2(\Omega) \cap H_0^1(\Omega)$ for a fixed φ in $H^2(\Omega) \cap H_0^1(\Omega)$ and f in $L^2(\Omega)$. In the sequel, we set $y^\delta := \mathcal{T}^\delta(\varphi)$. We recall the following continuity results [7]

Theorem 3.1. *For any pair (y_i, φ_i) in $\mathcal{U} \times \mathcal{U}$, that satisfies* (3.2) *where $i = 1, 2$. We get*

$$\|y_2 - y_1\|_{H^1(\Omega)} \leq L_\delta \|\varphi_2 - \varphi_1\|_{L^2(\Omega)},$$

where $L_\delta := \max\left\{1, \frac{2}{m\delta}\right\}$ and m is the coercivity constant of a.

Proof. From (3.4a), we obtain

$$a(y_2 - y_1, v) + (\beta_\delta(y_2 - \varphi_2) - \beta_\delta(y_1 - \varphi_2), v) = 0, \quad \forall v \in \mathcal{U}.$$

with $v = y_2 - y_1$, we write

$$a(y_2 - y_1, y_2 - y_1) + (\beta_\delta(y_2 - \varphi_2) - \beta_\delta(y_1 - \varphi_2), y_2 - y_1) = 0.$$

Since β_δ is nondecreasing, by the hypothesis (2.2) and (2.3), we deduce

$$\|y_2 - y_1\|_{H^1(\Omega)} \leq L_\delta \|\varphi_2 - \varphi_1\|_{L^2(\Omega)},$$

where $L_\delta := \max\left\{1, \frac{2}{m\,\delta}\right\}$. □

Theorem 3.2. *Let φ^δ in $H^2(\Omega) \cap H_0^1(\Omega)$ be a strongly convergent sequence in $H_0^1(\Omega)$ to some φ as δ tends to* 0. *Then the sequence $y^\delta := \mathcal{T}^\delta(\varphi^\delta)$ strongly converges to $y := \mathcal{T}(\varphi)$ in $H_0^1(\Omega)$.*

Proof. For every φ^δ in $H^2(\Omega) \cap H_0^1(\Omega)$, we set $y^\delta := \mathcal{T}^\delta(\varphi^\delta)$, then for any y^δ in $H_0^1(\Omega)$ the equation (3.2) is equivalent to

$$a(y^\delta, v) + (\beta_\delta(y^\delta - \varphi^\delta), v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{3.3}$$

In the equation (3.3), we choose $v = \varphi^\delta - y^\delta$, then we get

$$a(y^\delta, \varphi^\delta - y^\delta) + \int_\Omega \beta_\delta(y^\delta - \varphi^\delta)(\varphi^\delta - y^\delta)\, dx = \int_\Omega f(\varphi^\delta - y^\delta)\, dx.$$

We know by the definition of β_δ, thats if $y^\delta(x) - \varphi^\delta(x) \geq 0$, we have

$$\beta_\delta\left(y^\delta(x) - \varphi^\delta(x)\right) = 0,$$

otherwise, we get $\beta_\delta\left(y^\delta(x) - \varphi^\delta(x)\right) \leq 0$. Then we deduce that in all cases, we have

$$\beta_\delta\left(y^\delta - \varphi^\delta\right)\left(y^\delta - \varphi^\delta\right) \geq 0 \quad \text{a.e. in } \Omega,$$

that yields

$$a\left(y^\delta, y^\delta\right) \leq a\left(y^\delta, \varphi^\delta\right) + \left(f, y^\delta - \varphi^\delta\right),$$

with the hypothesis (2.2) and (2.3), we deduce (estimation regularity)

$$\left\|y^\delta\right\|_{H^1(\Omega)} \leq C\left\|\varphi^\delta\right\|_{H^1(\Omega)}, \tag{3.4}$$

where C is a constant only depending on f and a. We know that if φ^δ is strongly convergent in $H_0^1(\Omega)$ then φ^δ is bounded in $H_0^1(\Omega)$, and by (3.4) we deduce that y^δ is convergent to some y as δ tends to 0 weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$.
Let v in $\mathcal{K}(\varphi)$, and choose $v^\delta = \max\left(v, \varphi^\delta\right)$. We have v^δ in $\mathcal{K}\left(\varphi^\delta\right)$ and that v^δ is convergent to v strongly in $H_0^1(\Omega)$. Equation (3.3) with $v = v^\delta - y^\delta$ gives

$$a\left(y^\delta, v^\delta - y^\delta\right) + \int_\Omega \beta_\delta\left(y^\delta - \varphi^\delta\right)\left(v^\delta - y^\delta\right) dx = \int_\Omega f\left(v^\delta - y^\delta\right) dx.$$

- If $y^\delta \leq \varphi^\delta$, therefore $\beta_\delta\left(y^\delta - \varphi^\delta\right) < 0$ and $\left(v^\delta - y^\delta\right) \geq 0$, we deduces that $\beta_\delta\left(y^\delta - \varphi^\delta\right)\left(v^\delta - y^\delta\right) \leq 0$.
- If $y^\delta \geq \varphi^\delta$, then $y^\delta - \varphi^\delta \geq 0$, therefore $\beta_\delta\left(y^\delta - \varphi^\delta\right) = 0$.

So we deduce that in all cases we have $\beta_\delta\left(y^\delta - \varphi^\delta\right)\left(v^\delta - y^\delta\right) \leq 0$, and we get

$$a\left(y^\delta, y^\delta\right) \leq a\left(y^\delta, v^\delta\right) - \left(f, v^\delta - y^\delta\right).$$

Passing to the limit and using the lower semi-continuity of a gives

$$a(y, y) \leq \liminf_{\delta\to 0}\ a\left(y^\delta, y^\delta\right) \leq \liminf_{\delta\to 0}\ a\left(y^\delta, v^\delta\right) - \left(f, v^\delta - y^\delta\right) = a(y, v) - (f, v - y),$$

and

$$a(y, v - y) \geq (f, v - y), \quad \forall v \in \mathcal{K}(\varphi).$$

It remains to prove that y^δ tends to y, strongly in $H_0^1(\Omega)$. By using the fact that $w^\delta = \max\left(y, \varphi^\delta\right)$ converge to y strongly in $H_0^1(\Omega)$ it is sufficient to prove that $w^\delta - y^\delta$ converge to 0 strongly in $H_0^1(\Omega)$. From equation (3.3) we get

$$\begin{aligned} a\left(w^\delta - y^\delta, w^\delta - y^\delta\right) &= a\left(w^\delta, w^\delta - y^\delta\right) - a\left(y^\delta, w^\delta - y^\delta\right) \\ &= a\left(w^\delta, w^\delta - y^\delta\right) + \int_\Omega \beta_\delta\left(y^\delta - \varphi^\delta\right)\left(w^\delta - y^\delta\right) dx - \int_\Omega f\left(w^\delta - y^\delta\right) dx. \end{aligned}$$

As previously we deduce that

$$a\left(w^\delta - y^\delta, w^\delta - y^\delta\right) \leq a\left(w^\delta, w^\delta - y^\delta\right) - \left(f, w^\delta - y^\delta\right),$$

from the hypothesis (2.3), we get

$$m\left\|w^\delta - y^\delta\right\|_{H^1(\Omega)}^2 \leq a\left(w^\delta, w^\delta - y^\delta\right) - \left(f, w^\delta - y^\delta\right).$$

□

As a consequence of the previous theorem, we obtain the following corollaries

Corollary 3.1. *For any φ^δ in $H^2(\Omega) \cap H_0^1(\Omega)$, $y^\delta := \mathcal{T}^\delta(\varphi^\delta)$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. Since $\beta_\delta(y^\delta - \varphi^\delta)$ and f belongs to $L^2(\Omega)$, then $Ay^\delta \in L^2(\Omega)$ and $y^\delta \in H^2(\Omega)$. ☐

Corollary 3.2. *For any φ in $\mathcal{U}_{ad}$, the sequence $y^\delta := \mathcal{T}^\delta(\varphi)$, converges to $y := \mathcal{T}(\varphi)$, strongly in $H_0^1(\Omega)$.*

Corollary 3.3. *There exists a constant C depending only on f and a, such that for any φ in $\mathcal{U}$, we get*

$$\|\mathcal{T}(\varphi)\|_{H_0^1(\Omega)} \leq C(a,f)\,\|\varphi\|_{H_0^1(\Omega)}.$$

Proof. We choose $\varphi^\delta = \varphi$, and $y^\delta := \mathcal{T}^\delta(\varphi)$, as we know that y^δ converges to $\mathcal{T}(\varphi)$ strongly in $H_0^1(\Omega)$, we pass to the limit in (3.4). ☐

Theorem 3.3. *$\mathcal{T}$ is continuous from $\mathcal{U}$ endowed with the sequential weak topology of $H^2(\Omega)$ to $H_0^1(\Omega)$ endowed with the sequential weak topology.*

Proof. Let φ_k be a sequence that converges to φ weakly in $H^2(\Omega)$. Then the sequence φ_k converges strongly in $H_0^1(\Omega)$. We set $y_k := \mathcal{T}(\varphi_k)$. Let v in $\mathcal{K}(\varphi)$ and set $v_k = \sup(v, \varphi_k) \in \mathcal{K}(\varphi_k)$. The sequence v_k converges to v strongly in $H_0^1(\Omega)$. As $y_k := \mathcal{T}(\varphi_k)$, we get $a(y_k, v_k - y_k) \geq (f, v_k - y_k)$, i.e.

$$a(y_k, y_k) \leq a(y_k, v_k) - (f, v_k - y_k).$$

By Corollary 3.3 the sequence y_k is bounded and weakly converges in $H_0^1(\Omega)$ to some y. Using the lower semi-continuity of a, the previous relation gives

$$a(y,y) \leq a(y,v) - (f, v - y).$$

As $y_k \geq \varphi_k$, this implies that $y \geq \varphi$, therefore $y := \mathcal{T}(\varphi)$. ☐

We obtain the main result of this section.

Theorem 3.4. *Problem $(\mathcal{P})$ admits at least one optimal solution $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. Let φ_k a minimizing sequence. As $J(\varphi_k)$ is bounded, φ_k is H^2-bounded and converges to φ weakly in $H^2(\Omega)$. By Theorem 3.3, the sequence $y_k := \mathcal{T}(\varphi_k)$ converges to $y^* := \mathcal{T}(\varphi^*)$ weakly in $H_0^1(\Omega)$ and using the norms semi-continuity we obtain

$$J(\varphi^*) := \tfrac{1}{2}\int_\Omega (\mathcal{T}(\varphi^*) - z)^2\,dx + \tfrac{\nu}{2}\left(\int_\Omega (\Delta\varphi^*)^2\,dx\right) \leq \liminf_{k\to\infty} J(\varphi_k) = \inf(\mathcal{P}).$$

☐

3.2 An Approximated problem $(\mathcal{P}_\delta)$

We use a trick of Barbu [5], and add adapted penalization terms to the approximated functional J_δ (here we add $\frac{1}{2}\|\varphi - \varphi^*\|_0^2$) to force the relaxed obstacle family φ to converge to a desired solution φ^* of $(\mathcal{P})$. So for any $\delta > 0$, we define

$$J_\delta(\varphi) := \tfrac{1}{2}\left[\int_\Omega \left(\mathcal{T}^\delta(\varphi) - z\right)^2 dx + \nu\left(\int_\Omega (\Delta\varphi)^2\,dx\right) + \|\varphi - \varphi^*\|_{L^2(\Omega)}^2\right].$$

The approximated optimal control problem $(\mathcal{P}^\delta)$ stands

$$\min\{J_\delta(\varphi),\ \varphi \in \mathcal{U}_{ad}\}. \qquad (\mathcal{P}^\delta)$$

Theorem 3.5. *Problem $(\mathcal{P}^\delta)$ admits at least one solution φ^δ. Moreover, when δ go to 0, the family φ^δ converges to φ^* weakly in $H^2(\Omega)$, and $y^\delta := \mathcal{T}^\delta(\varphi^\delta)$ converges to $y^* := \mathcal{T}(\varphi^*)$, strongly in $H_0^1(\Omega)$.*

Proof. The functional J_δ is coercive, and lower semi-continuous on $\mathcal{U}$. Therefore, the problem $\left(\mathcal{P}^\delta\right)$ admits at least one solution φ^δ. We set $y^\delta := \mathcal{T}^\delta\left(\varphi^\delta\right)$, and note, that for any $\delta > 0$,

$$J_\delta\left(\varphi^\delta\right) \leq J_\delta\left(\varphi^*\right) := \tfrac{1}{2}\left[\int_\Omega \left(\mathcal{T}^\delta\left(\varphi^*\right) - z\right)^2 dx + \nu\left(\int_\Omega \left(\Delta\varphi^*\right)^2 dx\right)\right]. \tag{3.1}$$

By Theorem 3.2, we know that $\mathcal{T}^\delta\left(\varphi^*\right)$ converges to $\mathcal{T}\left(\varphi^*\right)$ strongly in $H_0^1\left(\Omega\right)$, so that $J_\delta\left(\varphi^*\right)$ converges to $J\left(\varphi^*\right)$ as $\delta \to 0$. Consequently, there exist $\delta_0 > 0$ and a constant j^*, such that

$$\forall\ \delta \leq \delta_0, \quad J_\delta\left(\varphi^\delta\right) \leq j^* < +\infty.$$

Consequently φ^δ is H^2-bounded uniformly, for any $\delta \leq \delta_0$. We use the Theorem 3.2, we get φ^δ converge to $\widetilde{\varphi}$ weakly in $H^2\left(\Omega\right)$ and strongly in $H_0^1\left(\Omega\right)$ and y^δ converge to $\widetilde{y} := \mathcal{T}\left(\widetilde{\varphi}\right)$ strongly in $H_0^1\left(\Omega\right)$. As $\mathcal{U}_{ad}$ is weakly closed, we have $\widetilde{\varphi}$ in $\mathcal{U}_{ad}$. By (3.1) and the lower semi-continuity of J_δ, we get

$$\begin{aligned} J\left(\widetilde{\varphi}\right) + \tfrac{1}{2}\left\|\widetilde{\varphi} - \varphi^*\right\|_0^2 \leq\ & \liminf_{\delta\to 0} J_\delta\left(\varphi^\delta\right) \\ \leq\ & \limsup_{\delta\to 0} J_\delta\left(\varphi^\delta\right) \\ \leq\ & \lim_{\delta\to 0} J_\delta\left(\varphi^*\right) \leq \lim_{\delta\to 0} J\left(\varphi^*\right) \\ \leq\ & J\left(\widetilde{\varphi}\right). \end{aligned}$$

This yields that $\left\|\widetilde{\varphi} - \varphi^*\right\|_0^2 \leq 0$, then $\widetilde{\varphi} = \varphi^*$ and $\lim_{\delta\to 0} J_\delta\left(\varphi^\delta\right) = J\left(\varphi^*\right)$.

In addition this proves that any clusters points of φ^δ is equal to φ^*, so that the whole family converges. □

3.3 Optimality conditions for problem $\left(\mathcal{P}^\delta\right)$

We give first necessary optimality conditions for problem $\left(\mathcal{P}^\delta\right)$. Les us recall the following result on the Gteaux-derivative of the operator $\mathcal{T}^\delta$ [1].

Lemma 3.1. *The mapping $\mathcal{T}^\delta$ is Gteaux-derivative at any φ in $\mathcal{U}_{ad}$:*

$$\forall\xi \in H_0^1\left(\Omega\right), \quad \frac{\mathcal{T}^\delta\left(\varphi + \tau\xi\right) - \mathcal{T}^\delta\left(\varphi\right)}{\tau} \overset{w}{\rightharpoonup} v^\delta,\ \textit{in}\ H_0^1\left(\Omega\right),\ \textit{when}\ \tau \to 0,$$

where v^δ is the solution of the following equation

$$Av^\delta + \beta_\delta'\left(y^\delta - \varphi\right)v^\delta = \beta_\delta'\left(y^\delta - \varphi\right)\xi \quad \text{in } \Omega, \quad v^\delta = 0 \quad \text{on } \partial\Omega.$$

Proof. See [2]. □

We define the approximate adjoint state p^δ in $H_0^1\left(\Omega\right)$ as the solution of the following adjoint equation

$$A^*p^\delta + \beta_\delta'\left(y^\delta - \varphi\right)p^\delta = y^\delta - z \quad \text{in } \Omega, \quad p^\delta = 0 \quad \text{on } \partial\Omega,$$

where A^* is the adjoint operator of A. As φ^δ is the solution of the problem $\left(\mathcal{P}^\delta\right)$, we get

$$\forall\varphi \in \mathcal{U}_{ad}, \quad \frac{d}{dt}J_\delta\left(\varphi^\delta + t\left(\varphi - \varphi^\delta\right)\right)_{|t=0} \geq 0.$$

That is

$$\forall\varphi \in \mathcal{U}_{ad}, \quad \int_\Omega \left(\chi^\delta\left(y^\delta - z\right) + \nu\Delta\varphi^\delta\Delta\left(\varphi - \varphi^\delta\right)\right)\ dx + \int_\Omega \left(\varphi^\delta - \varphi^*\right)\left(\varphi - \varphi^\delta\right)\ dx \geq 0,$$

where $\chi^\delta \in H_0^1(\Omega)$ and satisfies

$$A\chi^\delta + \beta_\delta'\left(y^\delta - \varphi^\delta\right)\chi^\delta = \beta_\delta'\left(y^\delta - \varphi^\delta\right)\left(\varphi - \varphi^\delta\right) \quad \text{in } \Omega.$$

from the definition of p^δ, we obtain

$$\int_\Omega \chi^\delta A^* p^\delta dx + \int_\Omega \beta_\delta'\left(y^\delta - \varphi^\delta\right) p^\delta \chi^\delta dx + \nu \int_\Omega \Delta\varphi^\delta \Delta\left(\varphi - \varphi^\delta\right) dx + \int_\Omega \left(\varphi^\delta - \varphi^*\right)\left(\varphi - \varphi^\delta\right) dx \geq 0,$$

where A^* denotes the adjoint operator of de A. Then

$$\int_\Omega A\chi^\delta p^\delta dx + \int_\Omega \beta_\delta'\left(y^\delta - \varphi^\delta\right) p^\delta \chi^\delta\, dx + \nu \int_\Omega \Delta\varphi^\delta \Delta\left(\varphi - \varphi^\delta\right) dx + \int_\Omega \left(\varphi^\delta - \varphi^*\right)\left(\varphi - \varphi^\delta\right) dx \geq 0,$$

we obtain

$$\int_\Omega \beta_\delta'\left(y^\delta - \varphi^\delta\right) p^\delta \left(\varphi - \varphi^\delta\right) dx + \nu \int_\Omega \Delta\varphi^\delta \Delta\left(\varphi - \varphi^\delta\right) dx + \int_\Omega \left(\varphi^\delta - \varphi^*\right)\left(\varphi - \varphi^\delta\right) dx \geq 0.$$

In the sequel, we set

$$\mu^\delta := \beta_\delta'\left(y^\delta - \varphi^\delta\right) p^\delta \in L^2(\Omega). \tag{3.1}$$

Finally, we obtain

Theorem 3.6. *If φ^δ is an optimal solution of $\left(\mathcal{P}^\delta\right)$ and $y^\delta := \mathcal{T}^\delta\left(\varphi^\delta\right)$, there exists p^δ in $H^2(\Omega) \cap H_0^1(\Omega)$ and μ^δ in $L^2(\Omega)$ such that the following system holds*

$$Ay^\delta + \beta_\delta\left(y^\delta - \varphi^\delta\right) = f \quad \textit{in } \Omega, \quad y^\delta = 0 \quad \textit{on } \partial\Omega, \tag{3.2a}$$

$$A^* p^\delta + \mu^\delta = y^\delta - z \quad \textit{in } \Omega, \quad p^\delta = 0 \quad \textit{on } \partial\Omega, \tag{3.2b}$$

$$\left(\mu^\delta + \varphi^\delta - \varphi^*, \varphi - \varphi^\delta\right) + \nu\left(\Delta\varphi^\delta, \Delta\left(\varphi - \varphi^\delta\right)\right) \geq 0, \quad \forall \varphi \in \mathcal{U}_{ad}. \tag{3.2c}$$

In the case $\mathcal{U}_{ad} := L^2(\Omega)$, we make this optimality system more precise.
Let χ in $\mathcal{U}$ and choose $\varphi = \varphi^\delta \pm \chi$; by the equation (3.2c), we obtain

$$\left(\mu^\delta + \varphi^\delta - \varphi^*, \chi\right) + \nu\left(\Delta\varphi^\delta, \Delta\chi\right) = 0, \quad \forall \chi \in \mathcal{U}. \tag{3.3}$$

Set $h^\delta = \Delta\varphi^\delta$ in $L^2(\Omega)$, so that for any χ in $\mathscr{D}(\Omega)$, the relation (3.3) gives

$$\left(\mu^\delta + \varphi^\delta - \varphi^*, \chi\right) + \nu\left(h^\delta, \Delta\chi\right) = 0 \text{ (in the distribution sens),}$$

that is

$$-\nu\Delta h^\delta = \mu^\delta + \varphi^\delta - \varphi^* \in \mathscr{D}'(\Omega).$$

Using the same techniques as in [6], we deduce that $h^\delta_{|\partial\Omega} = 0$. Consequently, $h^\delta \in \mathcal{U}$, and it is the unique solution of

$$-\nu\Delta h^\delta = \mu^\delta + \varphi^\delta - \varphi^* \text{ in } L^2(\Omega), \quad h^\delta = 0 \quad \text{on } \partial\Omega.$$

The last relation may be written as

$$-\nu\left(\Delta^2\varphi^\delta, u\right) = \left(\mu^\delta, u\right) - \left(\varphi^\delta - \varphi^*, u\right) \quad \text{in } \Omega, \quad \varphi^\delta = 0 \quad \text{on } \partial\Omega.$$

Thanks to Green's formula, the previous relation reads

$$-\nu\int_\Omega \Delta^2\varphi\, u\, dx - \left(\mu^\delta, u\right) - \left(\varphi^\delta - \varphi^*, u\right) =$$
$$= \int_\Omega \Delta\varphi\Delta u\, dx - \int_\Gamma \left(\Delta\varphi\frac{\partial u}{\partial\eta} - u\frac{\partial\Delta\varphi}{\partial\eta}\right) - \left(\mu^\delta, u\right) - \left(\varphi^\delta - \varphi^*, u\right).$$

So $\Delta\varphi$ vanishes on the boundary $\partial\Omega$, and we conclude that φ^δ belongs to $\mathcal{W} := \{u \,|\, u \in H^2(\Omega) \cap H_0^1(\Omega)\, \text{et}\, \Delta u_{|\partial\Omega} = 0\}$. Finally we have:

Corollary 3.4. *Assume conditions of Theorem 3.6 are fulfilled, and $\mathcal{U}_{ad} = \mathcal{U}$, then the optimality system $(\mathcal{S}^\delta)$ reads*

$$a\left(y^\delta, v\right) + \left(\beta_\delta\left(y^\delta - \varphi^\delta\right), v\right) = (f, v), \quad \forall v \in \mathcal{U}, \tag{3.4a}$$

$$a^*\left(p^\delta, w\right) + \left(\mu^\delta, w\right) = \left(y^\delta - z, w\right), \quad \forall w \in \mathcal{U}, \tag{3.4b}$$

$$\nu\left(\Delta^2\varphi^\delta, u\right) - \left(\mu^\delta, u\right) = \left(\varphi^\delta - \varphi^*, u\right), \quad \forall u \in \mathcal{W}. \tag{3.4c}$$

Here a^* denotes the adjoint form of a (associated with the adjoint operator A^*).

4 First order necessary optimality conditions for $(\mathcal{P})$

In this section, we have to estimate p^δ, and gives more convergence results, then we may pass to the limit in the system (3.4) as $\delta \to 0$.

Theorem 4.1. *When δ goes to 0, p^δ converges to p^* weakly in $H_0^1(\Omega)$ and μ^δ converges to μ^* weakly satr in $H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$, and*

$$\langle\mu^*, p^*\rangle \geq 0.$$

where $\mathcal{M}(\Omega)$ is the set of all regular signed measures in Ω.

Proof. Using (3.4b), we obtain

$$a^*\left(p^\delta, p^\delta\right) + \int_\Omega \beta'_\delta\left(y^\delta - \varphi^\delta\right)\left(p^\delta\right)^2 dx = \left(y^\delta - z, p^\delta\right). \tag{4.1}$$

As $\beta'(\cdot) \geq 0$, and thanks to hypothesis (2.2) and (2.3), we get

$$\left\|p^\delta\right\|_{H^1(\Omega)} \leq C\left\|y^\delta - z\right\|_{H^1(\Omega)},$$

which implies that p^δ converges to p^* weakly in $H_0^1(\Omega)$. Consequently $A^* p^\delta$ is uniformly bounded in $H^{-1}(\Omega)$ and

$$\mu^\delta = -A^* p^\delta + y^\delta - z. \tag{4.2}$$

Let $\gamma_\varepsilon \in \mathscr{C}^1(\mathbb{R})$ be a family of smooth approximations to the sign function and satisfy the following [17]:

$$\gamma'_\varepsilon(r) \geq 0 \quad \forall r \in \mathbb{R},$$

and

$$\gamma'_\varepsilon(r) := \begin{cases} 1 & \text{if } r > \varepsilon \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < -\varepsilon, \end{cases}$$

Then we can multiply (4.2) by $\gamma_\varepsilon\left(p^\delta\right)$ and integrate it over Ω. As a result, we get

$$\int_\Omega \mu^\delta \gamma_\varepsilon\left(p^\delta\right) dx \le C.$$

Letting $\varepsilon \to 0$, we have

$$\left\|\mu^\delta\right\|_{L^1(\Omega)} \le C.$$

Hence μ^δ is bounded in $L^1(\Omega)$ and consequently it is also bounded in $\mathcal{M}(\Omega)$, thus, μ^δ converge to μ^* weakly star in $H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$, such that

$$A^* p^* + \mu^* = y^* - z \quad \text{in} \quad \Omega, \quad p^* = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

As $0 \le \beta' \le 1$, by using (4.1) we get

$$a^*\left(p^\delta, p^\delta\right) \le \left(y^\delta - z, p^\delta\right).$$

And by the lower semi-continuity of a^*

$$\langle A^* p^*, p^* \rangle = a^*\left(p^*, p^*\right) \le \liminf_{\delta\to\infty} \left(y^\delta - z, p^\delta\right) = \left(y^* - z, p^*\right).$$

From (4.3), we obtain

$$0 \le \langle A^* p^*, p^* \rangle = \left(y^* - z, p^*\right) - \langle \mu^*, p^* \rangle \le \left(y^* - z, p^*\right),$$

so that

$$\langle \mu^*, p^* \rangle \ge 0.$$

□

In the sequel, we set $\xi^\delta := \beta_\delta\left(y^\delta - \varphi^\delta\right)$, then we obtain the following results.

Theorem 4.2. *When δ goes to 0, ξ^δ converges to ξ^* weakly in $L^2(\Omega)$, where ξ^* is negative and the state equation* (3.2a) *gives*

$$Ay^* + \xi^* = f.$$

Proof. From (3.4a), we obtain with $v = \beta_\delta\left(y^\delta - \varphi^\delta\right)$

$$\left\langle A\left(y^\delta - \varphi^\delta\right), \beta_\delta\left(y^\delta - \varphi^\delta\right)\right\rangle + \left(\beta_\delta\left(y^\delta - \varphi^\delta\right), \beta_\delta\left(y^\delta - \varphi^\delta\right)\right) = \left(f - A\varphi^\delta, \beta_\delta\left(y^\delta - \varphi^\delta\right)\right).$$

For the seek of simplicity, we set $r^\delta := y^\delta - \varphi^\delta$; with (2.1), this gives

$$\sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial r^\delta}{\partial x_i} \frac{\partial r^\delta}{\partial x_j} \beta'_\delta\left(r^\delta\right) dx + \int_\Omega a_0 r^\delta \beta_\delta\left(r^\delta\right) dx + \left\|\beta_\delta\left(r^\delta\right)\right\|^2_{L^2(\Omega)} + \\ + \sum_{i=1}^n \int_\Omega a_i \frac{\partial r^\delta}{\partial x_i} \beta_\delta\left(r^\delta\right) dx = \left(f - A\varphi^\delta, \beta_\delta\left(r^\delta\right)\right). \tag{4.4}$$

With hypothesis (**H**), we get

$$\sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial r^\delta}{\partial x_i} \frac{\partial r^\delta}{\partial x_j} \beta'_\delta\left(r^\delta\right) dx \ge \int_\Omega m \sum_{i=0}^n \left(\frac{\partial r^\delta}{\partial x_i}\right)^2 \beta'_\delta\left(r^\delta\right) dx. \\ \ge 0. \tag{4.5}$$

From (4.4), (4.5) and (**H**), we obtain

$$\begin{aligned}\left\|\beta_\delta\left(r^\delta\right)\right\|_{L^2(\Omega)}^2 &\leq \left\|f - A\varphi^\delta\right\|_{L^2(\Omega)}\left\|\beta_\delta\left(r^\delta\right)\right\|_{L^2(\Omega)} + \sum_{i=1}^{n}\left\|a_i\right\|_{L^\infty(\Omega)}\left(\left\|\nabla r^\delta\right\|_{L^2(\Omega)}\left\|\beta_\delta\left(r^\delta\right)\right\|_{L^2(\Omega)}\right)\\ &\leq \max\left\{1, \sum_{i=1}^{n}\left\|a_i\right\|_{L^\infty(\Omega)}\right\}\left(\left\|f - A\varphi^\delta\right\|_{L^2(\Omega)} + \left\|\nabla r^\delta\right\|_{L^2(\Omega)}\right)\left\|\beta_\delta\left(r^\delta\right)\right\|_{L^2(\Omega)}.\end{aligned}$$

Finally, we get

$$\left\|\beta_\delta\left(y^\delta - \varphi^\delta\right)\right\|_{L^2(\Omega)} \leq \alpha\left(\left\|f - A\varphi^\delta\right\|_{L^2(\Omega)} + \left\|\nabla\left(y^\delta - \varphi^\delta\right)\right\|_{L^2(\Omega)}\right),$$

where $\alpha := \max\left\{1, \sum_{i=1}^{n}\left\|a_i\right\|_{L^\infty(\Omega)}\right\}$, so that

$$\left\|\beta_\delta\left(y^\delta - \varphi^\delta\right)\right\|_{L^2(\Omega)} \leq \alpha\left(\left\|f - A\varphi^\delta\right\|_{L^2(\Omega)} + \left\|y^\delta - \varphi^\delta\right\|_{H^1(\Omega)}\right).$$

Since φ^δ and y^δ are respectively bounded in $H^2(\Omega) \cap H_0^1(\Omega)$ and $H_0^1(\Omega)$, we deduce that ξ^δ is bounded in $L^2(\Omega)$, by passing to the limit where $\delta \to 0$, we obtain that ξ^δ converge to ξ^* weakly in $L^2(\Omega)$. Passing to the limit in (3.2a), gives

$$Ay^* + \xi^* = f.$$

where ξ^*, is negative and we get $y^* \in H^2(\Omega) \cap H_0^1(\Omega)$. □

Corollary 4.1. *As φ^* is in $H^2(\Omega) \cap H_0^1(\Omega)$, $y^* := \mathcal{T}^*(\varphi^*)$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. As ξ^* and f belongs to $L^2(\Omega)$, then $Ay^* \in L^2(\Omega)$ and $y^* \in H^2(\Omega)$. □

Now, we give some Lemmas, the proof the below Theorem 4.3.

Lemma 4.1. *When δ goes to 0, $\left(\mu^\delta, \left(y^\delta - \varphi^\delta\right)^+\right) \to \langle\mu^*, y^* - \varphi^*\rangle$, and $\langle\mu^*, y^* - \varphi^*\rangle = 0$.*

Proof. By the definition of β and μ^δ (3.1), we get

$$\left(\mu^\delta, \left(y^\delta - \varphi^\delta\right)^+\right) = 0,$$

where $v^+ := \max\{0, v\}$, by Theorem 3.2 $\left(y^\delta - \varphi^\delta\right)^+$ converges strongly to $(y^* - \varphi^*)$ in $H_0^1(\Omega)$, then

$$\left(\mu^\delta, \left(y^\delta - \varphi^\delta\right)^+\right) \to \langle\mu^*, y^* - \varphi^*\rangle,$$

and

$$\langle\mu^*, y^* - \varphi^*\rangle = 0.$$

□

Lemma 4.2. *When δ goes to 0, $\left(\xi^\delta, p^\delta\right) \to \langle\xi^*, p^*\rangle$, and $\langle\xi^*, p^*\rangle = 0$.*

Proof. As befor, we set $r^\delta := y^\delta - \varphi^\delta$, so that $\xi^\delta = \beta_\delta\left(r^\delta\right)$. From the definition of β and β', we get respectively

$$\begin{aligned}\left(\xi^\delta, p^\delta\right) &= \left(\beta_\delta\left(r^\delta\right), p^\delta\right) = \\ &= \tfrac{1}{\delta}\left[\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \left(r^\delta + \tfrac{1}{4}\right) p^\delta dx - \int_{\left\{-\frac{1}{2} \le r^\delta \le 0\right\}} \left(r^\delta\right)^2 p^\delta dx\right],\end{aligned}$$

and

$$\begin{aligned}\left(\mu^\delta, r^\delta\right) &= \left(\beta'_\delta\left(r^\delta p^\delta\right), p^\delta\right) = \\ &= \tfrac{1}{\delta}\left[\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} p^\delta r^\delta dx - 2\int_{\left\{-\frac{1}{2} \le r^\delta \le 0\right\}} \left(r^\delta\right)^2 p^\delta dx\right].\end{aligned}$$

from that, we get

$$\left(\xi^\delta, p^\delta\right) - \tfrac{1}{2}\left(\mu^\delta, r^\delta\right) = \tfrac{1}{\delta}\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \tfrac{1}{2}\left(r^\delta + \tfrac{1}{2}\right) p^\delta d\boldsymbol{x},$$

Then, we obtain

$$\left|\left(\xi^\delta, p^\delta\right) - \tfrac{1}{2}\left(\mu^\delta, r^\delta\right)\right| \le \tfrac{1}{\delta}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \tfrac{1}{4}\left(r^\delta + \tfrac{1}{2}\right)^2 dx\right)^{1/2}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \left(p^\delta\right)^2 dx\right)^{1/2}.$$

As $H^1(\Omega) \hookrightarrow L^q(\Omega)$ with $2 < q \le 6$, we have

$$\begin{aligned}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} (p^\delta)^2 dx\right)^{1/2} &\le \left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \left(p^\delta\right)^q dx\right)^{1/q}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} dx\right)^{(q-2)/2q} \\ &\le C\left\|p^\delta\right\|_{H^1(\Omega)}\left(\text{meas}\left\{r^\delta \le -\tfrac{1}{2}\right\}\right)^{(q-2)/2q},\end{aligned}$$

where meas$\{\mathbb{A}\}$ is the measure of the set $\mathbb{A}$. Then we write

$$\left|\left(\xi^\delta, p^\delta\right) - \tfrac{1}{2}\left(\mu^\delta, r^\delta\right)\right| \le C\left\|p^\delta\right\|_{H^1(\Omega)} \tfrac{1}{\delta}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \tfrac{1}{4}\left(r^\delta + \frac{1}{2}\right)^2 dx\right)^{1/2}\left(\text{meas}\left\{r^\delta \le -\tfrac{1}{2}\right\}\right)^{(q-2)/2q}. \tag{4.6}$$

We have

$$\tfrac{1}{\delta}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \tfrac{1}{4}\left(r^\delta + \tfrac{1}{2}\right)^2 dx\right)^{1/2} \le \tfrac{1}{\delta}\left(\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \left(r^\delta + \tfrac{1}{4}\right)^2 dx\right)^{1/2}. \tag{4.7}$$

As $\|\beta_\delta\left(r^\delta\right)\|_{L^2(\Omega)}$ is bounded, we deduce that

$$\tfrac{1}{\delta^2}\int_{\left\{r^\delta \le -\frac{1}{2}\right\}} \left(r^\delta + \tfrac{1}{4}\right)^2 dx \le C,$$

and

$$\text{meas}\left\{r^\delta \le -\tfrac{1}{2}\right\} \le C\delta^2.$$

then, by passing to the limit when δ goes to 0, we have

$$\lim_{\delta\to 0}\left(\text{meas}\left\{r^\delta \leq -\tfrac{1}{2}\right\}\right) = 0.$$

Since $\|p^\delta\|_{H^1(\Omega)}$ is bounded, from (4.6) and (4.7), we obtain

$$\left|\left(\xi^\delta, p^\delta\right) - \tfrac{1}{2}\left(\mu^\delta, r^\delta\right)\right| \leq C\left(\text{meas}\left\{r^\delta \leq -\tfrac{1}{2}\right\}\right)^{(q-2)/2q},$$

and

$$\lim_{\delta\to 0}\left|\left(\xi^\delta, p^\delta\right) - \tfrac{1}{2}\left(\mu^\delta, r^\delta\right)\right| = 0,$$

so when δ goes to 0, $\left(\xi^\delta, p^\delta\right)$ converges to 0, and with Lemma 4.2, we get

$$\langle \xi^*, p^* \rangle = 0.$$

□

Finally, from the previous convergence results, we obtain the main result of this section

Theorem 4.3. *Let φ^*, be an optimal solution of problem $(\mathcal{P})$. Then $\Delta\varphi^*$ belong to $H_0^1(\Omega)$ and there exists p^* in $H_0^1(\Omega)$, $\xi^* \leq 0$ in $L^2(\Omega)$ and μ^* in $H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$, such that the following optimality $(\mathcal{S})$ system holds*

$$Ay^* + \xi^* = f \quad \textit{in } \Omega, \quad y^* = 0 \quad \textit{on } \partial\Omega, \tag{1.a}$$

$$A^*p^* + \mu^* = y^* - z \quad \textit{in } \Omega, \quad p^* = 0 \quad \textit{on } \partial\Omega, \tag{2.a}$$

$$-\nu\Delta^2\varphi^* + \mu^* = 0 \quad \textit{in } \Omega, \quad \Delta\varphi^* = \varphi^* = 0 \quad \textit{on } \partial\Omega, \tag{3.a}$$

$$\langle \mu^*, y^* - \varphi^* \rangle = 0, \tag{4.a}$$

$$\langle \xi^*, p^* \rangle = 0, \tag{5.a}$$

$$a^*(p^*, p^*) - (z - y^*, p^*) \leq 0, \tag{6.a}$$

$$\langle p^*, \mu^* \rangle \geq 0. \tag{7.a}$$

Remark 4.1. *In [12], Ito et Kunish had obtained the following optimality condition system $(\widetilde{\mathcal{S}})$*

$$Ay^* + \xi^* = f, \quad \xi^* = \max(0, \xi^* + y^* - \varphi^*), \tag{1.b}$$

$$A^*p + \mu^* = y^* - z \quad \textit{in } H^{-1}(\Omega), \tag{2.b}$$

$$\langle -\nu\Delta\varphi^* + \nu\varphi^* + \mu^*, \chi - \varphi^* \rangle \geq 0 \quad \textit{for all } \chi \in \mathcal{U}_{ad}, \tag{3.b}$$

$$\mu^*(y^* - \varphi^*) = 0 \quad \textit{a.e. in } \Omega, \tag{4.b}$$

$$p^*\xi^* = 0 \quad \textit{a.e. in } \Omega, \tag{5.b}$$

$$a^*(p^*,p^*) - (z-y^*,p^*) \leq 0, \tag{6.a}$$

$$\langle \mu^*, p^*\phi\rangle \geq 0 \quad \text{for all } \phi \in W^{1,\overline{q}}(\Omega), \quad \text{with } \phi \geq 0, \text{ and } \overline{q} > n. \tag{7.b}$$

Indeed, they studied the following optimal control problem $(\widetilde{\mathcal{P}})$

$$\langle Ay - f, \phi - y\rangle \geq 0 \quad \text{for all} \quad \phi \in \mathcal{K}(\varphi),$$

with

$$f \in L^2(\Omega), \quad \varphi \in \mathcal{U}_{ad} \quad \text{with } \varphi \leq 0 \quad \text{in} \quad \partial\Omega,$$

such that the cost functional is given by

$$\widetilde{J}(\varphi) := \tfrac{1}{2}\int_\Omega (\mathcal{T}(\varphi) - z)^2\, dx + \tfrac{\nu}{2}\left(\int_\Omega \left(|\varphi|^2 + |\nabla\varphi|^2\right) dx\right),$$

We notices a likeness between the two systems $(\mathcal{S})$ *and* $(\widetilde{\mathcal{S}})$ *, exepted for the equations (3.a) and (3.b) (are respectively the differential of the objective function* J *and* $\widetilde{J}$*); i.e. in [12], the authors treated the optimal control problem* $(\widetilde{\mathcal{P}})$*, such that* φ *belong to* $\mathcal{U}_{ad} := \{\varphi \in X : \varphi(x) \geq 0, \text{ on } \partial\Omega \text{ and } -a(\varphi, v) + (f, v) \leq (\overline{\lambda}, v) \text{ for all } v \in V \text{ with } v \geq 0\}$ *with* H^1*-obstacle (where* $\overline{\lambda} \in L^2(\Omega)$ *satisfying* $\overline{\lambda} \geq 0$ *a.e. on* Ω*), and in our work, we had study the optimal control problem* $(\mathcal{P})$*, where* φ *is in* $\mathcal{U}_{ad} := H^2(\Omega) \cap H_0^1(\Omega)$*, with* H^2*-obstacle.*

Conclusion

In this work, we treated the theoretical aspect of the problem $(\mathcal{P})$, we proved the existence of optimal solutions and constructed a necessary optimality conditions system. Additional optimal obstacle regularity has been also provided. Currently we study the numerical aspect of the problem $(\mathcal{P})$, via a numerical strategy based on the direct resolution of the optimality system and using a fixed point algorithm.
The author is grateful to Prof. M. Bergounioux for their instructive suggestions.

References

[1] R.A. Adams and J.F. Fournier , *Sobolev Spaces*, Academic Press, Paris, (second edition), 2003.

[2] D.R. Adams and S. Lenhart, *An obstacle control problem with a source term*, Applied Mathematics and Optimization, (**47**), (2002), 79-95.

[3] D.R. Adams and S. Lenhart, and J. Yong, *Optimal control of the obstacle for an elliptic variational inequality*, Applied Mathematics and Optimization, (**38**), (1998), 121-140.

[4] V. Barbu and P. Korman, *Approximating optimal controls for elliptic obstacle problem by monotone iteration schems*, Numerical functional analysis and Optimization, (**12**), (1991), 429-442.

[5] V. Barbu, *Optimal control of varitional inequalities*, Pitman, London, 1984.

[6] M. Bergounioux and S. Lenhart, *Optimal control of the bilateral obstacle problems*, SIAM Journal on Control and Optimization, (**43**), (2004), 249-255.

[7] M. Bergounioux and S. Lenhart, *Optimal control of the obstacle in semilinear variational inequalities,* Positivity, (**8**), (2004), 229-242.

[8] M. Bergounioux,*Optimal control of semilinear elliptic obstacle problems*, Journal of Nonlinear and Convex Analysis, (**3**), (2002), 25-39.

[9] H. Brezis and G. Stampacchia, *Sur La régularité de la solution d'inéquations elliptiques*, Bulletin de la Socit Mathmatique de France, (**96**), (1968), 153-180.

[10] A. Friedman, *Variational principles and free boundary problems*, John Wiley, New York, 1982.

[11] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, (second edition), Springer-Verlag, Berlin, 1983.

[12] K. Ito and K. Kunisch, *Optimal control of obstacle problems by H^1-obstacles*, Applied Mathematics and Optimization, (**56**), (2007), 1-17.

[13] J. L. Lions and E. Magens, *Problèmes aux limites non homogènes et applivations*, (**1**), Dunod, Paris, 1968.

[14] H. Lou, *On the regularity of an obstacle control problem*, Journal of Mathematical Analysis and Applications, (**258**), (2001), 32-51.

[15] F. Mignot, *Contrôle dans les inéquations variationelles elliptiques*, Journal of Functional Analysis, (**22**), (1976), 466-476.

[16] F. Mignot and J.P. Puel, *Optimal control in some variational inequalities*, SIAM Journal on Control and Optimization, (**22**), (1984), 466-476.

[17] Y. Ye and Q. Chen, *Optimal control of the obstacle in a quasilinear elliptic variational inequality*, Journal of Mathematical Analysis and Applications, (**294**), (2004), 258-272.