A new test procedure of independence in Copula models via χ^2-divergence

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Key Words: dependence function; Gumbel copula; pseudo-likelihood; asymptotic theory; multivariate rank statistics; semiparametric estimation.

Mathematics Subject Classification :62F03, 62F10, 62F12, 62H12, 62H15.

Abstract. We introduce a new test procedure of independence in the framework of parametric copulas with unknown marginals. The method is based essentially on the dual representation of χ^2-divergence on signed finite measures. The asymptotic properties of the proposed estimate and the test statistic are studied under the null and alternative hypotheses, with simple and standard limit distributions both when the parameter is an interior point or not.

1 Introduction and motivations

Parametric models for copulas have been intensively investigated during the last decades. *Copulas* have become of great interest in applied statistics, because of the fact that they constitute a flexible and robust way to model dependence between the marginals of random vectors. The reader may refer to the following books for excellent expositions of the basics of copula theory : Nelsen (1999) and Joe (1997). In this framework, semiparametric inference methods, based on *pseudo-likelihood*, have been applied to copulas by a number of authors (see, e.g., Shih and Louis (1995), Wang and Ding (2000), Tsukahara (2005) and the references therein). Throughout the available literature, investigations on the asymptotic properties of parametric estimators, as well as the relevant test statistics, have privileged the case where the parameter is an interior point of the admissible domain. However, for most parametric copula models of interest, the boundaries of the admissible parameter spaces include some important parameter values, typically among which, that corresponding to the independence of margins. This paper concentrates on this specific problem. We aim, namely, to investigate parametric inference procedures, in the case where the parameter belongs to the boundary of the admissible domain. In particular, the usual limit laws both for parametric copula estimators and test statistics become invalid under these limiting cases, and, in particular, under marginal independence. Motivated by this observation, we will introduce a new semiparametric inference procedure based on χ^2-*divergence* and *duality* technique. We will show that the proposed estimator remains asymptotically normal, even under the marginal independence assumption. This will allow us to introduce test statistic of independence, his study will be made, both under the null and alternative hypotheses.

It is well known since the work of Sklar (1959) that the joint behavior of a bivariate vector (X_1, X_2) with d.f. $\mathbf{F}(x_1, x_2) := \mathbf{P}(X_1 \leq x_1, X_2 \leq x_2)$, and continuous marginal d.f.'s $F_i(x_i) := P(X_i \leq x_i)$, $i = 1, 2$, is characterized by the copula (or dependence function) $C(\cdot, \cdot)$ associated with $\mathbf{F}(\cdot, \cdot)$. The copula function is defined, for all $(u_1, u_2) \in (0, 1)^2$, through the identity

$$C(u_1, u_2) := \mathbf{P}\left\{F_1(X_1) \leq u_1, F_2(X_2) \leq u_2\right\}.$$

Many useful multivariate models for dependence between X_1 and X_2 turn out to be generated by *parametric* families of copulas of the form $\{C_\theta;\ \theta \in \Theta\}$, typically indexed by a vector valued parameter $\theta \in \Theta \subseteq \mathbb{R}^p$ (see, e.g., Kimeldorf and Sampson (1975a), Kimeldorf and Sampson (1975b), and Joe (1993)). The nonparametric approach to copula estimation has been initiated by Deheuvels (1979b), who introduced and investigated the *empirical copula process*. In addition, Deheuvels (1980, 1981) described the limiting behavior of this empirical process (see, also Fermanian *et al.* (2004a) and the references therein). In this paper, we consider semiparametric copula models with unknown marginals.

In order to estimate the unknown *true* value of the parameter $\theta \in \Theta$, which we denote, throughout the sequel, by $\theta_T \in \Theta$, some *semiparametric estimation* procedures, based on the maximization, on the parameter space Θ, of properly chosen *pseudo-likelihood* criterion, have been proposed by Oakes (1994), and studied by Genest *et al.* (1995), Shih and Louis (1995), Wang and Ding (2000) and Tsukahara (2005) among others. In each of these papers, some asymptotic normality properties are established for $\sqrt{n}\big(\tilde{\theta} - \theta_T\big)$, where $\tilde{\theta} = \tilde{\theta}_n$ denotes a properly chosen estimator of θ_T. This is achieved, provided that θ_T lies in the *interior*, denoted by $\mathring{\Theta}$, of the parameter space $\Theta \subseteq \mathbb{R}^p$. On the other hand, the case where $\theta_T \in \partial\Theta := \overline{\Theta} - \mathring{\Theta}$ is a *boundary value* of Θ, has not been studied in a systematical way until present. Moreover, it turns out that, for the above-mentioned estimators, the asymptotic normality of $\sqrt{n}\big(\tilde{\theta}-\theta_T\big)$, may fail to hold for $\theta_T \in \partial\Theta$; indeed, under some regularity conditions, when θ is univariate, we can prove that the limit law is the distribution of $Z\mathbb{1}_{(Z\geq 0)}$ where Z is a centred normal variable, and that the limit law of the generalized pseudo-likelihood ratio statistic is a mixture of chi-square laws with one degree of freedom and Dirac measure at zero; see Bouzebda and Keziou (2008). Furthermore, when the parameter is multivariate, the derivation of the limit distributions under the null hypothesis of independence, becomes much more complex; see Self and Liang (1987). Also, the limit distributions are not standard which yields formidable numerical difficulties to calculate the critical value of the test. We cite below some examples of parametric copulas, for which marginal independence is verified for some specific values of the parameter θ, on the boundary $\partial\Theta$ of the admissible parameter set Θ. We start with examples for which θ varies within subsets of $\mathbb{R}$. Such is the case for the extreme value copulas, namely

$$C_A(u_1,u_2) := \exp\left\{\log u_1u_2 A\left(\frac{\log u_1}{\log u_1u_2}\right)\right\}, \tag{1.1}$$

where $A(\cdot)$ is a convex function on $[0,1]$, satisfying

- $A:[0,1]\mapsto[1/2,1]$ such that $\max(t,1-t)\leq A(t)\leq 1$ for all $0\leq t\leq 1$.

For

$$A(t) := A_\theta(t) = (t^\theta+(1-t)^\theta)^{1/\theta};\quad \theta\in[1,\infty[\tag{1.2}$$

we have Gumbel (1960) family of copulas, which is one of the most popular model used to model bivariate extreme values. For

$$A_\theta(t) = 1-(t^{-\theta}+(1-t)^{-\theta})^{-1/\theta};\quad \theta\in[0,\infty[\tag{1.3}$$

we obtain Galambos (1975) family of copulas. Finally for

$$A_\theta(t) = t\Phi\left(\theta^{-1}+\frac{1}{2}\theta\log\left(\frac{t}{1-t}\right)\right)+(1-t)\Phi\left(\theta^{-1}-\frac{1}{2}\theta\log\left(\frac{t}{1-t}\right)\right), \tag{1.4}$$

where $\theta\in[0,\infty[$ and $\Phi(\cdot)$ denoting the standard normal $N(0,1)$ distribution function, we obtain the Hüsler and Reiss (1989) family of copulas. A useful family of copulas, due to Joe (1993), is given, for $0<u_1,u_2<1$, by

$$C_\theta(u_1,u_2) := 1-\left[(1-u_1)^\theta+(1-u_2)^\theta-(1-u_1)^\theta(1-u_2)^\theta\right]^{1/\theta};\quad \theta\in[1,\infty[. \tag{1.5}$$

The Gumbel-Barnett copulas are given, for $0<u_1,u_2<1$, by

$$C_\theta(u_1,u_2) := u_1u_2\exp\left\{-(1-\theta)(\log u_1)(\log u_2)\right\};\quad \theta\in[0,1]. \tag{1.6}$$

The Clayton copulas of positive dependence are such that, for $0 < u_1, u_2 < 1$,

$$C_\theta(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}; \quad \theta \in]0, \infty[. \tag{1.7}$$

Parametric families of copulas with parameter θ varying in $\mathbb{R}^p$, for some $p \geq 2$, include the following classical examples. Below, we set $\theta = \left(\theta_1, \theta_2\right)^\top \in \mathbb{R}^2$.

$$C_\theta(u_1, u_2) := \left\{1 + \left[(u_1^{-\theta_1} - 1)^{\theta_2} + (u_2^{-\theta_1} - 1)^{\theta_2}\right]^{1/\theta_2}\right\}^{-1/\theta_1}, \ \theta \in]0, \infty[\times[1, \infty[; \tag{1.8}$$

$$\begin{aligned} C_\theta(u_1, u_2) := \exp\Big\{ - \Big[\theta_2^{-1} \log\Big(& \exp\left(-\theta_2(\log u_1)^{\theta_1}\right) \\ & + \exp\left(-\theta_2(\log u_2)^{\theta_1}\right) - 1\Big)\Big]^{1/\theta_1}\Big\}, \ \theta \in [1, \infty[\times]0, \infty[. \end{aligned} \tag{1.9}$$

For other examples of the kind, we refer to Joe (1997).

For each of the above examples, the independence case $C_{\theta_T}(u_1, u_2) = u_1 u_2$ (or $A(t) = 1$) occurs at the boundary of the parameter space Θ, i.e., when $\theta_T = 1$ for the models (1.2), (1.5) and (1.6), $\theta_T = 0$ for the models (1.3), (1.4) and (1.7), $\theta_T = (0, 1)^\top$ for the bivariate parameter model (1.8), and $\theta_T = (1, 0)^\top$ for the bivariate parameter model (1.9). In the sequel, we will denote by θ_0 the value of the parameter (when it exists), corresponding to the independence of the marginals, i.e., the value of the parameter for which we have

$$C_{\theta_0}(u_1, u_2) := u_1 u_2, \quad \text{for all } (u_1, u_2) \in (0, 1)^2.$$

Hence, $\theta_0 = 1$ for the models (1.2), (1.5) and (1.6), $\theta_0 = 0$ for the models (1.3), (1.4) and (1.7), $\theta_0 = (0, 1)^\top$ for the model (1.8), and $\theta_0 = (1, 0)^\top$ for the model (1.9). Note that for the models (1.3), (1.4), (1.7), (1.8) and (1.9), $C_{\theta_0}(u_1, u_2) = u_1 u_2$ is naturally defined to be the limit of $C_\theta(\cdot, \cdot)$ when θ tends to θ_0 with values in Θ. We denote $c_\theta(\cdot, \cdot) := \frac{\partial^2}{\partial u_1 \partial u_2} C_\theta(\cdot, \cdot)$ the density of $C_\theta(\cdot, \cdot)$ and we define $c_{\theta_0}(\cdot, \cdot)$ to be the limit of $c_\theta(\cdot, \cdot)$ when θ tends to θ_0 with values in Θ. Hence, we can show that for all the above models $c_{\theta_0}(u_1, u_2) = 1$ for all $0 < u_1, u_2 < 1$.

In contrast with the preceding examples, where $\theta_0 \in \partial\Theta$ is a boundary value of Θ, the case where θ_0 is an interior point of Θ may, at times, occur, but is more seldom. An example where $\theta_0 \in \mathring{\Theta}$ is given by the Farlie-Gumbel-Morgenstern (FGM) copula, defined by

$$C_\theta(u_1, u_2) := u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \ \theta \in \Theta := [-1, 1], \tag{1.10}$$

and for which $\theta_0 = 0 \in \mathring{\Theta} =]-1, 1[$.

In the present article, we will treat parametric estimation of θ_T, and tests of the independence assumption $\theta_T = \theta_0$. We consider both the case where $\theta_0 \in \mathring{\Theta}$ is an interior point of Θ, and the case where $\theta_0 \in \partial\Theta$ is a boundary value of Θ. To treat this case, we propose a new inference procedure, based on an estimation of χ^2-divergence by *duality* technique. This method may be applied independently of the dimension of the parameter space. Also the limit law of the estimate of the parameter is normal and the limiting distribution of the proposed test statistic is χ^2 under independence, either when θ_0 is an interior point, or when θ_0 is a boundary point of Θ. The idea is to include the parameter domain Θ into an enlarged space, say Θ_e, in order to render θ_0 an interior point of the new parameter space, Θ_e. The conclusion is then obtained through an application of χ^2-divergence and duality technique. Our methods rely on the fact that, under appropriate assumptions, the definition of the density $c_\theta(\cdot,\cdot) := \frac{\partial^2}{\partial u_1 \partial u_2} C_\theta(\cdot,\cdot)$ of $C_\theta(\cdot,\cdot)$, pertaining to the models we consider, may be extended beyond the *standard* domain of variation Θ of θ. On the other hand, the definition of $c_\theta(\cdot,\cdot)$ which corresponds to these extensions, is then, in general, no longer a density, and may, at times, become negative. For example, such is the case for the parametric models (1.2), (1.5), (1.3) and (1.4), for which $c_\theta(\cdot,\cdot)$ is meaningful for some $\theta \notin \Theta$, but then, becomes negative over some non-negligible (with respect to Lebesgue's measure) subsets of $(0,1)^2$. This implies that the log-likelihood of the data is not properly defined on the whole space Θ_e. For this reason, we will use the χ^2-divergence between signed finite measures. We will discuss this problem in more details, below, in section 2.

The remainder of the present paper is organized as follows. In section 2, we present our semiparametric inference procedure, based upon optimization of the χ^2-divergence between the model $(C_\theta, \theta \in \Theta_e)$ and the empirical copula associated to the data, and by using the *dual representation* of χ^2-divergence. We then derive the asymptotic limiting distribution of the proposed estimator. It will become clear later on from our results, that the asymptotic normality of the estimate holds, even under the independence assumption, when, either, θ_0 is an interior, or a boundary point of Θ. The proposed test statistic of independence is also studied, under the null hypothesis $\mathscr{H}_0$ of independence, as well as under the alternative hypothesis. The limiting asymptotic distribution of the test statistic under the alternative hypothesis is used to derive an approximation to the power function. An application of the forthcoming results will allow us to evaluate the sample size necessary to guarantee a pre-assigned power level, with respect to a specified alternative. Finally, section 4 reports a short simulation results, to illustrate the performance of the proposed test statistic. The proofs of these results will be postponed to the appendix.

2 A semiparametric estimation procedure through χ^2-divergence

As mentioned earlier, the problem of estimating θ, when $\theta \in \partial\Theta$, has not been systematically considered in the scientific literature; and the classical asymptotic normality property of the estimators is no longer satisfied. To overcome this difficulty, in what follows, we enlarge the parameter space Θ into a wider space $\Theta_e \supset \Theta$. This is tailored to let θ_0 become an interior point of Θ_e. Naturally, we assume that the definition of the function $c_\theta(\cdot,\cdot)$ may be extended to Θ_e. The difficulty associated with this construction is that, subject to a proper definition, the densities $c_\theta(\cdot,\cdot)$ of $C_\theta(\cdot,\cdot)$ with respect to the Lebesgue's measure, may become negative on some non negligible subsets of $I := (0,1)^2$ (in this case $c_\theta(\cdot,\cdot)$ becomes the density of a signed measure, see remark 3.2). Note that just as Deheuvels's empirical copula is not a copula, $c_\theta(\cdot,\cdot)$ for $\theta \in \Theta_e$ is not necessarily a copula density and fail to integrate to 1. When such is the case, a semiparametric estimation of θ_T via log-likelihood cannot be used. To circumvent this difficulty, we introduce a new inference procedure, based on χ^2-divergence method, and duality technique. Recall that the χ^2-divergence between a bounded signed measure $\mathbf{Q}$, and a probability $\mathbf{P}$ on $\mathscr{D}$, when $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$, is defined by

$$\chi^2(Q,P) := \int_{\mathscr{D}} \varphi\left(\frac{d\mathbf{Q}}{d\mathbf{P}}\right)\, d\mathbf{P}, \text{ where } \varphi \;:\; x \in \mathbb{R} \mapsto \varphi(x) := \frac{1}{2}(x-1)^2. \tag{2.1}$$

In the sequel, we denote by $\chi^2(\theta_0,\theta_T)$ the χ^2-divergence between $C_{\theta_0}(\cdot,\cdot)$ and $C_{\theta_T}(\cdot,\cdot)$. Applying the dual representation of ϕ-divergence obtained by Broniatowski and Keziou (2006) Theorem 4.4, we readily obtain that $\chi^2(\theta_0,\theta_T)$ can be rewritten as

$$\chi^2(\theta_0,\theta_T) := \sup_{f\in\mathscr{F}} \left\{ \int_I f\; dC_{\theta_0} - \int_I \varphi^*(f)\; dC_{\theta_T} \right\}, \tag{2.2}$$

where $\varphi^*(\cdot)$ is used to denote the convex conjugate of $\varphi(\cdot)$, namely, the function defined by

$$\varphi^* \;:\; t \in \mathbb{R} \mapsto \varphi^*(t) := \sup_{x\in\mathbb{R}} \{tx - \varphi(x)\} = \frac{t^2}{2} + t,$$

and $\mathscr{F}$ is an arbitrary class of measurable functions, fulfilling the following conditions $\forall f \in \mathscr{F};\ \int |f|\, dC_{\theta_0}$ is finite and $\varphi'(dC_{\theta_0}/dC_{\theta_T}) = \varphi'(1/c_{\theta_T}) \in \mathscr{F}$. Furthermore, the sup in (2.2) is unique and achieved at $f = \varphi'(1/c_{\theta_T})$.

Define the new parameter space Θ_e of θ as follows. Set

$$\Theta_e := \left\{ \theta \in \mathbb{R}^d \text{ such that } \int |\varphi'(1/c_\theta(u_1,u_2))|\; du_1du_2 < \infty \right\}. \tag{2.3}$$

By choosing the class of functions $\mathscr{F}$, via

$$\mathscr{F} := \{(u_1, u_2) \in I \mapsto \varphi'(1/c_\theta(u_1, u_2)) - 1 \ ; \ \theta \in \Theta_e\},$$

we infer from (2.2) the relation

$$\chi^2(\theta_0, \theta_T) := \sup_{\theta \in \Theta_e} \left\{ \int_I \left(\frac{1}{c_\theta(u_1, u_2)} - 1 \right) du_1 du_2 - \int_I \left(\frac{1}{2} \frac{1}{c_\theta(u_1, u_2)^2} - \frac{1}{2} \right) dC_{\theta_T}(u_1, u_2) \right\}. \tag{2.4}$$

It turns out that the supremum in (2.4) is reached iff $\theta = \theta_T$. Moreover, in general, θ_T is an interior point of the new parameter space Θ_e, especially under the null hypothesis of independence, namely, when $\theta_T = \theta_0$. Set

$$m(\theta, u_1, u_2) := \int_I \left(\frac{1}{c_\theta(u_1, u_2)} - 1 \right) \ du_1 du_2 - \left\{ \frac{1}{2} \frac{1}{c_\theta(u_1, u_2)^2} - \frac{1}{2} \right\}. \tag{2.5}$$

Consider a random sample $\{(X_{1k}, X_{2k}); k = 1, \ldots, n\}$ from the distribution of (X_1, X_2) denoted by $\mathbf{F}_{\theta_T}(x_1, x_2) := C_{\theta_T}(F_1(x_1), F_2(x_2)) = \mathbf{P}(X_1 \leq x_1, X_2 \leq x_2)$. In what follows, we propose to estimate the χ^2-divergence $\chi^2(\theta_0, \theta_T)$ between $C_{\theta_0}(\cdot, \cdot)$ and $C_{\theta_T}(\cdot, \cdot)$, by

$$\widehat{\chi}^2(\theta_0, \theta_T) := \sup_{\theta \in \Theta_e} \int_I m(\theta, u_1, u_2) \ dC_n(u_1, u_2), \tag{2.6}$$

and to estimate the parameter θ_T by

$$\widehat{\theta}_n := \arg \sup_{\theta \in \Theta_e} \left\{ \int_I m(\theta, u_1, u_2) \ dC_n(u_1, u_2) \right\}, \tag{2.7}$$

where $C_n(\cdot, \cdot)$ is the modified empirical copula, defined by

$$C_n(u_1, u_2) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{F_{1n}(X_{1k}) \leq u_1\}} \mathbb{1}_{\{F_{2n}(X_{2k}) \leq u_2\}}, \ (u_1, u_2) \in I, \tag{2.8}$$

where

$$F_{jn}(t) := \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{]-\infty, t]}(X_{jk}), \ j = 1, 2,$$

and $\mathbb{1}_A$ stands for the indicator function of the event A.

Remark 2.1 *The choice of the χ^2-divergence among various divergences is motivated by the following statements:*

- *Recall that the ϕ-divergences between a bounded signed measure* $\mathbf{Q}$*, and a probability* $\mathbf{P}$ *on* $\mathscr{D}$*, when* $\mathbf{Q}$ *is absolutely continuous with respect to* $\mathbf{P}$*, is defined by*

$$D_\phi(\mathbf{Q}, \mathbf{P}) := \int_{\mathscr{D}} \phi\left(\frac{d\mathbf{Q}}{d\mathbf{P}}\right)\, d\mathbf{P},$$

where ϕ *is a proper closed convex function from* $]-\infty, \infty[$ *to* $[0, \infty[$ *with* $\phi(1) = 0$ *and such that the domain* $\mathrm{dom}\phi := \{x \in \mathbb{R} : \phi(x) < \infty\}$ *is an interval with end points* $a_\phi < 1 < b_\phi$*.The Kullback-Leibler, modified Kullback-Leibler,* χ^2*, modified* χ^2*, and Hellinger divergences are examples of* ϕ*-divergences; they are obtained respectively for* $\phi(x) = x \log x - x + 1$*,* $\phi(x) = -\log x + x - 1$*,* $\phi(x) = \frac{1}{2}(x-1)^2$*,* $\phi(x) = \frac{1}{2}\frac{(x-1)^2}{x}$*, and* $\phi(x) = 2(\sqrt{x} - 1)^2$*. We observe that Kullback-Leibler, modified Kullback-Leibler, modified* χ^2*, and Hellinger divergences are infinite when* $d\mathbf{Q}/d\mathbf{P}$ *takes negative values on non negligible (with respect to* $\mathbf{P}$*) subset of* $\mathscr{D}$*, since the corresponding* $\phi(\cdot)$ *is infinite on* $(-\infty, 0)$*. This problem does not hold in the case of* χ^2*-divergence, indeed the corresponding* $\phi(\cdot)$ *is finite on* $\mathbb{R}$*.*

- *We give an example of copulas for which the likelihood-based procedure fails. We consider the Gumbel copulas* $C_\theta(\cdot, \cdot)$ *given in (1.2), it's corresponding density copula is defined by*

$$c_\theta(u_1, u_2) := C_\theta(u_1, u_2)(u_1 u_2)^{-1} \frac{(\widetilde{u_1}\widetilde{u_2})^{(\theta-1)}}{(\widetilde{u_1}^{\theta} + \widetilde{u_2}^{\theta})^{(2-1/\theta)}} \left[(\widetilde{u_1}^{\theta} + \widetilde{u_2}^{\theta})^{(1/\theta)} + \theta - 1\right], \quad (2.9)$$

where $\widetilde{x} = -\log x$*. We can show that* $c_\theta(\cdot, \cdot)$ *may takes negative values for some* $\theta \in \Theta_e$*. In fact* $c_{0.7}(u_1, u_2)$ *is negative for* $(u_1, u_2) \in [0.9, 1]^2$*, hence the likelihood function is not well defined.*

Remark 2.2 *The set* Θ_e *defined in (2.3) is generally with non empty interior* $\mathring{\Theta}_e$*. In particular, we may check that* θ_0 *(the value corresponding to independence) belongs to* $\mathring{\Theta}_e$*, since the integral in (2.3) is finite; it is equal to zero when* $\theta = \theta_0$*, for any copula density* $c_\theta(\cdot, \cdot)$*. For example, in the case of FGM copulas (1.10), it is easy to show that* $\Theta_e = \mathbb{R}$*. However, it is hard to determine the whole set* Θ_e *for some copulas, but in order to test the independence, we need only to prove the existence of a neighborhood* $N(\theta_0)$ *of* θ_0 *for which the integral in (2.3) is finite since we calculate the estimate* $\widehat{\theta}_n$ *in (2.7) by Newton-Raphson algorithm using* θ_0 *as initial point. The explicit calculation of the integral in (2.3) may be complicated for some copulas, in such cases we use the Monte Carlo method to compute this integral.*

Statistics of the form

$$\Psi_n := \int_I \psi(u_1, u_2)\, dC_n(u_1, u_2),$$

belong to the general class of *multivariate rank statistics*. Their asymptotic properties have been investigated at length by a number of authors, among whom we may cite Ruymgaart *et al.* (1972), Ruymgaart (1974) and Rüschendorf (1976). In particular, the previous authors have provided regularity conditions, imposed on $\psi(\cdot,\cdot)$, which imply the asymptotic normality of Ψ_n. The corresponding arguments have been modified by Genest *et al.* (1995), Tsukahara (2005) and Fermanian *et al.* (2004b), as to establish almost sure convergence of the estimators they consider. In the same spirit, the limiting behavior, as n tends to the infinity, of our estimator and test statistic $\mathbf{T}_n$, will make an instrumental use of the general theory of multivariate rank statistics. Using some similar arguments as in Qin and Lawless (1994), the existence and consistency of our estimator and test statistic will be established through an application of the law of the iterated logarithm for empirical copula processes, in combination with general arguments from multivariate rank statistics theory. In the sequel, without loss of generality, we will limit ourselves to the case where the parameter is univariate. The extensions of our result in a multivariate framework may be achieved, at the price of additional technicalities, and under similar assumptions. We will make use of the following definitions.

Definition 2.1 *(i) Let $\mathscr{Q}$ be the set of continuous functions q on $[0,1]$ which are positive on $(0,1)$, symmetric about $1/2$, increasing on $[0,1/2]$ and satisfy $\int_0^1 \{q(t)\}^{-2}dt < \infty$.*

(ii) A function $r:(0,1)\longrightarrow(0,\infty)$ is called u-shaped if it is symmetric about $1/2$ and increasing on $(0,1/2]$.

(iii) For $0<\beta<1$ and u-shaped function r, we define

$$r_\beta(t) := \begin{cases} r(\beta t) & if \quad 0 < t \leq 1/2; \\ r\{1-\beta(1-t)\} & if \quad 1/2 < t \leq 1. \end{cases}$$

If for $\beta > 0$ in a neighborhood of 0, there exists a constant M_β, such that $r_\beta \leq M_\beta r$ on $(0,1)$, then r is called a reproducing u-shaped function. We denote by $\mathscr{R}$ the set of reproducing u-shaped functions.

Typical examples of elements in $\mathscr{Q}$ and $\mathscr{R}$ are given by

$$q(t) = [t(1-t)]^{\zeta}, \ 0 < \zeta < 1/2, \qquad r(t) = \varrho\,[t(1-t)]^{-\varsigma}, \ \varsigma \geq 0, \ \varrho \geq 0.$$

We will describe the asymptotic properties of the proposed estimate $\widehat{\theta}_n$ under the following conditions.

(C.1) There exist functions $r_{1,k}, r_{2,k} \in \mathscr{R}$ such that

$$\Big|\frac{\partial^k}{\partial\theta^k}\, m(\theta_T, u_1, u_2)\Big| \leq r_{1,k}(u_1) r_{2,k}(u_2), \quad \text{for} \quad k = 1, 2,$$

and

$$\Big|\frac{\partial^3}{\partial\theta^3}\, m(\theta, u_1, u_2)\Big| \leq r_{1,3}(u_1) r_{2,3}(u_2) \text{ on a neighborhood } N(\theta_T) \text{ of } \theta_T,$$

where $\int_I \{r_{1,k}(u_1) r_{2,k}(u_2)\}^2\, dC_{\theta_T}(u_1, u_2) < \infty$ for $k = 1, 2, 3$;

(C.2) The function $(u_1, u_2) \in I \mapsto \frac{\partial}{\partial\theta}\, m(\theta_T, u_1, u_2)$ is of bounded variation on I;

(C.3) For each θ, the function $\frac{\partial}{\partial\theta}\, m(\theta, u_1, u_2) : I \longrightarrow \mathbb{R}$ is continuously differentiable, and there exist functions $r_i \in \mathscr{R}$, $\widetilde{r}_i \in \mathscr{R}$ and $q_i \in \mathscr{Q}$ for $i = 1, 2$, such that

$$\Big|\frac{\partial^2}{\partial\theta\partial u_i}\, m(\theta, u_1, u_2)\Big| \leq \widetilde{r}_i(u_i) r_j(u_j), i, j = 1, 2 \text{ and } i \neq j,$$

and

$$\int_I \{q_i(u_i)\widetilde{r}_i(u_i) r_j(u_j)\}\, dC_{\theta_T}(u_1, u_2) < \infty,\ \ i, j = 1, 2 \text{ and } i \neq j.$$

Set

$$\Xi := -E\left[\tfrac{\partial^2}{\partial\theta^2} m\left(\theta_T, F_1(X_{1k}), F_2(X_{2k})\right)\right], \tag{2.10}$$

and

$$\Sigma^2 := \operatorname{Var}\left[\tfrac{\partial}{\partial\theta} m\left(\theta_T, F_1(X_1), F_2(X_2)\right) + W_1(\theta_T, X_1) + W_2(\theta_T, X_2)\right], \tag{2.11}$$

where

$$W_i(\theta_T, X_i) := \int_I \mathbb{1}_{\{F_i(X_i) \leq u_i\}} \tfrac{\partial^2}{\partial\theta\partial u_i} m\left(\theta_T, u_1, u_2\right) c_{\theta_T}(u_1, u_2)\ du_1 du_2,\ \ i = 1, 2.$$

We can see that Ξ and $W_i(X_i)$ can be defined, respectively, by

$$\Xi = E\left[\left(\tfrac{\partial}{\partial\theta} m\left(\theta_T, F_1(X_{1k}), F_2(X_{2k})\right)\right)^2\right]$$

and, for $i = 1, 2$,

$$W_i(\theta_T, X_i) = -\int_I \mathbb{1}_{\{F_i(X_i) \leq u_i\}} \tfrac{\partial}{\partial\theta} m\left(\theta_T, u_1, u_2\right) \tfrac{\partial}{\partial u_i} m\left(\theta_T, u_1, u_2\right) c_{\theta_T}(u_1, u_2)\ du_1 du_2.$$

The following theorem describes the asymptotic behavior of the estimate $\widehat{\theta}_n$ given in (2.7). From now on, $\xrightarrow{d}$ denotes the convergence in distribution.

Theorem 2.1 *Assume that the conditions C.1-C.3 hold.*

(a) *Let* $B(\theta_T, n^{-1/3}) := \{\theta \in \Theta_e, |\theta - \theta_T| \le n^{-1/3}\}$. *Then, as* $n \to \infty$*, with probability one, the function* $\theta \mapsto \int m(\theta, u_1, u_2)\ dC_n(u_1, u_2)$ *reaches its maximum value at some point* $\widehat{\theta}_n$ *in the interior of the interval* $B(\theta_T, n^{-1/3})$*. As a consequence, the estimate is consistent almost surely and satisfies*

$$\int_I \frac{\partial}{\partial\theta} m\left(\widehat{\theta}_n, u_1, u_2\right)\ dC_n(u_1, u_2) = 0.$$

(b) *As* $n \to \infty$*,*

$$\sqrt{n}(\widehat{\theta}_n - \theta_T) \xrightarrow{d} N(0, \Sigma^2/\Xi^2).$$

The proof of Theorem 2.1 is postponed to section 6.

3 A test based on "χ^2-divergence"

One of the motivations of the present work is to build a statistical test of independence, based on χ^2-divergence. In the framework of the parametric copula model, the null hypothesis, namely, the independence case $C_\theta(u_1, u_2) = u_1 u_2$ corresponds to the condition that $\mathscr{H}_0 : \theta_T = \theta_0$. We consider the alternative composite hypothesis $\mathscr{H}_1 : \theta_T \neq \theta_0$. The corresponding generalized pseudo-likelihood ratio statistic is then given by

$$\mathbf{S}_n(\theta_0, \tilde{\theta}) = 2 \log \frac{\sup_{\theta\in\Theta} \prod_{k=1}^n c_\theta(\widehat{F}_{1n}(X_{1k}), \widehat{F}_{2n}(X_{2k}))}{\prod_{k=1}^n c_{\theta_0}(\widehat{F}_{1n}(X_{1k}), \widehat{F}_{2n}(X_{2k}))},$$

where, for $j = 1, 2$, $\widehat{F}_{jn}$ stands for $n/(n+1)$ times the marginal empirical distribution function of the j-th variable X_j. The rescaling by the factor $n/(n+1)$, avoids difficulties arising from potential unboundedness of $\log c_\theta(u_1, u_2)$, when either u_1 or u_2 tends to one. Since, θ_0 is a boundary value of the parameter space Θ, we can see, that the convergence in distribution of $\mathbf{S}_n$ to a χ^2 random variable is not likely to hold. In order to bring a solution to this problem, we introduce the statistic

$$\mathbf{T}_n(\theta_0, \widehat{\theta}_n) := 2n \sup_{\theta\in\Theta_e} \int_I m(\theta, u_1, u_2)\ dC_n(u_1, u_2). \tag{3.1}$$

Below, we will show that, under the null hypothesis $\mathscr{H}_0$, the just-given statistic $\mathbf{T}_n(\theta_0, \widehat{\theta}_n)$ converges in distribution to a χ^2 random variable. This property allows us to build a test of $\mathscr{H}_0$ against $\mathscr{H}_1$, asymptotically of level α. The limit law of $\mathbf{T}_n(\theta_0, \widehat{\theta}_n)$ will also be given under the alternative hypothesis $\mathscr{H}_1$. The following additional conditions will be needed for the statement of our results.

(C.4) We have

$$\lim_{\theta\to\theta_0} \frac{\partial}{\partial u_i} m(\theta, u_1, u_2) = 0,$$

and there exist $M_1 > 0$ and $\delta_1 > 0$ such that, for all θ in some neighborhood of θ_0, one has, for $i = 1, 2$,

$$\left|\frac{\partial}{\partial\theta} m(\theta, u_1, u_2)\frac{\partial}{\partial u_i} m(\theta, u_1, u_2)c_\theta(u_1, u_2)\right| < M_1 r(u_i)^{-1.5+\delta_1} r(u_{3-i})^{0.5+\delta_1},$$

where $r(u) := u(1-u)$ for $u \in (0, 1)$;

(C.5) There exist a neighborhood $N(\theta_T)$ of θ_T, and functions $r_i \in \mathscr{R}$ such that for all $\theta \in N(\theta_T)$, we have

$$\left|\frac{\partial}{\partial\theta} m(\theta, u_1, u_2)\right| \le r_1(u_1)r_2(u_2) \quad \text{with} \quad \int_I \{r_1(u_1)r_2(u_2)\}^2 \, dC_{\theta_T}(u_1, u_2) < \infty;$$

(C.6) There exist functions $r_i, \widetilde{r}_i \in \mathscr{R}$, $q_i \in \mathscr{Q}$, $i = 1, 2$ such that

$$\left| m(\theta_T, u_1, u_2)\right| \le r_1(u_1)r_2(u_2),$$

$$\left|\frac{\partial}{\partial u_i} m(\theta, u_1, u_2)\right| \le \widetilde{r}_i(u_i)r_j(u_j), i, j = 1, 2 \text{ and } i \ne j,$$

with

$$\int_I \{r_1(u_1)r_2(u_2)\}^2 \, dC_{\theta_T}(u_1, u_2) < \infty,$$

and

$$\int_I \{q_i(u_i)\widetilde{r}_i(u_i)r_j(u_j)\} \, dC_{\theta_T}(u_1, u_2) < \infty, \; i, j = 1, 2 \text{ and } i \ne j.$$

Theorem 3.1 *Assume that conditions C.1-C.4 hold. Then, under the null hypothesis $\mathscr{H}_0$, the statistic $\mathbf{T}_n$ converges in distribution to a χ_1^2 random variable (with 1 degree of freedom).*

The proof of Theorem 3.1 is postponed until section 6.

Theorem 3.2 *Assume that conditions C.1-C.3 and C.5-C.6 hold. Then, under the alternative hypothesis $\mathscr{H}_1$, we have*

$$\sqrt{n}\left(\frac{\mathbf{T}_n}{2n} - \chi^2(\theta_0, \theta_T)\right)$$

converges to a centered normal random variable with variance

$$\sigma_{\chi^2}^2 := \operatorname{Var}\left[m\left(\theta_T, F_1(X_{1k}), F_2(X_{2k})\right) + Y_1(\theta_T, X_1) + Y_2(\theta_T, X_2)\right], \tag{3.2}$$

where

$$Y_i(\theta_T, X_i) := \int_I \mathbb{1}_{\{F_i(X_i)\le u_i\}} \tfrac{\partial}{\partial u_i} m\left(\theta_T, u_1, u_2\right) c_{\theta_T}(u_1, u_2) \; du_1 du_2.$$

The proof of Theorem 3.2 is postponed until section 6.

Remark 3.1 *An application of Theorem 3.1, leads to reject the null hypothesis* $\mathscr{H}_0 : \theta_T = \theta_0$, *whenever the value of the statistic* $\mathbf{T}_n$ *exceeds* $q_{1-\alpha}$, *namely, the* $(1-\alpha)$*-quantile of the* χ_1^2 *law. The test corresponding to this rejection rule is then, asymptotically of level* α, *when* $n \to \infty$. *Accordingly, the critical region is given by*

$$CR := \{\mathbf{T}_n > q_{1-\alpha}\}. \tag{3.3}$$

The fact that this test is consistent follows from Theorem 3.2. Further, this theorem can be used to give an approximation to the power function $\theta_T \mapsto \beta(\theta_T) := P_{\theta_T}\{CR\}$ *in a similar way to Keziou and Leoni-Aubin (2005) and Keziou and Leoni-Aubin (2007). We so obtain that*

$$\beta(\theta_T) \approx 1 - \Phi\left(\frac{\sqrt{n}}{\sigma_{\chi^2}}\left(\frac{q_{1-\alpha}}{2n} - \chi^2(\theta_0, \theta_T)\right)\right), \tag{3.4}$$

where Φ *denotes, as usual, the distribution function of* $N(0,1)$ *standard normal random variable. A useful consequence of (3.4) is the possibility of computing an approximate value of the sample size ensuring a specified power* $\beta(\theta_T)$, *with respect to some pre-assigned alternative* $\theta_T \neq \theta_0$. *Let* n_0 *be the positive root of the equation*

$$\beta = 1 - \Phi\left(\frac{\sqrt{n}}{\sigma_{\chi^2}}\left(\frac{q_{1-\alpha}}{2n} - \chi^2(\theta_0, \theta_T)\right)\right), \tag{3.5}$$

which can be rewritten as

$$n_0 = \frac{(a+b) - \sqrt{a(a+2b)}}{2\chi^2(\theta_0, \theta_T)}, \tag{3.6}$$

where $a := \sigma_{\chi^2}^2 \left(\Phi^{-1}(1-\beta)\right)^2$ *and* $b := q_{1-\alpha}\chi^2(\theta_0, \theta_T)$. *The sought-after approximate value of the sample size is then given by*

$$n^* := \lfloor n_0 \rfloor + 1,$$

where $\lfloor u \rfloor$ *denote the integer part of* u.

Remark 3.2 *If the parameter* θ *does not belong to* Θ, *i.e.,* $\theta \in \Theta_e - \Theta$, *the densities* $c_\theta(\cdot,\cdot)$, *with respect to the Lebesgue's measure, associated to* $C_\theta(\cdot,\cdot)$, *may become negative on some non negligible subsets of* I. *So, in this case*

$$C_\theta(u_1, u_2) := \int_{(0,u_1)\times(0,u_2)} c_\theta(u_1, u_2)\, d\lambda(u_1, u_2);\ \forall u \in I, \tag{3.7}$$

may be a signed measure (and not a copula) for some $\theta \in \Theta_e - \Theta$. Hence, the formula (2.7) may lead to a value of the estimator $\widehat{\theta}_n$ belonging to $\Theta_e - \Theta$, and the corresponding $C_{\widehat{\theta}_n}$ is not necessarily a copula. So, for point estimation, the pseudo-maximum likelihood estimator (restricted to vary in the admissible domain) should be used instead of (2.7). The latter estimator may not be meaningful, and is likely to have a larger mean square error. The main advantage of our formula (2.6), where the supremum is taken on the extended space Θ_e, instead of the admissible domain Θ, is that it permits easily to build a test of independence even when the dimension of the parameter space Θ is larger than one. We can show that the proposed test statistic, based on the formula (2.6), has a $\chi^2(p)$ limit law with $p = dim(\Theta)$ degrees of freedom.

Remark 3.3 *The asymptotic variances (2.11) and (3.2) may be consistently estimated respectively by the sample variances of*

$$\frac{\partial}{\partial\theta} m\left(\widehat{\theta}_n, F_{1n}(X_{1,k}), F_{2n}(X_{2,k})\right) + W_1(\widehat{\theta}_n, X_{1,k}) + W_2(\widehat{\theta}_n, X_{2,k}), \quad k = 1, \ldots, n, \tag{3.8}$$

$$m\left(\widehat{\theta}_n, F_{1n}(X_{1,k}), F_{2n}(X_{2,k})\right) + Y_1(\widehat{\theta}_n, X_{1,k}) + Y_2(\widehat{\theta}_n, X_{2,k}), \quad k = 1, \ldots, n, \tag{3.9}$$

as was done in Genest et al. (1995). Similarly, the parameter Ξ in (2.10) may be consistently estimated by the sample mean of

$$\left[\frac{\partial}{\partial\theta} m\left(\widehat{\theta}_n, F_{1n}(X_{1,k}), F_{2n}(X_{2,k})\right)\right]^2, \quad k = 1, \ldots, n. \tag{3.10}$$

4 Simulation results

In this section, we present some simulation results aiming to illustrate the theoretical results of Theorem 3.1 and Theorem 3.2.

4.1 The chi square approximation

We illustrate the accuracy of the approximation of the statistic $\mathbf{T}_n$ by its limit law χ_1^2 under the null hypothesis of independence of marginals; see Theorem 3.1. We consider the Clayton, FGM and Gumbel copulas given in (1.7), (1.10) and (1.2) respectively. We use the Q-Q plot of the empirical quantiles of the proposed statistic $\mathbf{T}_n$ versus the quantiles of the χ_1^2 law. In Figure 1, the Q-Q plots are obtained from 1000 independent runs of samples with sizes $n = 100$ and $n = 500$ for Clayton and FGM copulas, and from 500 independent runs of samples with sizes $n = 100$ and $n = 500$ for Gumbel copula. We observe that the approximation is good even for moderate sample sizes. The integral in the expression of $m(\theta, \cdot, \cdot)$ is calculated by the Monte Carlo method, and the supremum in (3.1) is considered on the extended space Θ_e; it has been computed on a neighborhood of $\theta_0 := 0$ by the Newton-Raphson algorithm taking $\theta_0 := 0$ as an initial point.

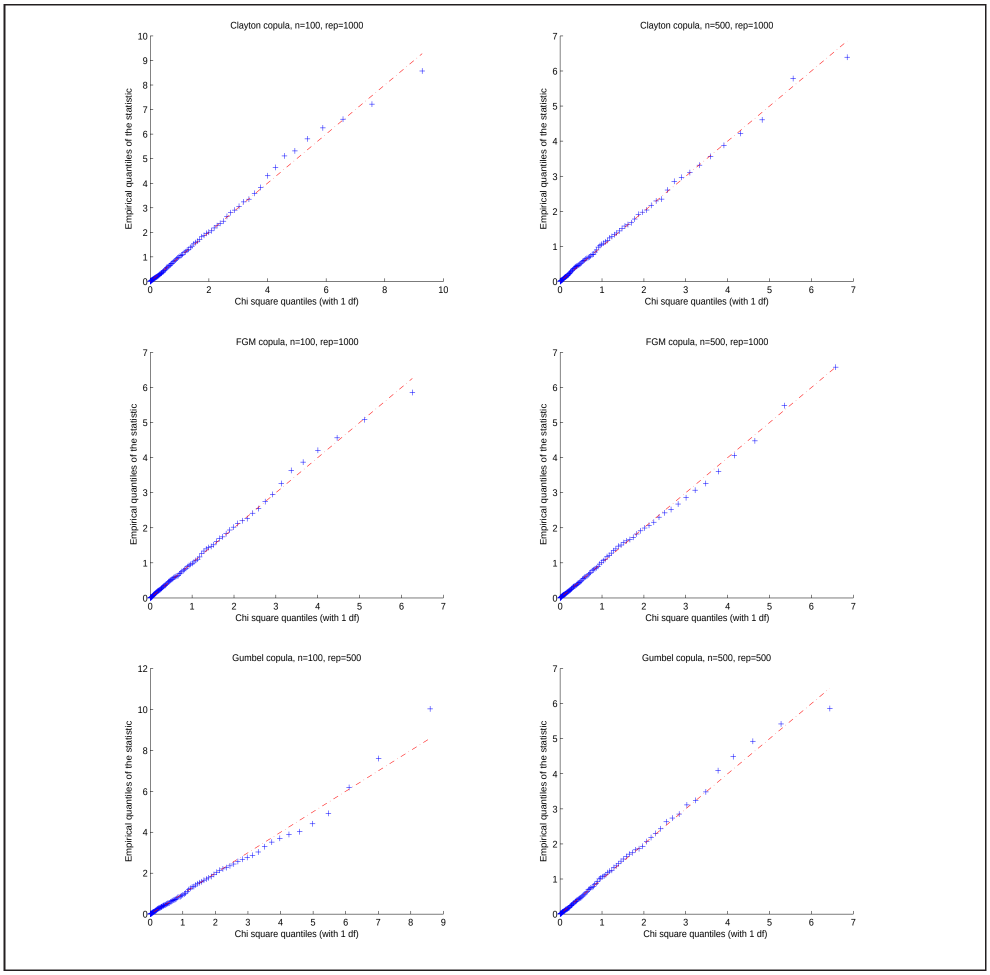

Figure 1: The Q-Q Plots of the statistic $\mathbf{T}_n$ versus its limit law χ_1^2

4.2 Power comparison

Figure 2 graphically compares the power of three different tests for bivariate independence based on the statistics $\mathbf{T}_n$, τ_n and ρ_n, where $\mathbf{T}_n$ is defined in (3.1), τ_n is the Kendall τ statistic,

$$\tau_n := \frac{2}{n^2 - n} \sum_{1 \le i < j \le n} \operatorname{sign}(R_i - R_j)\operatorname{sign}(S_i - S_j).$$

Here, $(R_1, S_1), \dots, (R_n, S_n)$ are the paired rank statistics pertaining to the sample $(X_{1,1}, X_{2,1})$, $\dots, (X_{1,n}, X_{2,n})$. ρ_n is the Spearman ρ statistic, defined by

$$\rho_n := -3\frac{n+1}{n-1} + \frac{12}{n^3 - n}\sum_{i=1}^{n} R_i S_i.$$

We consider the FGM copula and the marginals are exponential with parameter one. The power functions of the three statistics are plotted as a function of $\theta \in [0, 0.5]$ for sample sizes $n = 100$ and $n = 500$. Each power entry was obtained from 1000 independent runs. Looking at Figure 2, we can see that the test based on $\mathbf{T}_n$ is superior to τ_n and ρ_n. On the right panel of Figure 2, for large samples (n=500), we see that the power of the tests based on Kendall's tau and $\mathbf{T}_n$ are almost equal.

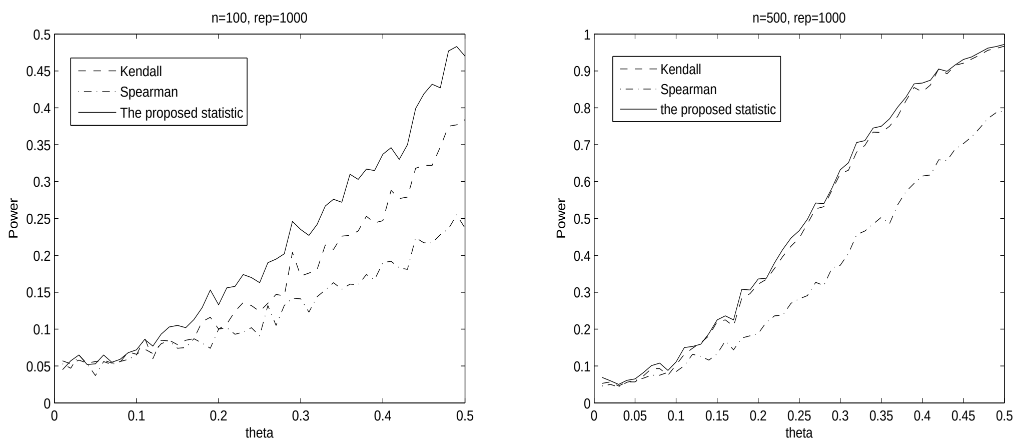

Figure 2: Power comparison

5 Conclusion

We have proposed a new test procedure based on χ^2-divergence and duality technique in the framework of parametric copulas. It seems that the procedure introduced here is particularly well adapted to the boundary problem. For point estimation, the estimator based on χ^2-divergence when we extend the parameter space, may not have a meaningful interpretation and most probably has a larger mean square error. However, Theorem 2.1 may be useful to construct the confidence region for the parameter of interest in connection with the intersection method (see Feng and McCulloch (1992)) which is easy to implement, while the asymptotic distribution of the maximum likelihood estimator is difficult to use, since the limiting distributions are complex when incorporating boundary constraints.

For clarity, our arguments are presented in the bivariate context. Note, however, that a d-variate generalization is possible with obvious changes of notation and modification assumptions. We mention that the popular multivariate Archimedean copulas

$$C_\theta(u_1, \ldots, u_d) = \phi^{-1}\left\{\phi(u_1) + \cdots + \phi(u_d)\right\}$$

can only admit positive dependence, so that as long as the independence copula belongs to a given Archimedean family, θ_0 is on the boundary. Note, also that the proposed test of independence is not *omnibus* but depends on the hypothesis that the dependence structure belong to a certain family.

6 Appendix

First we give a technical Lemma which we will use to prove our results.

Lemma 6.1 *Let $\mathbf{F}(\cdot,\cdot)$ be continuous and $C(\cdot,\cdot)$ have continuous partial derivatives. Assume that $\jmath$ is a continuous function, with bounded variation on I. Then*

$$\int_I \jmath(u_1, u_2)\ d\left(C_n(u_1, u_2) - C(u_1, u_2)\right) = O\left(n^{-1/2}(\log\log n)^{1/2}\right) \quad (a.s.). \tag{6.1}$$

Proof of Lemma 6.1. Recall that the *modified empirical copula* $C_n(\cdot,\cdot)$, is slightly different from the *empirical copula* $\mathbb{C}_n(\cdot,\cdot)$, introduced by Deheuvels (1979a), and defined by

$$\mathbb{C}_n(u_1, u_2) = \mathbf{F}_n\Big(F_{1n}^{-1}(u_1), F_{2n}^{-1}(u_2)\Big) \quad \text{for} \quad (u_1, u_2) \in (0,1)^2, \tag{6.2}$$

where $F_{1n}^{-1}(\cdot)$ and $F_{2n}^{-1}(\cdot)$ denote the empirical quantile functions, associated with $F_{1n}(x_1) = \mathbf{F}_n(x_1, \infty)$ and $F_{2n}(x_2) = \mathbf{F}_n(\infty, x_2)$, respectively, and defined by

$$F_{jn}^{-1}(t) := \inf\{x \in \mathbb{R} \mid F_{jn}(x) \geq t\}, \quad j = 1, 2. \tag{6.3}$$

Here, $\mathbf{F}_n(\cdot,\cdot)$ denotes the joint empirical distribution function, associated with the sample $\{(X_{1k}, X_{2k});\ k = 1, \ldots, n\}$, defined by

$$\mathbf{F}_n(x_1, x_2) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_{1k} \leq x_1\}} \mathbb{1}_{\{X_{2k} \leq x_2\}}, \quad -\infty < x_1, x_2 < \infty. \tag{6.4}$$

We know that $\mathbb{C}_n(\cdot,\cdot)$ and $C_n(\cdot,\cdot)$ coincide on the grid $\{(i/n, j/n)\,,\ 1 \leq i \leq j \leq n\}$. The subtle difference lies in the fact that $\mathbb{C}_n(\cdot,\cdot)$ is left-continuous with right-hand limits, whereas

$C_n(\cdot,\cdot)$ on the other hand is right continuous with left-hand limits. The difference between $\mathbb{C}_n(\cdot,\cdot)$ and $C_n(\cdot,\cdot)$, however, is small

$$\sup_{\mathbf{u}\in I}|\mathbb{C}_n(\mathbf{u})-C_n(\mathbf{u})|\leq\max_{1\leq i,j\leq n}\left|\mathbb{C}_n\left(\frac{i}{n},\frac{j}{n}\right)-\mathbb{C}_n\left(\frac{i-1}{n},\frac{j-1}{n}\right)\right|\leq\frac{2}{n}. \tag{6.5}$$

Using integration by parts, as in Fermanian *et al.* (2004a), we see that

$$\begin{aligned}\sqrt{n}\int_I \jmath(u_1,u_2)\ d(C_n-C)(u_1,u_2) &= \int_I\sqrt{n}(C_n-C)(u_1,u_2)\ d\jmath(u_1,u_2)\\ &-\int_I\sqrt{n}(C_n-C)(u_1,1)\ d\jmath(u_1,u_2)-\int_I\sqrt{n}(C_n-C)(1,u_2)\ d\jmath(u_1,u_2)\\ &-\int_{[0,1]}\sqrt{n}(C_n(u_1,1)-u_1)\ d\jmath(u_1,1)-\int_{[0,1]}\sqrt{n}(C_n(1,u_2)-u_2)\ d\jmath(1,u_2).\end{aligned}$$

Hence,

$$\left|\sqrt{n}\int_I \jmath(u_1,u_2)\ d(C_n-C)(u_1,u_2)\right|\leq 5\sqrt{n}\sup_{\mathbf{u}\in I}|(C_n-C)(\mathbf{u})|\int_I d\,|\jmath(\mathbf{u})|\,.$$

From this and (6.5), applying Theorem 3.1 in Deheuvels (1979a), we obtain the following result

$$\int_I \jmath(u_1,u_2)\ d(C_n-C)(u_1,u_2)=O\left(n^{-1/2}(\log\log n)^{1/2}\right)\quad (a.s.).$$

■

Proof of Theorem 2.1. (a) Under the assumption (C.1), a straightforward calculus yields

$$\int_I \tfrac{\partial}{\partial\theta}m(\theta_T,u_1,u_2)\ dC_{\theta_T}(u_1,u_2)=0. \tag{6.6}$$

Under the assumptions (C.1), and by applying Proposition 2.2 in Genest *et al.* (1995), we can see that, as $n\to\infty$,

$$\int_I \tfrac{\partial}{\partial\theta}m(\theta_T,u_1,u_2)\ dC_n(u_1,u_2)\longrightarrow\int_I \tfrac{\partial}{\partial\theta}m(\theta_T,u_1,u_2)\ dC_{\theta_T}(u_1,u_2)=0, \tag{6.7}$$

and

$$\int_I \tfrac{\partial^2}{\partial\theta^2}m(\theta_T,u_1,u_2)\ dC_n(u_1,u_2)\longrightarrow\int_I \tfrac{\partial^2}{\partial\theta^2}m(\theta_T,u_1,u_2)\ dC_{\theta_T}(u_1,u_2):=-\Xi<0, \tag{6.8}$$

almost surely. Now, for any $\theta=\theta_T+vn^{-1/3}$, with $|v|\leq 1$, consider a Taylor expansion of $\int m(\theta,u_1,u_2)\ dC_n(u_1,u_2)$ in θ in a neighborhood of θ_T, using (C.1) part 2, one finds

$$\begin{aligned}&n\int_I m(\theta,u_1,u_2)\ dC_n(u_1,u_2)-n\int_I m(\theta_T,u_1,u_2)\ dC_n(u_1,u_2) \qquad (6.9)\\ &=\ n^{2/3}v\int_I \tfrac{\partial}{\partial\theta}m(\theta_T,u_1,u_2)\ dC_n(u_1,u_2)+n^{1/3}\frac{v^2}{2}\int_I \tfrac{\partial^2}{\partial\theta^2}m(\theta_T,u_1,u_2)\ dC_n(u_1,u_2)+O(1)\quad (a.s.),\end{aligned}$$

uniformly in v with $|v| \leq 1$. On the other hand, under condition (C.2), by Lemma 6.1, we have

$$\int_I \tfrac{\partial}{\partial\theta} m(\theta_T, u_1, u_2) dC_n(u_1, u_2) = O(n^{-1/2}(\log\log n)^{1/2}) \quad (a.s.).$$

Therefore, using (6.8) and (6.9), we obtain for any $\theta = \theta_T + vn^{-1/3}$ with $|v| = 1$,

$$\begin{aligned} n\int_I m(\theta, u_1, u_2)\ dC_n(u_1, u_2) - n\int_I m(\theta_T, u_1, u_2)\ dC_n(u_1, u_2) \\ = \quad O(n^{1/6}(\log\log n)^{1/2}) - 2^{-1}\Xi n^{1/3} + O(1) \ \ (a.s.). \end{aligned} \tag{6.10}$$

Observe that the right-hand side vanishes when $\theta = \theta_T$, and that the left-hand side, by (6.8), becomes negative for all n sufficiently large. Thus, by the continuity of $\theta \mapsto \int m(\theta, u_1, u_2)\ dC_n(u_1, u_2)$, it holds that as $n \to \infty$, with probability one,

$$\theta \mapsto \int m(\theta, u_1, u_2)\ dC_n(u_1, u_2)$$

reaches its maximum value at some point $\widehat{\theta}_n$ in the interior of the interval $B(\theta_T, n^{-1/3})$. Therefore, the estimate $\widehat{\theta}_n$ satisfies

$$\int_I \tfrac{\partial}{\partial\theta} m(\widehat{\theta}_n, u_1, u_2)\ dC_n(u_1, u_2) = 0 \text{ and } \mid \widehat{\theta}_n - \theta_T \mid = O(n^{-1/3}). \tag{6.11}$$

(b) Making use of the first part of Theorem 2.1, and once more, by a Taylor expansion of

$$\int_I \tfrac{\partial}{\partial\theta} m(\widehat{\theta}_n, u_1, u_2)\ dC_n(u_1, u_2),$$

with respect to $\widehat{\theta}_n$, in the neighborhood of θ_T, we obtain that

$$\begin{aligned} 0 = \int_I \tfrac{\partial}{\partial\theta} m(\widehat{\theta}_n, u_1, u_2)\ dC_n(u_1, u_2) &= \int_I \tfrac{\partial}{\partial\theta} m(\theta_T, u_1, u_2)\ dC_n(u_1, u_2) \\ &+ (\widehat{\theta}_n - \theta_T)\int_I \tfrac{\partial^2}{\partial\theta^2} m(\theta_T, u_1, u_2)\ dC_n(u_1, u_2) + o(n^{-1/2}). \end{aligned}$$

Hence,

$$\sqrt{n}(\widehat{\theta}_n - \theta_T) = \left(-\int_I \tfrac{\partial^2}{\partial\theta^2} m(\theta_T, u_1, u_2)\ dC_n(u_1, u_2)\right)^{-1} \sqrt{n} W_n(\theta_T) + o(1), \tag{6.12}$$

where

$$W_n(\theta_T) := \int_I \tfrac{\partial}{\partial\theta} m(\theta_T, u_1, u_2)\ dC_n(u_1, u_2).$$

Applying Proposition 3 page 362 in Tsukahara (2005), under assumptions (C.1) part 1 and (C.3), as $n \to \infty$, we have

$$\sqrt{n} W_n(\theta_T) \xrightarrow{d} N\left(0, \Sigma^2\right). \tag{6.13}$$

Finally, by combining (6.13) and (6.8) in connection with Slutsky's Theorem, as $n \to \infty$,we conclude that

$$\sqrt{n}(\widehat{\theta}_n - \theta_T) \xrightarrow{d} N(0, \Sigma^2/\Xi^2). \tag{6.14}$$

This completes the proof. ∎

Proof of Theorem 3.1. Assume that $\theta_T = \theta_0$. From (6.12), using (6.8) and (6.13), we obtain

$$\sqrt{n}\left(\widehat{\theta}_n - \theta_T\right) = -\frac{1}{\Sigma}\sqrt{n} W_n(\theta_T) + o_P(1). \tag{6.15}$$

Expanding in Taylor series $\mathbf{T}_n(\theta_0, \widehat{\theta}_n)$ in $\widehat{\theta}_n$ around θ_T, we get

$$\mathbf{T}_n(\theta_0, \widehat{\theta}_n) = 2n W_n(\theta_T)(\widehat{\theta}_n - \theta_T) - \Sigma n(\widehat{\theta}_n - \theta_T)^2 + o_P(1). \tag{6.16}$$

Now, use (6.15) combined with (6.16) to obtain

$$\mathbf{T}_n(\theta_0, \widehat{\theta}_n) = \frac{1}{\Sigma} n W_n(\theta_T)^2 + o_P(1). \tag{6.17}$$

By (6.13), as $n \to \infty$, we have

$$\sqrt{n} W_n(\theta_T) \xrightarrow{d} N\left(0, \Sigma^2\right) \tag{6.18}$$

in distribution. When $\theta_T = \theta_0$, under Assumption (C.4), we can see that Σ^2 in (6.18) is equal to $1/\Xi$; see Proposition 2.2 in Genest *et al.* (1995). Combining this with (6.17) to conclude that

$$\mathbf{T}_n \xrightarrow{d} \chi_1^2,$$

under the null hypothesis $\mathscr{H}_0$. ∎

Proof of Theorem 3.2. Rewriting $\frac{\mathbf{T}_n}{2n}$ as

$$\frac{\mathbf{T}_n(\theta_0, \widehat{\theta}_n)}{2n} = \sup_{\theta \in \Theta_e} \int_I m(\theta, u_1, u_2)\, dC_n(u_1, u_2), \tag{6.19}$$

and making use of a Taylor expansion of $\frac{\mathbf{T}_n(\theta_0, \widehat{\theta}_n)}{2n}$, with respect to $\widehat{\theta}_n$, in a neighborhood of θ_T, under (C.5) to obtain

$$\frac{\mathbf{T}_n(\theta_0, \widehat{\theta}_n)}{2n} = \int_I m(\theta_T, u_1, u_2)\, dC_n(u_1, u_2) + o_P(n^{-1/2}).$$

Hence, one finds

$$\sqrt{n}\left(\frac{\mathbf{T}_n}{2n} - \chi^2(\theta_0, \theta_T)\right) = \sqrt{n}\left(\int_I m(\theta_T, u_1, u_2)\ dC_n - \int_I m(\theta_T, u_1, u_2)\ dC_{\theta_T}\right) + o_P(1).$$

Finally, under condition (C.6), application once more of Proposition 3 page 362 in Tsukahara (2005), concludes the proof. ■

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