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EXISTENCE OF TRAVELLING-WAVE SOLUTIONS AND LOCAL WELL-POSEDNESS OF THE FOWLER EQUATION

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Abstract. We study the existence of travelling-waves and local well-posedness in a subspace of $C_b^1(\mathbb{R})$ for a nonlinear evolution equation recently proposed by Andrew C. Fowler to describe the dynamics of dunes.

1. Introduction

1.1. **General setting.** Dunes are land formations of sand which are subject to different forms and sizes based on their interaction with the wind or water or some other mobile medium. In the case of dunes in the desert their shapes depend mainly on the amount of sand available and on the change of the direction of the wind with time (see Herrmann and Sauermann [6]). Some examples of dune patterns are longitudinal, transverse, star and Barchan dunes, however, there are more than 100 categories of dunes. Dunes also occur under rivers, for similar reasons, but their shapes are less exotic in this case, because the flow is mainly uni-directional.

An interesting topic is to try to understand if the shape of a dune is maintained when it moves. With regard to Barchan dunes, for example, Herrmann and Sauermann [6] have given some arguments against the hypothesis that Barchan dunes are solitary waves, mainly because they constantly lose sand at the two horns and tend to disappear if not supplied with new sand. Recently, Durán, Schwämmle and Herrmann [2] considered a minimal model for dunes consisting of three coupled equations of motion to study numerically the mechanisms of dune interactions for the case when a small Barchan dune collides with a bigger one; four different cases were observed, depending only on the relative sizes of the two dunes, namely, coalescence, breeding, budding, and solitary wave behavior.

In this paper, we are concerned with the following evolution equation proposed by Fowler (see [3], [4] and [5] for more details) to study nonlinear dune formation:

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\Big[\frac{u^2}{2}(x,t) - \frac{\partial u}{\partial x}(x,t) + \int_0^{+\infty} \xi^{-1/3}\frac{\partial u}{\partial x}(x-\xi,t)d\xi\Big] = 0, \qquad (1.1)$$

where $u = u(x,t)$ represents the dune amplitude, $x \in \mathbb{R}$, and $t \geq 0$. The second and fourth terms of equation (1.1) correspond to the nonlinear and nonlocal terms respectively, while the third term is the dissipative term.

Let us give a brief description of the model derivation. For more details, we refer to Fowler [3, 4, 5], which we follow closely. The model stems from the Exner law,

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which is the conservation of mass for the sediment:

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

where the bedload transport $q = q(\tau)$ is assumed, in the case of dunes, to depend only on the stress τ exerted by the fluid on the erodible bed. We assume a two-dimensional flow, where x is the horizontal direction and the second direction is the upwards coordinate orthogonal to x. This should account for transverse dunes, but obviously not for other types of dunes. The nonlocal term in equation (1.1) arises from a subtle modelling of the basal shear stress τ_b. Roughly speaking, the turbulent bottom shear stress is given by $\tau_b \approx f\rho v^2$, where ρ is the fluid density, f is a dimensionless friction coefficient and v is the mean fluid velocity (vertically averaged). By performing an asymptotic expansion with respect to the aspect ratio ϵ of the evolving bedform, $\epsilon = \frac{\text{bed thickness}}{\text{fluid depth}} \ll 1$, and a perturbation analysis of a basic Poiseuille flow (Orr-Sommerfeld equation), Fowler [3, 4, 5] was able to obtain the following expression:

$$\tau_b \approx f\rho v^2 \left\{ 1 - u + \alpha \int_0^{+\infty} \xi^{-1/3} \frac{\partial u}{\partial x}(x - \xi, t) d\xi \right\},$$

where α is a positive constant proportional to $Re^{1/3}$, Re being the Reynolds number. Due to the bed slope $\frac{\partial u}{\partial x}$, there is an additional force generated by gravity g. Therefore, the net stress causing motion is actually $\tau = \tau_b - (\rho_s - \rho) g D_s \frac{\partial u}{\partial x}$, where ρ_s is the sediment density and D_s the mean diameter of a sediment particle. As long as u is small, the shallow water approximation applies to the velocity v and, for small Froude number, the (dimensionless) mean fluid velocity can be approximated by $v \approx \frac{1}{1-u}$. Thus, the mean fluid velocity and the bottom shear stress depend on the motion of the dune profile u, and therefore there is a feedback between the dune profile and the motion of the fluid. In dimensionless variables, taking all physical constants equal to 1, the resulting net stress is then given by

$$\tau \approx 1 + u + u^2 + \int_0^{+\infty} \xi^{-1/3} \frac{\partial u}{\partial x}(x - \xi, t)\, d\xi - \frac{\partial u}{\partial x}.$$

Notice that the nonlinear nonlocal term $2u \int_0^{+\infty} \xi^{-1/3} \frac{\partial u}{\partial x}(x - \xi, t) d\xi$ has been discarded. By a Taylor expansion, up to order 2, we get $q(\tau) \approx q(1) + q'(1)(\tau - 1) + \frac{1}{2} q''(1)(\tau - 1)^2$. Now, considering a moving spatial coordinate, i.e. replacing x by the new variable $x - q'(1)t$, plugging q into the Exner equation, after a suitable rescaling, we obtain the canonical equation (1.1).

Some numerical computations have been performed by Fowler [4, 5] and Alibaud, Azerad and Isèbe [1]. Fowler mentions the fact that the numerical solution, computed with a pseudo-spectral method in a large domain, starting from random initial data, converges to a final state consisting of one travelling-wave. Alibaud et al., using a finite difference scheme valid for a bounded time interval, starting from a compactly supported nonnegative initial data, showed that the numerical solution of the Fowler equation (1.1) quickly evolves to a solution with a non zero negative part, showing the erosive effect of the nonlocal term. They also establish theoretically the non monotone property of (1.1), namely the violation of the maximum principle (see also Remark 3.2 below).

To the authors' knowledge, ours is the first study to report a rigorous mathematical proof of the existence of travelling-waves for dune morphodynamics. We

notice that we have not found nontrivial travelling-waves of the solitary-wave type for this model (see Remark 2.5 below), however we could not exclude the possibility that they exist. What we obtain is more bore-like travelling-waves. This type of travelling dunes has not been observed yet, to the author's knowledge. This may put under question the validity of the Fowler equation to faithfully describe dune morphodynamics. The authors hope that these results could be of interest for geographers, geologists, oceanographers and others.

1.2. **Organization of the paper.** In Section 2 we study the existence of travelling-wave solutions to equation (1.1). The main result of this section is Theorem 2.1 which implies that for each wave speed $d > 0$, and η in a neighborhood of zero, $\eta \in \mathbb{R}$, there exists a travelling-wave solution $u(x,t) = \phi(x - dt)$ to the following version of equation (1.1)

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\Big[\frac{u^2}{2}(x,t) - \frac{\partial u}{\partial x}(x,t) + \eta \int_0^{+\infty} \xi^{-1/3} \frac{\partial u}{\partial x}(x-\xi,t) d\xi\Big] = 0,$$

where $\phi \in C_b^1(\mathbb{R})$; the idea of its proof is to use the implicit function theorem on suitable Banach spaces. Then, by a scaling argument and considering a suitable translation of the travelling-wave, we extend this result for any $\eta \in \mathbb{R}$ and any wave speed $d \in \mathbb{R}$.

Section 3 is devoted to proving local well-posedness (LWP) for the integral equation associated to the initial value problem (IVP) for equation (1.1). Inspired by the regularity of the travelling-wave obtained in Section 2, we consider a suitable subspace of $C_b^1(\mathbb{R})$. The analysis of the linear equation associated to equation (1.1) is addressed in Sub-section 3.1. Next, in Sub-section 3.2, the main result of this section is stated in Theorem 3.1; it gives local-in-time existence of the solution of the integral equation associated to the IVP for equation (1.1), with initial data belonging to the subspace X of $C_b^1(\mathbb{R})$, where

$$X := \{f \in C_b^1(\mathbb{R}); f' \text{ is uniformly continuous}\}.$$

1.3. **Notations.** - We denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of all real and complex numbers respectively. $\mathbb{N}$ denotes the set of all natural numbers.
- We denote by $C(c_1, c_2, \ldots)$ a constant which depends on the parameters $c_1, c_2, \ldots$ C is assumed to be a non-decreasing function of its arguments.
- The norm of a measurable function $f \in L^p(\Omega)$, for Ω a subset of $\mathbb{R}$, is written $\|f\|_{L^p(\Omega)}^p = \int_\Omega |f|^p dx$ for $1 \leq p < +\infty$, and $\|f\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |f|$. The inner product of two functions $f, g \in L^2(\Omega)$ is written as $(f,g) = \int_\Omega f\bar{g}dx$. We will often omit set Ω when context is clear.
- We denote by $\hat{f} = \mathcal{F}f$ the Fourier transform of f ($\mathcal{F}^{-1}$ and $\check{}$ are used to denote the inverse of the Fourier transform), where $\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) dx$ for $f \in L^1(\mathbb{R})$ (it follows that $\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$ for $f, g \in L^1(\mathbb{R})$).
- The Schwartz space of rapidly decreasing functions on $\mathbb{R}$ is denoted $\mathcal{S}(\mathbb{R})$.
- We denote $\Lambda := (1 - \partial_x^2)^{1/2}$ and $H^s(\mathbb{R})$ ($s \in \mathbb{R}$) the usual Sobolev space $H^s(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}), \|u\|_{H^s} < \infty\}$, where $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}$.
- Let $\Omega \subset \mathbb{R}$. $C^0(\Omega) = C(\Omega)$ is used to denote the space of all continuous complex-valued functions on Ω. Moreover, $C^k(\Omega) = \{u : \Omega \mapsto \mathbb{C} \ ; \ u, u', \ldots, u^{(k)} \in C^0(\Omega)\}$, for $k \in \mathbb{N}$. We write $C^\infty(\Omega)$ to denote the set of infinitely differentiable complex-valued functions on Ω. Similarly, we use the notations $C^0(\Omega; Y) =$

$C(\Omega;Y), C^k(\Omega;Y), C^\infty(\Omega;Y)$ when functions take values in the Banach space Y.
- We write $C_\infty(\mathbb{R})$ to denote the space of all continuous complex-valued functions defined on $\mathbb{R}$ which tend to zero at infinity.
- We denote by $C_b(\mathbb{R}) = C_b^0(\mathbb{R})$ the space of all bounded continuous real-valued functions on $\mathbb{R}$ with the norm $\|\cdot\|_{L^\infty}$. Moreover, for every $k \in \mathbb{N}$, we write

$$C_b^k(\mathbb{R}) := \{f \in C^k(\mathbb{R}) \,;\, f, f', \dots, f^{(k)} \in C_b(\mathbb{R})\},$$

where $\|f\|_{C_b^k} := \sum_{i=0}^k \|f^{(i)}\|_{L^\infty}$, for all $f \in C_b^k(\mathbb{R})$.
- If X and Y are two Banach spaces, we denote by $\mathfrak{L}(X,Y)$ the set of all continuous linear mappings defined on X with values in Y; if $X = Y$, we denote by $\mathfrak{L}(X)$.

2. Existence of Travelling-Wave Solutions of the Fowler Equation

We begin this section with some notations and preliminary results. We define

$$\psi(x) := \chi_{(0,\infty)}(x) \cdot x^{-1/3}, \quad \text{for all } \ x \in \mathbb{R}, \tag{2.1}$$

where χ_A is used to denote the characteristic function of the set A. We also define

$$g[u] := \psi * \partial_x u. \tag{2.2}$$

We note that, since $\psi \in \mathcal{S}'(\mathbb{R})$, it follows that for $\phi \in \mathcal{S}(\mathbb{R})$, one has that $\psi * \phi \in C^\infty(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ and $\widehat{\psi * \phi} = \sqrt{2\pi}\hat{\psi}\hat{\phi}$ (see Rudin [8]). Then, for $\varphi \in \mathcal{S}(\mathbb{R})$, $g[\varphi](\cdot) = \psi * \partial_x\varphi(\cdot) = \int_0^{+\infty} \xi^{-1/3}\partial_x\varphi(\cdot - \xi)d\xi$. Next lemma gives the Fourier transform of function ψ.

Lemma 2.1. *For the function ψ defined by (2.1) we have*

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}}\Gamma\Big(\frac{2}{3}\Big)\Big(\frac{1}{2} - i\frac{\sqrt{3}}{2}sgn(\xi)\Big)|\xi|^{-2/3}, \tag{2.3}$$

where

$$sgn(\xi) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0, \end{cases}$$

and Γ is the gamma function.

Proof. We define the function $\psi_n(x) := \chi_{(0,n)}(x)x^{-1/3}$, for all $x \in \mathbb{R}$, and $n \in \mathbb{N}$. It is not difficult to see that $\psi_n \to \psi$ in $\mathcal{S}'(\mathbb{R})$ as n goes to infinity. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} \langle\hat{\psi}_n, \varphi\rangle &= \frac{1}{\sqrt{2\pi}}\int\Big[\int_0^n \frac{\cos(\xi x)}{\xi^{1/3}}d\xi - i\int_0^n \frac{\sin(\xi x)}{\xi^{1/3}}d\xi\Big]\varphi(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int x^{-2/3}\Big[\int_0^{nx} \frac{\cos(u)}{u^{1/3}}du - i\int_0^{nx} \frac{\sin(u)}{u^{1/3}}du\Big]\varphi(x)dx. \end{aligned}$$

Since

$$\int_0^{+\infty} \frac{\cos x}{x^{1/3}}dx = \frac{1}{2}\Gamma\Big(\frac{2}{3}\Big), \ \text{ and } \ \int_0^{+\infty} \frac{\sin x}{x^{1/3}}dx = \frac{\sqrt{3}}{2}\Gamma\Big(\frac{2}{3}\Big),$$

it follows that

$$\Big|x^{-2/3}\int_0^{nx} \frac{e^{-iu}}{u^{1/3}}du\ \varphi(x)\Big| \le C|x|^{-2/3}|\varphi(x)|,$$

for all $n \in \mathbb{N}$, and $x \in \mathbb{R}$. Therefore, the dominated convergence theorem implies that

$$\lim_{n\to\infty}\langle\hat{\psi}_n, \varphi\rangle = \frac{1}{\sqrt{2\pi}}\int x^{-2/3}\Gamma\Big(\frac{2}{3}\Big)\Big(\frac{1}{2} - i\frac{\sqrt{3}}{2}\mathrm{sgn}(x)\Big)\varphi(x)dx.$$

This completes the proof of the lemma. □

REMARK 2.1. *Let $s \in \mathbb{R}$. If $u \in H^s(\mathbb{R})$, one can define $g[u]$ through its Fourier transform by*

$$\widehat{g[u]}(\xi) := \Gamma\Big(\frac{2}{3}\Big)\Big(\frac{\sqrt{3}}{2} sgn(\xi) + \frac{i}{2}\Big)\xi^{1/3}\hat{u}(\xi), \tag{2.4}$$

for almost every $\xi \in \mathbb{R}$. Thus, if $u \in H^s(\mathbb{R})$, it follows that $g[u] \in H^{s-1/3}(\mathbb{R})$ and $\|g[u]\|_{H^{s-1/3}} \leq \Gamma\Big(\frac{2}{3}\Big)\|u\|_{H^s}$.

In this section we consider the following, more general, version of equation (1.1):

$$\partial_t u(x,t) + \partial_x\Big(\frac{u^2}{2} - \partial_x u + \eta\, g[u]\Big)(x,t) = 0, \tag{2.5}$$

where $\eta \in \mathbb{R}$. We will show existence of travelling-wave solutions to equation (2.5), for any $\eta \in \mathbb{R}$. First, we consider the case $\eta = 0$. For any $d \in \mathbb{R}$ (see Johnson [7]),

$$u_d(x,t) = \frac{d}{2}\Big[1 - \tanh\Big(\frac{d}{4}(x - \frac{d}{2}t)\Big)\Big] \tag{2.6}$$

is a solution to equation (2.5) with $\eta = 0$.

REMARK 2.2. *Let $\lambda > 0$. We define*

$$u_\lambda(x,t) := \frac{1}{\lambda}u\Big(\frac{x}{\lambda}, \frac{t}{\lambda^2}\Big), \quad \textit{for } x \in \mathbb{R}, \quad \textit{and } t \geq 0. \tag{2.7}$$

It is straightforward to check that if u is a solution to the equation

$$\partial_t u(x,t) + \partial_x\Big(\frac{u^2}{2} - \partial_x u + \lambda^{2/3}\eta\, g[u]\Big)(x,t) = 0, \tag{2.8}$$

then u_λ satisfies equation (2.5). Hence, if ϕ is a travelling-wave solution of equation (2.8) with speed c, then $\phi_\lambda(\cdot) = \frac{1}{\lambda}\phi(\frac{1}{\lambda}\cdot)$ is a travelling-wave solution of equation (2.5) with speed c/λ.

We define, for $c \in \mathbb{R}$, the functions

$$g_c(x) := c\Big(1 - \tanh\Big(\frac{c}{2}x\Big)\Big), \quad \text{and} \quad h_c(x) := g_c'(x) = -\frac{c^2}{2}\mathrm{sech}^2\Big(\frac{c}{2}x\Big). \tag{2.9}$$

REMARK 2.3. *Let $c \in \mathbb{R}$. We see that $g[g_c] = I_1 + I_2$, where $I_j := \psi_j * h_c$ for $j = 1, 2$, with $\psi_1 := \psi\cdot\chi_{(0,1)}$, and $\psi_2 := \psi\cdot\chi_{(1,+\infty)}$. Now, we state some immediate properties of the function $g[g_c]$.*

a.) *Let $p > 3$. Since $\psi_1 \in L^1(\mathbb{R})$, and $\psi_2 \in L^p(\mathbb{R})$, it follows from the Young inequality for convolution that $g[g_c] \in L^p(\mathbb{R})$.*

b.) *Furthermore, $g[g_c] \in C_\infty(\mathbb{R})$. In fact, it follows from the dominated convergence theorem that I_1 is continuous and $I_1(x) \to 0$ as $|x| \to \infty$. Moreover, Hölder's inequality and the dominated convergence theorem imply that*

$$I_2(x) \leq C(c)\Big(\int_1^{+\infty} \frac{sech^4(\frac{c}{2}(x-\xi))}{\xi^{4/3}} d\xi\Big)^{1/4} \to 0, \ \textit{as } |x| \to +\infty.$$

The continuity of I_2 is shown similarly to the continuity of I_1.

Let $c \in \mathbb{R}$. In the sequel, we will consider the following spaces:

$$\begin{aligned} \mathfrak{X} = \mathfrak{X}_c &:= \Big\{\varphi \in C_b^1(\mathbb{R}) \ ; \ \int \varphi' h_c'\, dx = 0\Big\}, \\ \tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}_c &:= \Big\{g_c + \varphi \ ; \ \varphi \in \mathfrak{X}\Big\}, \end{aligned}$$

where $\|\cdot\|_{\mathfrak{X}} = \|\cdot\|_{C_b^1}$. One sees that $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a Banach space.

REMARK 2.4. *Assume that $\varphi \in C_b^1(\mathbb{R})$. By integration by parts one has that*

$$g[\varphi](x) = \int_0^1 \frac{1}{\xi^{1/3}}\varphi'(x-\xi)d\xi + \varphi(x-1) - \frac{1}{3}\int_1^{+\infty} \frac{1}{\xi^{4/3}}\varphi(x-\xi)d\xi. \tag{2.10}$$

Then $g[\varphi] \in C_b(\mathbb{R})$. Moreover,

$$\|g[\varphi]\|_{L^\infty} \leq C\,\|\varphi\|_{C_b^1}. \tag{2.11}$$

*Hence, if $\phi \in \tilde{\mathfrak{X}}$, it follows from Remark 2.3-***b.)** *above that $g[\phi] \in C_b(\mathbb{R})$.*

Suppose now that $u(x,t) = \phi(x-ct)$ is a solution to equation (2.5), where $\phi \in \tilde{\mathfrak{X}}$. Then $-c\phi' + \frac{d}{dx}(\frac{\phi^2}{2} - \phi' + \eta g[\phi]) = 0$. Thus, a sufficient condition to guarantee that ϕ satisfies the last equation is

$$F(\eta,\phi) = F_c(\eta,\phi) := c\phi - \frac{\phi^2}{2} + \phi' - \eta g[\phi] = 0. \tag{2.12}$$

We denote by τ_c the function given by $\tau_c(x) := c - g_c(x) = c\tanh(\frac{c}{2}x)$, for $x \in \mathbb{R}$. We now define the function $G = G_c$, which is well defined on $\mathbb{R}\times\mathfrak{X}$ by Remarks 2.3 and 2.4 above, as

$$\begin{aligned} G : \mathbb{R}\times\mathfrak{X} &\mapsto C_b(\mathbb{R}) \\ (\eta,\varphi) &\mapsto G(\eta,\varphi) = \tau_c\varphi - \frac{\varphi^2}{2} + \varphi' - \eta g[\varphi] - \eta g[g_c]. \end{aligned} \tag{2.13}$$

Assume that $\phi = \varphi + g_c \in \tilde{\mathfrak{X}}$. Since $F(0,g_c) = 0$, it follows that $F(\eta,\phi) = G(\eta,\varphi)$. Hence, ϕ satisfies equation (2.12) if and only if φ verifies the equation $G(\eta,\varphi) = 0$.

The following theorem implies the existence of a travelling-wave solution, $u(x,t) = \phi(x-ct)$ with $c > 0$ and $\phi \in \tilde{\mathfrak{X}}$, to equation (2.5) for η in a neighborhood of zero; its proof uses the implicit function theorem.

THEOREM 2.1. *Suppose $c > 0$. Then there exist $\delta, \delta_0 > 0$ such that for every $\eta \in (-\delta,\delta)$, there is exactly one $\varphi_\eta = \varphi_{\eta,c} \in \mathfrak{X}$ for which $\|\varphi_\eta\|_{\mathfrak{X}} \leq \delta_0$ and $G(\eta,\varphi_\eta) = 0$. Moreover, the mapping $\eta \mapsto \varphi_\eta$ is a C^∞-map on a neighborhood of 0.*

Proof. Let $c > 0$. The mapping $G = G_c$ is defined on the Banach space $\mathbb{R}\times\mathfrak{X}$ taking values in the Banach space $(C_b(\mathbb{R}), \|\cdot\|_{L^\infty})$, and satisfies $G(0,0) = 0$.

We now claim that $\partial_1 G$ and $\partial_2 G$ exist as partial F-derivatives (Fréchet derivative) on $\mathbb{R}\times\mathfrak{X}$ and that the partial F-derivative $\partial_2 G(0,0) : \mathfrak{X} \mapsto C_b(\mathbb{R})$ is bijective. In fact, let us take $(\eta,\varphi) \in \mathbb{R}\times\mathfrak{X}$. One can see that

$$\partial_1 G(\eta,\varphi)\cdot = -(g[\varphi] + g[g_c])\cdot$$

and

$$\partial_2 G(\eta,\varphi)\cdot = (\tau_c - \varphi)\cdot + \partial_x \cdot - \eta g[\cdot]. \tag{2.14}$$

Then $\partial_1 G(\eta,\varphi) \in \mathfrak{L}(\mathbb{R}, C_b(\mathbb{R}))$, and $\partial_2 G(\eta,\varphi) \in \mathfrak{L}(\mathfrak{X}, C_b(\mathbb{R}))$. Moreover, we obtain that $\|\partial_1 G(\eta,\varphi)\|_{\mathfrak{L}(\mathbb{R},C_b(\mathbb{R}))} \leq C\cdot(\|\varphi\|_{C_b^1} + \|g[g_c]\|_{L^\infty})$, and $\|\partial_2 G(\eta,\varphi)\|_{\mathfrak{L}(\mathfrak{X},C_b(\mathbb{R}))} \leq C\cdot(1 + |\eta| + \|\tau_c - \varphi\|_{L^\infty})$, where we have used inequality (2.11). Hence, $\partial_1 G, \partial_2 G$ exist as partial F-derivatives on $\mathbb{R}\times\mathfrak{X}$.

We will now show that the partial F-derivative $\partial_2 G(0,0) = \tau_c + \partial_x : \mathfrak{X} \mapsto C_b(\mathbb{R})$ is bijective. We begin with the injectivity; we emphasize here that the definition of the space $\mathfrak{X} \subset C_b^1(\mathbb{R})$ was chosen to ensure the injectivity of the mapping $\partial_2 G(0,0)$.

Let f be an element of $\mathfrak{X}$ such that $\tau_c f + f' = 0$. By solving the last ordinary differential equation, one gets

$$f(x) = f(0) \cdot e^{-\int_0^x \tau_c(s)ds} = f(0) \cdot \operatorname{sech}^2\Big(\frac{c}{2}x\Big).$$

Since $f \in \mathfrak{X}$, it follows that

$$\int f'(x)h_c'(x)dx = -f(0)\frac{2}{c^2}\int (h_c')^2(x)dx = 0.$$

Then $f(0) = 0$, and therefore $f = 0$.
We will now show that the mapping $\partial_2 G(0,0)$ is onto. Let y be an element of $C_b(\mathbb{R})$. By the method of variation of parameters, we obtain that the function

$$g(x) := \lambda l_c(x) + l_c(x)\int_0^x \frac{y(s)}{l_c(s)}ds \tag{2.15}$$

is a solution to the equation $\tau_c g + g' = y$, for any $\lambda \in \mathbb{R}$, where $l_c := -\frac{2}{c^2}h_c = \operatorname{sech}^2(\frac{c}{2}x)$. We will prove that $g \in \mathfrak{X}$ for a suitably chosen real number λ. First, we remark that there exists a unique $\lambda = \lambda_{y,c} \in \mathbb{R}$ such that $\int g'h_c'dx$=0. In fact take

$$\lambda := \frac{c^2}{2\int (h_c')^2(x)dx}\int \Big[(h_c')^2(x)\int_0^x \frac{y(s)}{h_c(s)}ds + y(x)h_c'(x)\Big]dx, \tag{2.16}$$

where we note that

$$0 < \int (h_c')^2(x)dx \le \frac{c^6}{4}\int \operatorname{sech}^4\Big(\frac{c}{2}x\Big)dx = C(c), \quad \int |h_c'(x)|dx = c^2,$$

and

$$\int (h_c')^2(x)\Big|\int_0^x \frac{y(s)}{h_c(s)}ds\Big|dx \le \frac{c^4}{2}\|y\|_{L^\infty}\int \frac{\sinh^2(\frac{c}{2}x)}{\cosh^6(\frac{c}{2}x)}\Big|\int_0^x \frac{1+\cosh(cs)}{2}ds\Big|dx$$
$$\le \frac{c^3}{4}\|y\|_{L^\infty}\int \Big[c|x|\operatorname{sech}^4\Big(\frac{c}{2}x\Big) + 2\operatorname{sech}^2\Big(\frac{c}{2}x\Big)\Big]dx \le C(c)\|y\|_{L^\infty}.$$

It remains to show that g given by (2.15) and (2.16) belongs to $C_b^1(\mathbb{R})$. It is immediate to see that $g \in C(\mathbb{R})$, we need to show that g is bounded. We have that

$$\operatorname{sech}^2\Big(\frac{c}{2}x\Big)\Big|\int_0^x \frac{y(s)}{\operatorname{sech}^2(\frac{c}{2}s)}ds\Big| \le \frac{\|y\|_{L^\infty}}{1+\cosh(cx)}\Big|\int_0^x (1+\cosh(cs))ds\Big|$$
$$\le \|y\|_{L^\infty}\Big(\frac{|x|}{1+\cosh(cx)} + \frac{1}{c}|\tanh(cx)|\Big) \le C(c)\|y\|_{L^\infty}.$$

Then $g \in C_b(\mathbb{R})$. Moreover, since g satisfies the equation $\tau_c g + g' = y$, it follows that $g \in C_b^1(\mathbb{R})$. Hence, $g \in \mathfrak{X}$. Therefore, $\partial_2 G(0,0)$ is a surjective mapping.

It is not difficult to see, by using inequality (2.11), that G, $\partial_1 G$ and $\partial_2 G$ are continuous on $\mathbb{R} \times \mathfrak{X}$. Then, the implicit function theorem implies the first part of the theorem. Furthermore, from (2.13) one can see that function G is quadratic in φ and linear in η, therefore it is not difficult to verify that $\partial^2_{i,j}G(\eta,\varphi)$ is independent of $(\eta,\varphi) \in \mathbb{R}\times\mathfrak{X}$, for all $i,j \in \{1,2\}$. Hence, $\partial^k_{i_1,\dots,i_k}G(\eta,\varphi) = 0$ for all $k \ge 3$, where $i_1,\dots,i_k \in \{1,2\}$, and $(\eta,\varphi) \in \mathbb{R}\times\mathfrak{X}$. Finally, the second part of the theorem is then a consequence of the fact that the mapping G is a C^∞-map on $\mathbb{R}\times\mathfrak{X}$.

□

Corollary 2.1. *Let $\eta \in \mathbb{R}$ and $d \in \mathbb{R}$. Then there is a travelling-wave solution $\tilde{\phi} \in C_b^1(\mathbb{R})$ of equation (2.5) with speed d.*

Proof. Let $c>0$. By Theorem 2.1 there exists $\lambda_0=\lambda_0(\eta,c)>0$ such that for every $\lambda\in(0,\lambda_0)$, there is a $\phi=\phi_{\lambda,\eta,c}\in C_b^1(\mathbb{R})$ such that $u(x,t)=\phi(x-ct)$ is a solution to equation (2.8). Now we can see, by using Remark 2.2, that $\phi^\dagger(\cdot)=\frac{1}{\lambda}\phi(\frac{1}{\lambda}\cdot)$ is a travelling-wave solution of equation (2.5) with speed $c/\lambda\in(\frac{c}{\lambda_0},+\infty)$. The result now follows from the fact that if $\phi^\dagger(x-\tilde{c}t)$ is a solution of equation (2.5) for some $\tilde{c}>0$, then $\tilde{\phi}(x,t):=\phi^\dagger(x-(\tilde{c}+k)t)+k$ is also a solution for all $k\in\mathbb{R}$. □

Remark 2.5. *Let us comment about the existence of solitary travelling waves. Let us proceed formally at first. By multiplying the equation* $-c\phi'+\frac{d}{dx}(\frac{\phi^2}{2}-\phi'+\eta g[\phi])=0$ *by* ϕ*, then integrating between* $-\infty$ *and* x*, assuming that* $\phi,\phi'\to 0$ *as* $|x|\to+\infty$*, we get*

$$-c\frac{\phi^2(x)}{2}+\frac{\phi^3(x)}{3}-\int_{-\infty}^{x}\phi(y)\phi''(y)dy+\eta\int_{-\infty}^{x}\frac{dg[\phi]}{dy}(y)\phi(y)dy=0.$$

Making $x\to+\infty$*, integrating by parts, and then applying Parseval's relation and Remark 2.1, we obtain*

$$\int_{-\infty}^{+\infty}\Big(\xi^2-\frac{\eta}{2}\Gamma\Big(\frac{2}{3}\Big)\xi^{4/3}\Big)|\hat{\phi}(\xi)|^2d\xi=0. \tag{2.17}$$

These formal steps can be justified by assuming for instance that $\phi\in H^2(\mathbb{R})$*. Thus, equation (2.17) implies that if* $\eta\leq 0$*, then* $\phi=0$*. We can then conclude that there are no nontrivial travelling-waves of the solitary-wave type for equation (2.5) when* $\eta\leq 0$*. However, in the physical case, that is to say when* $\eta=1$ *or more generally when* $\eta>0$*, equation (2.17) does not preclude the possibility that they may exist.*

3. Local Theory in a subspace of $C_b^1(\mathbb{R})$

In Section 2, we proved the existence of a travelling-wave solution $u(x,t)=\phi(x-ct)$ to equation (2.5) for any $\eta\in\mathbb{R}$, where c is an appropriate positive number and $\phi\in C_b^1(\mathbb{R})$. Motivated by this last result, we will consider in this section the local well-posedness theory for the following initial value problem (IVP)

$$\begin{cases}\partial_t u(x,t)+\partial_x\big(\frac{1}{2}u^2-\partial_x u+g[u]\big)(x,t)=0,\\ u(0)=u_0,\end{cases} \tag{3.1}$$

where $g[u]$ is given by (2.2), and u_0 belongs to a suitable subspace of $C_b^1(\mathbb{R})$. The Cauchy problem associated to the IVP (3.1) for initial data $u_0\in L^2(\mathbb{R})$ was recently studied by Alibaud, Azerad and Isèbe [1].

3.1. **The Linear Equation.** First, we consider the linear part associated to the IVP (3.1), namely

$$\begin{cases}\partial_t u(x,t)-\partial_x^2 u(x,t)+\partial_x g[u](x,t)=0,\\ u(0)=u_0.\end{cases} \tag{3.2}$$

By formally taking the Fourier transform of the last expression, we get

$$\hat{u}(\xi,t)=\hat{K}(\xi,t)\hat{u}_0(\xi), \tag{3.3}$$

where

$$\hat{K}(\xi,t)=e^{-t[\xi^2-\xi^{4/3}(a+ib\,\mathrm{sgn}(\xi))]}, \tag{3.4}$$

for $\xi\in\mathbb{R}$ and $t\geq 0$, with $a:=\frac{1}{2}\Gamma(\frac{2}{3})$ and $b:=-\frac{\sqrt{3}}{2}\Gamma(\frac{2}{3})$. For $\xi\in\mathbb{R}$, we define

$$\Phi(\xi):=(a+ib\,\mathrm{sgn}(\xi)). \tag{3.5}$$

We note that $|\Phi(\xi)| = \Gamma(\frac{2}{3})$, for all $\xi \in \mathbb{R}$.

REMARK 3.1. *The non local term $\partial_x g[u]$ is anti-dissipative of order $4/3$.*

REMARK 3.2. *For every $t > 0$, the kernel $K(\cdot, t)$ is not a nonnegative function. Indeed, by contradiction, if $K(\cdot, t)$ would be nonnegative, one could bound*

$$|\hat{K}(\xi, t)| \leq \big|\frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} K(x, t) dx\big| \leq \hat{K}(0, t) = 1.$$

But, on the other hand, $|\hat{K}(\xi, t)| = e^{-t[\xi^2 - a\xi^{4/3}]} > 1$, for $0 < |\xi| < a^{3/2}$. Hence, for every $t > 0$, there exists $x \in \mathbb{R}$ such that $K(x, t) < 0$. This fact implies, in particular, that the IVP (3.1) is non-monotone (see [1] *for more details).*

For $t \geq 0$, we define the operator $E(t)$ by

$$\begin{cases} E(t)\phi(x) = \frac{1}{\sqrt{2\pi}}\big(K(\cdot, t) * \phi\big)(x), \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ E(0)\phi = \phi, \end{cases} \tag{3.6}$$

where $\phi \in C_b(\mathbb{R})$ (see Lemma 3.11 below). Now, we define the following spaces

$$Y := \{g \in C_b(\mathbb{R}); g \text{ is uniformly continuous}\}; \tag{3.7}$$

$$X := \{f \in C_b^1(\mathbb{R}); f' \text{ is uniformly continuous}\}. \tag{3.8}$$

One can see that $(Y, \|\cdot\|_{C_b(\mathbb{R})})$, and $(X, \|\cdot\|_{C_b^1(\mathbb{R})})$ are Banach spaces and that $X \hookrightarrow Y$. In Sub-section 3.2 we will show local-in-time well-posedness of the IVP (3.1), with initial data $u_0 \in X$.

The following lemma contains a calculus result.

LEMMA 3.1. *Let $h : \mathbb{R} \mapsto \mathbb{C}$ be a function which satisfies the following conditions:*
i.) *$h \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$;*
ii.) *$h' \in L^1(\mathbb{R})$, $|h'(x)| \to 0$ as $|x| \to +\infty$. Moreover, there exist $\lim_{x \downarrow 0} h'(x) = h'(0^+)$, and $\lim_{x \uparrow 0} h'(x) = h'(0^-)$;*
iii.) *$h'' \in L^1(\mathbb{R})$.*
Then $\hat{h} \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$, and

$$\|\hat{h}\|_{L^1} \leq \sqrt{\frac{2}{\pi}}\Big[\|h\|_{L^1} + |h'(0^+) - h'(0^-)| + \|h''\|_{L^1}\Big]. \tag{3.9}$$

Proof. Since $h \in L^1(\mathbb{R})$, it follows from the Riemann-Lebesgue lemma that $\hat{h} \in C_\infty(\mathbb{R})$. After using integration by parts twice, we see that

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{+\infty} h''(x)\frac{e^{-i\xi x}}{(i\xi)^2} dx + \frac{h'(0^+) - h'(0^-)}{(i\xi^2)}\Big], \quad \text{for } \xi \neq 0.$$

Expression (3.9) follows from the last equation and from the fact that $\|\hat{h}\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}}\|h\|_{L^1}$. □

REMARK 3.3. *It is well-known that $W^{1,1}(\mathbb{R}) \subset C_\infty(\mathbb{R}) \cap AC(\mathbb{R})$, where $AC(\mathbb{R})$ denotes the space of all complex-valued functions, which are absolutely continuous on $\mathbb{R}$. Therefore, it follows from Lemma 3.1 above that if $f \in W^{2,1}(\mathbb{R})$, then $\hat{f} \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$ and*

$$\|\hat{f}\|_{L^1} \leq \sqrt{\frac{2}{\pi}}\Big[\|f\|_{L^1} + \|f''\|_{L^1}\Big]. \tag{3.10}$$

REMARK 3.4. *Suppose now that $t \in (0,1)$. Since*

$$\begin{aligned} K(x,t) &= \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} e^{-t[\xi^2 - \xi^{4/3}\Phi(\xi)]} d\xi \\ &= \frac{t^{-1/2}}{\sqrt{2\pi}} \int e^{i(t^{-1/2}x)\xi}\ e^{-[\xi^2 - \xi^{4/3}\Phi(\xi)]}\ e^{-(1-t^{1/3})\xi^{4/3}\Phi(\xi)} d\xi, \end{aligned}$$

it follows that

$$K(x,t) = t^{-1/2}\big(K(\cdot,1) * G(\cdot, 1-t^{1/3})\big)(t^{-1/2}x), \ \ \textit{for}\ \ x \in \mathbb{R}, \tag{3.11}$$

where

$$G(\cdot, 1-t^{1/3}) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(e^{-(1-t^{1/3})\xi^{4/3}(a+ib\ sgn(\xi))})(\cdot). \tag{3.12}$$

The next three lemmas are elementary calculus results which will be used in the sequel.

LEMMA 3.2. *Suppose that $\alpha > -1$, and $\beta > 0$. Then*

$$I(\alpha,\beta) := \int |\xi|^\alpha e^{-\beta|\xi|^{4/3}} d\xi = C(\alpha)\beta^{-\frac{3}{4}(\alpha+1)}.$$

Proof. The assertion of the lemma follows from the fact that

$$I(\alpha,\beta) = \beta^{-\frac{3}{4}(\alpha+1)} \int |\tau|^\alpha e^{-|\tau|^{4/3}} d\tau.$$

□

LEMMA 3.3. *Suppose that $\alpha > -1$, $\beta > 0$, and $t > 0$. Then*

$$I(\alpha,\beta,t) := \int |\xi|^\alpha e^{-t[\xi^2 - \beta|\xi|^{4/3}]} d\xi \le C(\alpha,\beta)\Big[e^{\frac{4}{27}\beta^3 t} + t^{-\frac{\alpha+1}{2}}\Big].$$

Proof. It is elementary to check that $\xi^2 - \beta\xi^{4/3} \ge -\frac{4}{27}\beta^3$ for all $\xi \in \mathbb{R}$, and $\xi^2 - \beta\xi^{4/3} \ge \xi^2/2$ for $\xi \ge (2\beta)^{3/2}$. Then

$$\begin{aligned} I(\alpha,\beta,t) &\le 2\Big[\int_0^{(2\beta)^{3/2}} \xi^\alpha e^{\frac{4}{27}\beta^3 t} d\xi + \int_{(2\beta)^{3/2}}^{+\infty} \xi^\alpha e^{-\frac{t}{2}\xi^2} d\xi\Big] \\ &\le C(\alpha,\beta)\Big[e^{\frac{4}{27}\beta^3 t} + \int_0^{+\infty} \Big(\frac{2}{t}\Big)^{\frac{\alpha}{2}} u^\alpha e^{-u^2} \sqrt{\frac{2}{t}} du\Big], \end{aligned}$$

where in the last inequality we have used the fact that $\alpha > -1$. The result now follows. □

LEMMA 3.4. *Suppose that $g \in W^{1,1}(\mathbb{R})$, and $l \in L^\infty(\mathbb{R})$. If $f := g * l$, then $f \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, and $f'(x) = (g' * l)(x)$ for all $x \in \mathbb{R}$.*

Proof. Since $g \in L^1(\mathbb{R})$ and $l \in L^\infty(\mathbb{R})$, it follows from Young's inequality that $f \in L^\infty(\mathbb{R})$. Moreover, since $|f(x+h) - f(x)| \le \|g(\cdot+h) - g(\cdot)\|_{L^1} \|l\|_{L^\infty}$ for all $x \in \mathbb{R}$, and $\|g(\cdot+h) - g(\cdot)\|_{L^1} \to 0$ as h tends to zero, it follows that $f \in C(\mathbb{R})$.

Let $x \in \mathbb{R}$. Since $W^{1,1}(\mathbb{R}) \subset AC(\mathbb{R})$, we see that

$$\Big|\frac{f(x+h)-f(x)}{h} - \int g'(x-y)l(y)dy\Big| =$$
$$\Big|\int\int_0^1 \big(g'(x-y+th)-g'(x-y)\big)l(y)dtdy\Big|$$
$$\leq \|l\|_{L^\infty}\int_0^1 \|g'(\cdot+th)-g'(\cdot)\|_{L^1}dt \to 0 \quad \text{as} \quad h \to 0,$$

where the last expression is a consequence of the dominated convergence theorem. □

The following five lemmas provide more explicit estimates than the corresponding results mentioned in [1]. The next lemma gives an upper bound, which goes to infinity as t tends to 1, for $\|G(\cdot, 1-t^{1/3})\|_{L^1}$ when $t \in [0,1)$.

LEMMA 3.5. *Let $t_0 \in (0,1)$. Then, for all $t \in [0,t_0]$, the function $G(\cdot,1-t^{1/3})$ given by (3.12) belongs to $L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$. Moreover,*

$$\|G(\cdot,1-t^{1/3})\|_{L^1} \leq C\big[(1-t^{1/3})^{3/4} + (1-t^{1/3})^{-3/4}\big] \leq C\cdot(1-t_0^{1/3})^{-3/4}, \quad (3.13)$$

for all $t \in [0,t_0]$, where C is a positive constant independent of t.

Proof. Let $t \in [0,1)$. We define $g(\xi,t) := e^{-(1-t^{1/3})\xi^{4/3}\Phi(\xi)}$ for $\xi \in \mathbb{R}$. It follows that $g(\cdot,t)$ is continuous. Furthermore,

$$\partial_\xi g(\xi,t) = -\frac{4}{3}(1-t^{1/3})\xi^{1/3}\Phi(\xi)e^{-(1-t^{1/3})\xi^{4/3}\Phi(\xi)}, \quad \text{for } \xi \neq 0. \quad (3.14)$$

Then $|\partial_\xi g(\xi,t)| \to 0$ as $|\xi| \to +\infty$, and $\partial_\xi g(0^+,t) = 0 = \partial_\xi g(0^-,t)$. Moreover,

$$\begin{aligned}\partial_\xi^2 g(\xi,t) &= \Big[-\frac{4}{9}(1-t^{1/3})\xi^{-2/3}\Phi(\xi) + \Big(\frac{4}{3}(1-t^{1/3})\xi^{1/3}\Phi(\xi)\Big)^2\Big] \\ &\quad \times e^{-(1-t^{1/3})\xi^{4/3}\Phi(\xi)}, \quad \text{for } \xi \neq 0.\end{aligned}$$

We see that $g(\cdot,t) \in C_\infty(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$. In addition, $\|g(\cdot,t)\|_{L^1} = C\cdot(1-t^{1/3})^{-3/4}$, and $\|\partial_\xi g(\cdot,t)\|_{L^1} = 4$. Furthermore,

$$\begin{aligned}\|\partial_\xi^2 g(\cdot,t)\|_{L^1} &\leq C\cdot(1-t^{1/3})\int |\xi|^{-2/3}e^{-(1-t^{1/3})a\xi^{4/3}}d\xi \\ &\quad + C\cdot(1-t^{1/3})^2\int |\xi|^{2/3}e^{-(1-t^{1/3})a\xi^{4/3}}d\xi \\ &\leq C\cdot(1-t^{1/3})^{3/4},\end{aligned}$$

where the last inequality is a consequence of Lemma 3.2 above. The result now follows from Lemma 3.1. □

Lemmas 3.6 and 3.9 below provide estimates for $\|K(\cdot,t)\|_{L^1}$ and $\|\partial_x K(\cdot,t)\|_{L^1}$, for any $t > 0$.

LEMMA 3.6. *Suppose that $t > 0$. Then the function $K(\cdot,t) \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$, and*

$$\|K(\cdot,t)\|_{L^1} \leq C\cdot\big(1 + t^2 e^{\frac{4}{27}a^3 t}\big), \quad (3.15)$$

where C is a positive constant independent of t.

Proof. Let $t>0$. It follows from (3.4) that

$$\partial_\xi \hat{K}(\xi,t) = -t\Big[2\xi - \frac{4}{3}\xi^{1/3}\Phi(\xi)\Big]\hat{K}(\xi,t), \quad \text{for } \xi\neq 0,$$

and

$$\partial_\xi^2 \hat{K}(\xi,t) = \Big\{ -t\Big[2-\frac{4}{9}\xi^{-2/3}\Phi(\xi)\Big] + t^2\Big[2\xi - \frac{4}{3}\xi^{1/3}\Phi(\xi)\Big]^2\Big\}\hat{K}(\xi,t), \text{ for } \xi\neq 0.$$

Then $\hat{K}(\cdot,t)\in C_\infty(\mathbb{R})\cap C^2(\mathbb{R}\setminus\{0\})$. Moreover, $|\partial_\xi\hat{K}(\xi,t)|\to 0$ as $|\xi|\to+\infty$, and $\partial_\xi\hat{K}(0^+,t)=0=\partial_\xi\hat{K}(0^-,t)$. Furthermore,

$$\|\hat{K}(\cdot,t)\|_{L^1} = 2\int_0^{+\infty} e^{-t[\xi^2-a\xi^{4/3}]}d\xi \leq C\Big[e^{\frac{4}{27}a^3t}+\frac{1}{\sqrt{t}}\Big],$$

where the last inequality is a consequence of Lemma 3.3. Again using Lemma 3.3, we see that

$$\begin{aligned}
\|\partial_\xi\hat{K}(\cdot,t)\|_{L^1} &\leq C\big[1+t^{1/3}+t\,e^{\frac{4}{27}a^3t}\big], \quad \text{and}\\
\|\partial_\xi^2\hat{K}(\cdot,t)\|_{L^1} &\leq C\big[\sqrt{t}+t^{5/6}+t^{7/6}+(t+t^2)e^{\frac{4}{27}a^3t}\big] \leq C\big[\sqrt{t}+t^2e^{\frac{4}{27}a^3t}\big].
\end{aligned}$$

Lemma 3.1, applied to $h(\cdot)=\hat{K}(\cdot,t)$, implies that

$$\|K(\cdot,t)\|_{L^1}\leq C\Big[\frac{1}{\sqrt{t}}+t^2e^{\frac{4}{27}a^3t}\Big], \quad \text{for all } t>0. \tag{3.16}$$

Suppose now that $t\in(0,1)$. Then

$$\begin{aligned}
\int|K(x,t)|dx &= \frac{1}{\sqrt{t}}\int\big|K(\cdot,1)*G(\cdot,1-t^{1/3})\big|(x/\sqrt{t})dx\\
&= \int(1+y^2)^{1/2}\frac{\big|K(\cdot,1)*G(\cdot,1-t^{1/3})\big|(y)}{(1+y^2)^{1/2}}dy\\
&\leq C\,\big\|\hat{K}(\cdot,1)\hat{G}(\cdot,1-t^{1/3})\big\|_{H^1},
\end{aligned}\tag{3.17}$$

where the first equality above comes from (3.11). From the fact that $e^{-(1-h^{1/3})a\xi^{4/3}}\leq 1$, for all $h\in[0,1)$, $\xi\in\mathbb{R}$, and equation (3.14) we have that $|\hat{G}(\xi,1-h^{1/3})|\leq C$, and $|\partial_\xi\hat{G}(\xi,1-h^{1/3})|\leq C|\xi|^{1/3}$ for all $\xi\in\mathbb{R}$. Now, from Lemma 3.3, we obtain

$$\begin{aligned}
\|\hat{K}(\cdot,1)\hat{G}(\cdot,1-h^{1/3})\|_{H^1} &\leq \|\hat{K}(\cdot,1)\hat{G}(\cdot,1-h^{1/3})\|_{L^2}\\
&+\|\partial_\xi\hat{K}(\cdot,1)\hat{G}(\cdot,1-h^{1/3})\|_{L^2}+\|\hat{K}(\cdot,1)\partial_\xi\hat{G}(\cdot,1-h^{1/3})\|_{L^2}\leq C,
\end{aligned}\tag{3.18}$$

for all $h\in[0,1)$. The assertion of the lemma now follows from (3.16)-(3.18). □

The following result gives an upper bound for $\|\partial_xK(\cdot,t)\|_{L^1}$ when $t\in(0,1)$.

Lemma 3.7. *Let $t\in(0,1)$. Then, $K(\cdot,t)\in L^1(\mathbb{R})\cap C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$. In addition, $\partial_xK(\cdot,t)(x)=t^{-1}\big(\partial_xK(\cdot,1)*G(\cdot,1-t^{1/3})\big)(t^{-1/2}x)$ for all $x\in\mathbb{R}$, $\partial_xK(\cdot,t)\in L^1(\mathbb{R})\cap C_\infty(\mathbb{R})$, and*

$$\|\partial_xK(\cdot,t)\|_{L^1}\leq\frac{C}{\sqrt{t}}\big[(1-t^{1/3})^{3/4}+(1-t^{1/3})^{-3/4}\big], \tag{3.19}$$

where C is a positive constant independent of t.

Proof. Let f denote the function given by $f(\xi) := \xi\hat{K}(\xi,1)$ for all $\xi \in \mathbb{R}$. Then $f \in C_\infty(\mathbb{R}) \cap C^2(\mathbb{R})$. Furthermore,

$$f'(\xi) = \Big[1 - \xi\Big(2\xi - \frac{4}{3}\xi^{1/3}\Phi(\xi)\Big)\Big]\hat{K}(\xi,1), \quad \text{for } \xi \neq 0,$$

and

$$f''(\xi) = \Big\{\Big[-4\xi + \frac{16}{9}\xi^{1/3}\Phi(\xi)\Big] - \Big[1 - \xi\Big(2\xi - \frac{4}{3}\xi^{1/3}\Phi(\xi)\Big)\Big]\Big[2\xi - \frac{4}{3}\xi^{1/3}\Phi(\xi)\Big]\Big\}\hat{K}(\xi,1),$$

for $\xi \neq 0$. By using Lemma 3.3, it follows that $\|f\|_{L^1} \leq C$, $\|f'\|_{L^1} \leq C$, and $\|f''\|_{L^1} \leq C$. Moreover, $|f'(\xi)| \to 0$ as $|\xi| \to +\infty$, and $f'(0^+) = 1 = f'(0^-)$. Thus, Lemma 3.1 implies that $\partial_x K(\cdot,1) \in C_\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Therefore, using Lemma 3.6, we have that $K(\cdot,1) \in W^{1,1}(\mathbb{R})$. Let $t \in (0,1)$. Lemma 3.2 implies that

$$\|G(\cdot, 1-t^{1/3})\|_{L^\infty} \leq C \cdot (1-t^{1/3})^{-3/4}. \tag{3.20}$$

Thus, applying Lemma 3.4 to equation (3.11), taking into account (3.20), one sees that $K(\cdot,t) \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, and $\partial_x K(\cdot,t)(x) = t^{-1}\big(\partial_x K(\cdot,1) * G(\cdot, 1 - t^{1/3})\big)(t^{-1/2}x)$ for all $x \in \mathbb{R}$. Furthermore,

$$\begin{aligned}\|\partial_x K(\cdot,t)\|_{L^1} &= t^{-1/2}\int |\partial_x K(\cdot,1) * G(\cdot, 1-t^{1/3})|(y)dy \\ &\leq Ct^{-1/2}\big[(1-t^{1/3})^{3/4} + (1-t^{1/3})^{-3/4}\big],\end{aligned}$$

where in the last step we have used Young's inequality and Lemma 3.5. □

Next lemma will be useful to study $\|\partial_x K(\cdot,t)\|_{L^1}$ for $t \geq t_0$, where $t_0 > 0$.

LEMMA 3.8. *Suppose that $t > 0$. Then, $\partial_x K(\cdot,t) \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$, and*

$$\|\partial_x K(\cdot,t)\|_{L^1} \leq C\Big[\frac{1}{t} + t^2 e^{\frac{4}{27}a^3 t}\Big], \tag{3.21}$$

where C is a positive constant independent of t.

Proof. Let $t > 0$. For $\xi \in \mathbb{R}$, we define $h(\xi,t) := \xi\hat{K}(\xi,t)$. Then

$$\partial_\xi h(\xi,t) = \Big[1 - t\Big(2\xi^2 - \frac{4}{3}\xi^{\frac{4}{3}}\Phi(\xi)\Big)\Big]\hat{K}(\xi,t), \quad \text{for } \xi \neq 0,$$

and

$$\partial_\xi^2 h(\xi,t) = -t\Big\{\Big[4\xi - \frac{16}{9}\xi^{\frac{1}{3}}\Phi(\xi)\Big] + \Big[1 - t\Big(2\xi^2 - \frac{4}{3}\xi^{\frac{4}{3}}\Phi(\xi)\Big)\Big]\Big[2\xi - \frac{4}{3}\xi^{\frac{1}{3}}\Phi(\xi)\Big]\Big\}\hat{K}(\xi,t),$$

for $\xi \neq 0$. We see that $h(\cdot,t) \in C_\infty(\mathbb{R}) \cap C^2(\mathbb{R})$, $|\partial_\xi h(\xi,t)| \to 0$ as $|\xi| \to +\infty$, and $\partial_\xi h(0^+,t) = 1 = \partial_\xi h(0^-,t)$. Moreover, by Lemma 3.3, we have that

$$\begin{aligned}\|h(\cdot,t)\|_{L^1} &\leq C\Big[\frac{1}{t} + e^{\frac{4}{27}a^3 t}\Big], \\ \|\partial_\xi h(\cdot,t)\|_{L^1} &\leq C\Big[\frac{1}{t^{1/2}} + \frac{1}{t^{1/6}} + (1+t)e^{\frac{4}{27}a^3 t}\Big], \quad \text{and} \\ \|\partial_\xi^2 h(\cdot,t)\|_{L^1} &\leq C\Big[1 + t^{1/3} + t^{2/3} + (t+t^2)e^{\frac{4}{27}a^3 t}\Big].\end{aligned}$$

The proof of the lemma is now completed by applying Lemma 3.1. □

The next result provides a unified upper bound for $\|\partial_x K(\cdot,t)\|_{L^1}$ for any $t > 0$, which takes the best of the corresponding bounds obtained in Lemmas 3.7 and 3.8.

LEMMA 3.9. *Suppose that $t > 0$. Then the function $\partial_x K(\cdot,t) \in L^1(\mathbb{R}) \cap C_\infty(\mathbb{R})$. Moreover,*

$$\|\partial_x K(\cdot,t)\|_{L^1} \leq C\Big[\frac{1}{\sqrt{t}} + t^2 e^{\frac{4}{27}a^3 t}\Big], \tag{3.22}$$

where C is a positive constant independent of t.

The following lemma will be used in the proof of Lemma 3.11 below.

LEMMA 3.10.

$$\lim_{A\to+\infty} \int_{|y|>A} \big|K(\cdot,1) * G(\cdot,1-h^{1/3})\big|(y)dy = 0, \quad \text{uniformly in } h \in [0,1). \tag{3.23}$$

Proof. We recall that

$$\hat{K}(\xi,1) = e^{-[\xi^2 - \xi^{4/3}\Phi(\xi)]}, \quad \text{and} \quad \hat{G}(\xi,1-h^{1/3}) = \frac{1}{\sqrt{2\pi}} e^{-(1-h^{1/3})\xi^{4/3}\Phi(\xi)}.$$

From (3.18) we see that

$$\begin{aligned}
&\int_{|y|>A} \big|K(\cdot,1) * G(\cdot,1-h^{1/3})\big|(y)dy \\
&\leq \Big(\int_{|y|>A} \frac{dy}{1+y^2}\Big)^{\frac{1}{2}} \Big(\int (1+y^2)\big|K(\cdot,1) * G(\cdot,1-h^{1/3})(y)\big|^2 dy\Big)^{\frac{1}{2}} \\
&\leq \frac{C}{\sqrt{A}}\, \|\hat{K}(\cdot,1)\hat{G}(\cdot,1-h^{1/3})\|_{H^1} \leq \frac{C'}{\sqrt{A}}.
\end{aligned}$$

This concludes the proof. □

The next lemma shows that $(E(t))_{t\geq 0}$ is a C^0-semigroup on the Banach space Y and also on the Banach space X.

LEMMA 3.11. **i.)** *If $u_0 \in C_b(\mathbb{R})$, then $u(t) := E(t)u_0 \in C_b(\mathbb{R})$ for every $t \geq 0$. In addition,*

$$\|E(t)\|_{\mathfrak{L}(C_b(\mathbb{R}))} \leq C \cdot \big(1 + t^2 e^{\frac{4}{27}a^3 t}\big), \quad \text{for all } t > 0. \tag{3.24}$$

Moreover, $(E(t))_{t\geq 0}$ is a C^0-semigroup on Y.
ii.) *$(E(t))_{t\geq 0}$ is a C^0-semigroup on X. Furthermore, (3.24) remains true if the space $C_b(\mathbb{R})$ is replaced by $C_b^1(\mathbb{R})$.*

Proof. **i.)** Let $u_0 \in C_b(\mathbb{R})$, and $t > 0$. Since $u(x,t) = E(t)u_0(x) = \frac{1}{\sqrt{2\pi}} \int K(x-y,t)u_0(y)dy$, it follows that

$$\|u(t)\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \|u_0\|_{L^\infty} \|K(\cdot,t)\|_{L^1} \leq C \cdot \big(1 + t^2 e^{\frac{4}{27}a^3 t}\big) \|u_0\|_{L^\infty},$$

where the last inequality is a consequence of Lemma 3.6. Moreover,

$$|u(x+h,t) - u(x,t)| \leq \frac{\|u_0\|_{L^\infty}}{\sqrt{2\pi}} \|K(\cdot+h,t) - K(\cdot,t)\|_{L^1} \quad \to 0 \text{ as } h \to 0.$$

Thus, we have proved that if $u_0 \in C_b(\mathbb{R})$, then $u(t) \in C_b(\mathbb{R})$ for all $t \geq 0$. In addition, one can see that $E(t+s)\phi = E(t)E(s)\phi$, for all $t, s \geq 0$, and $\phi \in C_b(\mathbb{R})$.

Suppose now that $t = 0$, $u_0 \in Y \setminus \{0\}$, and $h \in (0,1)$. Since $\hat{K}(0,h) = \frac{1}{\sqrt{2\pi}} \int K(z,h)dz = 1$, and using (3.11) we have that

$$\begin{aligned}
|u(x,h) - u_0(x)| &= \frac{1}{\sqrt{2\pi}} \Big| \int K(z,h)\big(u_0(x-z) - u_0(x)\big)dz \Big| \\
&= \frac{1}{\sqrt{2\pi}} \Big| \int h^{-1/2}\big(K(\cdot,1) * G(\cdot, 1-h^{1/3})\big)(h^{-1/2}z)\big(u_0(x-z) - u_0(x)\big)dz \Big| \\
&= \frac{1}{\sqrt{2\pi}} \Big| \int \big(K(\cdot,1) * G(\cdot, 1-h^{1/3})\big)(y)\big(u_0(x-y\sqrt{h}) - u_0(x)\big)dy \Big|. \qquad (3.25)
\end{aligned}$$

Let $\epsilon > 0$. By Lemma 3.10, there exists $A > 0$, such that for every $h \in [0,1)$,

$$\int_{|y|>A} \big|K(\cdot,1) * G(\cdot, 1-h^{1/3})\big|(y)dy < \frac{\sqrt{2\pi}\epsilon}{4\|u_0\|_{L^\infty}}. \qquad (3.26)$$

Since u_0 is uniformly continuous, there exists $\delta > 0$ such that for all $z, w \in \mathbb{R}$,

$$\text{if } |z-w| < \delta, \text{ then } |u_0(z) - u_0(w)| < \sqrt{2\pi}\epsilon/(2C\|K(\cdot,1)\|_{L^1}), \qquad (3.27)$$

where C is a positive constant such that $\|G(\cdot, 1-h^{1/3})\|_{L^1} \le C$ for all $h \in [0, 1/2)$ (see Lemma 3.5). Let $h \in (0, \min\{\frac{1}{2}, \frac{\delta^2}{A^2}\})$. Using (3.25)-(3.27), we get

$$\begin{aligned}
|u(x,h) - u_0(x)| &\le \frac{2\|u_0\|_{L^\infty}}{\sqrt{2\pi}} \int_{|y|>A} \big|K(\cdot,1) * G(\cdot, 1-h^{1/3})(y)\big|dy \\
&+ \frac{1}{\sqrt{2\pi}} \int_{|y|\le A} \big|K(\cdot,1) * G(\cdot, 1-h^{1/3})(y)\big| \ \big|u_0(x-y\sqrt{h}) - u_0(x)\big|dy \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2C\|K(\cdot,1)\|_{L^1}} \|K(\cdot,1) * G(\cdot, 1-h^{1/3})\|_{L^1} \le \epsilon,
\end{aligned}$$

for all $x \in \mathbb{R}$, where the last inequality is a consequence of Young's inequality. Therefore,

$$\lim_{h\to 0} \|u(h) - u_0\|_{L^\infty} = 0. \qquad (3.28)$$

We notice that if $u_0 \in Y$, then $u(t) = E(t)u_0 \in Y$ for all $t \ge 0$. In fact, assume $t > 0$ and let $\epsilon > 0$ be given. Since u_0 is uniformly continuous, there exists $\delta > 0$ such that if $|h| < \delta$, then $|u_0(x+h) - u_0(x)| < \epsilon\sqrt{2\pi}/\|K(\cdot,t)\|_{L^1}$, for any $x \in \mathbb{R}$. Suppose $|h| < \delta$, then

$$|u(x+h,t) - u(x,t)| \le \frac{1}{\sqrt{2\pi}} \int \big|K(y,t)\big|\big|u_0(x-y+h) - u_0(x-y)\big|dy \le \epsilon, \quad \text{for all } x \in \mathbb{R}.$$

Hence, $u(t)$ is uniformly continuous, for all $t > 0$.

Assume now that $t > 0$, and $u_0 \in Y$. It follows from (3.28) and the semigroup property that

$$\lim_{h\downarrow 0} \|E(t+h)u_0 - E(t)u_0\|_{L^\infty} = 0.$$

On the other hand, for $h > 0$, one can see that

$$\begin{aligned}
|u(x,t-h) - u(x,t)| &= \frac{1}{\sqrt{2\pi}} \Big| \int K(x-y, t-h)\big(u_0(y) - u(y,h)\big)dy \Big| \\
&\le \frac{1}{\sqrt{2\pi}} \|K(\cdot, t-h)\|_{L^1} \|u(h) - u_0\|_{L^\infty} \\
&\le C \cdot \big(1 + (t-h)^2 e^{\frac{4}{27}a^3(t-h)}\big) \|u(h) - u_0\|_{L^\infty},
\end{aligned}$$

where the last inequality is a consequence of Lemma 3.6. Equation (3.28) and the last inequality imply that

$$\lim_{h\downarrow 0}\|E(t-h)u_0 - E(t)u_0\|_{L^\infty} = 0.$$

ii.) Let $u_0 \in X$. By item **i.)** above, we already know that $u \in C([0,+\infty);Y)$, where $u(t) = E(t)u_0$ for all $t \geq 0$. We will now prove that $\partial_x u(t) \in C_b(\mathbb{R})$, $\partial_x u(t)$ is uniformly continuous, and $\lim_{h\to 0}\|\partial_x u(t+h) - \partial_x u(t)\|_{L^\infty} = 0$, for all $t \geq 0$. Suppose first that $t > 0$. Then

$$\begin{aligned}
&\Big|\frac{u(x+h,t)-u(x,t)}{h} - \frac{1}{\sqrt{2\pi}}K(\cdot,t) * u_0'(x)\Big| \\
&= \frac{1}{\sqrt{2\pi}}\Big|\Big(K(\cdot,t) * \frac{u_0(\cdot+h)-u_0(\cdot)}{h}\Big)(x) - K(\cdot,t) * u_0'(x)\Big| \\
&\leq \frac{1}{\sqrt{2\pi}}\int \big|K(x-y,t)\big|\Big|\frac{u_0(y+h)-u_0(y)}{h} - u_0'(y)\Big|dy.
\end{aligned}$$

The dominated convergence theorem and Lemma 3.6 imply that the last expression tends to zero as h goes to zero. Then there exists

$$\partial_x u(x,t) = \frac{1}{\sqrt{2\pi}}K(\cdot,t) * u_0'(x), \quad \text{for all } x \in \mathbb{R}, \text{ and } t > 0. \tag{3.29}$$

It is easy to see that the last expression is also valid if we only require that $u_0 \in C_b^1(\mathbb{R})$. It follows from (3.29) and Lemma 3.6 that

$$\|\partial_x u(\cdot,t)\|_{L^\infty} \leq C \cdot \big(1 + t^2 e^{\frac{4}{27}a^3 t}\big)\|u_0'\|_{L^\infty}, \quad \text{for all } t > 0.$$

Using the fact that u_0' is uniformly continuous, (3.29), and Lemma 3.6, it follows that $\partial_x u(\cdot,t)$ is uniformly continuous, for all $t > 0$.

Finally, since $(E(t))_{t\geq 0}$ is a C^0-semigroup on the space Y and using (3.29), we see that $\lim_{h\to 0}\|\partial_x u(t+h) - \partial_x u(t)\|_{L^\infty} = 0$, for all $t \geq 0$. □

3.2. **Local Theory in the Space X.** In this Sub-section we will use the Banach fixed-point theorem on an appropriate complete metric space to find a local-in-time solution to the integral equation associated to the IVP (3.1). The following lemma will be helpful during the proof of Theorem 3.1 below.

LEMMA 3.12. *Suppose that $u \in C([0,T];X)$. We define*

$$D(\cdot,t) := \int_0^t K(\cdot,t-s) * \frac{1}{2}\partial_x u^2(\cdot,s)ds, \quad \text{for } t \in [0,T]. \tag{3.30}$$

Then $D \in C([0,T];X)$.

Proof. **i.)** Let $t \in (0,T]$. Now we first prove that $D(t) \in X$. In fact,

$$\begin{aligned}
\|D(\cdot,t)\|_{L^\infty} &\leq \sup_{s\in[0,T]}\|u(\cdot,s)\|_{C_b^1}^2 \int_0^t \|K(\cdot,t-s)\|_{L^1}ds \\
&\leq C\|u\|_{C([0,T];X)}^2 \int_0^t \big(1 + (t-s)^2 e^{\frac{4}{27}a^3(t-s)}\big)ds,
\end{aligned}$$

where the last inequality is a consequence of Lemma 3.6. Then

$$\|D(\cdot,t)\|_{L^\infty} \leq C'\,\|u\|_{C([0,T];X)}^2\,\nu(t), \tag{3.31}$$

where

$$\nu(r) := r + r^2 e^{\frac{4}{27}a^3 r}, \quad \text{for all } r \geq 0. \tag{3.32}$$

Moreover,

$$\|D(\cdot+h,t)-D(\cdot,t)\|_{L^\infty} \le \int_0^t \|K(\cdot,t-s)\|_{L^1}\|\frac{1}{2}\partial_x u^2(\cdot+h,s)-\frac{1}{2}\partial_x u^2(\cdot,s)\|_{L^\infty}ds.$$

Using the fact that $\partial_x u(\cdot,s)u(\cdot,s)$ is uniformly continuous on $\mathbb{R}$, for all $s\in[0,T]$, Lemma 3.6, and the dominated convergence theorem, it follows from the last inequality that $D(t)$ is uniformly continuous on $\mathbb{R}$. Now we claim that there exists $\frac{\partial D}{\partial x}(x,t)$ (in the classical sense), for all $x\in\mathbb{R}$, and

$$\begin{aligned}\frac{\partial D}{\partial x}(x,t) &= \int_0^t \partial_x\big(K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\big)(x)ds\\ &= \int_0^t \partial_x K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)(x)ds, \quad \text{for all } x\in\mathbb{R}. \quad (3.33)\end{aligned}$$

We now establish the last claim. It follows from Lemmas 3.6, 3.9, and 3.4 that $K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, and

$$\partial_x\big(K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\big)(x)=\big(\partial_x K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\big)(x), \quad (3.34)$$

for all $x\in\mathbb{R}$ and $s\in[0,t)$. Moreover,

$$\begin{aligned}&\Big|\frac{D(x+h,t)-D(x,t)}{h}-\int_0^t \partial_x K(\cdot,t-s)*\frac{1}{2}\partial_x u^2(\cdot,s)(x)ds\Big|\\ &=\Big|\int_0^t\Big(\frac{K(\cdot+h,t-s)-K(\cdot,t-s)}{h}-\partial_x K(\cdot,t-s)\Big)*\frac{1}{2}\partial_x u^2(\cdot,s)(x)ds\Big|. \quad (3.35)\end{aligned}$$

In addition,

$$\begin{aligned}&\Big\|\Big(\frac{K(\cdot+h,t-s)-K(\cdot,t-s)}{h}-\partial_x K(\cdot,t-s)\Big)*\frac{1}{2}\partial_x u^2(\cdot,s)\Big\|_{L^\infty}\\ &\le \|u\|^2_{C([0,T];X)}\ \Big\|\frac{K(\cdot+h,t-s)-K(\cdot,t-s)}{h}-\partial_x K(\cdot,t-s)\Big)\Big\|_{L^1}\\ &\le \|u\|^2_{C([0,T];X)}\ \Big(\frac{1}{|h|}\Big|\int_0^h \|\partial_x K(\cdot+y,t-s)\|_{L^1}dy\Big|+\|\partial_x K(\cdot,t-s)\|_{L^1}\Big)\\ &\le C\|u\|^2_{C([0,T];X)}\ \Big(\frac{1}{\sqrt{t-s}}+(t-s)^2e^{\frac{4}{27}a^3(t-s)}\Big)\in L^1((0,t),ds), \quad (3.36)\end{aligned}$$

where the last inequality is a consequence of Lemma 3.9. The claim now follows from (3.34)-(3.36) and the dominated convergence theorem.
It follows directly from (3.33) and Lemma 3.9 that

$$\|\partial_x D(\cdot,t)\|_{L^\infty}\le C\ \|u\|^2_{C([0,T];X)}\ \mu(t), \quad (3.37)$$

where

$$\mu(r):=\sqrt{r}+r^2e^{\frac{4}{27}a^3r}, \quad \text{for all } r\ge 0. \quad (3.38)$$

The fact that $\partial_x D(\cdot,t)$ is uniformly continuous on $\mathbb{R}$ can be shown similarly to the analogous result for $D(\cdot,t)$, using Lemma 3.9 instead of Lemma 3.6.

ii.) We will now prove that $D\in C([0,T];X)$. Let $t\in[0,T)$. We first assume that $h>0$. Then

$$\|D(\cdot,t+h)-D(\cdot,t)\|_{L^\infty}\le I_1(t,h)+I_2(t,h),$$

where

$$I_1(t,h) := \int_0^t \big\|\big(K(\cdot,t+h-s)-K(\cdot,t-s)\big)*\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty} ds, \quad \text{and}$$
$$I_2(t,h) := \int_t^{t+h} \big\|K(\cdot,t+h-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty} ds.$$

We see that

$$\begin{aligned}&\big\|\big(K(\cdot,t+h-s)-K(\cdot,t-s)\big)*\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty}\\ &\le \|u\|^2_{C([0,T];X)}\ \|K(\cdot,t+h-s)-K(\cdot,t-s)\|_{L^1}\\ &\le C\|u\|^2_{C([0,T];X)}\ \big(1+T^2e^{\frac{8}{27}a^3T}\big)\ \in L^1((0,t),ds),\end{aligned}$$

where the last inequality follows from Lemma 3.6 and the fact that $h\in(0,T)$. Thus, using Lemma 3.11 and the dominated convergence theorem we have that

$$I_1(t,h)=\sqrt{2\pi}\int_0^t \big\|\big(E(h)-1\big)E(t-s)\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty} ds\to 0, \quad \text{as } h\downarrow 0.$$

Moreover, using Lemma 3.6 we get like in (3.31) that

$$I_2(t,h)\le C\ \|u\|^2_{C([0,T];X)}\ \nu(h)\to 0, \quad \text{as } h\downarrow 0,$$

where $\nu(\cdot)$ is given by (3.32). Hence,

$$\lim_{h\downarrow 0}\|D(\cdot,t+h)-D(\cdot,t)\|_{L^\infty}=0. \tag{3.39}$$

On the other hand, it follows from (3.33) that

$$\|\partial_x D(\cdot,t+h)-\partial_x D(\cdot,t)\|_{L^\infty}\le J_1(t,h)+J_2(t,h),$$

where

$$J_1(t,h) := \int_0^t \big\|\big(\partial_x K(\cdot,t+h-s)-\partial_x K(\cdot,t-s)\big)*\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty} ds, \quad \text{and}$$
$$J_2(t,h) := \int_t^{t+h} \big\|\partial_x K(\cdot,t+h-s)*\frac{1}{2}\partial_x u^2(\cdot,s)\big\|_{L^\infty} ds.$$

It follows directly from Lemma 3.9 that

$$\begin{aligned}J_2(t,h) &\le \|u\|^2_{C([0,T];X)}\ \int_0^h \|\partial_x K(\cdot,\tau)\|_{L^1}d\tau\\ &\le C\ \|u\|^2_{C([0,T];X)}\ \mu(h)\to 0, \quad \text{as } h\downarrow 0,\end{aligned}$$

where $\mu(\cdot)$ is given by (3.38). To estimate $J_1(t,h)$ we first extend $\partial_x K$ for all times in the following way:

$$H(\cdot,s):=\begin{cases}\partial_x K(\cdot,s), \text{ if } s\in[0,T],\\ 0, \text{ if } s\in\mathbb{R}\setminus[0,T].\end{cases}$$

We note that $H\in L^1(\mathbb{R}^2)$. In fact, by Lemma 3.9 we get

$$\int\int |H(x,s)|dxds=\int_0^T \|\partial_x K(\cdot,s)\|_{L^1}ds\le C\ \mu(T).$$

Then

$$\begin{aligned} J_1(t,h) &\leq \|u\|^2_{C([0,T];X)} \int_0^t \|\partial_x K(\cdot,\tau+h) - \partial_x K(\cdot,\tau)\|_{L^1} d\tau \\ &\leq \|u\|^2_{C([0,T];X)} \int\int |H(x,\tau+h) - H(x,\tau)| dx d\tau \to 0, \quad \text{as } h \downarrow 0, \end{aligned}$$

where the last assertion follows from the continuity of translations in $L^1(\mathbb{R}^2)$. Hence,

$$\lim_{h\downarrow 0} \|\partial_x D(\cdot,t+h) - \partial_x D(\cdot,t)\|_{L^\infty} = 0. \tag{3.40}$$

It follows from (3.39) and (3.40) that $\lim_{h\downarrow 0} \|D(\cdot,t+h) - D(\cdot,t)\|_{C_b^1} = 0$.

The case when $t \in (0,T]$ and $h < 0$ can be shown similarly to the previous case. This finishes the proof of the lemma. □

The next theorem is the main result of this section, it states local-in-time existence of the solution of the integral equation associated to the IVP (3.1).

THEOREM 3.1. *Suppose $u_0 \in X$. Then there exist $T = T(\|u_0\|_{C_b^1}) > 0$ and a unique function $u \in C([0,T];X)$ satisfying the integral equation*

$$u(\cdot,t) = E(t)u_0(\cdot) - \frac{1}{2}\int_0^t E(t-s)\partial_x u^2(\cdot,s) ds, \tag{3.41}$$

where $E(t)$ is defined by (3.6).

Proof. Let $M := 1 + 2\|u_0\|_{C_b^1}$. Let $T > 0$ be fixed. T will be suitably chosen later. We now consider the nonlinear operator A given by

$$(Af)(\cdot,t) := E(t)u_0(\cdot) - \frac{1}{2}\int_0^t E(t-s)\partial_x f^2(\cdot,s) ds,$$

defined on the complete metric space

$$\Theta_T^M := \Big\{ f \in C([0,T];X);\ \sup_{t\in[0,T]} \|f(\cdot,t)\|_{C_b^1} \leq M \Big\}.$$

Let $f \in \Theta_T^M$. It follows from Lemmas 3.11 and 3.12 that $Af \in C([0,T];X)$.

We will now prove that we can choose $T = \tilde{T} > 0$ small enough such that $A(\Theta_{\tilde{T}}^M) \subset \Theta_{\tilde{T}}^M$. Suppose $f \in \Theta_T^M$. By Lemma 3.11 we know that $\lim_{h\downarrow 0} \|(E(h) - 1)u_0\|_{C_b^1} = 0$. Then there exists $\delta = \delta(\|u_0\|_{C_b^1}) > 0$ such that if $0 \leq h \leq \delta$, then $\|E(h)u_0\|_{C_b^1} \leq \frac{1}{2}\big(1 + 3\|u_0\|_{C_b^1}\big)$. If $T \leq \delta$, using Lemmas 3.6 and 3.9, and (3.33), we get

$$\begin{aligned} &\|(Af)(\cdot,t)\|_{C_b^1} \leq \frac{1}{2}\big(1 + 3\|u_0\|_{C_b^1}\big) \\ &+ \frac{1}{2\sqrt{2\pi}} \int_0^t \big(\|K(\cdot,t-t')\|_{L^1} + \|\partial_x K(\cdot,t-t')\|_{L^1}\big)\|f\|^2_{C([0,T];X)} dt' \\ &\leq \frac{1}{2}\big(1 + 3\|u_0\|_{C_b^1}\big) + M^2 C \int_0^t \Big[\frac{1}{\sqrt{\tau}} + \tau^2 e^{\frac{4}{27}a^3\tau}\Big] d\tau \\ &\leq \frac{1}{2}\big(1 + 3\|u_0\|_{C_b^1}\big) + M^2 C\, \mu(T), \end{aligned}$$

for all $t \in [0,T]$, where $\mu(\cdot)$ is given by (3.38). Take $T^\dagger > 0$ such that $M^2 C\, \mu(T^\dagger) \leq \frac{1}{2}\big(1 + \|u_0\|_{C_b^1}\big)$. Thus, if $\tilde{T} \in (0, \min\{\delta, T^\dagger\}]$, then $\|(Af)(\cdot,t)\|_{C_b^1} \leq M$ for all $t \in [0,\tilde{T}]$.

Finally, we will prove that there exists $T' \in (0, \tilde{T}]$ such that A is contractive on $\Theta^M_{T'}$. Suppose that $f, g \in \Theta^M_{\tilde{T}}$. Let $t \in [0, \tilde{T}]$. Then

$$\begin{aligned}
&\|(Af)(\cdot,t) - (Ag)(\cdot,t)\|_{C^1_b} \\
&\leq C \int_0^t \big(\|K(\cdot,t-t')\|_{L^1} + \|\partial_x K(\cdot,t-t')\|_{L^1}\big)\|\partial_x f^2(\cdot,t') - \partial_x g^2(\cdot,t')\|_{L^\infty} dt' \\
&\leq C \int_0^t \big(\|K(\cdot,t-t')\|_{L^1} + \|\partial_x K(\cdot,t-t')\|_{L^1}\big) \\
&\times \big[\|f(\cdot,t')\|_{L^\infty}\|\partial_x(f(\cdot,t') - g(\cdot,t'))\|_{L^\infty} + \|f(\cdot,t') - g(\cdot,t')\|_{L^\infty}\|\partial_x g(\cdot,t')\|_{L^\infty}\big] dt' \\
&\leq CM\|f-g\|_{C([0,\tilde{T}];X)}\, \mu(t).
\end{aligned}$$

Taking $T' \in (0, \tilde{T}]$ such that $CM\ \mu(T') < 1$, it follows that A is a contraction on $\Theta^M_{T'}$. Therefore, the mapping A has a unique fixed point $u \in \Theta^M_{T'}$ which satisfies equation (3.41) with $T' = T'(\|u_0\|_{C^1_b}) > 0$. The uniqueness of the solution of equation (3.41) in the class $C([0,T'];X)$ is a consequence of Proposition 3.1 below. □

The next proposition shows the continuous dependance of the solutions of equation (3.41) on the initial data.

PROPOSITION 3.1. *Suppose that $u, v \in C([0,T];X)$ are solutions of equation (3.41) with initial data $u_0, v_0 \in X$ respectively. Then for all $t \in [0,T]$ we have*

$$\|u(\cdot,t) - v(\cdot,t)\|_{C^1_b} \leq Ce^{\alpha t}\|u_0 - v_0\|_{C^1_b}, \tag{3.42}$$

where C and α are positive constants depending on T, $\|u\|_{C([0,T];X)}$, and $\|v\|_{C([0,T];X)}$.

Proof. Let $t \in [0,T]$. We write $w(\cdot,t) := u(\cdot,t) - v(\cdot,t)$. Then

$$\|w(\cdot,t)\|_{C^1_b} \leq \|E(t)(u_0 - v_0)\|_{C^1_b} + \frac{1}{2}\Big\| \int_0^t E(t-t')\big(\partial_x u^2(\cdot,t') - \partial_x v^2(\cdot,t')\big)dt'\Big\|_{C^1_b}. \tag{3.43}$$

It follows from Lemma 3.11 that

$$\|E(t)(u_0 - v_0)\|_{C^1_b} \leq C'\|u_0 - v_0\|_{C^1_b}, \tag{3.44}$$

where

$$C' = C \cdot \big(1 + T^2 e^{\frac{4}{27}a^3 T}\big).$$

Moreover, by Lemmas 3.6 and 3.9, we get

$$\begin{aligned}
&\frac{1}{2}\Big\| \int_0^t E(t-t')\big(\partial_x u^2(\cdot,t') - \partial_x v^2(\cdot,t')\big)dt'\Big\|_{C^1_b} \\
&\leq \frac{\|u\|_{C([0,T];X)} + \|v\|_{C([0,T];X)}}{\sqrt{2\pi}} \\
&\times \int_0^t \big(\|K(\cdot,t-t')\|_{L^1} + \|\partial_x K(\cdot,t-t')\|_{L^1}\big)\|w(\cdot,t')\|_{C^1_b} dt' \\
&\leq C \cdot \big(\|u\|_{C([0,T];X)} + \|v\|_{C([0,T];X)}\big) \\
&\times \int_0^t \Big[1 + (t-t')^2 e^{\frac{4}{27}a^3(t-t')} + \frac{1}{\sqrt{t-t'}}\Big]\|w(\cdot,t')\|_{C^1_b} dt' \\
&\leq \tilde{C}\int_0^t \frac{\|w(\cdot,t')\|_{C^1_b}}{\sqrt{t-t'}} dt',
\end{aligned} \tag{3.45}$$

where

$$\tilde{C} := C \cdot (1 + T^{5/2} e^{\frac{4}{27} a^3 T}) \left(\|u\|_{C([0,T];X)} + \|v\|_{C([0,T];X)} \right).$$

Thus, it follows from (3.43), (3.44) and (3.45) that

$$\|w(\cdot,t)\|_{C_b^1} \leq C' \|u_0 - v_0\|_{C_b^1} + \tilde{C} \int_0^t \frac{\|w(\cdot,t')\|_{C_b^1}}{\sqrt{t-t'}} dt'.$$

Then

$$\begin{aligned}
&\|w(\cdot,t)\|_{C_b^1} \leq C' \|u_0 - v_0\|_{C_b^1} \\
&+ \tilde{C} \int_0^t \frac{1}{\sqrt{t-t'}} \Big[C' \|u_0 - v_0\|_{C_b^1} + \tilde{C} \int_0^{t'} \frac{\|w(\cdot,r)\|_{C_b^1}}{\sqrt{t'-r}} dr \Big] dt' \\
&\leq C'(1 + 2\tilde{C}\sqrt{T}) \|u_0 - v_0\|_{C_b^1} + \tilde{C}^2 \int_0^t \int_r^t \frac{\|w(\cdot,r)\|_{C_b^1}}{\sqrt{t-t'}\sqrt{t'-r}} dt' dr \\
&= C \|u_0 - v_0\|_{C_b^1} + \tilde{C}^2 B\big(\frac{1}{2}, \frac{1}{2}\big) \int_0^t \|w(\cdot,r)\|_{C_b^1} dr,
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function defined by $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$, for $\Re(x), \Re(y) > 0$. The proposition now follows by applying Gronwall's inequality to the last expression. □

3.3. **Future Work.** Some interesting problems remain, though: the study of the global well-posedness for the IVP (3.1) with initial data belonging to the space X, and the nonlinear stability theory of the travelling-wave solution of equation (1.1). These two problems will be addressed elsewhere.

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