Local beta-Crusts for Simple Curves Reconstruction

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Abstract

In this paper, we consider the problem of curve reconstruction from a finite planar set of points. To solve this problem, we propose to use a family of neighborhood graphs included in the Gabriel graph. The neighborhood that we use is the β-neighborhood, initially defined in the context of circle-based β-skeletons, but applied to edges of the Voronoi diagram. This family of graphs includes the local crust. This formulation enables us to design effective algorithms to reconstruct curves, by using as a prior knowledge that the curves to be reconstructed are without intersections. We show, through several examples, that the proposed algorithms improve the results obtained with the local crust, when the set of points is of low density.

1. Introduction

Reconstruct the shape of objects from a finite set of points P, measured on the boundary of the objects, is an important step in many application areas, such as computer vision, image analysis, or shape modeling. In this article, the set of points P is supposed to be a sample of simple curves of $\mathbb{R}^2$ (curves without intersections), and the reconstruction consists in finding a polygonal interpolation of P, topologically equivalent to the sampled curves.

Among the methods which solve the problem of curve reconstruction, the ones based on neighborhood graphs are known to be efficient, both in practice and theory. All of them provide an adequate interpolation if the set of points P is a sufficiently dense sample of the curves. Conceptually, neighborhood graphs connect the points of P that are close relatively to a given measure. The neighborhood, which allows to generate the edges, can be interpreted as the structuring element of the graphs. First works on curve reconstruction, based on neighborhood graphs, were inspired by point cloud clustering and description methods. The main graphs used by these methods are the Gabriel graph, the k-nearest neighbors graphs, the minimum spanning tree, the relative neighborhood graph, or the Delaunay triangulation (see [20] for a recent survey of these methods).

An important class of these graphs allows to describe the shape of the point cloud at several levels of details, by using an union and/or an intersection of discs as a structuring element. Two points of P are connected by an edge if the neighborhood of the edge does not contain any points of P. Among these graphs, one can distinguish the family of α-shapes [10], which is generated with empty discs of radius α, and the family of β-skeletons [16], which is formed with two empty discs of same radius and local scale β. Extensions of these two descriptors were proposed by using γ-neighborhood graphs [21]. Since the Delaunay triangulation is generated with the empty discs of maximum radius, it includes most of the graphs mentioned above. While the neighborhood of these graphs is defined by using a global parameter, the neighborhood of the A-shapes [19, 18], is defined by using a set of control points A. The edges of this family of graphs are generated by discs empty of points of $P \cup A$. The A-shapes are also subgraphs of the Delaunay triangulation. The choice of control points is related to the medial axis of the shapes to be reconstructed. As the medial axis can be approximated by a subset of the edges of the Voronoi diagram [6], the vertices of the Voronoi diagram are good candidates for the set A.

In the context of simple closed curve reconstruction, a similar concept has been proposed to define the crust of P [1]. The crust is a particular case of the A-shapes, where A corresponds to the whole set of Voronoi vertices. It always gives a reconstruction topologically equivalent to the sampled curves, if P is sufficiently dense and if the curves are twice differentiable (or smooth) [1]. The calculation of the crust requires to compute the Voronoi diagram of P, as well as the Delaunay triangulation of $P \cup A$. In order to improve the complexity of this calculation, a local version of the crust, called the local crust, has been proposed in [13], and studied beforehand in another form in [2]. It only requires to compute the Delaunay triangulation of P, and provides equivalent results to those obtained with the crust. Moreover, the Voronoi edges, whose dual Delaunay edges do not belong to the local crust, approximate the medial axis of the reconstructed curves [13]. This subgraph of the Voronoi diagram of P is called the anti-crust of P.

Other curve reconstruction methods use neighborhood graphs. It is the case of the methods using the α-shapes [4], the β-skeletons [1] or the minimum spanning tree [11]. In [7], the nearest neighbors crust is proposed as an alternative to the crust and the local crust. Due to undesirable effects when the sampled curves are open or non-smooth, the ideas in [7] have been modified by incorporating parameters that control the sharpness of corners and the local density [8, 9]. These ideas have also been extended to take into account curves with intersections [17].

The principal objective of this paper is to describe a hierarchical family of graphs which is generated by a structuring element defined on the edges of the Voronoi diagram. Our work takes as a starting point two properties of the local crust and the anti-crust that we demonstrate. Firstly, the local crust is a subgraph of the Gabriel graph. Secondly, the edges of the anti-crust are generated by using Gabriel discs that are circumscribed to Voronoi vertices and empty of points of P. Based on these properties, we extend the Gabriel neighborhood of the Voronoi edges to the circle-based β-neighborhood, initially defined for β-skeletons [16]. When β is fixed, this neighborhood enables to define two types of graphs, the first one is a subgraph of the Voronoi diagram, and the second one is a subgraph of the Gabriel graph. We call respectively these graphs the β-medial axis and the local β-crust. Beyond that, we show that certain properties of the family of local β-crusts allow to design efficient nonparametric algorithms that extract simple curves from the Delaunay triangulation. These algorithms exploit as a prior knowledge that the curves to be reconstructed are without intersections. A similar assumption has also been considered in [15].

The rest of this article is organized as the following. The next section points out the concepts on which the graphs and the algorithms we propose are based. Section 3 presents the β-neighborhood of Voronoi edges, the β-medial axes and the local β-crusts. Then, we discuss the differences with the local crust. In Section 4 we propose an algorithm which computes the maximum value of β for which there exists a local β-crust that reconstructs simple curves. The results of this algorithm are improved through an extension described in Section 5. We show experimentally that the proposed algorithms improve the results obtained with the local crust [13] and the nearest neighbors crust [7], when the set of points is not dense.

2. Related concepts

Let P be a finite set of N distinct points of the Euclidean plane. We assume P to be in general position. Here, we recall some definitions and properties needed for the comprehension of the graphs and the algorithms presented in Section 3, Section 4 and Section 5.

2.1. Voronoi diagram and Delaunay Triangulation

In this paper, the Voronoi diagram and the Delaunay triangulation of the point set P are the two fundamental data structures used to compute a polygonal interpolation of P.

Let $\|p-q\|$ be the Euclidean distance between two points p and q of $\mathbb{R}^2$. The *Voronoi polygon* of a point $p_i \in P$, denoted by $V(p_i,\, P)$, is the set of points $p \in \mathbb{R}^2$ such that $\|p - p_i\| \leq \|p - p_j\|$, for all $p_j \in P$. The *Voronoi diagram* of P is the set of Voronoi polygons $V(P) = \{V(p_i,\, P),\, \forall p_i \in P\}$. Two points are neighbors in $V(P)$ if the intersection of their Voronoi polygons is not empty.

The topological and geometrical dual of the Voronoi diagram of P is the Delaunay triangulation of P, noted $D(P)$. Two points of the $D(P)$ are neighbors if there exists a circle passing through these points and such that its interior is empty of points of P. The dual of a Delaunay triangle is a Voronoi vertex, the dual of a Delaunay edge is a Voronoi edge, and the dual of a Delaunay vertex is a Voronoi polygon.

For the detail of these two concepts, on can refer to [3]. In particular, the edges of the convex hull of P, noted *conv*(P), correspond to the boundary edges of $D(P)$, i.e. to the edges incident to one triangle of $D(P)$. The dual Voronoi edges of the convex hull edges are infinite. To simplify the notations, we suppose that all the edges of $V(P)$ are finite. The case of infinite Voronoi edges is discussed in Section 3.4.

2.2. Gabriel graph and β-skeletons

The *Gabriel Graph* of P [12], noted *GG*(P), is the graph that connects two points p_i and p_j of P if the diametral disk of the edge p_ip_j contains no points of $P \setminus \{p_i, p_j\}$. It is a subgraph of the Delaunay triangulation. The disk of diameter p_ip_j, called the *Gabriel disk* of p_ip_j and noted $B(p_ip_j)$, represents the neighborhood of p_ip_j. The edges of the Gabriel graph have also the property to cut their dual Voronoi edges.

Also based on the notion of empty neighborhood, the family of circle-based β-*skeletons* of P [16] describes a hierarchy of graphs indexed by a real positive valued parameter β. The β-*neighborhood* of a pair of points p_i and p_j of P, noted $N_\beta(p_ip_j)$, is defined as the union of the two disks of radius $\beta\|p_i - p_j\|/2$, for $\beta \geq 1$, circumscribed to the points p_i and p_j. When $\beta \leq 1$, $N_\beta(p_ip_j)$ is defined to be the intersection of the two disks of radius $\|p_i - p_j\|/2\beta$ circumscribed to p_i and p_j. Given a value of β, the edge p_ip_j is in the β-skeleton of P if $N_\beta(p_ip_j)$ is empty of points of $P \setminus \{p_i, p_j\}$. The family of β-skeletons of P includes the Gabriel graph of P for $\beta = 1$. Moreover, for all $\beta \geq 1$ the β-skeleton of P is a subgraph of the Delaunay triangulation. This is not necessarily the case when $\beta < 1$.

2.3. Medial axis

The *medial axis* of a collection of curves $\Gamma \subset \mathbb{R}^2$ is defined as the closure of the set of points of $\mathbb{R}^2$ that have two or more closet points in Γ. The medial axis, which can be interpreted as a continuous version of the Voronoi diagram, is one of the most important descriptors of shapes [5]. It can be approximated by a subset of the edges of the Voronoi diagram of a dense sample of the curves [6]. When the curves are smooth and without intersections, the medial axis never intersects the curves. These properties are the main ideas exploited in smooth curve sampling and reconstruction methods [2, 1, 7].

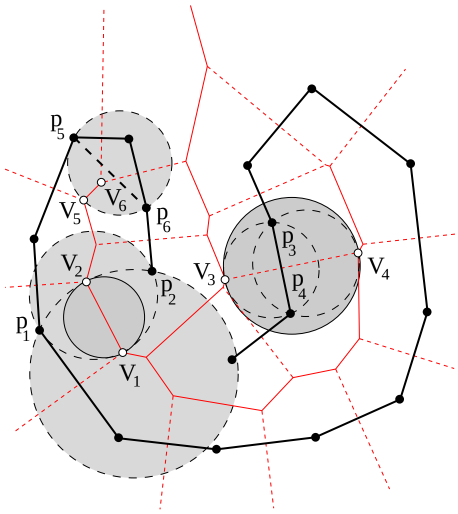


Figure 1. Illustration of the proofs of Observation 1 and Lemma 1.

2.4. Local crust and anti-crust

The *local crust* of P, noted $LC(P)$, is a subgraph of the Delaunay triangulation of P computed from a relation between Delaunay and Voronoi edges. Let p_ip_j be an edge of $D(P)$ and v_iv_j be its dual Voronoi edge. The edge p_ip_j is in the local crust if and only if there exists a circle circumscribed to p_ip_j and whose interior is empty of the Voronoi vertices v_i and v_j [13]. This local test can be formalized by considering the two open disks circumscribed to the points p_i, p_j, v_i and p_i, p_j, v_j [14]:

$$p_ip_j \in LC(P) \Leftrightarrow \begin{cases} b(p_ip_jv_i) \cap \{v_j\} = \emptyset, \text{ and} \\ b(p_ip_jv_j) \cap \{v_i\} = \emptyset. \end{cases} \quad (1)$$

Due to the relation of duality between the edge p_ip_j and the edge v_iv_j, one can show that the test of emptiness $b(p_ip_jv_i) \cap \{v_j\} = \emptyset$ is equivalent to $b(p_ip_jv_i) \cap \{v_j\} = \emptyset$. Thus, only one of the two tests is needed in Eq. (1) to determine if the edge p_ip_j is in $LC(P)$ [14].

The dual Voronoi edges of the Delaunay edges that are not in the local crust constitute the *anti-crust* of P [13], noted $AC(P)$. This graph provides an approximation of the medial axis of the curves that are reconstructed by $LC(P)$.

3. Local β-crust and β-medial axis

In this section, we introduce the graphs which are used by the algorithms presented in Section 4 and Section 5.

3.1. Motivations

The proposed graphs are inspired by two properties of the local crust and the anti-crust. Firstly, the local crust of P is a subgraph of the Gabriel graph of P. This property, indirectly mentioned in [13], implies that the edges of $LC(P)$ never cut the edges of $AC(P)$. So, the local crust and the anti-crust are consistent with the definitions of a smooth curve and its medial axis, or any curve which does not intersect its medial axis.

Observation 1 $LC(P) \subset GG(P)$.

Proof. Let p_ip_j be an edge that is not in $GG(P)$. Let v_iv_j be the dual Voronoi edge of p_ip_j. In order to get a contradiction, suppose that $p_ip_j \in LC(P)$. By definition of $GG(P)$, we have $p_ip_j \cap v_iv_j = \emptyset$ (or one of the vertices of v_iv_j), i.e. the vertices v_i and v_j are located on the same side of p_ip_j. Let v_i be the nearest vertex to p_i and p_j. As illustrated in Fig. 1 (vertices p_5, p_6 and v_5, v_6) , this implies that $v_i \in b(p_ip_jv_j)$, and from Eq. (1), $p_ip_j \notin LC(P)$. $\square$

Secondly, the anti-crust of P can be defined using Gabriel disks on Voronoi edges. Indeed, an edge $v_iv_j \in V(P)$ is included in $AC(P)$ if and only if the closed disk of diameter v_iv_j is empty of points of P.

Lemma 1 $v_iv_j \in AC(P) \Leftrightarrow B(v_iv_j) \cap P = \emptyset$.

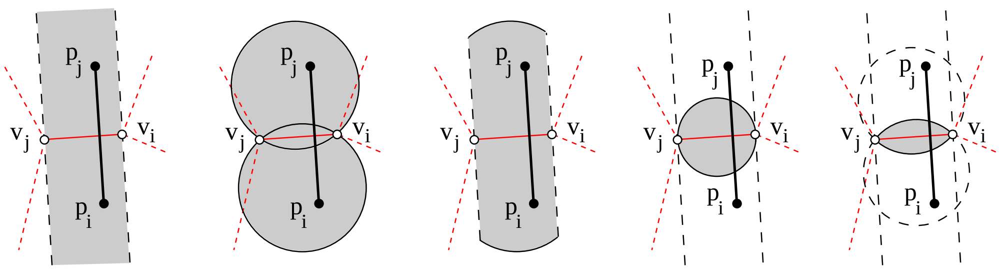
Proof. Let p_ip_j be the dual Delaunay edge of a Voronoi edge v_iv_j. By definition of the anti-crust and by Eq. (1), we have $v_iv_j \in AC\text{-}(P) \Leftrightarrow v_j \in b(p_ip_jv_i)$ (and $v_i \in b(p_ip_jv_j)$). As illustrated in Fig. 1 (points p_1, p_2, p_3 and p_4), this is equivalent to $B(v_iv_j) \subset b(p_ip_jv_i)$ (and respectively $B(v_iv_j) \subset b(p_ip_jv_j)$). Then the vertices p_i and p_j are not in $B(v_iv_j)$. Since they are the two nearest neighbors of the edge v_iv_j, there exists no points of P in $B(v_iv_j)$. $\square$

From Observation 1 and Lemma 1, one can deduce that an edge of the Delaunay triangulation is in the local crust if it is entirely included in the diametral disk of its dual. Based on this property, we generalize the definitions of the local crust and the anti-crust in order to obtain a family of graphs, and thus a range of possible solutions to the problem of curve reconstruction.

3.2. Main definitions and properties

The basic idea of the extension is to use β-disks defined on Voronoi edges instead of Gabriel disks (Observation 1). Let v_iv_j be an edge of $V(P)$ and p_ip_j be its dual Delaunay edge. Let $\beta \in]0, +\infty[$ be a given parameter, and $N_\beta(v_iv_j)$ be the β-neighborhood of the edge v_iv_j (see Section 2.2 for the definition of N_β).

In order to preserve the properties related to the Gabriel graph, the edge p_ip_j must be located in the part of $\mathbb{R}^2$ limited by the two lines, parallel with p_ip_j, which pass through the vertices of v_iv_j (see Fig. 2(a)). This part of $\mathbb{R}^2$, noted $H(v_iv_j)$, can be written as the set of points $p \in \mathbb{R}^2$ such that $\angle v_iv_jp < \pi/2$ and $\angle v_jv_ip < \pi/2$. Then, we define the neighborhood of v_iv_j as the intersection of $N_\beta(v_iv_j)$ and $H(v_iv_j)$ (see Fig. 2(c,d,e)). According to whether this intersection is empty or not of p_i (and p_j), the edge v_iv_j belongs to the β-*medial axis* of P, noted β-*MA*(P), or the edge p_ip_j belongs to the *local* β-*crust* of P, noted β-*LC*(P):



(a) $H(v_iv_j)$ (b) $N_\beta(v_iv_j)$, $\beta > 1$ (c) $\beta > 1$ (d) $\beta = 1$ (e) $\beta < 1$

Figure 2. Construction of the neighborhood used to form the local β-crust and the β-medial axis, according to the value of β.

$$v_iv_j \in \beta\text{-}MA(P) \Leftrightarrow \{p_i, p_j\} \not\subset N_\beta(v_iv_j) \cap H(v_iv_j) \tag{2}$$

$$p_ip_j \in \beta\text{-}LC(P) \Leftrightarrow \{p_i, p_j\} \subset N_\beta(v_iv_j) \cap H(v_iv_j) \tag{3}$$

The local β-crust of P is the set of Delaunay edges entirely included in the β-neighborhood of their dual Voronoi edges:

$$p_ip_j \in \beta\text{-}LC(P) \Leftrightarrow p_ip_j \subset N_\beta(v_iv_j). \tag{4}$$

By construction of the neighborhood, the local β-crusts are subgraphs of the Gabriel graph.

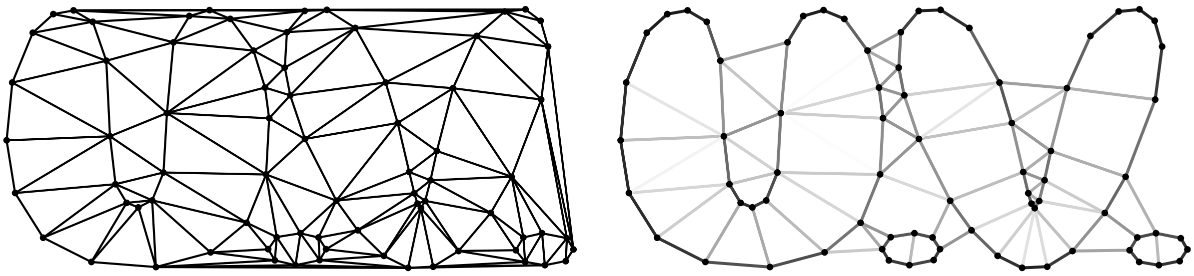


Figure 3. Spectrum of local β-crusts (bottom) computed from the Delaunay triangulation (top). The weight of Delaunay edges is represented by gray levels, from black to white according to the apparition order of the edges in the family of local β-crusts.

Theorem 1 β-*LC*$(P) \subseteq$ *GG*(P), $\forall \beta \in]0, +\infty[$.

Thus, one can deduce that β-*LC*(P) is the set of Gabriel edges whose vertices are included in the β-neighborhood of their dual edges. While varying β, the local β-crusts of P (and the β-medial axes) describe a hierarchical family of graphs. This is due to the relation of inclusion which exists between the β-neighborhoods.

Theorem 2 $\forall \beta, \beta' \in]0, +\infty[$, $\beta < \beta' \Leftrightarrow \beta$-*LC*$(P) \subseteq \beta'$-*LC*$(P) \Leftrightarrow \beta'$-*MA*$(P) \subseteq \beta$-*MA*$(P)$.

Proof. Let p_ip_j be an edge of β-*LC*(P) and v_iv_j be its dual Voronoi edge. From the equation (4), we have $p_ip_j \subset N_\beta(v_iv_j)$. That is, $\beta' > \beta$. By definition, $N_\beta(v_iv_j) \subset N_{beta'}(v_iv_j)$, and thus $p_ip_j \subset N_{beta'}(v_iv_j)$. $\square$

Moreover, the family of local β-crusts of P, and that of β-medial axes, are finite. The number of graphs from these two families is limited by the number of edges of the Gabriel graph of P. Indeed, for each edge p_ip_j of *GG*(P), there exists a value $\beta > 0$ such that p_ip_j belongs to β-*LC*(P), and p_ip_j does not belong to β'-*LC*(P), for all $\beta' < \beta$. In other words, the value of β from which the edge p_ip_j is included in the family of local β-crusts. By duality, it is also the value of β from which the dual edge of p_ip_j is excluded of the family of the β-medial axes. This allows to associate one weight to each edge of $D(P)$. The *weight* of an edge $p_ip_j \in D(P)$, noted $w(p_ip_j)$, is defined to be:

$$w(p_ip_j) = \begin{cases} \beta : \{p_i\} \subset \partial N_\beta(v_iv_j) & \text{if } p_ip_j \in GG(P), \\ +\infty & \text{otherwise,} \end{cases} \tag{5}$$

where ∂N_β represents the boundary of N_β. Based on the relation of inclusion stated by theorem 2, the edges of $D(P)$ are ordered according to the ascending order of their weights : $p_ip_j \le p_kp_l \Leftrightarrow w(p_ip_j) \le w(p_kp_l) \Leftrightarrow w(p_ip_j)$-*LC*$(P) \subseteq w(p_kp_l)$-*LC*$(P)$. Then, we define the *spectrum* of the local β-crusts of P, noted *spectrum*(P), as the set of edges of $D(P)$ ordered according to the preceding relation. Let $\beta_{\min}$ and $\beta_{\max}$ be the minimal and the maximal values of the weights of the edges of *GG*(P). Given a value $\beta \in [\beta_{\min}, \beta_{\max}]$, this value separates the spectrum of P in two disjoined subsets of edges : those which have a weight lower or equal to β belong to β-*LC*(P), and those which have a weight higher than β have their dual edges in β-*MA*(P). The local β-crust coincide with the Gabriel graph when $\beta \ge \beta_{\max}$. This shows that the maximum number of local β-crusts of P corresponds to the number of edges of *GG*(P) (certain edges can have equal weights). An spectrum example is illustrated in Fig. 3.

3.3. Alternative definitions

The β-neighborhood can be expressed with angles. Let $\theta \in (0, \pi)$ be the angle defined by:

$$\theta(\beta) = \begin{cases} \pi - \arcsin(\beta) & \text{if } \beta \in (0, 1], \\ \arcsin(1/\beta) & \text{if } \beta \in [1, +\infty). \end{cases} \tag{6}$$

This notation implies several alternative definitions of the local β-crust of P. Let p_ip_j be an edge of $D(P)$, and v_iv_j be its dual Voronoi edge. Let $\angle p_iv_iv_j$ be the angle between the vector v_ip_i and the vector v_iv_j.

Lemma 2 $p_ip_j \in \beta$-*LC*(P) *if and only if:*
(i) $\angle p_iv_iv_j < \pi/2$ *and* $\angle p_iv_jv_i < \pi/2$, *and*
(ii) $\angle p_iv_iv_j + \angle v_iv_jp_i \leq \pi - \theta(\beta)$.

Proof. The property (i) ensures p_ip_j to be in $GG(P)$ and it comes from the definition of $GG(P)$. To demonstrate the property (ii), we consider the triangle $p_iv_iv_j$. From Eq. (6), Eq. (3) and the definition of N_β, we obtain $p_ip_j \in \beta$-*LC*(P) iff $p_ip_j \in GG(P)$ and $\angle v_ip_iv_j \geq \theta(\beta)$. As the sum of the angles in $p_iv_iv_j$ is π, $\angle v_ip_iv_j \geq \theta(\beta)$ is equivalent to the property (ii). $\square$

From Lemma 2, the local β-crust can also be expressed with the angles of the Delaunay triangles. Let $p_ip_jp_k$ and $p_ip_jp_l$ be the two triangles incident to the edge p_ip_j.

Lemma 3 $p_ip_j \in \beta$-*LC*(P) *if and only if:*
(i) $\angle p_ip_kp_j < \pi/2$ *and* $\angle p_jp_lp_i < \pi/2$, *and*
(ii) $\angle p_ip_kp_j + \angle p_jp_lp_i \leq \pi - \theta(\beta)$.

Proof. Let v_iv_j be the Voronoi dual of p_ip_j. We have $\angle p_ip_kp_j = \angle p_iv_iv_j$ and $\angle p_jp_lp_i = \angle v_iv_jp_i$. Then, property (ii) is equivalent to property (ii) of Lemma 2. $\square$

The conditions (i) and (ii) of Lemma 3 allow to redefine the weights of the edges of $D(P)$ using the angles of the Delaunay triangles. Eq. (5) is then replaced by the following equation:

$$w(p_ip_j) = \begin{cases} \angle p_ip_kp_j + \angle p_jp_lp_i & \text{if } p_ip_j \in GG(P), \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

The value of β, which corresponds to the weight of an edge, is obtained by the opposite equation of Eq. (6):

$$\beta(\theta) = \begin{cases} \sin(\pi - \theta) & \text{if } \theta > \pi/2, \\ 1/\sin(\pi - \theta) & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 2 and Lemma 3 allow to treat the case of the convex hull edges.

3.4. Case of convex hull edges

For instance, we supposed that all the edges of $V(P)$ are finite, i.e. all the edges of $D(P)$ are incident to exactly two triangles of $D(P)$. However, edges of *conv*(P) are incident to only one triangle of $D(P)$. Their dual edges have one vertex in $V(P)$. In this case, the β-neighborhood of the Voronoi edges cannot be directly defined from $V(P)$.

In order to define the β-neighborhood of the infinite edges of $V(P)$, we consider the Voronoi diagram of $P \cup P'$, where P' is a finite set of points satisfying : $D(P) \subset D(P \cup P')$. The points of P' are selected on the infinite edges of $V(P)$. Thus, to take into account the case of infinite edges of $V(P)$, the local β-crusts and the β-medial axes of P are defined by using $V(P \cup P')$ and $D(P \cup P')$, and by considering the edges of which the vertices belong to P.

By using Lemma 2 and Lemma 3, the local β-crusts and the β-medial axes of P can be completely computed from $D(P)$ and $V(P)$. For that, only one of the two angles is considered in the conditions (i) and (ii) if the Delaunay edge belongs to *conv*(P). Conceptually, this is equivalent to add to P' a point at the infinity on each infinite edge of $V(P)$.

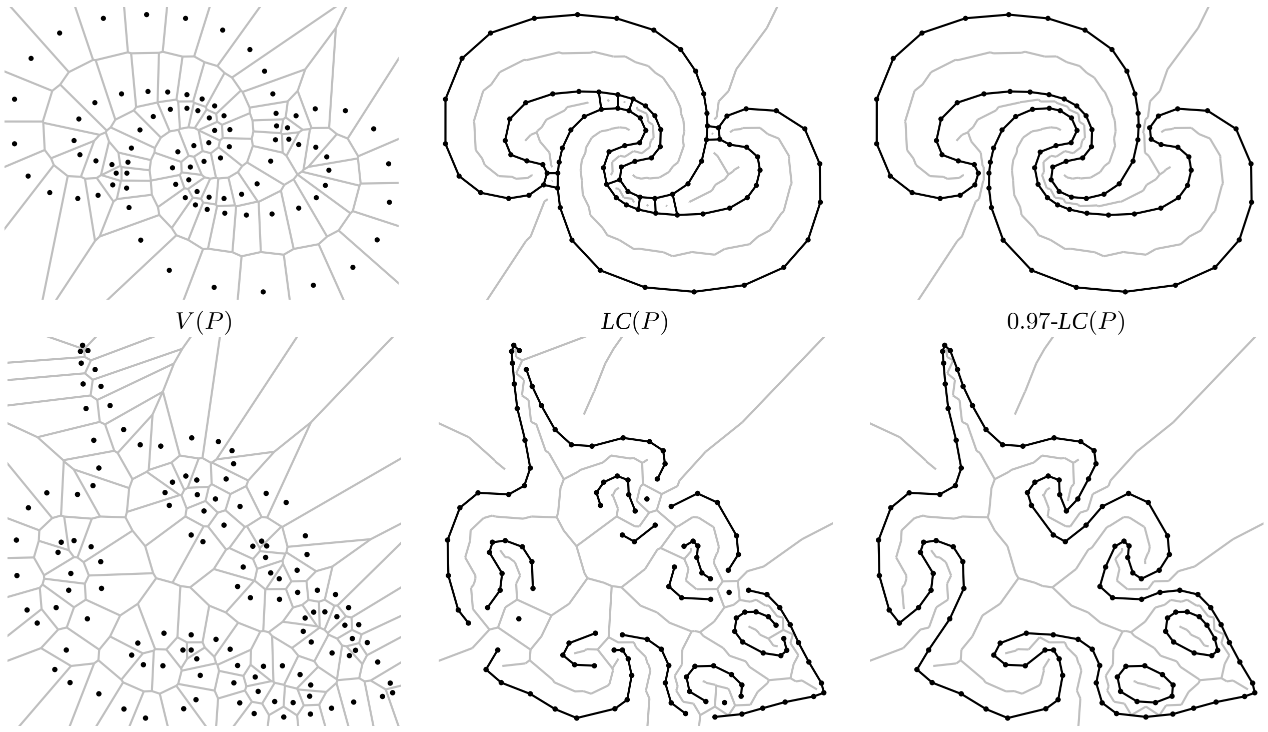


Figure 4. Comparison between local β-crusts and the local crust.

3.5. Differences with the local crust

Given a value of β, the local β-crust is the set of edges of $D(P)$ which are completely included in the β-neighborhood of their dual edges. When $\beta = 1$, the β-neighborhood of an edge v_iv_j of $V(P)$ is the Gabriel disc $B(v_iv_j)$. According to Theorem 1, the local 1-crust of P corresponds to the local crust of P. Moreover, the relation of inclusion between the local β-crusts (Theorem 2), when β is varying, implies the following theorem.

Theorem 3 *The local crust is related to the family of local β-crusts by the three following properties:*
(i) $LC(P) = 1\text{-}LC(P)$,
(ii) $\forall\beta \in (0, 1]$, $\beta\text{-}LC(P) \subseteq LC(P)$.
(iii) $\forall\beta \in [1, +\infty)$, $LC(P) \subseteq \beta\text{-}LC(P)$.
The property (ii) of Theorem 3 shows that the local crust contains the local β-crusts for which $\beta \leq 1$. The property (iii) shows that for $\beta > 1$, the local β-crusts are at least made up of the edges of the local crust. Base upon these properties, we can study the differences between the local crust and the local β-crusts, in the case of samples of curves.

Let Γ be a collection of curves of $\mathbb{R}^2$ and P be a sample of Γ. In some cases, as illustrated in Fig. 4, the local crust cannot correctly reconstruct Γ from P. These cases can be encountered if one of the three following configurations holds:
- All the correct edges are present in $LC(P)$ and at least one edge is incorrect.
- None incorrect edge is present in $LC(P)$ but at least one correct edge is missing.
- $LC(P)$ includes at the same time correct and incorrect edges.

In the two first configurations, there exists one local β-crust that is able to correctly reconstruct Γ if the weight of all the correct edges of $D(P)$ is lower than the weight of the incorrect edges of $D(P)$. Section 4 presents an algorithm which computes such a graph in the case of simple curves. In the third configuration, it cannot exist a local β-crust which correctly reconstruct Γ. We discuss this problem in Section 5.

4. Optimal local β-crust for simple curves

In this section, we propose a method to compute the maximum value of β for which there is a graph, of the family of local β-crusts of P, which reconstructs a collection of closed or opened curves, without intersections. We call this value the optimal value of β, noted β_{opt}.

According to the definition of the spectrum of P, β_{opt}-*LC*(P) corresponds to the set of edges of $D(P)$ that have a weight lower or equal to β_{opt}. To compute β_{opt}-*LC*(P), the idea is to add the edges of $D(P)$ to a graph G, initially empty, in the order defined by the spectrum of P. The edges are added while the degree of their vertices in G is strictly lower than two. In the contrary case, at least one vertex of G would have a degree equal to three after the insertion of the edge in G. Thus, this would represent a collection of curves with an intersection, which contradicts the assumption concerning the topology of the solution (curves without intersections). During the traversal of the spectrum, several edges of $D(P)$ can have the same weight. In this case, if one of them has a vertex of degree two, then none of these edges is added to G.

The computation of the spectrum of P is summarized by Algorithm 1. For each edge of $D(P)$, its weight is given by Eq. (7). The spectrum is then represented by a dictionary L whose key is a weight value, and whose elements are lists of edges having a same weight. The elements of L are sorted according to the ascending order of the weights.

Algorithm 1 *spectre*(P, D)

Input: the Delaunay triangulation D of $P \subset \mathbb{R}^2$.
Output: the dictionary $L = \{(w_1, E_1), \ldots, (w_m, E_m)\}$ such that $w_1 \leq \ldots \leq w_m$.

$L \leftarrow \emptyset$
for each edge $p_ip_j \in D$ **do**
  if $p_ip_j \in conv(P)$ **then**
    $p_k \leftarrow p \in P$ such that the triangle $p_ip_jp \in D$
    if $\angle p_ip_kp_j < \pi/2$ **then**
      $w \leftarrow \angle p_ip_kp_j$
    else $w \leftarrow +\infty$
  else
    $p_k, p_l \leftarrow \{p \in P$ such that the triangle $p_ip_jp \in D\}$
    if $\angle p_ip_kp_j < \pi/2$ and $\angle p_jp_lp_i < \pi/2$ **then**
      $w \leftarrow \angle p_ip_kp_j + \angle p_jp_lp_i$
    else $w \leftarrow +\infty$
  insert p_ip_j in L at the key w
return L

The complete calculation of β_{opt}-*LC*(P) is summarized by Algorithm 2. The graph β_{opt}-*MA*(P) is also extracted from $V(P)$ with this algorithm. It corresponds to the set of edges whose weight is strictly higher than β_{opt}.

Algorithm 2 can be seen as a thresholding method, where the threshold is automatically computed by using the topological properties of the desired solution. Several reconstruction examples are illustrated in Fig. 5 on samples of closed simple curves. As the β_{opt}-local crust correctly reconstructs when the local crust correctly reconstructs (Theorem 3), the examples show the differences between these two graphs. The results are also compared with those obtained with the nearest neighbors crust of P [7], noted *NNC*(P) in the examples.

Algorithm 2 β_{opt}-$LC(P)$

Input: $P \subset \mathbb{R}^2$.
Output: $G = (V, E)$ corresponding to β_{opt}-*LC*(P), $G' = (V', E')$ corresponding toβ_{opt}-*MA*(P), and β_{opt}.

1: $V \leftarrow P$
2: $E, V', E' \leftarrow \emptyset$
3: $w \leftarrow 0$
4: $D \leftarrow D(P)$
5: $L \leftarrow$ *spectre*$(P,\ D)$
6: $k \leftarrow 0$
7: **for** $k = 1, \ldots,$ *sizeof*(L) **do**
8:     $E_k \leftarrow$ the element of L at the key w_k
9:     **for each** edge p_ip_j in E_k **do**
10:         **if** (*deg*$(p_i, G) < 2$) and (*deg*$(p_j, G) < 2$) **then**
11:             *deg*$(p_i, G) \leftarrow$ *deg*$(p_i, G) + 1$
12:             *deg*$(p_j, G) \leftarrow$ *deg*$(p_j, G) + 1$
13:         **else** go to line 16
14:     $E \leftarrow E \cup E_k$
15:     $w \leftarrow w_k$
16: **for** $l = k, \ldots,$ *sizeof*(L) **do**
17:     $E_k \leftarrow$ the element of L at the key w_k
18:     **for each** edge p_ip_j in E_k **do**
19:         $E' \leftarrow E' \cup \{$*dual*$(p_ip_j)\}$
20: **return** G, G' et $\beta(w)$ (Eq. 8)

5. Improvement of the results

Algorithm 2 allows to reconstruct curves without intersections by using the family of the local β-crusts. According to the distribution of the points of P on the curves, the results obtained with this algorithm can be bad. As illustrated in Fig. 6, many correct edges are missing. This is because the spectrum cannot be separated in two disjoined subsets of edges such that the reconstruction is correct. At least one edge of $D(P)$ has a weight weaker than the maximum value of the weight of the correct edges. In this case, one can notice that the local crust does not reconstruct the curves correctly (Theorem 3).

The reconstruction can be considerably improved by slightly modifying Algorithm 2. Instead of stopping the addition of Delaunay edges to G when one of them has a vertex of degree two (line 13), the dual edge of the current edge is added to the graph G', and the spectrum is traversed to its last element. Thus, all the edges of $D(P)$ are traversed in the increasing order of their weights, and added to the reconstruction if they do not imply an intersection (vertex of degree three). Here, the spectrum is represented by a the list L of the edges of $D(P)$ such that $L = \{e_1, \ldots, e_m\ :\ e_i \in D(P),\ w(e_1) \leq \ldots \leq w(e_m)\}$. The whole method is summarized by Algorithm 3. Its time complexity is $O(N \log N)$. The graphs G and G', obtained with Algorithm 3, do not necessarily correspond to a local β-crust or to a β-medial axis. On the other hand, they include the β_{opt}-local crust and the β_{opt}-medial axis (Algorithm 2). A result obtained with Algorithm 3 is illustrated in Fig. 7 on sparse samples of closed curves. One can notice that it improves the results obtained with the local crust.

Algorithm 3 *recons*(P)

Input: $P \subset \mathbb{R}^2$.

Output: $G = (V, E)$ corresponding to the curves, $G' = (V, E')$ corresponding to the approximation of the medial axis of the curves

$V \leftarrow P$
$E, V', E' \leftarrow \emptyset$
$w \leftarrow 0$
$D \leftarrow D(P)$
$L \leftarrow$ *spectre*$(P,\ D)$
$k \leftarrow 0$
for $k = 1, \ldots,$ *sizeof*(L) **do**
  $pq \leftarrow L[k]$
  if (*deg*$(p, G) < 2$) and (*deg*$(q, G) < 2$) **then**
    $E \leftarrow E \cup \{pq\}$
    deg$(p, G) \leftarrow$ *deg*$(p, G) + 1$
    deg$(q, G) \leftarrow$ *deg*$(q, G) + 1$
  else $E' \leftarrow E' \cup \{$*dual*$(pq)\}$
return G et G'

6. Conclusion

In this article, the problem of curve reconstruction, from a set of points of $\mathbb{R}^2$, is formulated by using a hierarchical family of neighborhood graphs. The neighborhood is defined on the edges of the Voronoi diagram, as the union or the intersection of two discs of same radius, parameterized by a ratio of size β. The graphs of this family, which we call the local β-crusts, are subgraphs of the Gabriel graph. Moreover, when $\beta = 1$, the local β-crust corresponds to the local crust.

Based on properties of the local β-crusts, two algorithms are proposed to extract simple curves from the Delaunay triangulation. The first one calculates the maximum value of β for which a local β-crust reconstructs simple curves. The second algorithm improves the results obtained with the first algorithm, while simplifying the method. Each of these algorithms also provides an approximation of the medial axis of the reconstructed curves.

The main ongoing work is to establish the relation between the value of β and the value of the parameter ϵ which occurs in the ϵ-sampling [1] of curves.

References

[1] N. Amenta, M. W. Bern, and D. Eppstein. The crust and the β-skeleton: combinatorial curve reconstruction. *Graphical Models and Image Processing*, 60(2):125–135, 1998.

[2] D. Attali. r-regular shape reconstruction from unorganized points. In *Proc. of the 13th annual Symp. on Computational Geometry*, pages 248–253. ACM Press, 1997.

[3] F. Aurenhammer and R. Klein. Voronoi diagrams. In J. Sack and G. Urrutia, editors, *Handbook of Computational Geometry*, chapter V, pages 201–290. 2000.

[4] F. Bernardini and C. L. Bajaj. Sampling and reconstructing manifolds using alpha-shapes. In *Proc. of the 9th Canadian Conf. on Computational Geometry*, pages 193–198, 1997.

[5] H. Blum. A transformation for extracting new descriptors of shape. In W. Wathen-Dunn, editor, *Models for the Perception of Speech and Visual Form*, pages 362–380. 1967.

[6] J. W. Brandt and V. R. Algazi. Continuous skeleton computation by Voronoi diagram. *CVGIP: Image Understanding*, 55(3):329–337, 1992.

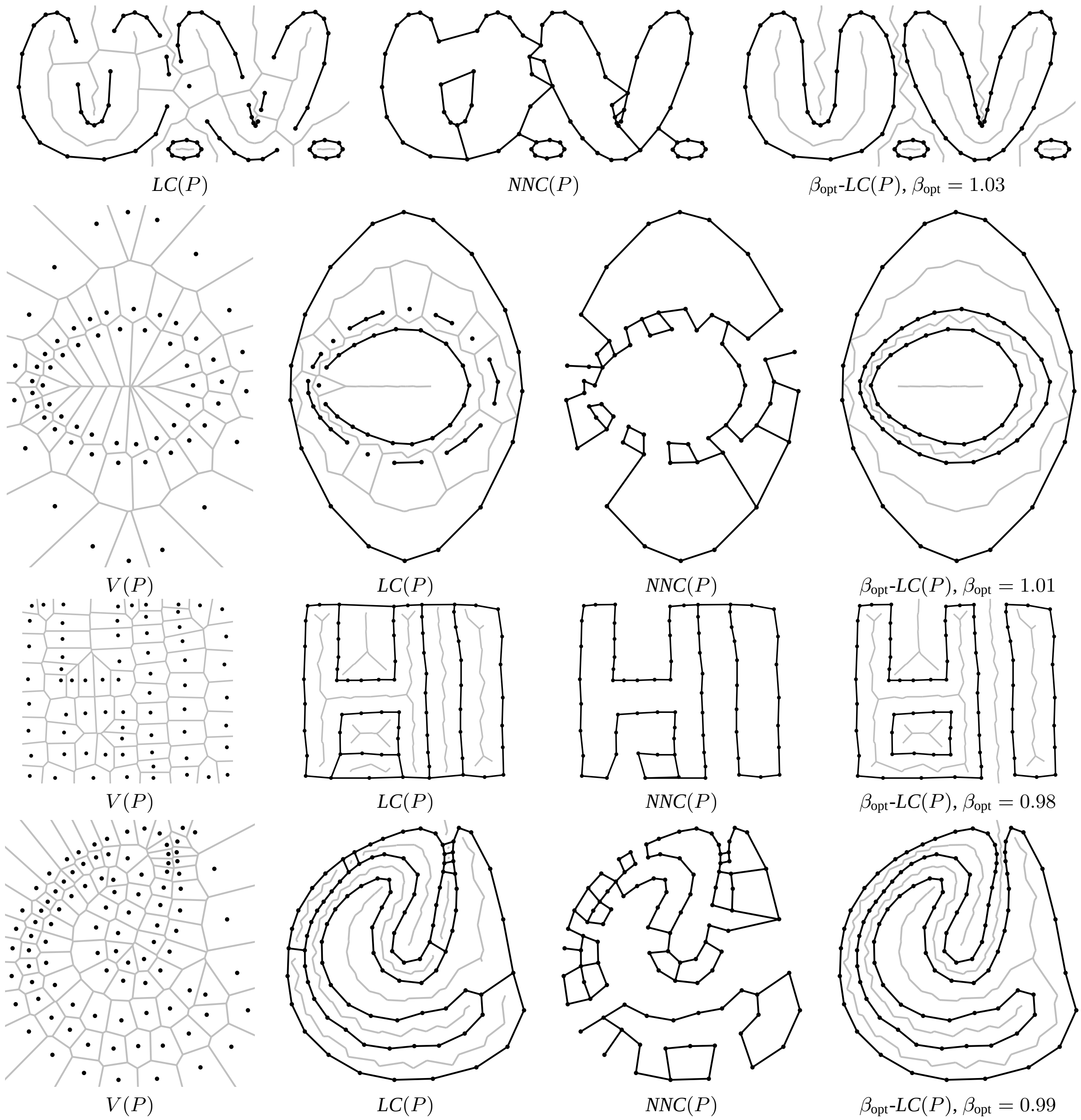


Figure 5. Reconstruction examples with Algorithm 2 and comparisons with the local crust and the nearest neighbors crust.

[7] T. K. Dey and P. Kumar. A simple provable algorithm for curve reconstruction. In *Proc. of the 10th Symposium on Discrete Algorithms*, pages 893–894. ACM/SIAM, 1999.

[8] T. K. Dey, K. Mehlhorn, and E. A. Ramos. Curve reconstruction: Connecting dots with good reason. *Computational Geometry*, 15(4):229–244, 2000.

[9] T. K. Dey and R. Wenger. Reconstructing curves with sharp corners. *Computational Geometry*, 19(2-3):89–99, 2001.

[10] H. Edelsbrunner, D. G. Kirkpatrick, and R. Seidel. On the shape of a set of points in the plane. *IEEE Transactions on Information Theory*, IT-29:65–75, 1983.

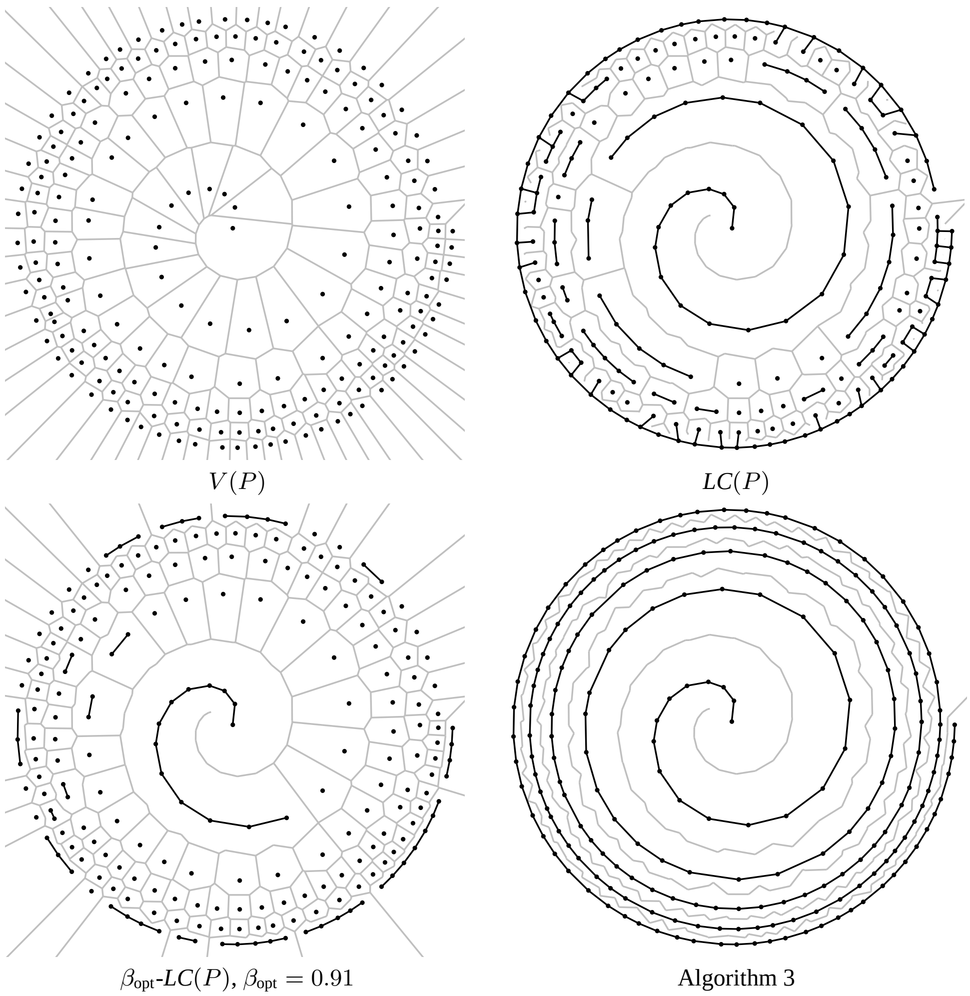


Figure 6. Reconstruction with Algorithm 2 and with the local crust is incorrect meanwhile the reconstruction with Algorithm 3 is correct.

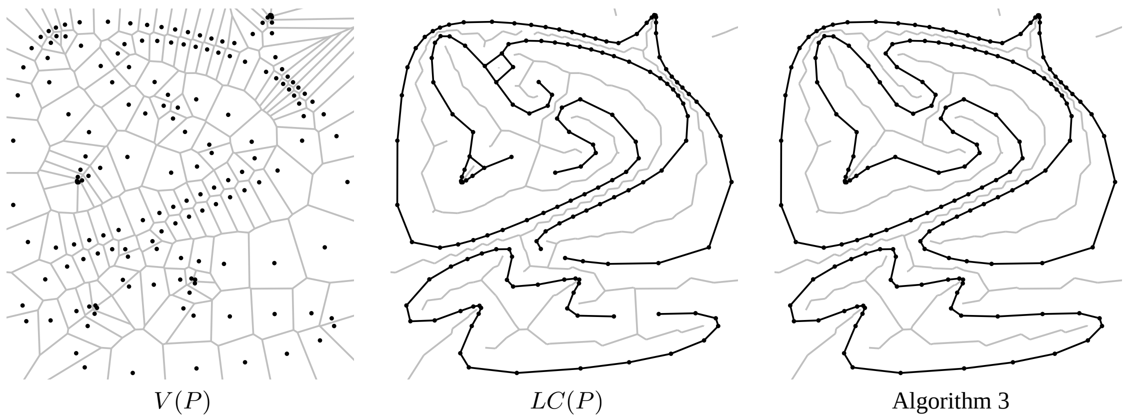


Figure 7. Reconstruction with Algorithm 3 and comparison with the local crust.

[11] L. H. D. Figueiredo and J. D. M. Gomez. Computational morphology of curves. *Visual Computing*, 11:105–112, 1995.

[12] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18(3):259–278, 1969.

[13] C. Gold. Crust and anti-crust: a one-step boundary and skeleton extraction algorithm. In *Proc. of the 15th annual Symp. on Computational Geometry*, pages 189–196. ACM Press, 1999.

[14] C. Gold and J. Snoeyink. A one-step crust and skeleton extraction algorithm. *Algorithmica*, 30(2):144–163, 2001.

[15] H. Hiyoshi. Closed curve reconstruction from unorganized sample points. In M. L. Gavrilova, editor, *Proc. of the 3rd Int. Symp. on Voronoi Diagrams in science and engineering*, pages 122–131. IEEE Computer Society, 2006.

[16] D. G. Kirkpatrick and J. D. Radke. A framework for computational morphology. In G. T. Toussaint, editor, *Computational Geometry*, pages 217–248. Elsevier, 1985.

[17] T. Lenz. How to sample and reconstruct curves with unusual features. In *EWCG: Proc. of the 22nd European Workshop on Computational Geometry*, pages 29–32, 2006.

[18] M. Melkemi. A-shapes of a finite point set. In *Proc. of the 13th annual Symposium on Computational Geometry*, pages 367–369. ACM Press, 1997.

[19] M. Melkemi and L. Melkemi. An algorithm for detecting dot patterns. In R. A. Earnshaw and J. A. Vince, editors, *Computer graphics: developments in virtual environments*, pages 161–169. Academic Press Ltd., 1995.

[20] J. O'Rourke and G. Toussaint. Pattern recognition. In J. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 51, pages 1135–1162. Chapman & Hall, second edition, 2004.

[21] R. C. Veltkamp. The γ-neighborhood graph for computational morphology. In *Proc. of Computing Science in the Netherlands'88*, pages 451–462, 1988.