

Groups of Finite Morley Rank with Solvable Local Subgroups

Adrien Deloro, Eric Jaligot

▶ **To cite this version:**

Adrien Deloro, Eric Jaligot. Groups of Finite Morley Rank with Solvable Local Subgroups. Communications in Algebra, 2012, 40 (3), pp.1019-1068. 10.1080/00927872.2010.544695 . hal-00250070v4

HAL Id: hal-00250070
https://hal.science/hal-00250070v4

Submitted on 17 Apr 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Groups of finite Morley rank with solvable local subgroups

Adrien Deloro*and Eric Jaligot†

October 28, 2010

Abstract

We lay down the fundations of the theory of groups of finite Morley rank in which local subgroups are solvable and we proceed to the local analysis of these groups. We prove a main Uniqueness Theorem, analogous to the Bender method in finite group theory, and derive its corollaries. We also consider homogeneous cases and study torsion.

2000 Mathematics Subject Classification: Primary 20F11; Secondary 20E25, 20E34
Keywords: Cherlin-Zilber Conjecture; Morley Rank; Feit-Thompson Theorem, Bender Method.

1 Introduction

1.1 Generalities

In the Classification of the Finite Simple Groups [GLS94], the study of minimal simple groups has been a fundamental minimal case for the whole process. The local analysis of these finite simple groups, in which each proper subgroup is solvable, has been delineated by J. Thompson, originally for the Odd Order Theorem [FT63, BG94]. This work has later been used to get a classification of minimal simple groups in presence of elements of order 2, and this classification has then been slightly generalized to the case of finite groups in which each normalizer of a nontrivial solvable subgroup is also solvable. Furthermore, the simplicity assumption was replaced by a mere nonsolvability assumption in this last stage. This full classification, with only very few extra groups in addition to the minimal simple ones, has been published in the series [Tho68, Tho70, Tho71, Tho73].

*Institut de Mathématiques de Jussieu, UMR 7586, Université Pierre et Marie Curie (Paris 6), 4 place Jussieu, case 247, 75252, Paris Cedex 05, France
†Corresponding author — Institut Fourier, CNRS, Université Grenoble I, 100 rue des maths, BP 74, 38402 St Martin d'Hères cedex, France

The present paper is the first of a series containing the transfer of arguments and results between two classes of groups of finite Morley rank very analogous to the finite minimal simple groups and, respectively, the nonsolvable finite groups in which normalizers of solvable subgroups are solvable. A very large body of ideas from the classification of the finite simple groups, including in particular J. Thompson's original ones, has already been used in the context of groups of finite Morley rank [ABC08]. Here we are going to concentrate on the specific case of "small" groups of finite Morley rank, where smallness is witnessed by similar properties. In fact, a large body of very specific work has been accomplished in the last years about *minimal connected simple* groups of finite Morley rank, that is connected simple groups of finite Morley rank in which every proper definable connected subgroup is solvable. We propose here to transfer this specific work to the more general class of $*$-*locally solvable* groups of finite Morley rank, that is groups of finite Morley rank in which $N(A)$ is solvable for each nontrivial definable abelian subgroup A.

In finite group theory, and following Alperin, a subgroup is called *p-local* if it is the normalizer of a nontrivial p-subgroup for some prime p. Thompson's classification was stated for groups in which each p-local subgroup is solvable, which is equivalent to requiring that normalizers of nontrivial solvable (or just abelian) subgroups are solvable. As we will see in Section 3.1 below, our $*$-local solvability assumption can similarly be characterized by the solvability of $\tilde{p}$-local subgroups, where $\tilde{p}$-subgroups are in our context specific subgroups sharing certain characteristic properties very similar to those of finite p-groups (see Fact 2.3 below). In particular, the "$*$" in our notation aims at reminding this notion of p-local or $\tilde{p}$-local solvability.

Since, when working with groups of finite Morley rank, one prefers to deal with the connected category, we will actually weaken this definition of $*$-local solvability in the following three possible ways, by assuming the solvability of the *connected components* only of normalizers of nontrivial definable abelian subgroups A, in which case we will speak of $*$-*locally*$^\circ$ *solvable* groups, and/or by considering nontrivial definable *connected* abelian subgroups A only, which will be called $*$-*local*$_\circ$ *solvability*. In fact, we will most of the time work here with the weakest definition of $*$-*local*$^\circ_\circ$ *solvability*, i.e., assuming only that $N^\circ(A)$ is solvable for each nontrivial definable connected abelian subgroup A of the ambient group. We refer to Definition 3.1 below.

The only known infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields and a long-standing conjecture postulates that there are no other such groups, which would imply also that any nonsolvable connected group is nicely built up from simple algebraic groups. In particular, the only known nonsolvable connected $*$-locally$^\circ$ solvable groups of finite Morley rank are of the form PSL_2 over some algebraically closed field K, and of the form SL_2 in the slightly more general $*$-locally$^\circ_\circ$ solvable case. For example, if

we consider in $\mathrm{SL}_3(K)$ the definable connected abelian subgroup

$$A = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} : t \in K^\times \right\},$$

then $N^\circ(A)$ is a central product $A \cdot E$ where E is a definable connected subgroup isomorphic to $\mathrm{SL}_2(K)$, so that $N^\circ(A)$ is not solvable. More precisely, for connected $*$-locally$^\circ$ solvable and $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank there are in the classical algebraic case no other groups than PSL_2 and SL_2, and in particular no groups of Lie rank 2 and more.

All the classes of $*$-locally solvable groups of finite Morley rank defined here contain of course all solvable groups of finite Morley rank, groups of the form PSL_2 or SL_2, but also many hypothetic configurations of semisimple weak analogs of so-called bad groups of finite Morley rank which appear as potential counterexamples to the main conjecture on simple groups. All the results of the present series of papers tend to lead, very roughly, to a kind of trichotomy for $*$-locally$^\circ_\circ$ solvable groups as follows.

- Solvable groups.
- PSL_2 or SL_2.
- Semisimple weak analogs of bad groups.

We refer for example to [DJ10, Theorem 5] in the subsequent series for a theorem leading, in presence of infinite elementary abelian 2-subgroups, exactly to one of the two first cases as above. For nonsolvable groups, the dichotomy between algebraic groups and "bad" groups appeared early in the seminal [Che79] on groups of Morley rank 3, or in a more general form in [Jal01a]. There is in fact a kind of general dichotomy between configurations in which a local analysis in Thompson's style can be achieved, and those in which it rapidly stops, and we will keep track of both aspects throughout.

In the present first paper of our series, we are essentially going to recast the theory of solvable and of minimal connected simple groups into the more general context of $*$-locally solvable groups, taking also care of presenting the various aspects of the theory in a very linear and coherent way. Throughout, we will insist on the differences between $*$-local$^\circ$ solvability, which offers no new substantial phenomena compared to the minimal connected simple case, and the weaker $*$-local$^\circ_\circ$ solvability, where many new phenomena can occur. This is at least explained by the alternative SL_2 to PSL_2. While the present paper serves as a large fishpond of results for smooth proofs in subsequent papers, it also contains no result easily stated in this introduction. In the present paper we will never assume the presence of involutions, and we refer to [DJ10] and [DJ07a] in our series for the case of groups with involutions.

1.2 Historical context

Before passing to more comments on the actual contents of the present paper, it seems necessary to make more precise historical remarks. First of all, solvable groups of finite Morley have been highly investigated, notably by Nesin and Frécon, and their understanding is at the core of the analysis of groups of finite Morley rank, especially of those studied here.

With the ongoing work on simple groups of finite Morley rank with involutions, it became clear as corollaries of [Jal99] and [Jal01b] that there were no "small" simple groups of finite Morley rank of mixed type, and that the only specimen in even type was $\mathrm{PSL}_2(K)$, with K an algebraically closed field of characteristic 2.

Then it was time to start the study of "small" simple groups of odd type, even though there was almost nothing to start with. The fundations, notably the notion of a minimal connected simple group, were laid down in the preprint [Jal00] which remained unpublished. It contained the first recognition result of PSL_2 in characteristic different from 2 in this context, though under strong assumptions at that time. It also contained the embryo of the local analysis of minimal connected simple groups of finite Morley rank. The original lemma, which turned out later to be an analog of the Bender method in finite group theory, was there given in any characteristic. It has unfortunately been disseminated between different characteristics later, and we will give here global forms and the general Uniqueness Theorem in Section 4.1.

Because of the absence of a unipotence theory in characteristic zero at that time, and in order to reduce the size of an overambitious project to manageable size, the second author adopted the so-called "tameness" assumption for the recognition of PSL_2 with the weakest expectable hypothesis in this context. The nonalgebraic configurations were also studied in this tame context, and the full analysis algebraic/nonalgebraic appeared in [CJ04].

In the meantime Cherlin suggested to develop a robust unipotence theory in characteristic 0 for attacking certain problems concerning large groups of odd type without the tameness assumption. This became the main tool in Burdges' thesis [Bur04a] and this application corresponds to [Bur04b]. This new abstract unipotence theory then allowed one to develop the local analysis of minimal connected simple groups where the above mentioned uniqueness theorem fails [Bur07]. It was also Cherlin's idea to use this in presence of involutions to study other nonalgebraic configurations without tameness [BCJ07, Case II].

With a nice unipotence theory then available in any characteristic, the recognition of PSL_2 started again in the context of minimal connected simple groups of odd type without tameness, in the thesis of the first author [Del07b]. The recognition of PSL_2 has then been obtained as in the tame case under the weakest expectable assumptions and appeared in [Del07a]. Using this new experience for the algebraic case, the nonalgebraic configurations were studied in [Del08], reaching essentially all the conclusions of [CJ04] in the general case. The paper [DJ07a] will at the same time improve and linearize the sequence of arguments represented by [BCJ07, Del07a, Del08], and also greatly simplify

those in [Del08].

The final generalization from minimal connected simple groups to $*$-locally solvable groups that we accomplish here has been suggested by Borovik by analogy with finite group theory.

1.3 Contents of the paper

We now wish to comment on the contents of the present paper. Section 2 will contain background material, with notably an emphasis on the abstract unipotence theory in groups of finite Morley rank in Section 2.1 in continuation of [Bur06] and [FJ08]. We shall also formalize the notion of a *soapy* subgroup, the finest approximation of unipotent subgroups where all the finest computations will be done in [DJ07a].

Section 3 will lay down the basic fundations concerning $*$-locally solvable groups of finite Morley rank. In Sections 3.3 and 3.4 we will mainly focus on the new phenomena which can occur in the $*$-locally$^\circ_\circ$ solvable case in comparison to the $*$-locally$^\circ$ solvable one, and give a very detailed description of them.

Section 4 will concern the local analysis of $*$-locally solvable groups of finite Morley rank, with in Section 4.1 the main Uniqueness Theorem (usually called "Jaligot's Lemma") corresponding to the Bender method in finite group theory. This theorem, whose numerous corollaries are at the core of almost all applications "towards algebraicity" in the theory, is the heart of the present paper. In technical terms and in the very general form given here, it will be stated as follows.

Uniqueness Theorem 4.1 *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and U a Sylow $\tilde{p}$-subgroup of G. Assume that U_1 is a nontrivial definable $\tilde{p}$-subgroup of U containing a nonempty (possibly trivial) subset X of G such that $d_p(C^\circ(X)) \leq r$. Then U is the unique Sylow $\tilde{p}$-subgroup of G containing U_1, and in particular $N(U_1) \leq N(U)$.*

When the above theorem cannot be applied, typically in highly nonalgebraic configurations, the lenghty analysis of a maximal pair of Borel subgroups from [Bur07] can be applied in the case of minimal connected simple groups. In Section 4.2 we will see that this parallel technique generalizes without (too much) pain to $*$-locally$^\circ_\circ$ solvable groups.

Section 5 eventually concludes the present paper with several specific aspects concerning homogeneous cases as well as torsion. For instance we apply the Uniqueness Theorem, in its most straightforward forms, in Section 5.3 when the ambient group contains nontrivial p-unipotent subgroups, to get structural results similar to what occurs in PSL_2 in characteristic p. When analyzing certain homogeneous groups, we also take the opportunity to examine further possible variations of the notion of "bad" groups in the $*$-locally$^\circ_\circ$ solvable context.

1.4 General background and notations

For the basic background on groups of finite Morley rank we generally refer to [BN94]. The more recent [ABC08] is also a very complete source. We will try to refer as much as possible to these when needed, but we assume the reader familiar with certain background facts such as the descending chain condition on definable subgroups, the existence of connected components, the uniqueness of generic types in connected groups [Che79], and the following crucial fact.

Fact 1.1. *A connected group acting definably on a finite set fixes it pointwise.*

We will also assume the reader familiar with Zilber's generation lemma and its corollaries [Zil77] [BN94, §5.4], notably the definability of subgroups generated by definable connected subgroups and corollaries on commutator subgroups [BN94, Corollary 5.29]. The other results used here which are less classical, or in a form not directly available in the literature, will be explicitely mentioned as "Facts" below.

If X is a subset, or a single element, of a group of finite Morley rank, we denote by $H(X)$ the *definable hull* of X, that is the smallest definable subgroup containing X. In the literature the notation "$d(\cdot)$" is more commonly used, but we prefer to keep the latter for certain integer-valued unipotence "d"egrees, and instead use "$H(\cdot)$" for "H"ulls which are definable subgroups.

If x and y are elements of a group, we write x^y for $y^{-1}xy$, and if X and Y are two subsets we denote by X^Y the set of elements x^y, with (x, y) varying in $X \times Y$. We denote by $N(X)$ the set of elements g such that $X^g = X$, with a subscript if one wants to specify in which particular subset elements g are taken.

2 Background

2.1 Unipotence theory

For the following abstract unipotence theory in groups of finite Morley rank [Bur04a, Bur04b, Bur06], we follow essentially the general exposition of [FJ08], using in particular the global notation of [Jal08b]. We denote by $\mathcal{P}$ the set of all prime numbers.

A *decent torus* is a divisible abelian group of finite Morley rank which coincides with the definable hull of its (divisible abelian) torsion subgroup. The latter is known to be in the finite Morley rank context a direct product, with p varying in $\mathcal{P}$, of finite products of the Prüfer p-group $\mathbb{Z}_{p^\infty}$ [BP90], and by divisibility decent tori are connected.

If p is a prime, a *p-unipotent* group of finite Morley rank is a definable connected nilpotent p-group of bounded exponent.

A *unipotence parameter* is a pair

$$\tilde{p} = (\textit{characteristic } p, \textit{unipotence degree } r) \in (\{\infty\} \cup \mathcal{P}) \times (\mathbb{N} \cup \{\infty\})$$

satisfying $p < \infty$ if and only if $r = \infty$. A group of finite Morley rank is a *$\tilde{p}$-group* if it is nilpotent and of the following form, depending on the value of $\tilde{p}$.

- if $\tilde{p} = (\infty, 0)$, a decent torus.
- if $\tilde{p} = (\infty, r)$, with $0 < r < \infty$, a group generated by its definable indecomposable subgroups A such that $A/\Phi(A)$ is torsion-free and of rank r. Here a group of finite Morley rank is *indecomposable* if it is abelian and not the sum of two proper definable subgroups. An indecomposable group A must be connected [Bur06, Lemma 1.2], and $\Phi(A)$ denotes its maximal proper definable conected subgroup.
- if $\tilde{p} = (p, \infty)$, with p prime, a p-unipotent subgroup.

We note that nilpotence of $\tilde{p}$-groups is imposed by the definition, and that these groups are in any case generated by definable connected subgroups, and hence always connected by Zilber's generation lemma [BN94, Corollary 5.28]. A *Sylow* $\tilde{p}$-subgroup of a group of finite Morley rank is a maximal definable (connected) $\tilde{p}$-subgroup.

The term "characteristic" for p in a unipotence parameter (p, r) clearly refers to the characteristic of the ground field for p-unipotent groups in algebraic groups when p is finite. When p is infinite and $0 < r < \infty$, it refers to nontrivial torsion-free groups, which are potentially additive groups of fields of characteristic 0. When p is infinite and $r = 0$, i.e., for decent tori, it conveys no special meaning. The term "unipotence degree" (one can also speak of "weight") is explained in Fact 2.7 below by the constraints on actions of such groups on others.

A group of finite Morley rank is (p, r)-*homogeneous* if every definable connected nilpotent subgroup is a (p, r)-group. We say that it is *homogeneous* if it is (p, r)-homogeneous for some unipotence parameter (p, r). Following [Che05], a divisible abelian $(\infty, 0)$-homogeneous group of finite Morley rank is usually called a *good torus*. We refer to [FJ08, Lemma 2.17] for homogeneity of $\tilde{p}$-groups.

Fact 2.1. [FJ08, Theorem 2.18] *Let G be a connected group of finite Morley rank acting definably on a $\tilde{p}$-group H. Then $[G, H]$ is a definable $\tilde{p}$-homogeneous subgroup of H.*

Proof. The main point is when the unipotence degree r of H satisfies $0 < r < \infty$ and is proved in [Fré06a, Theorem 4.11]. When the unipotence degree of H is infinite, this is just [FJ08, Lemma 2.17-c]. Decent tori are centralized by any connected group acting on them: this is an easy consequence of Fact 1.1 called *rigidity* of decent tori (see Fact 2.7 (1) below). Hence $[G, H]$ is trivial when $r = 0$. □

Corollary 2.2. *Let G be any $\tilde{p}$-group. Then G^n and $G^{(n)}$ are definable homogeneous $\tilde{p}$-subgroups for any $n \geq 1$.*

If G is a group of finite Morley rank and $\tilde{\pi}$ is a set of unipotence parameters, we define

$$U_{\tilde{\pi}}(G) = \langle \Sigma \mid \tilde{p} \in \pi \text{ and } \Sigma \text{ is a definable } \tilde{p}\text{-subgroup of } G \rangle.$$

The latter subgroup is always definable and connected by Zilber's generation lemma. When $\tilde{\pi}$ is empty it is trivial and when $\tilde{\pi}$ consists of a single unipotence parameter $\tilde{p}$ we simply write $U_{\tilde{p}}(G)$. If $\tilde{p} = (p, \infty)$ for some prime p, we also write $U_p(G)$ for $U_{\tilde{p}}(G)$. A *$U_{\tilde{\pi}}$-group* is a group G such that $U_{\tilde{\pi}}(G) = G$. We refer to [FJ08, Lemma 2.13] for properties of push-forwards and pull-backs of such groups.

Fact 2.3. [Bur06, §3] *Let G be a nilpotent group of finite Morley rank.*

(1) *G is the central product of its Sylow p-subgroups and its Sylow (∞, r)-subgroups.*

(2) *If G is connected, then G is the central product of its Sylow $\tilde{p}$-subgroups.*

Proof. The connected case corresponds to [FJ08, Theorem 2.7]. Without connectedness we refer to the decomposition of G as the central product of a definable divisible (connected) subgroup D and a definable subgroup B of bounded exponent [Nes91] [BN94, Theorem 6.8], and to the decomposition of a nilpotent group of bounded exponent as the central product of its (definable) Sylow p-subgroups. □

Fact 2.4. *A $\tilde{p}$-group of finite Morley rank cannot be a $\tilde{q}$-group when $\tilde{q} \neq \tilde{p}$.*

Proof. It suffices to use the commutation provided by Fact 2.3 (2) to reduce the problem to abelian groups. Then it follows easily from the definitions. □

We will also frequently use a variation for $\tilde{p}$-groups on the usual *normalizer condition* in finite nilpotent groups [Bur06, Lemma 2.4] (see [FJ08, Proposition 2.8]).

Definition 2.5. *Let G be a group of finite Morley rank.*

(1) *We say that G* admits *the unipotence parameter $\tilde{p}$ if $U_{\tilde{p}}(G) \neq 1$.*

(2) *We denote by $d_\infty(G)$ the* maximal unipotence degree in characteristic ∞, *i.e., the maximal integer $r \in \mathbb{N}$ such that G admits the unipotence parameter (∞, r), and -1 if G admits none such.*

(3) *If p is a prime, we denote by $d_p(G)$ the* maximal unipotence degree in characteristic p, *i.e., the symbol ∞ if G admits the unipotence parameter (p, ∞), and -1 otherwise.*

(4) *A unipotence parameter $\tilde{p} = (p, r)$ is* maximal in its characteristic *for G if $d_p(G) = r$ (notice here that the characteristic p can be ∞ or prime). This is equivalent to saying that r is the maximal unipotence parameter in characteristic p.*

(5) *Finally, we define the* absolute unipotence degree $d(G)$ *of G as the maximum of $d_\infty(G)$ and $\max_{p \in \mathcal{P}}\{d_p(G)\}$.*

We say that a unipotence parameter (p, r) is *absolutely maximal* for G if $d(G) = d_p(G) = r$, i.e., if G contains nontrivial p-unipotent subgroups if $p < \infty$ and otherwise admits (∞, r) and contains no nontrivial definable connected nilpotent subgroup of bounded exponent and no nontrivial definable (∞, r')-subgroup with $r' > r$.

We say that a unipotence parameter (p, r) is *maximal* for G if $d(G) = 0$ whenever $r = 0$, or $d_p(G) = r$ otherwise. This has essentially the effect of not considering good tori of PSL_2 over a pure field of positive charateristic as having maximal unipotence degree. We will often mention this special example separately.

The following lemma makes known facts more transparent in our notation, or rather makes our notation more transparent via known facts.

Lemma 2.6. *Let G be a group of finite Morley rank.*

(1) *G is finite if and only if $d(G) = -1$.*

(2) *G is a good torus if and only if G is connected solvable and $d(G) \leq 0$.*

Proof. If $d(G) \geq 0$, then G has a nontrivial definable connected nilpotent subgroup, and hence it cannot be finite. Conversely, if G is infinite, then its minimal infinite definable subgroups are abelian by Reineke's Theorem [BN94, Theorem 6.4]. As such subgroups are also connected, they contain a nontrivial Sylow $\tilde{p}$-subgroup for some unipotence parameter $\tilde{p} = (p, r)$ by Fact 2.3 (2), and hence $d(G) \geq r \geq 0 > -1$.

If G is a good torus, then it is abelian and connected, and any definable connected subgroup is a good torus, in particular a decent torus, and by Fact 2.4 $d(G) \leq 0$. Conversely, if G is a connected solvable group which admits no unipotence parameter $\tilde{p} = (p, r)$ with $r \geq 1$, then G is a good torus by [Bur04b, Theorem 2.15]. □

For any group G of finite Morley rank we define, similarly to $U_p(G)$, the *unipotent radical in characteristic* ∞ as

$$U_\infty(G) = U_{(\infty, d_\infty(G))}(G).$$

One can also define the *absolute unipotent radical* $U(G)$ as

$$U(G) = \langle U_p(G) \mid p \text{ prime} \rangle \text{ if it is nontrivial and } U_\infty(G) \text{ otherwise.}$$

Finally, a unipotent radical $U_{(p,r)}(G)$ is *maximal* for G if (p, r) is maximal for G.

2.2 Carter and soapy subgroups

The preceding abstract unipotence theory in groups of finite Morley rank gives important approximations of semisimple and unipotent subgroups of algebraic groups. On the one hand it gives a good approximation of maximal tori in any group of finite Morley rank via the notion of a Carter subgroup. On the other

hand it detects, and it is a more difficult task, approximations of unipotent subgroups in $*$-locally solvable groups via the notion of a soapy subgroup.

All this is available thanks to a good understanding of possible actions of $\tilde{p}$-subgroups onto each other in groups of finite Morley rank. These constraints can be summarized as follows. The first item is often called *rigidity* of decent tori.

Fact 2.7. *Let G be a group of finite Morley rank, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ two sets of unipotence parameters, and $r \in \mathbb{N} \cup \{\infty\}$.*

(1) *Assume $G = TH$ where T is a definable decent torus of G and H is a definable connected subgroup normalizing T. Then $T \leq Z(G)$. In particular, if T is a definable decent torus in a group of finite Morley rank, then $C^\circ(T) = N^\circ(T)$.*

(2) *Assume $G = U_1U_2$ where each $U_i = U_{\tilde{\pi}_i}(U_i)$ is a definable nilpotent subgroup and U_1 is normal. Assume that all unipotence degrees involved in $\tilde{\pi}_1$ are $\leq r$ and that all unipotence degrees involved in $\tilde{\pi}_2$ are $\geq r$. Then U_1U_2 is nilpotent.*

(3) *Assume $G = H_1H_2$ where each $H_i = U_{\tilde{\pi}_i}(H_i)$ is definable and H_1 is normal and nilpotent. Assume that all unipotence degrees involved in $\tilde{\pi}_1$ are $\leq r$ and that all unipotence degrees involved in $\tilde{\pi}_2$ are $> r$. Then $G = H_1C^\circ(H_1)$.*

(4) *Assume $G = U_1U_2$ where U_1 is a normal nilpotent subgroup such that $U_1 = U_{\tilde{\pi}_1}(U_1)$, will all unipotence degrees involved in $\tilde{\pi}_1$ infinite, and $U_2 = U_{\tilde{\pi}_2}(U_2)$, where all unipotence degrees r involved in $\tilde{\pi}_2$ satisfy $0 < r < \infty$. Then $U_2 \leq C(U_1)$.*

Proof. The first item, which was the main key tool in [Che05], is a mere application of Fact 1.1 together with the fact that Prüfer p-ranks of decent tori are finite for any prime p [BP90].

The second item is [FJ08, Proposition 2.10]. See also [FJ05, §3] and [Bur06, §4] for earlier versions of the same fact.

For the third item, we notice that if $\tilde{p} \in \tilde{\pi}_2$ and Σ is any definable connected $\tilde{p}$-subgroup of H_2, then $H_1 \cdot \Sigma$ is nilpotent by the second point, and both factors commute by our assumption on the unipotence degrees involved and Fact 2.3 (2). In particular $U_{\tilde{p}}(H_2) \leq C^\circ(H_1)$ and as $H_2 = \langle U_{\tilde{p}}(H_2) \mid \tilde{p} \in \tilde{\pi}_2 \rangle$, our claim follows.

For the last item we refer to [Bur06, Lemma 4.3] for the fact that an (∞, r)-group, with $0 < r < \infty$, which normalizes a p-unipotent group must centralize it. This is essentially a corollary of [Wag01, Corollary 8]. Then one can argue as in the third point. □

Fact 2.7 has as a general consequence the existence of a very good approximation of semisimple subgroups of algebraic groups in the context of groups of finite Morley rank. If $\tilde{\pi}$ is a set of unipotence parameters, a *Carter $\tilde{\pi}$-subgroup* of a group of finite Morley rank is a definable connected nilpotent subgroup

$Q_{\tilde{\pi}}$ such that $U_{\tilde{\pi}}(N(Q)) = Q$. A *Carter* subgroup of a group of finite Morley rank is a definable connected nilpotent subgroup Q such that $N^\circ(Q) = Q$. By [FJ08, Lemma 2.13] this corresponds to a Carter $\tilde{\pi}$-subgroup for the set $\tilde{\pi}$ of *all* unipotence parameters, or merely the set of unipotence parameters admitted by the ambient group.

The existence of Carter subgroups in arbitrary groups of finite Morley rank, which appeared in [FJ05], had originally been looked for by the second author in the context of minimal connected simple groups in order to generalize [CJ04]. It follows essentially from Fact 2.7, by considering $\tilde{p}$-subgroups from the least to the most unipotent. We refer to [FJ08, Theorem 3.3].

A definable subset X of a group G of finite Morley rank is *generous* in G if the union X^G of its G-conjugates is generic in G. In simple algebraic groups maximal tori are generous. In groups of finite Morley rank we only have equivalent conditions to this property [Jal06, Corollary 3.8].

After semisimple groups, we now proceed to modelling unipotent subgroups. We denote by $F(G)$ the *Fitting* subgroup of any group G, i.e., the subgroup generated by all normal nilpotent subgroups. It is always definable and nilpotent in the finite Morley rank case [BN94, Theorem 7.3]. A consequence of Fact 2.7 dual to [FJ08, Theorem 3.3] is the following (see [FJ08, Lemma 2.11], and [Bur04b, Theorem 2.16] for the original version).

Fact 2.8. *Let H be a connected solvable group of finite Morley rank and $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$. Assume $d_p(H) \leq r$. Then $U_{\tilde{p}}(H) \leq F^\circ(H)$.*

We note that the assumption $r > 0$ is necessary in Fact 2.8. In the standard Borel subgroup B of PSL_2 in positive characteristic, $d_\infty(B) = 0$, but maximal tori of B are not in the unipotent radical of B.

Unipotent subgroups are usually not generous in linear algebraic groups, and thus in general more difficult to detect. Every *nontrivial* subgroup $U_{\tilde{p}}(H)$ as in Fact 2.8 is generally a good approximation of a unipotent radical, at least much finer than the Fitting subgroup. We will need even finer approximations when considering $*$-locally solvable groups of finite Morley rank, notably the property of being homogeneous and central in the Fitting subgroup. This issues from the minimal subgroups used originally in [Jal00], after the considerable reworking in [Del07b] using Burdges' unipotence theory and Frécon's homogeneity result (see [Del07a, §5.1] and [Del08, 5.46, 6.31...]).

Recall that, for every connected solvable group H of finite Morley rank, a unipotence parameter $\tilde{q} = (q, d)$ is *maximal* for H if $d(H) = 0$ whenever $d = 0$, or $d_q(H) = d$ otherwise. By Lemma 2.6, a nontrivial connected solvable group H is a good torus if and only if its unique maximal unipotence parameter is $(\infty, 0)$. Otherwise, maximal unipotence parameters are all the (p, ∞) such that $U_p(H) \neq 1$, and (∞, d) with $d \geq 1$ and $d_\infty(H) = d$ if it exists.

Definition 2.9. *Let H be a connected solvable group of finite Morley rank. A subgroup U of H is* soapy *(resp.* characteristically soapy*) in H if the two following conditions hold.*

(1) *U is a nontrivial definable connected subgroup of $Z(F^\circ(H))$, $\tilde{q}$-homogeneous for some unipotence parameter $\tilde{q}$ maximal for H.*

(2) *U is normal (resp. definably characteristic) in H.*

We will see in Section 4.1.4 that in $*$-locally$^\circ_\circ$ solvable groups these subgroups have a strong tendency to escape from intersections of distincts Borel subgroups, like unipotent subgroups in PSL_2 and like a soap between two hands. Another not less serious reason for this name is that these groups were born near Marseilles, which is famous for its soap.

We could also specify a set of maximal unipotence parameters for H, and define these interesting subgroups as products of the present ones. In practice only one unipotence parameter will suffice for us.

The next lemma says that the existence of soapy subgroups is not essentially weaker than that of characteristically soapy subgroups.

Lemma 2.10. *Let H be a connected solvable group of finite Morley rank and $\tilde{q}$ a unipotence parameter maximal for H. If H contains a $\tilde{q}$-homogeneous soapy subgroup, then it contains a $\tilde{q}$-homogeneous characteristically soapy subgroup as well.*

Proof. If $\tilde{q} = (\infty, 0)$ then H is a good torus, and H itself is the desired group.

In general one can proceed as follows. Let U be a $\tilde{q}$-homogeneous soapy subgroup of H. Let $\tilde{U}$ be the subgroup of $Z(F^\circ(H))$ generated by all $\tilde{q}$-homogeneous soapy subgroups of H. It is nontrivial, definable and connected as the product of finitely many soapy subgroups by Zilber's generation lemma, and one sees easily that it is $\tilde{q}$-homogeneous with [FJ08, Lemma 2.13] (see also [Fré06a, Corollary 3.5]). It is clearly definably characteristic in H. Hence $\tilde{U}$ is characteristically soapy in H. □

We finish this section with a general criterion for finding characteristically soapy subgroups.

Lemma 2.11. *Let H be a connected solvable group of finite Morley rank and $\tilde{q}$ a unipotence parameter maximal for H. If $U_{\tilde{q}}(Z(F^\circ(H)))$ is not central in H, then H contains a $\tilde{q}$-homogeneous characteristically soapy subgroup.*

Proof. Set $U = [U_{\tilde{q}}(Z(F^\circ(H))), H]$. By assumption U is nontrivial. It is a definable connected homogeneous $\tilde{q}$-subgroup by Fact 2.1, contained in $Z(F^\circ(H))$ as the latter is normal in H, and obviously definably characteristic in H. □

As far as unipotence theory in concerned, there are two general conjugacy theorems in groups of finite Morley rank, the conjugacy of *maximal* definable decent tori [Che05] (see also [FJ08, §6.2] for the non-maximal ones) and the conjugacy of *generous* Carter subgroups [Jal06]. For the sake of further references we mention the following corollary of the conjugacy of maximal decent tori which has been known for a long time in presence of 2-divisible torsion [BN94, Lemma 10.22].

Corollary 2.12 (Control of fusion)**.** *Let G be a group of finite Morley rank, p a prime, and T a p-torus of G. If X and Y are two G-conjugate subsets of G such that $C_T(X)$, $C_T(Y)$, and $C(Y)$ all have the same Prüfer p-ranks, then $Y = X^g$ for some g conjugating $C_T^\circ(X)$ to $C_T^\circ(Y)$. In particular if T is a maximal p-torus of G then any two G-conjugate subsets of $C(T)$ are $N(T)$-conjugate.*

Proof. First notice that there are always *maximal* p-tori, by finiteness of the Prüfer p-rank [BP90].

Assume $Y = X^g$ for some $g \in G$. Then $C_T^\circ(X)^g$ and $C_T^\circ(Y)$ are both contained in the definable subgroup $C^\circ(Y)$. By conjugacy of decent tori [Che05] and the assumption, $C_T^\circ(X)^g = C_T^\circ(Y)^\gamma$ for some $\gamma \in C^\circ(Y)$. Then $g\gamma^{-1}$ conjugates $C_T^\circ(X)$ to $C_T^\circ(Y)$ and as $Y^\gamma = Y = X^g$ the element $g\gamma^{-1}$ conjugates X to Y.

When X and Y are two subsets of $C(T)$ and T is maximal we can apply the preceding and the new element g conjugating X to Y will now normalize T. □

2.3 Solvable groups

Fact 2.13. [BP90] *Let p be a prime and S a p-subgroup of a solvable group of finite Morley rank, or more generally a locally finite p-subgroup of any group of finite Morley rank. Then*

(1) *S° is a central product of a p-torus and a p-unipotent subgroup.*

(2) *If S is infinite and of bounded exponent, then $Z(S)$ contains infinitely many elements of order p.*

Lemma 2.14. *Let H be a connected solvable group of finite Morley rank and p a prime. If $U_p(H) = 1$, then the Sylow p-subgroup of $F(H)$ is central in H.*

Proof. Assume $U_p(H) = 1$, and let S denote the Sylow p-subgroup of $F(H)$. By Fact 2.3, S is the product of a finite p-subgroup and of a p-torus. As each of these two subgroups is normal in H, each is central in H, by Facts 1.1 and 2.7 (1) respectively. □

Following [FJ08, §4-5] there are nice links between Carter $\tilde{\pi}$-groups and covering properties in connected solvable groups of finite Morley rank, the so-called connected subformation theory. In particular one knows that the collection $\mathcal{N}$ of connected nilpotent groups of finite Morley rank is a connected subformation. The main link between Carter subgroup theory and subformation theory in connected solvable groups is then a guarantee that Carter subgroups of a connected solvable group G of finite Morley rank are $\mathcal{N}$-covering subgroups of G; we refer to the important result [FJ08, Proposition 5.1]. The properties of $\mathcal{N}$-covering subgroups and $\mathcal{N}$-projectors which interest us are that these groups cover all nilpotent connected sections containing them.

We refer to [FJ08, Theorem 5.8] for conjugacy theorems. We note that when $\tilde{\pi}$ is a single unipotence parameter, Carter $\tilde{p}$-subgroups coincide with Sylow $\tilde{p}$-subgroups [FJ08, §3.2], so that Sylow $\tilde{p}$-subgroups are conjugate in connected

solvable groups of finite Morley rank. There is also structural information concerning Carter $\tilde{\pi}$-subgroups of connected solvable groups of finite Morley rank [FJ08, Corollary 5.9]; we will use this only with $\tilde{\pi} = \{\tilde{p}\}$ here.

2.4 Genericity

Lemma 2.15. *Let H be a connected solvable group of finite Morley rank generically covered by a uniformly definable family of finite subgroups. Then H is nilpotent and of bounded exponent.*

Proof. We first note that any group generically covered by a uniformly definable family of finite groups is generically of bounded exponent. In fact, by elimination of infinite quantifiers [BC02, Proposition 2.2], there is a uniform bound on the cardinals of the finite groups involved.

Now $H/F^\circ(H)$ is divisible abelian by [Nes90]. As Prüfer p-ranks are finite for each prime p, there is a finite subgroup of $H/F^\circ(H)$ containing all images modulo $F^\circ(H)$ of the finite groups. This shows by generic covering that $H/F^\circ(H)$ is trivial. Hence H is nilpotent. Now it suffices to use the generic covering again and Fact 2.3 (2) with [FJ08, Lemma 2.13 (1)]. □

The following lemma has its roots in [Jal00, Lemme 2.13] (see [CJ04, Fact 2.36]).

Lemma 2.16. *Let G be a connected group of finite Morley rank and X a nonempty definable G-invariant subset of G. If M is a definable subgroup of G such that $X \cap M$ is generic in X, then $X \cap M$ contains a definable G-invariant subset generic in X.*

Proof. By assumption X is a union of G-conjugacy classes. By assumption also, $X \cap M$ is nonempty.

Let Y_1 be a definable generic subset of $X \cap M$ consisting of elements of $X \cap M$ whose G-conjugacy classes have traces on $X \cap M$ of constant ranks. Let Y_2 be a definable generic subset of Y_1 consisting of elements of Y_1 whose G-conjugacy classes in G have constant ranks. Both exist as we have, by definability of the rank, finite definable partitions in each case. Now Y_2 is generic in Y_1 which is generic in $X \cap M$, so Y_2 is generic in $X \cap M$ and in X. Replacing X by Y_2^G, one can thus assume that G-conjugacy classes in G of elements of X, as well as their traces on M, are of constant ranks. We also have then that $x^G \cap M$ is nonempty for any x in X.

Now, as X is the union of the G-conjugacy classes of its elements in $X \cap M$ and since we have reduced to the situation where all relevant ranks are constant, the assumption that $X \cap M$ is generic in X implies easily by additivity of the rank that $x^G \cap M$ is generic in x^G for any x in X.

Let $N = \bigcap_{g \in G} M^g$. By descending chain condition on definable subgroups, $N = M^{g_1} \cap \cdots \cap M^{g_n}$ for finitely many elements g_1, ..., g_n of G. As G is connected, x^G, which is in definable bijection with $G/C(x)$, has Morley degree

1 for any x in X. By taking conjugates one also has $x^G \cap M^{g_i}$ generic in x^G for each x in X and each g_i. Hence $x^G \cap N$, which can be written as

$$(x^G \cap M^{g_1}) \cap \cdots \cap (x^G \cap M^{g_n}),$$

is also generic in x^G, for any x in X. Now the fact that all ranks involved are constant implies that $X \cap N$ is generic in X as well.

But $X \cap N$ is G-invariant as both sets involved are. Hence $X \cap N$ is the desired definable G-invariant subset of $X \cap M$ generic in X. □

3 $*$-Locally solvable groups

3.1 Fondations

Definition 3.1. *We say that a group of finite Morley rank is*

(1) $*$-locally solvable *if $N(A)$ is solvable for each nontrivial definable abelian subgroup A.*

(2) $*$-locally$_\circ$ solvable *if $N(A)$ is solvable for each nontrivial definable abelian connected subgroup A.*

(3) $*$-locally$^\circ$ solvable *if $N^\circ(A)$ is solvable for each nontrivial definable abelian subgroup A.*

(4) $*$-locally$^\circ_\circ$ solvable *if $N^\circ(A)$ is solvable for each nontrivial definable abelian connected subgroup A.*

Lemma 3.2. *Let G be a group of finite Morley rank.*

(1) *If G satisfies one of the Definitions 3.1* (1), (2), (3), *or* (4), *then so does any definable subgroup of G.*

(2) *If G is $*$-locally solvable, then it is $*$-locally$_\circ$ solvable and $*$-locally$^\circ$ solvable, and if G has any of the two latter properties, then it is $*$-locally$^\circ_\circ$ solvable.*

Proof. Obvious. □

Definition 3.3. *Let G be a group of finite Morley rank and H a subgroup of G. We say that a subgroup L of G is*

(1) H-local *if $L \leq N(H)$.*

(2) H-local$^\circ$ *if $L \leq N^\circ(H)$.*

Then we say that a subgroup L is *local* if it is H-local for some subgroup H, and *local*$^\circ$ if it is H-local$^\circ$ for some subgroup H. We can give a priori stronger conditions which turn out to be equivalent to Definitions 3.1 (1)–(4) in terms of local subgroups.

Lemma 3.4. *Let G be a group of finite Morley rank. Then G is*

(1) *$*$-locally solvable if and only if X-local subgroups are solvable for every nontrivial solvable subgroup X.*

(2) *$*$-locally$_\circ$ solvable if and only if X-local subgroups are solvable for every infinite solvable subgroup X.*

(3) *$*$-locally$^\circ$ solvable if and only if X-local$^\circ$ subgroups are solvable for every nontrivial solvable subgroup X.*

(4) *$*$-locally$^\circ_\circ$ solvable if and only if X-local$^\circ$ subgroups are solvable for every infinite solvable subgroup X.*

Proof. Clearly the right conditions are stronger than the left ones.

Assume now a left condition, and suppose X is some nontrivial solvable subgroup of G, and L is an X-local subgroup, i.e., $L \leq N(X)$. Then L normalizes the definable hull $H(X)$ of X, and its connected component $H^\circ(X)$ as well. Now a classical corollary of Zilber's generation lemma on derived subgroups ([BN94, Corollary 5.29]) implies that the last nontrivial term of the derived series of $H(X)$, as well as that of $H^\circ(X)$, is definable. It is abelian by definition, and as it is characteristic in $H(X)$ (resp. $H^\circ(X)$), it is normalized by L. Then one sees in each case which has to be considered that the latter is solvable by the left condition. □

Nontrivial solvable groups H of finite Morley rank contain some nontrivial definable characteristic Sylow $\tilde{p}$-subgroups or Sylow p-subgroups, by Fact 2.3 applied in $F(H)$. Hence H-local subgroups normalize nontrivial $\tilde{p}$-groups or p-groups, so that our definitions are consistent with the notion due to Alperin of a local subgroup in finite group theory [Tho68], as a subgroup normalizing some nontrivial p-subgroup. Before stating this a little bit more precisely in the $*$-locally$^\circ_\circ$ solvable case, we look at quotients.

Lemma 3.5. *Let G be a group of finite Morley rank and N a definable normal solvable subgroup.*

(1) *If G is $*$-locally solvable, then so is G/N.*

(2) *If G is $*$-locally$_\circ$ solvable, then so is G/N.*

(3) *If G is $*$-locally$^\circ$ solvable, then so is G/N.*

(4) *If G is $*$-locally$^\circ_\circ$ solvable, then so is G/N.*

Proof. We denote by $\overline{G}$ the quotient by N.

(1). Let $\overline{A}$ be a nontrivial definable abelian subgroup of $\overline{G}$. The preimage of $N_{\overline{G}}(\overline{A})$ normalizes AN, which is solvable and nontrivial, and hence it is solvable by $*$-local solvability of G. As N is solvable, $N_{\overline{G}}(\overline{A})$ is also solvable.

(2). One can proceed as in (1), taking A infinite modulo N, and looking at the normalizer of $(AN)^\circ$.

(3). One can proceed as in (1), taking connected components of normalizers throughout.

(4). It suffices to mix the two preceding cases. □

We continue with trivial remarks. In a group of finite Morley rank we call a *Borel* subgroup any maximal definable connected solvable subgroup.

Lemma 3.6. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Then a subgroup B is a Borel subgroup if and only if B is a maximal X-local$^\circ$ subgroup for some infinite solvable subgroup X. Furthermore X can be chosen to be any, and has to be some, infinite normal subgroup of B.*

Proof. Let B be a Borel subgroup of G. Then $B \leq N^\circ(X) \leq N^\circ(H(X))$ for any infinite normal subgroup X of B, and $N^\circ(H(X))$ is solvable by $*$-local$^\circ_\circ$ solvability of G. Hence we get equality by maximality of B, and B is a connected X-local$^\circ$ subgroup. If B is contained in a Y-local$^\circ$ subgroup L of G for some infinite solvable subgroup Y, then $B \leq N^\circ(H(Y))$ and as $N^\circ(H(Y))$ is solvable by $*$-local$^\circ_\circ$ solvability one gets $B = N^\circ(H(Y))$ again by maximality of B. As $B \leq L \leq N^\circ(H(Y))$, $B = L$.

Let now B be a maximal X-local$^\circ$ subgroup of G for some infinite solvable subgroup X. By $*$-local$^\circ_\circ$ solvability B is contained in a Borel subgroup B_1. Now B_1 is Y-local$^\circ$ for some infinite solvable subgroup Y, and the maximality of B implies that $B = B_1$.

Now if a Borel subgroup B normalizes an infinite solvable subgroup X, then $X \cdot B$ is solvable, as well as its definable hull, and by maximality $H^\circ(X) \leq B$, and $H^\circ(X)$ is an infinite normal subgroup of B. □

Lemma 3.7. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Then the following are equivalent.*

(1) *$N^\circ(A) < G^\circ$ for each nontrivial definable connected abelian subgroup A.*

(2) *G° is not solvable.*

(3) *G° has two distinct Borel subgroups.*

Proof. If G° is solvable, then $G^\circ \leq N^\circ(A)$ where A is the last nontrivial term of the derived series of G, which is definable and connected by [BN94, Corollary 5.29]. Hence the first condition implies the second one.

If G° has a unique Borel subgroup B, then $G^\circ = N^\circ(B)$ is solvable by $*$-local$^\circ_\circ$ solvability, so the second condition implies the third one.

Finally, assume $G^\circ = N^\circ(A)$ for some nontrivial definable connected abelian subgroup A. By $*$-local$^\circ_\circ$ solvability G° is then solvable, and hence cannot have two distinct Borel subgroups. Hence the last condition implies the first one. □

Lemma 3.7 can be refined as follows in the $*$-locally$^\circ$ solvable case.

Lemma 3.8. *Let G be a $*$-locally$^\circ$ solvable group of finite Morley rank. Then the following are equivalent.*

(1) *$N^\circ(A) < G^\circ$ for each nontrivial definable abelian subgroup A of G.*

(2) G° *is not solvable.*

(3) G° *has two distinct Borel subgroups.*

Proof. As in Lemma 3.7. If $N^\circ(A) = G^\circ$ for some nontrivial definable abelian subgroup A of G, then G° is now solvable by $*$-local$^\circ$ solvability. □

In PSL_2, connected normalizers of unipotent subgroups correspond to Borel subgroups. The following is a first approximation of this in $*$-locally$^\circ_\circ$ solvable groups.

Lemma 3.9. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Assume that for q prime or infinite $d_q(G) \geq 1$, and let U be a Sylow $(q, d_q(G))$-subgroup of G. Then $N^\circ(U)$ is a Borel subgroup of G.*

Proof. By $*$-local$^\circ_\circ$ solvability of G, $N^\circ(U) \leq B$ for some Borel subgroup B. Now Fact 2.8 implies $U \leq F^\circ(B)$, and in particular $B \leq N^\circ(U)$ by maximality of U. Hence $N^\circ(U) = B$ is a Borel subgroup of G. □

Lemma 3.10. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and B a Borel subgroup of G such that $d_p(B) = r$. Then $U_{\tilde{p}}(B)$ is a Sylow $\tilde{p}$-subgroup of G.*

Proof. By Fact 2.8, $U := U_{\tilde{p}}(B)$ is included in $F^\circ(B)$, and in particular is a $\tilde{p}$-group. It is obviously definably characteristic in B. If $U < V$ for some Sylow $\tilde{p}$-subgroup V of G, then $U < U_{\tilde{p}}(N_V(U))$ by normalizer condition, [FJ08, Proposition 2.8]. But as $N^\circ(U)$ is solvable by $*$-local$^\circ_\circ$ solvability of G, and contains B, it is B by maximality of B. Hence $U < U_{\tilde{p}}(N_V(U)) \leq U_{\tilde{p}}(B) = U$, a contradiction. □

When $r = 0$ Lemma 3.10 fails. For example, in the standard Borel subgroup B of PSL_2 over a pure algebraically closed field of positive characteristic, $U_{(\infty,0)}(B) = B$. However the lemma becomes true for $r = 0$ if one assumes that the absolute unipotence degree of B satisfies $d(B) = 0$.

3.2 Semisimple groups

Obviously with $*$-locally solvable groups one becomes quickly interested in normal solvable subgroups.

Fact 3.11. [BN94, Theorem 7.3] *Let G be a group of finite Morley rank. Then G has a largest normal solvable subgroup, which is definable. It is denoted by $R(G)$ and called the* solvable radical *of G.*

Definition 3.12. *Let G be a group of finite Morley rank. We say that*

(1) *G is* semisimple *if $R(G) = 1$, or equivalently if $N(A) < G$ for each nontrivial abelian subgroup A of G.*

(2) *G is* semisimple$^\circ$ *if $R^\circ(G) = 1$, or equivalently if $N(A) < G$ for each nontrivial connected abelian subgroup A of G.*

Of course, if G is any group of finite Morley rank, then $G/R(G)$ is semisimple and $G/R^\circ(G)$ is semisimple$^\circ$, as solvable-by-solvable groups are solvable. The following fact has remained implicit in previous arguments, and we just state it precisely.

Fact 3.13. *Let G be a connected group of finite Morley rank with $R(G)$ finite. Then $R(G) = Z(G)$ and $G/R(G)$ is semisimple.*

Proof. The connected group G acts by conjugation on its finite solvable radical $R(G)$, and thus by Fact 1.1 $R(G) \leq Z(G)$. As the center is always contained in the solvable radical one gets $R(G) = Z(G)$. The semisimplicity of $G/R(G)$ is always true. □

Lemma 3.14. *Let G be a group of finite Morley rank and H a nonsolvable definable connected subgroup of G.*

(1) *If G is $*$-locally$^\circ_\circ$ solvable, then H is semisimple$^\circ$, $R(H) = Z(H)$ is finite and $H/R(H)$ is semisimple.*

(2) *If G is $*$-locally$^\circ$ solvable, then H is semisimple.*

Proof. This is obvious by definitions and Fact 3.13. □

3.3 New configurations

All the work concerning minimal connected simple groups of finite Morley rank generalizes identically to the case of $*$-locally$^\circ$ solvable groups of finite Morley rank. The reason is that in the study of minimal connected simple groups every argument is based on the consideration of connected normalizers of nontrivial subgroups X. If such a subgroup X is finite, then its connected normalizer coincides with its connected centralizer by Fact 1.1.

When dealing with the more general class of $*$-locally$^\circ_\circ$ solvable groups, connected centralizers of elements of finite order might be nonsolvable. In the present papers we try to concentrate exclusively on the more general class of $*$-locally$^\circ_\circ$ solvable groups, and hence new phenomena can appear. In the present section we try to give an overview of the new pathological configurations which might occur in this context. We see these new configurations as some kind of "speed limits" when generalizing arguments from the minimal connected simple/$*$-locally$^\circ$ solvable case to the more general $*$-locally$^\circ_\circ$ solvable case.

Recall from Lemma 3.2 that

$$\{*\text{-locally}^\circ \text{ solvable groups}\} \subsetneq \{*\text{-locally}^\circ_\circ \text{ solvable groups}\},$$

the inclusion being strict. The main (and only) example in the algebraic category of a connected group which is $*$-locally$^\circ_\circ$ solvable but not $*$-locally$^\circ$ solvable is $\mathrm{SL}_2(K)$, with K an algebraically closed field of characteristic different from 2: its solvable radical consists of a cyclic group of order 2.

In the context of groups of finite Morley rank there might a priori be other configurations occuring, contradicting even the property that the solvable radical is cyclic of order 2. In what follows we just make a list of potential pathological configurations of connected $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank which are not $*$-locally$^\circ$ solvable, and which remain at the end of our classification.

A *full Frobenius* group is a group G with a proper subgroup H such that

$$H \text{ is malnormal in } G \text{ and } G = H^G.$$

The existence of such groups of finite Morley rank is the main obstacle to the Algebraicity Conjecture for simple groups of finite Morley rank. We refer to [Jal01a, Propositions 3.3 and 3.4] for basic properties of such groups, if they exist. Recall, in particular, that H is automatically definable and connected whenever G is connected. We often call such a group, with G connected and H nilpotent, a *bad* group — this definition is slightly more general than the one given in [BN94, Chapter 13], but given the analog full Frobenius structure it leads to it is much more natural to adopt the present general definition here. In any case these groups have no involutions, and hence their torsion can involve only odd primes.

We view the following potential configuration of $*$-locally$^\circ_\circ$ solvable group, or any of its natural variations, as a kind of "universal conterexample" to the algebraic case as far as torsion is concerned. Elements belonging to a decent torus are called *toral*.

Configuration 3.15. *G is a connected $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with a proper subgroup H such that*

(1) *G is a full Frobenius group, with malnormal subgroup H (which is automatically definable and connected by [Jal01a, Proposition 3.3]).*

(2) *$R(H) = Z(H)$ is finite and nontrivial, consisting of p-toral elements of H for some prime p.*

(3) *$H/Z(H)$ is a full Frobenius group for some proper definable connected solvable malnormal subgroup $B/Z(H)$.*

(4) *$B/Z(H)$ has nontrivial p-unipotent subgroups, for some prime p dividing $|Z(H)|$, and also nontrivial q-unipotent subgroups for other primes q.*

A group G as in Configuration 3.15 would have *p-mixed type*, i.e., containing both nontrivial p-tori and p-unipotent subgroups, and have nontrivial q-unipotent subgroups for several primes q.

In SL_2, a generic element belongs to a maximal torus, and in particular to the connected component of its centralizer. Here is another potential new pathological phenomenon in the $*$-locally$^\circ_\circ$ solvable setting.

Configuration 3.16. *G is a connected $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with a proper subgroup B such that*

(1) *G is a full Frobenius group, with malnormal subgroup B (which is automatically definable and connected by [Jal01a, Proposition 3.3]).*

(2) *B is a nilpotent group such that, for x generic in B, $x \notin C^\circ(x)$.*

A generic element x of a group G as in Configuration 3.16 would satisfy $x \notin C^\circ(x)$. We note that examples of connected nilpotent groups B of finite Morley rank as in clause (2) of Configuration 3.16 are provided by [BN94, §3.2.3] or the Baudisch 2-nilpotent group [Bau96]. With such subgroups B a group G as in Configuration 3.16 would be $*$-locally$^\circ$ solvable, but if G had the prescribed property modulo a nontrivial finite center, then it would not be $*$-locally$^\circ$ solvable.

Even with involutions and algebraic subgroups one can imagine the following configuration which seems to remain open at the end of [DJ07a].

Configuration 3.17. *G is a connected $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with an involution i such that $C(i) < G$ and $C(i) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field K of characteristic different from 2.*

In [CJ04] all nonalgebraic configurations are known to have nongenerous Borel subgroups. Even assuming all Borel subgroups generous does not seem to be helpful in [DJ07a] toward finding a contradiction in Configuration 3.17. This is a major new phenomenon possibly occuring in the $*$-locally$^\circ_\circ$ solvable case as opposed to the minimal connected simple/$*$-locally$^\circ$ solvable one.

3.4 $*$-Local$^\circ$/$*$-Local$^\circ_\circ$ solvability

In Section 3.3 we saw certain speed limits when considering immediate generalizations to the wider class of $*$-locally$^\circ_\circ$ solvable groups, which usually rely on the existence of certain semisimple$^\circ$ but not semisimple groups. We nevertheless intend in this section to start dealing with these aspects in the general class of $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank, bearing in mind the speed limits of Section 3.3. For this purpose it is useful to study systematically subgroups of the form $C^\circ(x)$ in $*$-locally$^\circ_\circ$ solvable groups. When such a subgroup is not solvable it has a finite solvable radical, which is then the center, and its quotient modulo the center is semisimple. This boils down to the study of semisimple $*$-locally$^\circ_\circ$ solvable groups.

We begin with some generalities.

Lemma 3.18. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. If H is a nonsolvable definable connected subgroup of G, then $C_G(H)$ is finite.*

Proof. Assume $C^\circ_G(H)$ infinite. Then it contains a nontrivial definable connected solvable subgroup B by Lemma 2.6. We then have $H \leq C^\circ(B) \leq N^\circ(B)$, which must be solvable by $*$-local$^\circ_\circ$ solvability of G. □

In a $*$-locally$^\circ_\circ$ solvable group G of finite Morley rank, we call a subset X *exceptional* in G if $C^\circ(X)$ is nonsolvable. Such sets are finite by Lemma 3.18, and as $C(X) = C(\langle X \rangle)$ any such subset X can always be identified with the

finite subgroup it generates. Dually, if H is a nonsolvable definable connected subgroup in G, then $C(H)$ centralizes the nonsolvable definable connected subgroup H and is an exceptional subset of G.

We denote by $\mathcal{E}_F$ and $\mathcal{E}_S$ the set of finite exceptional subgroups of G and the set of nonsolvable definable connected subgroups respectively ($\mathcal{E}$ stands for "e"xceptional, F for "f"inite, and S for "s"emisimple). Both sets are nonempty if and only if G° is nonsolvable. Of course both sets are naturally ordered by inclusion.

Taking connected centralizers $C^\circ(\cdot)$ from $\mathcal{E}_F$ to $\mathcal{E}_S$ and centralizers $C(\cdot)$ from $\mathcal{E}_S$ to $\mathcal{E}_F$ defines a *Galois connection* between $\mathcal{E}_F$ and $\mathcal{E}_S$ (see [Bir67]). That is, and following a similar exposition in [ABC08], they satisfy the following properties.

Lemma 3.19.

(1) *The mappings C° and C are order-reversing.*

(2) *If $X \in \mathcal{E}_F$ then $X \leq C(C^\circ(X))$ and if $H \in \mathcal{E}_S$ then $H \leq C^\circ(C(H))$.*

As in any Galois connection, this has the following consequence.

Proposition 3.20. *Let $X \in \mathcal{E}_F$ and $H \in \mathcal{E}_S$. Then $C^\circ(X) = C^\circ(C(C^\circ(X)))$ and $C(H) = C(C^\circ(C(H)))$.*

If we denote for X in $\mathcal{E}_F$ and H in $\mathcal{E}_S$

$$\overline{X} = C(C^\circ(X)) \text{ and } \overline{H} = C^\circ(C(H)),$$

then the two operations $\overline{\ }$ are *closure operations* on $\mathcal{E}_F$ and $\mathcal{E}_S$ respectively. That is, they satisfy the following.

Corollary 3.21.

(1) *For $X \in \mathcal{E}_F$ and $H \in \mathcal{E}_F$, we have $X \leq \overline{X} = \overline{\overline{X}}$ and $H \leq \overline{H} = \overline{\overline{H}}$.*

(2) *Monotonicity: For $X_1 \subseteq X_2$ in $\mathcal{E}_F$ and $H_1 \leq H_2$ in $\mathcal{E}_S$, we have $\overline{X_1} \leq \overline{X_2}$ and $\overline{H_1} \leq \overline{H_2}$.*

The closed elements of $\mathcal{E}_F$ and $\mathcal{E}_S$ are those of the form $\overline{X}$ and $\overline{H}$ respectively.

One can also refine Lemma 3.18 by giving a uniform bound on cardinals of elements of $\mathcal{E}_F$. We first note the following general combination of [BN94, Corollary 5.17] and of the elimination of infinite quantifiers [BC02, Proposition 2.2].

Lemma 3.22. *Let G be a group of finite Morley rank. Then there exists a natural number m such that, for any subset X of G, $|C(X)| \leq m$ or $C(X)$ is infinite.*

Lemma 3.23. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Then there exists a natural number m bounding uniformly the cardinals of finite exceptional subsets of G.*

Proof. Let m be as in Lemma 3.22. If X is a finite exceptional subset of G, then $C^\circ(X)$ is nonsolvable and $C(C^\circ(X))$ is finite by Lemma 3.18. As $C(C^\circ(X))$ is a finite centralizer, its cardinal is uniformly bounded by m. Now $X \subseteq C(C^\circ(X))$, and thus the cardinal of X is uniformly bounded by m. □

If G is a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, we call *exception index* and denote by $e(G)$ the maximal integer m such that G has a nonsolvable definable connected subgroup centralizing a subset X with m elements. Notice that X coincides with $\langle X \rangle$, so that $e(G)$ is the largest cardinal of an exceptional subgroup in $\mathcal{E}_F$.

Maximal exceptional subgroups of $\mathcal{E}_F$ correspond to minimal nonsolvable definable connected subgroups of $\mathcal{E}_S$, and vice-versa. A case of particular interest is the following.

Lemma 3.24. *Minimal nontrivial exceptional subgroups are cyclic of prime order.*

Proof. Obvious. □

One can clarify the structure of elements of $\mathcal{E}_S$ as follows.

Lemma 3.25. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank and H a nonsolvable definable connected subgroup. Then H is semisimple$^\circ$, $R(H) = Z(H)$ is finite and $H/R(H)$ is semisimple.*

Proof. Lemma 3.14 (1). □

Of course, the sets of closed sets in $\mathcal{E}_F$ and $\mathcal{E}_S$ are at most reduced to $\{1\}$ and $\{G^\circ\}$ in the $*$-locally$^\circ$ solvable case (in case G° is nonsolvable, and empty otherwise).

The next lemma seems to be the only way to get $*$-locally$^\circ$ solvable groups out of $*$-locally$^\circ_\circ$ solvable ones.

Lemma 3.26. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank and H a nonsolvable definable connected subgroup in G, which is minimal with respect to this property. Then $H/R(H)$ is $*$-locally$^\circ$ solvable.*

Proof. Denote by $\overline{\ }$ the quotients by $R(H)$, and let A be the preimage in H of a nontrivial definable abelian subgroup $\overline{A}$ of $\overline{H}$. Of course A is a definable solvable subgroup of H. Let N be the preimage in H of $N_{\overline{H}}(\overline{A})^\circ$. We have $N \leq N_H(A)$ and $\overline{N} = \overline{N}^\circ = \overline{N^\circ}$. As $R(H) < A$, $N^\circ \leq N_H(A) < H$. Hence N° is solvable by minimality of H, as well as $\overline{N}$. □

Lemma 3.26 seems to be a very rough indication that the new $*$-locally$^\circ_\circ$ solvable groups which are not $*$-locally$^\circ$ solvable are more or less as in Configuration 3.15.

We also note that nonsolvable definable connected subgroups attached to a nontrivial finite exceptional subgroup are of finite index in their normalizers.

Lemma 3.27. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. If X is an exceptional finite subset of G, then $N^\circ(C^\circ(X)) = C^\circ(X)$.*

Proof. $N^\circ(C^\circ(X))$ normalizes $C(C^\circ(X))$, which is finite and contains X. So it centralizes X by Fact 1.1, and we are done. □

In Lemma 3.27 we even have a uniform bound on the orders of the groups $N(C^\circ(X))/C^\circ(X)$, because these groups are uniformly definable.

A natural question is to know whether exceptional finite subsets X are contained in their attached nonsolvable definable connected subgroups, i.e., whether $X \subseteq C^\circ(X)$. This would follow from the more general, but similarly natural, question to know whether nonsolvable definable connected subgroups are self-normalizing. This is the kind of problem which seems optimistically trackable when $C^\circ(X)$ is generous in the ambient group since the intensive experience on Weyl groups from [CJ04], and we refer to [Jal09] for positive answers in the most interesting situations.

We are now going to look more closely at the interesting case in which an exceptional finite subgroup X of $\mathcal{E}_F$ satisfies $X \leq C^\circ(X)$. In this case $X \leq Z(C^\circ(X))$, and X is in particular an abelian finite subgroup. Typical finite abelian groups belonging to the connected component of their centralizers are the finite subgroups of decent tori. (And this is in general not true around groups of bounded exponent, as noticed after Configuration 3.16.)

Lemma 3.28. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank and T a maximal definable decent torus of G. Then the union of elements of $\mathcal{E}_F$ contained in T is finite and invariant under any automorphism of G leaving T invariant.*

Proof. For the finiteness we can use Lemma 3.23 to get a uniform bound, at most the exception index $e(G)$ of G, on the cardinals of the finite groups involved. Then, as Prüfer p-ranks are finite for any prime p in a decent torus, subgroups of order at most $e(G)$ must be contained in a finite subgroup of T.

The second point is obvious. □

A question, which might be difficult, is to know whether the union in Lemma 3.28 is necessarily a (finite) subgroup of T, and is itself exceptional. If this were the case, then calling this group E, one would have a nonsolvable group $C^\circ(E)/R(C^\circ(E))$ where nontrivial toral elements are not exceptional anymore. This is a desirable property for certain questions such as bounding Prüfer ranks in our treatment of odd type groups in [DJ07a] via [DJ07b]. This desirable property can however be obtained as follows.

Lemma 3.29. *Let G be a connected nonsolvable $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, and T a maximal definable decent torus of G. Then G has a nonsolvable definable connected subgroup H containing T and such that $C^\circ(\bar{t})$ is solvable for any nontrivial toral element $\bar{t}$ of $H/R(H)$.*

Proof. Let X be a maximal exceptional finite subgroup of T. Then $H = C^\circ(X)$ is nonsolvable. As $X \leq T$ and T is abelian and connected, $X \leq T \leq H$.

Let now $\bar{t}$ be a nontrivial toral element of $\overline{H} = H/R(H)$. By pullback of decent tori, [FJ08, Lemma 2.13 (2)] or rather [Fré06b, Lemma 3.1], and

[Che05], we may assume t in $TR(H)$, i.e., $t = t'r$ for some $t' \in T$ and some $r \in R(H)$. As in Lemma 3.26, or with Fact 1.1, one sees that the preimage of the connected centralizer of t modulo $R(H)$ cannot be nonsolvable: otherwise its connected component would centralize $t = t'r$, and as $r \in R(H) = Z(H)$ it would centralize t', so that $X\langle t'\rangle$ would be an exceptional finite subgroup of T containing X properly, a contradiction. This finishes our proof. □

Before moving ahead we close the present section by describing more precisely the set of exceptional subsets of a decent torus T as in Lemma 3.29, or more generally of an *arbitrary* subset T of a $*$-locally$^\circ_\circ$ solvable group G of finite Morley rank.

First we naturally consider the notion of closure relative to T. For X an exceptional subset of T, we say that X is closed *in* T if $X = C_T(C^\circ(X))$. Of course the notion of relative closedness is robust.

Remark 3.30. *Any set of the form $C_T(C^\circ(X))$ is closed in T.*

Proof. As $X \subseteq C_T(C^\circ(X)) \subseteq \overline{X}$, $\overline{X} = C(C^\circ(C_T(C^\circ(X))))$ by taking the closure in G, and $C_T(C^\circ(C_T(C^\circ(X)))) = C_T(C^\circ(X))$ by taking the intersection with T. □

The poset of exceptional subsets of T is best described as follows by the notion of minimal extensions of closed subsets. We say that (X_1, X_2) is a *minimal* extension of closed sets of T if $X_1 \subsetneq X_2$ are two exceptional subsets of T closed in T and any closed subset Y of T such that $X_1 \subseteq Y \subseteq X_2$ is either X_1 or X_2. The relation "(X_1, X_2) is a minimal extension of closed sets of T" defines an oriented graph on the set of closed sets of T, which is clearly irreflexive, antisymmetric, and loop-free, that is without cycles preserving the orientation (but possibly with cycles not preserving the orientation). We call this graph the *graph of exceptional subsets of T*. Its main properties are the following.

Lemma 3.31. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank and T an arbitrary subset of G.*

(1) *Assume (X_1, X_2) is a minimal extension in the graph of exceptional subsets of T and Y is a subset such that $X_1 \subsetneq Y \subseteq X_2$. Then $C_T(C^\circ(Y)) = X_2$. Moreover $C^\circ(X_2) < C^\circ(X_1)$.*

(2) *Assume (X, X_1) and (X, X_2) are two minimal extensions in the graph of exceptional subsets of T. Then either $X_1 = X_2$ or $X_1 \cap X_2 = X$.*

Proof. (1). $X_1 \subsetneq Y \subseteq C_T(C^\circ(Y)) \subseteq X_2$ and as $C_T(C^\circ(Y))$ is closed in T by Remark 3.30 it must be X_2 by minimality of the extension (X_1, X_2).

The claim that $C^\circ(X_2) < C^\circ(X_1)$ follows merely from the fact that $X_1 \neq X_2$ are closed in T.

(2). Let $Y = X_1 \cap X_2$. If $X \subsetneq Y$, then the first point implies that $X_1 = C_T(C^\circ(Y)) = X_2$. □

Finally, we note that the graph of exceptional subsets of T as in Lemma 3.31 always has a "minimal" element, namely $T \cap Z(G)$, and "maximal" elements,

corresponding to the maximal traces on T of exceptional sets in $\mathcal{E}_F$, which are of cardinal at most $e(G)$. We also note that the graph has a *finite height*: the length of a maximal chain of exceptional closed sets in T is at most $e(G)$.

When T is a nilpotent divisible subgroup of G (for example as in Lemmas 3.28 and 3.29), then exceptional subsets of T are necessarily in a same decent torus (the maximal decent torus of the definable hull of T) and by Lemma 3.28 applied in this decent torus the graph of exceptional subsets of T is finite.

3.5 Genericity

Fact 3.32. (see [FJ08, Theorem 7.3]) *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with a nontrivial decent torus T, and Q a Carter subgroup of G containing T. Then Q is generous in G°, and $T \leq \tilde{T} \leq Q$ for some maximal definable decent torus $\tilde{T}$ of G.*

Proof. The existence of Q is guaranteed by [FJ08, Theorem 3.3], as decent tori are of minimal unipotence degree.

By [Che05], $C^\circ(T)$ is generous in G°. Now $C^\circ(T)$ is solvable by $*$-local$^\circ_\circ$ solvability of G, and the Carter subgroup Q is generous in $C^\circ(T)$ by [FJ08, Theorem 3.11]. It follows that Q is generous in G° by the transitivity of generosity provided in [Jal06, Lemma 3.9].

Doing the same argument as above for a maximal definable decent torus $\tilde{T}$ containing T, one gets a generous Carter subgroup $\tilde{Q}$ of G° containing $\tilde{T}$, and as generous Carter subgroups are conjugate by [Jal06] one gets that Q contains a maximal definable decent torus, which necessarily contains T. □

We record here an application of Lemma 2.16 in the case of $*$-locally solvable groups of finite Morley rank. This will be the *clé de voûte* for a concentration argument in one of the most prominent theorems on odd type groups in [DJ07a].

Lemma 3.33. (see [Del07a, Corollaire 2.4]) *Let G be a group of finite Morley rank and X a nonempty definable G°-invariant subset of G°. Let M be a definable solvable subgroup of G° such that $X \cap M$ is generic in X.*

(1) *If G is $*$-locally$^\circ$ solvable and $X \neq \{1\}$, then G° is solvable.*

(2) *If G is $*$-locally$^\circ_\circ$ solvable and X is infinite, then G° is solvable.*

Proof. (1). Let Y be the definable G°-invariant subset of $X \cap M$ generic in X provided by Lemma 2.16. As X is nonempty, Y is also nonempty, and $G^\circ = N^\circ(\langle Y \rangle)$. Now $\langle Y \rangle$ is a subgroup of M, and hence is solvable. If it is nontrivial, then G° must be solvable by Lemma 3.4 (3). Otherwise, $\{1\}$ is a generic subset of X, and X must be finite. Hence X is a finite set of finite conjugacy classes, with one nontrivial by assumption. This nontrivial finite G°-conjugacy class must be central in G° by Fact 1.1, and as G° has then a nontrivial center it must again be solvable by $*$-local$^\circ$ solvability.

(2). One argues in the same way. Now, as X is infinite, Y is also infinite by genericity. As $G^\circ = N^\circ(\langle Y \rangle)$ is $\langle Y \rangle$-local$^\circ$ with $\langle Y \rangle$ infinite and solvable, as it is contained in M, Lemma 3.4 (4) now gives the solvability of G°. □

4 Local analysis

We now proceed to the *local analysis* of $*$-locally solvable groups of finite Morley rank, that is the analysis of intersections of their (most interesting) subgroups.

In Section 4.1 we deal with a series of results which correspond to the Bender method in finite group theory. In general these lemmas say in our context that sufficiently unipotent subgroups of $*$-locally solvable groups of finite Morley rank are disjoint, like unipotent subgroups in PSL_2 or SL_2. They are the main tool for analyzing $*$-locally solvable groups, notably the only trick involving unipotence in the recognition of PSL_2 in the algebraic part of [DJ07a]. The original form was first proved, both in positive and null characteristic, in the context of minimal connected simple groups in the unpublished [Jal00]. Then they appeared in the tame context in [CJ04, Section 3.4] where they were treated essentially as the positive characteristic case, i.e., involving no particular graduation in the unipotence theory. The positive characteristic case was recalled as the outline of [Bur07] and later the characteristic 0 case, using Burdges' general unipotence theory in characteristic 0, appeared in [Del07a, §3.2] for the recognition of PSL_2 in characteristic different from 2. We are going to give forms of these lemmas which are entirely uniform in the unipotence degrees, in particular independent of the characteristics, and in the most general context of $*$-locally$^\circ_\circ$ solvable groups.

Section 4.2 will then concern the situation in which a particular consequence of such uniqueness theorems fails. This is a priori a possibility when the subgroups considered are not unipotent enough with respect to the ambient group. The pathological situation appearing can be analyzed somehow by replacing the maximality in terms of unipotence degrees by a maximality for inclusion concerning a pair of Borel subgroups involved. The endless, but very precise, resulting description is the bulk of [Bur07], and in the context of $*$-locally$^\circ_\circ$ solvable groups we will follow the exposition of this paper *verbatim*. This full description is applied one time in a nonalgebraic situation in [DJ07a] and this is why we need to restate, slightly more generally but in its full detail, the analysis from [Bur07].

4.1 Uniqueness Theorem

4.1.1 The main theorem

The following Uniqueness Theorem is our analog of the Bender method in groups of finite Morley rank and is the main tool for analyzing $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank. There are various forms of this theorem but the present one seems to be the most relevant, at least for our applications in [DJ07a]. Its consequences on Borel subgroups in Section 4.1.2 below will be the closest analogs of the Uniqueness Theorem of Bender in finite group theory [Ben70a] [Ben70b] [Gag76, §5-7] [BG94, Chapter II].

Theorem 4.1. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and U a Sylow $\tilde{p}$-subgroup of G.*

Assume that U_1 is a nontrivial definable $\tilde{p}$-subgroup of U containing a nonempty (possibly trivial) subset X of G such that $d_p(C^\circ(X)) \leq r$. Then U is the unique Sylow $\tilde{p}$-subgroup of G containing U_1, and in particular $N(U_1) \leq N(U)$.

Before the proof, a few remarks.

(1) If G° is solvable and $r = d_p(G) > 0$, then assuming that $F^\circ(G^\circ)$ has a nontrivial normal definable $\tilde{p}$-subgroup U_1 one gets with Theorem 4.1 (applied with $X = 1$ for example!) that G° has a unique Sylow $\tilde{p}$-subgroup, which is thus normal and contained in $F^\circ(G^\circ)$. In the event of the absence of such a subgroup U_1 one easily gets the same conclusion with Fact 2.7 (3) and (4). Hence in some sense Theorem 4.1 can be seen as a generalization from solvable groups to $*$-locally$^\circ_\circ$ solvable groups of Fact 2.8.

(2) The nontriviality of U_1 is needed in Theorem 4.1, as in a hypothetic $\tilde{p}$-homogeneous semisimple bad group the trivial subgroup would be contained in infinitely many conjugates of the Sylow $\tilde{p}$-subgroup.

(3) Theorem 4.1 fails if $\tilde{p} = (\infty, 0)$. For exemple if G is of the form $T \times (U \rtimes T)$, with U p-unipotent for some prime p and T a good torus, whose second copy acts faithfully on U, then $d_\infty(G) = 0$, so that all assumptions of Theorem 4.1 are satisfied with U_1 the central copy of T, but the latter is contained in infinitely many conjugates of the maximal good torus $T \times T$. We will give in Lemma 4.2 below a version of Theorem 4.1 specific for the unipotence parameter $\tilde{p} = (\infty, 0)$, by replacing $d_p(C^\circ(X))$ by the absolute unipotence degree $d(C^\circ(X))$ but with no more $*$-local solvability assumption.

After these comments we pass to the proof of Theorem 4.1.

Proof. Assume V is a Sylow $\tilde{p}$-subgroup of G distinct from U and containing U_1, and chosen so as to maximize the rank of $U_{\tilde{p}}(U \cap V)$. Let T denote $U \cap V$. As $U_1 \leq T$, the subgroup T is infinite. As T is nilpotent, $N := N^\circ(T)$ is solvable by $*$-local$^\circ_\circ$ solvability of G and Lemma 3.4 (4). Notice that $T < U$, as otherwise $U = (U \cap V) \leq V$ and $U = V$ by maximality of U. Similarly $T < V$, as otherwise $V = (U \cap V) \leq U$ and $V = U$ by maximality of V. In particular by normalizer condition, [FJ08, Proposition 2.8], $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_U(T))$ and $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_V(T))$.

We claim that $d_p(N) = r$. If $d_p(N) > r$, then $r < \infty$, $p = \infty$, and N contains a nontrivial Sylow (∞, r')-subgroup Σ with $r' > r$. Notice that $d_p(T) \leq r$ by Corollary 2.2 and Fact 2.3 and our assumption that the subset X of T satisfies $d_p(C(X)) \leq r$. Then $T \cdot \Sigma$ is nilpotent by Fact 2.7 (3) and (4), and T commutes with Σ by Fact 2.3. In particular Σ commutes with X and $d_p(C^\circ(X)) \geq r' > r$, a contradiction to our assumption. Hence $d_p(N) \leq r$, and as N contains $U_{\tilde{p}}(N_U(T))$ (or $U_{\tilde{p}}(N_V(T))$) which is nontrivial and of unipotence degree r we get $d_p(N) = r$.

By Fact 2.8 and the assumption that $r \geq 1$ we get $U_{\tilde{p}}(N) \leq F^\circ(N)$. In particular $U_{\tilde{p}}(N)$ is nilpotent, and contained in a Sylow $\tilde{p}$-subgroup Γ of G.

Now $U_1 \leq U_{\tilde{p}}(T) < U_{\tilde{p}}(N_U(T)) \leq \Gamma$, so our maximality assumption on V implies that $\Gamma = U$. In particular $U_{\tilde{p}}(N_V(T)) \leq \Gamma = U$. But then $U_{\tilde{p}}(T) < U_{\tilde{p}}(N_V(T)) \leq U_{\tilde{p}}(U \cap V) = U_{\tilde{p}}(T)$, a contradiction which finishes the proof of our first statement.

The inclusion $N(U_1) \leq N(U)$ follows from the uniqueness. □

We conclude with a version of Theorem 4.1 specific for the unipotence parameter $\tilde{p} = (\infty, 0)$, and which indeed does not rely on $*$-local solvability.

Lemma 4.2. *Let G be a group of finite Morley rank, T a maximal definable decent torus of G, and x an element of T such that $C^\circ(x)$ is solvable and $d(C^\circ(x)) \leq 0$. Then T is the unique maximal definable decent torus of G containing x, and in particular $N(\langle x \rangle) \leq N(T)$.*

Proof. By assumption and Lemma 2.6 (2), $C^\circ(x)$ is a good torus. As $x \in T$ and T is connected abelian, $T \leq C^\circ(x)$, and $T = C^\circ(x)$ by maximality of T. Now any maximal definable decent torus containing x must be in $C^\circ(x)$ for the same reason, hence in T, and hence equal to T by maximality of T. Furthermore, $N(\langle x \rangle) \leq N(C^\circ(x)) = N(T)$. □

4.1.2 Consequences on Borel subgroups

Applied to the case of Borel subgroups Theorem 4.1 has the following corollaries. These can be seen as absolute approximations in the context of $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank of the fact that any unipotent subgroup of PSL_2 belongs to a unique Borel subgroup of the ambient group.

Corollary 4.3. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$, and B a Borel subgroup of G such that $d_p(B) = r$. Let U_1 be a nontrivial definable $\tilde{p}$-subgroup of $U_{\tilde{p}}(B)$ containing a nonempty subset X such that $d_p(C^\circ(X)) \leq r$. Then $U_{\tilde{p}}(B)$ is the unique Sylow $\tilde{p}$-subgroup of G containing U_1, and in particular $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$. Furthermore, B is the unique Borel subgroup containing U_1 and admitting $\tilde{p}$ as a unipotence parameter maximal in its characteristic.*

Proof. The fact that $U_{\tilde{p}}(B)$ is a Sylow $\tilde{p}$-subgroup of G is Lemma 3.10. The uniqueness of $U_{\tilde{p}}(B)$ among Sylow $\tilde{p}$-subgroups containing U_1, as well as the inclusion $N(U_1) \leq N(U_{\tilde{p}}(B))$, is then Theorem 4.1.

Let now B_1 be a Borel subgroup of G containing U_1 and admitting $\tilde{p} = (p, r)$ as maximal in its characteristic. Notice that $U_{\tilde{p}}(B_1)$ is a Sylow $\tilde{p}$-subgroup of G by Lemma 3.10. As it contains U_1, Theorem 4.1 now implies $U_{\tilde{p}}(B_1) = U_{\tilde{p}}(B)$. Now the connected normalizers of these (equal) groups are solvable by $*$-local$^\circ_\circ$ solvability of G, contain B_1 and B respectively, hence are equal to B_1 and B respectively by maximality, and are equal. Hence $B_1 = B$, as desired for our second claim. □

PSL_2 in positive characteristic offers a counterexample to Corollary 4.3 when $r = 0$. It suffices to consider for U_1 a maximal torus of the standard Borel

subgroup B, so that $N(T) \not\leq N(B)$ and $T \leq B^w$ where w is a nontrivial Weyl group element associated to T. For the case $r = 0$ we refer to Lemma 4.2.

Corollary 4.3 takes the following form when (p, r) is maximal in its characteristic over the whole ambient group G.

Corollary 4.4. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$ such that $d_p(G) = r$. Let B be a Borel subgroup of G such that $d_p(B) = r$. Then $U_{\tilde{p}}(B)$ is a Sylow $\tilde{p}$-subgroup of G, and if U_1 is a nontrivial definable $\tilde{p}$-subgroup of B, then $U_{\tilde{p}}(B)$ is the unique Sylow $\tilde{p}$-subgroup of G containing U_1, $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$, and B is the unique Borel subgroup of G containing U_1.*

Proof. Let $X = 1$. Then $d_p(C^\circ(X)) = d_p(G) = r$, so Corollary 4.3 applies with $X = 1$. Notice that $\tilde{p}$ is maximal in its characteristic for any Borel subgroup admitting it, and that when U_1 is a nontrivial definable $\tilde{p}$-group then any Borel subgroup containing it admits $\tilde{p}$. □

Like for Corollary 4.3, PSL_2 in positive characteristic offers a counterexample when $r = 0$ in Corollary 4.4.

The preceding uniqueness theorems are often used as follows to "fusion" Borel subgroups sharing too unipotent subgroups.

Lemma 4.5. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Assume that, for $i = 1$ and 2, $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters with $r_i > 0$ and B_i are two distinct Borel subgroups of G such that $d_{p_i}(B_i) = r_i$. Then there is no Borel subgroup B_3 of G such that $d_{p_i}(B_i \cap B_3) = d_{p_i}(B_3) = r_i$ and $d_{p_i}(C^\circ(U_{\tilde{p}_i}(B_i \cap B_3))) \leq r_i$.*

Proof. Assume the contrary. Applying Corollary 4.3 with $U_1 = X = U_{\tilde{p}_1}(B_1 \cap B_3)$ implies that $B_1 = B_3$, and with $U_1 = X = U_{\tilde{p}_2}(B_2 \cap B_3)$ that $B_2 = B_3$. Hence $B_1 = B_2$, a contradiction. □

We finish with a version of Lemma 4.5 concerning the case in which the unipotence degrees r_i's are maximized over the whole ambient group.

Lemma 4.6. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Assume that, for $i = 1$ and 2, $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters with $r_i > 0$ and B_i are two distinct Borel subgroups of G such that $d_{p_i}(G) = d_{p_i}(B_i) = r_i$. Then there is no Borel subgroup B_3 of G such that $d_{p_i}(B_i \cap B_3) = r_i$.*

Proof. Under the stated assumptions $d_{p_i}(B_i) = r_i$. If there was a contradicting Borel subgroup B_3, then $d_{p_i}(B_i \cap B_3) = r_i = d_{p_i}(B_i)$ and $d_{p_i}(C^\circ(U_{\tilde{p_i}}(B_i \cap B_3))) \leq r_i$, a contradiction to Lemma 4.5. □

Again PSL_2 in positive characteristic offers counterexamples to Lemmas 4.5 and 4.6 when $r_i = 0$, as we may take for B_1 and B_2 two distinct conjugates of the standard Borel subgroup B and for B_3 any of these two.

4.1.3 Consequences on Fitting subgroups

The first paragraph of the proof of the following lemma appeared as [Bur07, Corollary 2.2].

Lemma 4.7. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. If B_1 and B_2 are two distinct Borel subgroups and X denotes $F(B_1) \cap F(B_2)$, then X° is torsion free, $X = X^\circ \times S$ for a finite subgroup S, and for any subgroup S_1 of X, $C^\circ(S_1)$ is nonsolvable if and only if $S_1 \leq S$.*

Proof. Assume X° not torsion free. Then it contains a nontrivial decent torus T or a nontrivial p-unipotent subgroup U. In the first case, $T \leq Z(B_1) \cap Z(B_2)$ by Fact 2.7 (1), $B_1 = N^\circ(T) = B_2$ by $*$-local$^\circ_\circ$ solvability and Lemma 3.6, a contradiction. In the second case Corollary 4.4 with $\tilde{p} = (p, \infty)$ and $X = U$ yields $B_1 = B_2$, again a contradiction.

We now have $X = X^\circ \times S$ for some finite subgroup S of X by Fact 2.3.

Let S_1 be a subgroup of X. If $S_1 \not\leq S$, then S_1 contains an element of the form $s \cdot x$ for some s in S and some nontrivial element x in X°. As X° is torsion-free, x as infinite order, as well as $s \cdot x$, and $C^\circ(S_1) \leq C^\circ(H(s \cdot x)) \leq N^\circ(H^\circ(s \cdot x))$, which is solvable by $*$-local$^\circ_\circ$ solvability of G. Hence $C^\circ(S_1)$ nonsolvable implies $S_1 \leq S$.

We now want to show that if $S_1 \leq S$, then $C^\circ(S_1)$ is nonsolvable. It suffices to do it for S, so we assume toward a contradiction $C^\circ(S)$ solvable. Let B_3 be a Borel subgroup of G containing $C^\circ(S)$. Notice that the finite nilpotent group S is the product of its Sylow p-subgroups. If p_1 and p_2 are two (not necessarily distinct) primes dividing the order of S, then we claim that one cannot have $U_{p_1}(B_1) \neq 1$ and $U_{p_2}(B_2) \neq 1$. Assume the contrary. As Sylow subgroups for primes different from p_1 in $F(B_1)$ commute with $U_{p_1}(B_1)$ (by Fact 2.3 (1)!), $U_{p_1}(B_1 \cap C^\circ(S))$ is nontrivial by Fact 2.13 (2). Similarly, $U_{p_2}(B_2 \cap C^\circ(S))$ is nontrivial. Now Lemma 4.6 gives a contradiction, which proves our claim. It follows that all nontrivial p-unipotent subgroups of B_1 or B_2, for p dividing the order of S, are on one side, say they are all in B_1. Notice then that all p-unipotence blocks of B_2, for p dividing the order of S, are trivial. In particular $S \leq Z(B_2)$ by Lemma 2.14. Hence $B_2 \leq C^\circ(S) \leq B_3$, $B_3 = B_2$, and $C^\circ(S) = B_2$. So one cannot have $C^\circ(S) = B_1$, as $B_1 \neq B_2$. Hence S is not central in B_1. By Lemma 2.14, there is a prime p dividing the order of S and such that $U_p(B_1) \neq 1$. As above, $U_p(C^\circ_{B_1}(S))$ is nontrivial by Fact 2.3 (1) and Fact 2.13 (2), and Corollary 4.4 gives then $B_1 = B_2$, a contradiction. □

A subgroup S as in Lemma 4.7 could for example be the subgroup $Z(H)$ in the hypothetic Configuration 3.15.

A version of the following lemma has been baptized "Jaligot's lemma" in [Bur07, §2] (see [CJ04, §3.4] and [Bur07, Lemma 2.1]).

Lemma 4.8. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Assume that, for $i = 1$ and 2, $\tilde{p}_i = (p_i, r_i)$ are two unipotence parameters such that $d_{p_i}(G) = r_i$, and B_i are two distinct Borel subgroups such that $d_{p_i}(B_i) = r_i$. If*

X denotes $F(B_1) \cap F(B_2)$, then X is finite and $C^\circ(S_1)$ is nonsolvable for any subgroup S_1 of X.

Proof. Assume X° non-trivial. By $*$-local$^\circ_\circ$ solvability of G, $N^\circ(X)$ is solvable and hence contained in a Borel subgroup B_3 of G. As X° is torsion-free by Lemma 4.7, the assumption that $d_{p_i}(B_i) = r_i$ implies $r_i > 0$ for each i. By Fact 2.8, $U_{\tilde{p}_i}(B_i)$ is included in $F^\circ(B_i)$, and by Fact 2.3, Γ_i, the last nontrivial iterated term of the descending central series of $U_{\tilde{p}_i}(B_i)$, is central in $F^\circ(B_i)$. Hence $\Gamma_i \leq N^\circ(X) \leq B_3 \cap B_i$. Now by assumption each Γ_i is nontrivial, and a $\tilde{p}_i$-group by Corollary 2.2. Corollary 4.4 implies that each Γ_i is contained in a unique Borel subgroup of G, which gives $B_3 = B_1$ and $B_3 = B_2$, contradicting the assumption that $B_1 \neq B_2$. Hence X is finite.

Our last claim is contained in Lemma 4.7. □

In absence of $*$-local$^\circ$ solvability one might have $F(B_1) \cap F(B_2)$ (finite and) nontrivial in Lemma 4.8, as for example in Configuration 3.15 again.

4.1.4 Consequences on soapy subgroups

We continue as in Sections 4.1.2 and 4.1.3 with consequences of the Uniqueness Theorem 4.1, now on soapy subgroups. All these properties make us think of a soap sliding between two hands, exactly like a unipotent subgroup which cannot be contained in two distinct Borel subgroups in PSL_2. The following lemmas will be used in our most critical computations in [DJ07a].

Lemma 4.9. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B_1 and B_2 two Borel subgroups each having a soapy subgroup U_1 and U_2 respectively. Then*

(1) *B_1 is unique among Borel subgroups of G containing U_1 and admitting the unipotence parameter of U_1 as maximal.*

(2) *If $[U_1, U_2] = 1$, then $B_1 = B_2$.*

Proof. (1). By $*$-local$^\circ_\circ$ solvability of G, $N^\circ(U_1)$ is solvable. As U_1 is normal in B_1, the maximality of B_1 implies $N^\circ(U_1) = B_1$. If the unipotence parameter of U_1 is $(\infty, 0)$, then B_1 is a good torus, as well as any Borel subgroup admitting $(\infty, 0)$ as maximal. So any such Borel subgroup is contained in $C^\circ(U_1) = B_1$, and thus equal to B_1. Otherwise, as $C^\circ(U_1) \leq N^\circ(U_1)$, the first item is a mere application of Corollary 4.3.

(2). Again $N^\circ(U_1) = B_1$ and similarly $N^\circ(U_2) = B_2$. Hence U_1, $U_2 \leq B_1 \cap B_2$ under the assumption that U_1 and U_2 commute. If U_1 is a good torus, then as for the first item B_1 is a good torus as well, as well as its subgroup U_2, and similarly B_2 also. We then get $B_2 \leq C^\circ(U_1) \leq N^\circ(U_1) = B_1$, and equality of B_1 and B_2. One concludes symmetrically when U_2 is a good torus, so one can assume that both U_1 and U_2 are not good tori. As $U_1, U_2 \leq B_1 \cap B_2$, Corollary 4.4 gives $B_1 = B_2$ or $\max(d(U_1), d(U_2)) < \infty$. In any case Corollary 4.3 gives $B_1 = B_2$. □

The following lemma allows one to build soapy subgroups in presence of two Borel subgroups.

Lemma 4.10. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B_1 and B_2 two Borel subgroups, and U_1 a soapy subgroup of B_1. If $U_1 \leq B_2$, then B_2 contains a characteristically soapy subgroup.*

Proof. If $B_1 = B_2$, then U_1 is a soapy subgroup of B_2 and we may use Lemma 2.10.

Assume now $B_1 \neq B_2$, and let $\tilde{q}_1$ be the unipotence parameter attached to U_1. Let $\tilde{q}_2$ be a unipotence parameter maximal for B_2. If $\tilde{q}_2 = (\infty, 0)$, then B_2 is a good torus, as well as U_1, as well as B_1, and then one concludes as usual by $*$-local$^\circ_\circ$ solvability that $B_1 = B_2$. Hence $\tilde{q}_2$ is not $(\infty, 0)$. If $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ is not central in B_2, then we may apply Lemma 2.11.

So now assume toward a contradiction $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ central in B_2. In particular $U_{\tilde{q}_2}(Z(F^\circ(B_2))) \leq C^\circ(U_1) \leq N^\circ(U_1) = B_1$ by $*$-local$^\circ_\circ$ solvability of G. By Corollary 4.4, $\tilde{q}_1$ and $\tilde{q}_2$ do not represent subgroups of bounded exponent, as $B_1 \neq B_2$. The maximality of $\tilde{q}_1$ for B_1 and of $\tilde{q}_2$ for B_2 then yields $\tilde{q}_1 = \tilde{q}_2$. But Corollary 4.3 gives the uniqueness of B_2 among Borel subgroups containing $U_{\tilde{q}_2}(Z(F^\circ(B_2)))$ and admitting $\tilde{q}_2$ as maximal. Thus $B_1 = B_2$, a contradiction in the last case under consideration. □

4.1.5 Consequences on Carter subgroups

Theorem 4.1 also gives information on Carter subgroups possessing a subgroup sufficiently unipotent relatively to the ambient group.

Lemma 4.11. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, Q a Carter subgroup of G and $\tilde{p} = (p, r)$ a unipotence parameter admitted by Q. Assume Q contains a nontrivial definable central $\tilde{p}$-subgroup U_1 with a nonempty subset X such that $d_p(C(X)) \leq r$. Then exactly one of the following three cases occurs.*

(1) *Q is a generous Carter subgroup.*

(2) *For g generic in Q, $d_p(C^\circ(g)) > r$.*

(3) *The generic element of Q is exceptional, and in particular any element of Q has order at most $e(G)$.*

Proof. Notice that the assumption together with Corollary 2.2 and Fact 2.3 (2) implies that $\tilde{p}$ is maximal in its characteristic for Q.

If Q is generous, then $C^\circ(g) \leq Q$ holds for g generic in Q by [Jal06, Corollary 3.8] (see also [Jal06, Lemma 3.10]), so cases (2) and (3) cannot occur.

Assume Q not generous in G. By Fact 3.32, Q contains no nontrivial good torus, and thus $r > 0$ as $\tilde{p}$ is maximal in its characteristic for Q. By Theorem 4.1, U_1 is contained in a unique Sylow $\tilde{p}$ subgroup of G, say U, and $Q \leq N^\circ(U)$. Notice that $N^\circ(U)$ is solvable by $*$-local$^\circ_\circ$ solvability of G. By condition (d) in

[Jal06, Corollary 3.8] a generic element g of Q is in infinitely many conjugates of Q.

Suppose toward a contradiction $d_p(C^\circ(g)) \leq r$ and $C^\circ(g)$ solvable. Then $\tilde{p} \neq (\infty, 0)$ is a unipotence parameter maximal in its characteristic for the definable connected solvable subgroup $C^\circ(g)$. It follows that $C^\circ(g)$ contains a unique Sylow $\tilde{p}$-subgroup by Fact 2.8, which is necessarily a $\tilde{p}$-subgroup of U as it contains U_1. If γ is an element of G such that $g \in Q^\gamma$, then U_1 and U_1^γ are both contained in $U_{\tilde{p}}(C^\circ(g))$, and by uniqueness applied now to U_1^γ one gets $U = U^\gamma$. Hence all G-conjugates of Q containing g are actually $N^\circ(U)$-conjugate. But now Q is generous in the definable connected solvable subgroup $N^\circ(U)$, and thus a generic element of Q is in a unique $N^\circ(U)$-conjugate of Q by [Jal06, Corollary 3.8]. This is a contradiction. Hence when Q is not generous one of the two cases (2) or (3) must occur.

Notice that in case (3) a generic element of Q, being exceptional, has order at most $e(G)$, and then the exponent of Q is bounded by $e(G)$ by Fact 2.3 (2).

It just remains to show that cases (2) and (3) cannot occur simultaneously. But in case (2) r cannot be ∞, and in case (3) it must. □

Of course, by Corollary 2.2, Lemma 4.11 applies when $d_p(G) = d_p(Q) = r$. In particular a nongenerous Carter subgroup which is not divisible must be as in case (3) of Lemma 4.11.

The Uniqueness Theorem may also serve as in [CJ04] to get a quite general theory of Weyl groups in the specific context of groups considered here, as well as applications to finite exceptional subgroups. We do not include it here, and refer to the more general treatment in [Jal09].

4.2 Maximal pairs of Borel subgroups

When the absolute maximality assumptions concerning unipotence degrees fail in Lemma 4.8 one might have (or rather cannot exclude) pairs of Borel subgroups whose Fitting subgroups have an infinite intersection. This situation has been studied intensively in [Bur07]. In what follows, not only claim we no originality compared to this paper, but also we will tend to follow its analysis word by word. The only differences will appear in the notation used for unipotence parameters and in a special care needed for dealing here with our weakest assumption of $*$-local$^\circ_\circ$ solvability. Some additional results from [Del07a] will be mentioned in the process.

Definition 4.12. [Bur07, Definition 3.1] *Let G be a group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups. We say that (B_1, B_2) is a* maximal pair *(of Borel subgroups) if the definable connected subgroup $(B_1 \cap B_2)^\circ$ is maximal for inclusion among all definable connected subgroups of the form $(L_1 \cap L_2)^\circ$, with L_1 and L_2 two distinct Borel subgroups of G.*

Hypothesis 4.13. [Bur07, Hypothesis 3.2]

(1) *G is a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank.*

(2) (B_1, B_2) *is a maximal pair of Borel subgroups of* G.

(3) $[F(B_1) \cap F(B_2)]^\circ$ *is nontrivial.*

Notation 4.14. [Bur07, Notation 3.3] *Let* $H = (B_1 \cap B_2)^\circ$, $X = F(B_1) \cap F(B_2)$, *and* $r' = d_\infty(X)$.

Recall that X° is torsion free by Lemma 4.7. In particular $0 < r' < \infty$. In particular $0 < d_\infty(B_1) < \infty$ and $0 < d_\infty(B_2) < \infty$.

Notice that by Lemma 4.8 one cannot have $d(B_1) = d(B_2) = \infty$. So at least one of the two Borel subgroups B_1 and B_2, say B_i, has no bounded exponent subgroup. In particular $0 < d(B_i) < \infty$. The other Borel subgroup B_{i+1} might satisfy $0 < d(B_{i+1}) \leq \infty$ (the latter inequality will be shown to be also strict in the analysis below).

The proof of the steps below can be recovered from [Bur07]. This in general involves practically no changes, except taking connected components throughout when possible, and in particular considering X° instead of X, and using Lemma 3.4 (4) and Lemma 3.6 when considering normalizers of various connected solvable subgroups and their correlations with Borel subgroups. As for background facts used in the process, the reader can also use the following dictionary.

[Bur07, Fact 1.5] and [Bur07, Fact 1.9]: Fact 2.8

[Bur07, Fact 1.13]: [FJ08, Proposition 2.8]

[Bur07, Fact 1.14]: Fact 2.3

[Bur07, Fact 1.15]: [FJ08, Lemma 2.9]

[Bur07, Fact 1.16]: Fact 2.7

[Bur07, Fact 1.22]: [FJ08, Corollary 5.11]

[Bur07, Fact 1.19(4)]: [FJ08, Proposition 5.1], or [Nes90] ([BN94, Theorem 9.21]) and [FJ08, Proposition 5.1]

Besides, Corollary 4.4 implies that two distinct Borel subgroups cannot share a common nontrivial p-unipotent subgroup, and in particular all subgroups H considered here satisfy $d(H) = d_\infty(H)$.

Homogeneity of X. Observe that $H' \leq X \trianglelefteq H$. We show the asymmetry of the situation, i.e., $d_\infty(B_1) \neq d_\infty(B_2)$. We may assume in any case that $d_\infty(B_2) \leq d_\infty(B_1)$, and we indeed show that $d_\infty(B_2) < d_\infty(B_1)$. Notice that $d(H) = d_\infty(H)$.

4.15. [Bur07, Lemma 3.5] $d_\infty(H) < d_\infty(B_1)$.

4.16. [Bur07, Lemma 3.6] $d(H) = d_\infty(B_2)$.

4.17. $d(H) = d_\infty(H) = d_\infty(B_2) < d_\infty(B_1)$.

4.18. [Bur07, Proposition 3.7] If H is nonabelian, then B_1 and B_2 are the only Borel subgroups containing H.

4.19. [Bur07, Lemma 3.8] $F^\circ(B_i) \not\leq H$ for $i = 1$ and 2.

4.20. [Bur07, Lemma 3.9] If X_1 is an infinite definable subgroup of X normal in H, then $N^\circ(X_1) \leq B_1$.

4.21. $[X \cap Z(F(B_2))]^\circ = 1$.

4.22. [Bur07, Theorem 3.10] X° is a homogeneous (∞, r')-subgroup.

Proof. 4.15: As there is no nontrivial p-unipotent subgroup in H, $d(H) < \infty$, and the rest of the proof is similar.

4.17: This is a corollary of 4.16.

4.19: Since $d_\infty(H) < d_\infty(B_1)$ by 4.15, $F^\circ(B_1) \not\leq H$. Suppose toward a contradiction $F^\circ(B_2) \leq H$. Then $H \trianglelefteq B_2$ by [Nes90] ([BN94, Theorem 9.21]), and $H \leq B_1 \cap B_1^g$ for some $g \in B_2 \setminus N(B_1)$. By maximality of H, (B_1, B_1^g) is a maximal pair, and 4.17 applied to this maximal pair gives a contradiction.

4.20: By $*$-local$^\circ_\circ$ solvability $N^\circ(X_1)$ is solvable, and the rest of the proof is similar.

4.21: This is a corollary of 4.20. □

Fitting subgroup of B_2.

4.23. [Bur07, Lemmas 3.11 and 3.12] $F^\circ(B_2)$ is divisible (in particular $d(B_2) = d_\infty(B_2)$) and $U_{(\infty,r)}(F^\circ(B_2)) \leq Z(H)$ when $0 \leq r \leq d(B_2)$ and $r \neq r'$.

4.24. [Bur07, Lemma 3.13] $U_{(\infty,r')}(F^\circ(B_2))$ is not contained in H and not abelian.

4.25. [Bur07, Corollary 3.14] $U_{(\infty,r)}(F^\circ(B_2))$ is not abelian if and only if $r = r'$.

4.26. [Bur07, Lemma 3.15] $U_{(\infty,r)}(B_2) \leq F^\circ(B_2)$ for every $r > r'$.

Structure of H.

4.27. [Bur07, Lemma 3.16] $U_{(\infty,r')}(H) \leq F^\circ(B_2)$. In particular $Y := U_{(\infty,r')}(H)$ is nilpotent and the unique Sylow (∞, r')-subgroup of H. It is normal in H, and in $F^\circ(H)$.

4.28. [Del07a, Lemme 3.11] If a Carter subgroup of H is also a Carter subgroup of B_2, then $U_{(\infty,r')}(B_2)$ is nilpotent, included in $F^\circ(B_2)$, and the unique Sylow (∞, r')-subgroup of B_2.

4.29. [Bur07, Lemma 3.17] $N^\circ(Y) \leq B_2$ and $X^\circ < Y$. In addition we have $U_{(\infty,r')}(N_{F(B_2)}(Y)) \not\leq H$.

4.30. [Bur07, Theorem 3.18] Every definable connected nilpotent subgroup of H is abelian.

4.31. [Bur07, Lemma 3.19] If H is not abelian, then $N^\circ(H) = H$.

Proof. 4.28: Let Q be a Carter subgroup of H, which is also a Carter subgroup of B_2. Then $U_{(\infty,r')}(F^\circ(B_2)) \cdot U_{(\infty,r')}(Q)$ is a Sylow (∞, r')-subgroup of B_2 by [FJ08, Corollary 5.11]. By conjugacy of such subgroups in B_2, [FJ08, Theorem 5.8], it suffices to show that it is contained in $F^\circ(B_2)$. But the first factor clearly is, and the second also by 4.27. □

Structure of B_1. Let Q denote a Carter subgroup of H.

4.32. [Bur07, Lemma 3.20] $F^\circ(B_1)$ is divisible (and in particular $d(B_1) = d_\infty(B_1)$) and $U_{(\infty,0)}(F^\circ(B_1)) \leq Z^\circ(H)$.

4.33. [Bur07, Lemma 3.21] $X^\circ = U_{(\infty,r')}(F^\circ(B_1))$, and also $B_1 = N^\circ(X^\circ)$.

4.34. [Bur07, Corollary 3.22] $U_{(\infty,r')}(F^\circ(B_1))$ is abelian, and $F^\circ(B_1) \leq C^\circ(X^\circ)$.

4.35. [Bur07, Lemma 3.23] $U_{(\infty,r')}(Q) = U_{(\infty,r')}(Z(H))$, and this group is not trivial.

4.36. [Bur07, Theorem 3.24] $N^\circ(U_{(\infty,r')}(Q)) \leq B_2$. So $N^\circ(Q) \leq B_2$, and Q is a Carter subgroup of B_1.

4.37. [Bur07, Lemma 3.25] $U_{(\infty,r)}(F^\circ(B_1)) = 1$ for any $r \neq r'$ with $1 \leq r \leq d(B_2)$.

4.38. [Bur07, Corollary 3.26] r' is the minimal unipotence degree $1 \leq r < \infty$ such that $F(B_1)$ admits the unipotence parameter (∞, r).

4.39. [Bur07, Corollary 3.27] For $1 \leq r \leq d(B_2)$, a Sylow (∞, r)-subgroup of H is a Sylow (∞, r)-subgroup of B_1.

Proof. 4.35: By 4.29, $U_{(\infty,r')}(H/H')$ is not trivial. So $U_{(\infty,r')}(Q)$ is not trivial by [Nes90] ([BN94, Theorem 9.21]) and [FJ08, Proposition 5.1]. The rest of the proof is similar, using [FJ08, Proposition 5.1] again here.

4.36: One gets similarly $H \leq N^\circ(U_{(\infty,r')}(Q))$. By $*$-local$^\circ_\circ$ solvability of G the latter group is solvable. If it contains H properly, then it can grow only in one Borel subgroup B_1 or B_2, and must agree with H on the other. □

Nonabelian intersections.

Remark 4.40. $\mathrm{Tor}\,(X)$ *is toral and contained in* $Z(B_1) \cap Z(B_2)$*, and* $C^\circ(X) = C^\circ(X^\circ)$.

Proof. Let S be the (finite) torsion subgroup of X, as in Lemma 4.7. As $d(B_1) < \infty$ and $d(B_2) < \infty$, S is a toral subgroup of B_1 and B_2, and included in $Z(B_1) \cap Z(B_2)$ by Lemma 2.14. By 4.20, $C^\circ(X) \leq C^\circ(X^\circ) \leq B_1$, and as $X = X^\circ \times S$ with $S \leq Z(B_1)$, $C^\circ(X) = C^\circ(X^\circ)$. □

4.41. [Bur07, Lemma 3.28] The subgroup $C^\circ(X^\circ)$ is not nilpotent. If H is not abelian, then B_1 is the unique Borel subgroup of G containing $C^\circ(X^\circ)$.

4.42. [Bur07, Corollary 3.29] Suppose H not abelian. Then, for any infinite definable subgroup $X_1 \leq X$, B_1 is the unique Borel subgroup of G containing $C^\circ(X_1)$.

4.43. If H is nonabelian, then $C^\circ(Y) \leq C^\circ(X^\circ) \leq B_1$.

4.44. (Compare with [Del07a, Lemma 3.10]) If H is nonabelian, then any Sylow (∞, r')-subgroup of G containing Y is contained in B_2.

4.45. [Bur07, Lemma 3.30] Let B be a Borel subgroup of G, distinct from B_1. Suppose that (B, B_1) is a maximal pair, that $H_1 = (B \cap B_1)^\circ$ is not abelian, and that $d(B_1) \geq d(B)$. Then B is $F^\circ(B_1)$-conjugate to B_2.

Proof. 4.42: Recall that $C^\circ(X) = C^\circ(X^\circ)$. $C^\circ(X) \leq C^\circ(X_1)$, the latter being solvable by $*$-local$^\circ_\circ$ solvability of G, so 4.41 gives the desired result.

4.43: $X^\circ \leq Y$.

4.44: We want to show that $\Sigma \leq B_2$ for any Sylow (∞, r')-subgroup Σ of G containing Y. One can assume $Y < \Sigma$, and then $Y < U_{(\infty,r')}(N^\circ_\Sigma(Y))$ by

normalizer condition, [FJ08, Proposition 2.8]. By 4.29, $N^\circ(Y) \leq B_2$, and thus $U_{(\infty,r')}(N^\circ_\Sigma(Y)) \leq B_2$.

If $U_{(\infty,r')}(N^\circ_\Sigma(Y))$ is abelian, then it centralizes Y. But $C^\circ(Y) \leq C^\circ(X^\circ) \leq B_1$ by 4.41. Hence $U_{(\infty,r')}(N^\circ_\Sigma(Y)) \leq (B_1 \cap B_2)^\circ = H$ and then $U_{(\infty,r')}(N^\circ_\Sigma(Y)) = Y$, a contradiction.

Hence $U_{(\infty,r')}(N^\circ_\Sigma(Y))$ is nonabelian. Now it follows from 4.30 that in a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, a nonabelian definable connected nilpotent subgroup is contained in a unique Borel subgroup. As $U_{(\infty,r')}(N^\circ_\Sigma(Y))$ is in B_2 and in Σ, this gives $\Sigma \leq B_2$. □

Conclusions.

4.46. [Bur07, Proposition 4.1] Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G, and H a nontrivial definable connected subgroup of $B_1 \cap B_2$. Then the following hold:

(1) H' is a homogeneous (∞, r')-group for some $1 \leq r' < \infty$ (or trivial).

(2) Every definable connected nilpotent subgroup of H is abelian.

(3) $U_{(\infty,r')}(F^\circ(H)) = U_{(\infty,r')}(H)$ is the unique Sylow (∞, r')-subgroup of H.

(4) $U_{\tilde{q}}(F^\circ(H)) \leq Z(H)$ for any $\tilde{q} \neq (\infty, r')$.

(5) $0 \leq d_\infty(H) = d(H) \leq d(C(H')) \leq d(N(H')) \leq \infty$, all inequalities, except maybe the third one, being strict when H is not abelian.

Proof. We may assume H not abelian, as otherwise all statements are trivially true once one has noticed that $d_\infty(H) = d(H)$ by Corollary 4.4.

The proofs of the four first points are similar, considering a maximal pair (B_3, B_4) containing H with $d(B_3) \geq d(B_4)$, and for the fifth point one argues as follows.

By 4.17, $\infty > d(B_3) > d(B_4) \geq d(H) = d_\infty(H) > 0$ (be careful, this is not the same H, and one uses also the divisibility of $F^\circ(B_3)$ and of $F^\circ(B_4)$). By Fact 2.3, $U_{(\infty,d_\infty(B_3))}(B_3) \leq C(H')$, thus $d_\infty(C(H')) > d_\infty(H) = d(H)$. Hence for the last point it suffices to show that $d(N(H')) < \infty$. Otherwise, $U_p(N(H'))$ is nontrivial for some prime p; now the nontrivial group $U_{(\infty,d_\infty(B_3))}(B_3)$, which is also in $N(H')$, normalizes $U_p(N(H'))$, and centralizes it by Fact 2.7 (4), so that $U_p(N(H')) \leq N^\circ(U_{(\infty,d_\infty(B_3))}(B_3)) = B_3$ (by $*$-local$^\circ_\circ$ solvability and Lemma 3.6), a contradiction to the divisibility of $F^\circ(B_3)$. Hence $d(N(H')) < \infty$ and this completes the proof of the fifth point. □

4.47. [Bur07, Corollary 4.2] Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. Then a definable connected nonabelian nilpotent subgroup is contained in exactly one Borel subgroup of G.

4.48. [Bur07, Corollary 4.2'] Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank. If Q is a Carter subgroup of a Borel subgroup B, and if Q is not abelian, then Q is a Carter subgroup of G.

4.49. [Bur07, Lemma 4.4] Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G. Suppose that $H = (B_1 \cap B_2)^\circ$ is not abelian, and that $C^\circ(H') \leq B_1$. Then B_1 and B_2 are the only Borel subgroups containing H.

Proof. 4.48: $N^\circ(Q)$ is contained in a Borel subgroup B_1 of G by $*$-local$^\circ_\circ$ solvability. As $Q \leq B \cap B_1$, $B = B_1$ by 4.47, and $N^\circ_G(Q) \leq N^\circ_{B_1}(Q) = N^\circ_B(Q) = Q$ □

Maximal pairs with nonabelian intersections can then be characterized.

4.50. [Bur07, Theorem 4.3] Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B_1 and B_2 two distinct Borel subgroups of G. Suppose $H = (B_1 \cap B_2)^\circ$ nonabelian. Then the following are equivalent:

(1) B_1 and B_2 are the only Borel subgroups containing H.

(2) (B_1, B_2) is a maximal pair.

(3) If $B_3 \neq B_1$ is a Borel subgroup containing H, then $(B_1 \cap B_3)^\circ = H$.

(4) $C^\circ(H')$ is contained in B_1 or B_2.

(5) B_1 and B_2 are not conjugate under the action of $C^\circ(H')$.

(6) $d_\infty(B_1) \neq d_\infty(B_2)$.

We can now describe the maximal pairs having a nonabelian intersection$^\circ$, collecting the results from [Bur07] with the additional results from [Del07a]. We slightly change the presentation in comparison to [Bur07, Theorem 4.5], as we prefer to distinguish between a symmetric version and an asymmetric one.

Theorem 4.51. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, and (B_1, B_2) a maximal pair of Borel subgroups such that $H := (B_1 \cap B_2)^\circ$ is nonabelian. Let $r' = d_\infty(H')$.*

(1) *$0 < d(B_1) < \infty$ and $0 < d(B_2) < \infty$.*

(2) *$N^\circ(H) = H$.*

(3) *$[F(B_1) \cap F(B_2)]^\circ$ is (∞, r')-homogeneous, and $r' > 0$.*

Furthermore, if Q denotes a Carter subgroup of H and $Q_{r'}$ denotes $U_{(\infty,r')}(Q)$, then, (4) *$Q_{r'} \neq 1$, and exactly one of the following cases occurs:*

(4.*a*) *$N^\circ(Q_{r'}) = H$.*

(4.*b*) *$H < N^\circ_{B_1}(Q_{r'})$; furthermore $N^\circ_{B_2}(Q_{r'}) = H$ and B_1 is the unique Borel subgroup containing $N^\circ(Q_{r'})$.*

(4.*c*) *$H < N^\circ_{B_2}(Q_{r'})$; furthermore $N^\circ_{B_1}(Q_{r'}) = H$ and B_2 is the unique Borel subgroup containing $N^\circ(Q_{r'})$.*

Proof. (1): 4.7, 4.32, 4.23. (2): 4.31. (3): 4.7, 4.22. (4): 4.35, and proof of [Del07a, Lemme 3.9] for the trichotomy. □

Theorem 4.52. *Assume in addition to Theorem 4.51 that $d(B_1) \geq d(B_2)$. Then*

(1) $0 < d(B_2) < d(H) = d(B_1) < \infty$.

(2) *Q is a Carter subgroup of B_1.*

(3) $U_{(\infty,r')}(F(B_1)) = [F(B_1) \cap F(B_2)]^\circ$.

(4) *B_1 is the unique Borel subgroup containing $C^\circ(U_{(\infty,r')}(F(B_1)))$.*

(5) $N^\circ(Q) \leq B_2$.

(6) $U_{(\infty,r')}(H) \leq F^\circ(B_2)$, *and* $N^\circ(U_{(\infty,r')}(H)) \leq B_2$.

(7) $U_{\tilde{q}}(F(B_2)) \leq Z(H)$ *for any* $\tilde{q} \neq (\infty, r')$, *and* $U_{(\infty,r')}(F(B_2))$ *is nonabelian (in particular* $U_{\tilde{q}}(F(B_2))$ *is nonabelian iff* $\tilde{q} = (\infty, r')$*).*

(8) *Any Sylow (∞, r')-subgroup of G containing $U_{(\infty,r')}(H)$ is contained in B_2.*

(9) *If Q is a Carter subgroup of B_2, then $U_{(\infty,r')}(F(B_2))$ is the unique Sylow (∞, r')-subgroup of B_2, and in particular the unique Sylow (∞, r')-subgroup of G containing $U_{(\infty,r')}(H)$.*

Proof. (1): 4.51 (1), 4.17. (2): 4.36. (3): 4.33. (4): 4.33, 4.41. (5): 4.36. (6): 4.27, 4.29. (7): 4.23, 4.24. (8): 4.44. (9): 4.28, 4.44. □

Finally, we record a point about exceptional elements, which applies in particular in Theorems 4.51 and 4.52.

Theorem 4.53. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank and (B_1, B_2) a maximal pair of Borel subgroups such that $[F(B_1) \cap F(B_2)]^\circ$ is nontrivial. Then the finite subgroup S of $F(B_1) \cap F(B_2)$ as in Lemma 4.7 is toral and central, both in B_1 and B_2.*

Proof. $F^\circ(B_1)$ and $F^\circ(B_2)$ are divisible by 4.23 and 4.32, and Remark 4.40 applies. □

We finish with an extra homogeneity result.

Lemma 4.54. (see [Del07a, Lemme 3.8]) *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, B and B^g two distinct conjugates of a same Borel subgroup B. If $[F(B) \cap F(B^g)]^\circ$ is not homogeneous, then $F^\circ(B)$ is abelian.*

Proof. By assumption $[F(B) \cap F(B^g)]^\circ$ contains two nontrivial Sylow subgroups U_1 and U_2 with two distinct unipotence parameters, say $\tilde{p}$ for U_1 and $\tilde{q}$ for U_2.

By $*$-local$^\circ_\circ$ solvability of G, $N^\circ(U_1)$ is contained in a Borel subgroup B_1. If $B_1 \neq B$, then, by Fact 2.3 (2), 4.47 implies that Sylow subgroups of $F^\circ(B)$ of unipotence parameters different from $\tilde{p}$ are abelian. If $B_1 = B$, then $B_1 \neq B^g$ and one sees similarly that Sylow subgroups of unipotence parameters different from $\tilde{p}$ of $F^\circ(B^g)$, and thus also of $F^\circ(B)$, are also abelian.

Considering a Borel subgroup B_2 containing $N^\circ(U_2)$, one sees similarly that Sylow subgroups of $F^\circ(B)$ of unipotence parameters different from $\tilde{q}$ are abelian.

Now $F^\circ(B)$ is abelian by Fact 2.3 (2). □

4.3 Nonsolvable definable connected subgroups

Section 4.2 concerned the analysis of intersections of maximal pairs of Borel subgroups. In the present section we continue a little bit in this vein when one of the two subgroups involved is not necessarily solvable, a possibility in the context of $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank in comparison to the context of minimal connected simple groups.

Definition 4.55. *Let G be a group of finite Morley rank and K a definable connected subgroup of G. We say that a Borel subgroup B of G has* maximal intersection *with K if $B \not\leq K$ and $(K \cap B)^\circ$ is maximal for inclusion among groups of the form $(K \cap B_1)^\circ$ for some Borel subgroup B_1 of G such that $B_1 \not\leq K$.*

We note in Definition 4.55 that if K is solvable and not a Borel subgroup, then it has a maximal intersection with any Borel subgroup containing it. If K is a Borel subgroup of G, and a Borel subgroup B has a maximal intersection with K, then G° is not solvable.

Lemma 4.56. *Let G be a group of finite Morley rank, K a definable connected subgroup of G, and B a Borel subgroup of G having maximal intersection with K. Then any Borel subgroup B_1 of G such that $(K \cap B)^\circ < (K \cap B_1)^\circ$ is in K.*

Proof. This is immediate by definition. □

It follows that if K is a Borel subgroup of a $*$-locally$^\circ_\circ$ solvable group G and B is a Borel subgroup of G having maximal intersection with K, then if $(K \cap B)^\circ$ is nonabelian any Borel subgroup B_3 of G containing $(K \cap B)^\circ$ such that $(K \cap B)^\circ < (K \cap B_3)^\circ$ must be K, and hence (K, B) is a maximal pair of Borel subgroups of G by the equivalence provided in Theorem 4.50 (3).

In the general case of a $*$-locally$^\circ_\circ$ solvable group G a proper definable connected subgroup K can be nonsolvable, and we slightly clarify the situation in this general case.

Lemma 4.57. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, K a nontrivial definable connected subgroup of G, B a Borel subgroup of G having maximal intersection with K, and let $H = (K \cap B)^\circ$. Then assuming H nontrivial exactly one of the following cases occurs.*

(1) *H is an abelian Carter subgroup of K and of B.*

(2) *H is an abelian Carter subgroup of K and $H < N^\circ_B(H) \leq B$.*

(3) *H is an abelian Carter subgroup of B, and $H < N^\circ(H) \leq K$. In this case any Borel subgroup of K containing H is a Borel subgroup of G.*

(4) *H is a nonabelian Borel subgroup of K.*

(5) *H is nonabelian and not a Borel subgroup of K. In this case any Borel subgroup of K containing H is a Borel subgroup of G.*

Proof. Notice that $N^\circ(H)$ is solvable by $*$-local$^\circ_\circ$ solvability of G.

Assume first H abelian. If H has finite index in its normalizer in K and in B then we are in case (1).

Assume now $H < N^\circ_B(H)$. Then the maximality of the intersection forces $N^\circ_K(H) = H$, and H is an abelian Carter subgroup of K. Hence we are in case (2).

Assume now $H < N^\circ_K(H)$. Then the maximality of the intersection forces $N^\circ(H) \leq K$ with Lemma 4.56. Now $N^\circ_B(H) \leq (K \cap B)^\circ = H$, and H is an abelian Carter subgroup of B. Hence we are in case (3) by Lemma 4.56.

This treats all cases corresponding to the case H abelian, so we may now assume H nonabelian. If H is a Borel subgroup of K, then we are in case (4).

It remains only to consider the case in which H is not abelian and not a Borel subgroup of K. By Lemma 4.56, any Borel subgroup of K containing H is a Borel subgroup of G. We are in case (5). □

5 Homogeneous cases and torsion

In this final section we collect various additional results of specialized nature about $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank, generally depending on the Uniqueness Theorems of Section 4.1.

The first type of results concerns the homogeneous cases. Recall from [FJ08] or Section 2.1 that a group of finite Morley rank is homogeneous if it is $\tilde{p}$-homogeneous for some unipotence parameter $\tilde{p}$, that is every definable connected nilpotent subgroup is a $\tilde{p}$-group. (This is weaker than the definition in [Fré06a], which requires to consider all definable connected subgroups, not only the nilpotent ones.) In a $\tilde{p}$-homogeneous group one sees easily with Lemma 2.6 and Fact 2.8 that any Borel subgroup is a (homogeneous) $\tilde{p}$-group, and in particular nilpotent. Hence we will more generally consider the case in which all Borel subgroups are nilpotent, and look at the homogeneous cases at various levels of generality.

The torsion-free case will be fairly well understood in this context, and with torsion this connects to a bit of Sylow theory. As far as torsion is concerned, there is in general no Sylow theory as in Fact 2.13 available in an arbitrary group of finite Morley rank. However the following fact shows similarities with Fact 2.13 in the general case.

Fact 5.1. [BC07] *Let G be a connected group of finite Morley rank, t a π-element of G for some set π of primes. If $U_p(C(t)) = 1$ for every p in π, then t belongs to a, and in fact to any, maximal π-torus of G and of $C^\circ(t)$.*

5.1 Nilpotent Borel subgroups

In this section we consider $*$-locally$^\circ_\circ$ solvable groups in which each Borel subgroup is nilpotent. We start with a lemma concerning abelian Borel subgroups.

Lemma 5.2. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with an abelian Borel subgroup B. Let B_u denote the maximal definable connected subgroup of B of bounded exponent. Then B has a subgroup E finite modulo B_u such that $B \cap B^g \leq E$ for any element g of G not in $N(B)$, and one of the following two cases occurs.*

(1) *B is a generous abelian Carter subgroup.*

(2) *B is an abelian Carter subgroup of bounded exponent.*

Proof. For any g in $G \setminus N(B)$, $N^\circ(B \cap B^g)$ contains B and B^g, and if $B \cap B^g$ is infinite then $N^\circ(B \cap B^g)$ is solvable by $*$-local$^\circ_\circ$ solvability of G and one gets B, $B^g \leq N^\circ(B \cap B^g)$ and $B = B^g$ by maximality, a contradiction. Hence $B \cap B^g$ is finite for every $g \in G \setminus N(B)$.

The uniformly definable family of finite subgroups $B \cap B^g$, for $g \in G \setminus N(B)$, consists of subgroups of uniformly bounded cardinals by elimination of infinite quantifiers. As Prüfer p-ranks are finite for any prime p, all these subgroups must be contained modulo B_u in a finite subgroup of the maximal definable decent torus of B. Calling E the preimage in B of this group, this proves our first statement.

If $B_u < B$, then E is not generic in B and one can conclude that the Carter subgroup B of G is generous by the equivalence given in [Jal06, Corollary 3.8 (c)]. This proves our alternative. □

We note that the two cases in Lemma 5.2 are a priori not necessarily mutually exclusive. In the $*$-locally$^\circ$ solvable context E is necessarily trivial, and B is then necessarily generous in any case.

We now pass to nilpotent Borel subgroups, replacing the commutativity assumption by a nilpotence assumption on all Borel subgroups of the ambient group. We easily see with Lemmas 3.4 and 3.6 that a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank has only nilpotent Borel subgroups if and only if $N^\circ(A)$ is nilpotent for every nontrivial definable connected nilpotent subgroup A. For a more compact terminology, we will naturally call such groups *$*$-locally$^\circ_\circ$ nilpotent*. The first lemma is essentially the content of the first part of the proof of Lemma 5.2 and typical of earlier work on bad groups [BN94, Chapter 13].

Lemma 5.3. *Let G be a $*$-locally$^\circ_\circ$ nilpotent group of finite Morley rank. Then any two distinct Borel subgroups have a finite intersection.*

Proof. Assume the contrary. Let B_1 and B_2 contradict our claim, with $[B_1 \cap B_2]^\circ$ of maximal rank. Call the latter group H, and notice that $H < B_1$ and $H < B_2$. By normalizer condition in nilpotent groups, [BN94, Lemma 6.3], $H < N^\circ_{B_1}(H)$ and $H < N^\circ_{B_2}(H)$. Now $N^\circ(H)$ is nilpotent by $*$-local$^\circ_\circ$ nilpotency of G, and

contained in a Borel subgroup B_3. As $H < (B_1 \cap B_3)^\circ$ and $H < (B_2 \cap B_3)^\circ$, our maximality assumption forces $B_1 = B_3 = B_2$, a contradiction. □

We get in any case conclusions similar to those of Lemma 5.2.

Lemma 5.4. *Let G be a $*$-locally$^\circ_\circ$ nilpotent group of finite Morley rank and let B be a Borel subgroup of G. Then B has a definable subgroup E, finite modulo the bounded exponent part of B, such that $B \cap B^g \leq E$ for any intersection $B \cap B^g$ with $g \in G \setminus N(B)$. Moreover one of the following two cases occurs.*

(1) *B is a generous Carter subgroup.*

(2) *B is a Carter subgroup of bounded exponent.*

Proof. With Lemma 5.3 applied to distinct conjugates of B, the existence of E follows as in the proof of Lemma 5.2. The alternative follows as well. □

Like for Lemma 5.2, the two cases in Lemma 5.4 are a priori not mutually exclusive, and if the ambient group G is $*$-locally$^\circ$ solvable then distinct conjugates of B are necessarily pairwise disjoint by the same proof as in Lemma 5.3, and B is always generous.

5.2 The torsion-free homogeneous case

We shall now evacuate, or rather collect in Pandora's box of bad groups, $\tilde{p}$-homogeneous $*$-locally$^\circ_\circ$ solvable groups of finite Morley rank, with $\tilde{p}$ not of the form $(\infty, 0)$ or (p, ∞) for p a prime. In this case Borel subgroups are nilpotent and torsion-free by [FJ08, Lemma 2.17] and Fact 2.4. More generally, we have the following result for such groups.

Theorem 5.5. *Let G be a torsion-free $*$-locally$^\circ_\circ$ nilpotent group of finite Morley rank. Then Borel subgroups are conjugate and either*

(1) *G is nilpotent, or*

(2) *G is a full Frobenius group, with malnormal subgroup a Borel subgroup B.*

As far as torsion is concerned there is a classical lifting result.

Fact 5.6. **[BN92]** *Let G be a group of finite Morley rank, H a definable normal subgroup, and x a p-element modulo H, for some prime p. Then the definable hull $H(x)$ of x contains a p-element.*

Proof. Notice that G is connected by absence of torsion and Fact 5.6.

By Lemmas 5.3 and 5.4, distinct Borel subgroups have trivial intersections, and each Borel subgroup is generous. As G is connected it cannot have two disjoint generic subsets. If B_1 and B_2 are two Borel subgroups, then two conjugates of B_1 and B_2 must have a nontrivial intersection by generosity, and then are equal. This shows that Borel subgroups are conjugate.

If G is not nilpotent, then $B < G$ for some Borel subgroup B of G. By Fact 5.6, $N(B) = B$, and B is malnormal in G by disjointness of distinct Borel

subgroups. As B^G is generic, any element g of G has an infinite centralizer (this is also an easy consequence of the main result of [BBC07] in arbitrary connected groups), and in particular normalizes a Borel subgroup by Lemma 2.6 (1) and the disjointness of Borel subgroups. Hence $G = B^G$, and G is a full Frobenius group with malnormal subgroup B. □

We note that a connected (∞, r)-homogeneous group of finite Morley rank, with $0 < r < \infty$, is torsion-free by Facts 2.4 and 5.1, and in particular Theorem 5.5 applies to such homogeneous connected $*$-locally$^\circ_\circ$ solvable groups.

Otherwise in the torsion-free case all results of Section 4 still apply, where all definable subgroups are connected. In this case Carter subgroups are conjugate by the same proof as in [Fré08].

5.3 The bounded exponent case

In presence of bounded exponent torsion the uniqueness theorems of Section 4.1 can be applied in their most straightforward forms in order to deal with generosity.

Lemma 5.7. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank such that $U_p(G)$ is nontrivial for some prime p. Then one of the following three cases occurs.*

(1) *Maximal p-unipotent subgroups are conjugate in G° and $N^\circ(U)$ is a generous Borel subgroup of unbounded exponent for any maximal p-unipotent subgroup U (and in fact one may assume also $N^\circ(U) = UC^\circ(U)$).*

(2) *There is a maximal p-unipotent subgroup U normalized but not centralized by a nontrivial q-torus T for some (and in fact infinitely many) prime(s) $q \neq p$. Moreover T is contained in a generous Carter subgroup of G.*

(3) *$N^\circ(U)$ is a Carter subgroup of bounded exponent for some maximal p-unipotent subgroup U.*

Proof. First recall that $N^\circ(U)$ is a Borel subgroup of G for any maximal p-unipotent subgroup U of G by Lemma 3.9.

Assume case (3) does not occur. This means that for any maximal p-unipotent subgroup U, $N^\circ(U)$ is not nilpotent of bounded exponent. By Fact 2.8, this simply means that any such group $N^\circ(U)$ has unbounded exponent.

If $UC^\circ(U) < N^\circ(U)$ for some maximal p-unipotent subgroup U, then Wagner's theorem [Wag01, Corollary 8] gives a nontrivial q-torus in $N^\circ(U)$, for some prime $q \neq p$, acting nontrivially on U (see for example [FJ05, Fact 2.5] and Zilber's field theorem [BN94, §9.1]). The fact that there are infinitely many primes q occuring in the definable subgroup of the multiplicative group of the field of characteristic p is due to [Wag03]. Then Fact 3.32 shows that we are in case (2).

This leaves us with the case in which $N^\circ(U) = UC^\circ(U)$ is a Borel subgroup of unbounded exponent for any maximal p-unipotent subgroup U.

If $N^\circ(U) \cap N^\circ(U^g)$ has a nontrivial connected component X for some $g \in G$, then $N^\circ(X)$ is solvable by $*$-local$^\circ_\circ$ solvability of G. As $N^\circ(U) = UC^\circ(U)$, X centralizes a nontrivial p-unipotent subgroup of U by Fact 2.13 (2), and similarly a nontrivial p-unipotent subgroup of $N^\circ(U^g)$. Now, as $N^\circ(X)$ is contained in a Borel subgroup, Lemma 4.6 implies $N^\circ(U) = N^\circ(U^g)$. Hence distinct conjugates of $N^\circ(U)$ have finite intersections.

As $N^\circ(U)$ has unbounded exponent, these finite intersections cannot cover $N^\circ(U)$ generically by Lemma 2.15. In particular they land in a (definable) non-generic subset of $N^\circ(U)$, and one concludes easily that $N^\circ(U)$ is generous in G° (see for instance [CJ04, Lemma 3.3], bearing in mind that $N^\circ(U)$ is of finite index in its normalizer, as a Borel subgroup).

We thus have $N^\circ(U) = UC^\circ(U)$ a generous Borel subgroup of unbounded exponent for any maximal p-unipotent subgroup U.

Now let U_1 and U_2 be two maximal p-unipotent subgroups of G. By generosity of $N^\circ(U_1)$ and [Jal06, Proposition 2.1], a generic element g of G° is in a conjugate of $N^\circ(U_1)$, and in finitely many such. Similarly, g is in a conjugate of $N^\circ(U_2)$, say $N^\circ(U_2)$ after conjugacy, and in finitely many such. Now $Z^\circ(U_2)$ centralizes g as $N^\circ(U_2) = U_2C^\circ(U_2)$. So it permutes naturally by conjugation the finitely many conjugates of $N^\circ(U_1)$ containing g, and one can argue as in [Jal06, Fundamental Lemma 3.3]. By Fact 1.1, it fixes each of them, and in particular it normalizes a conjugate of U_1, say U_1 up to conjugacy. Hence $Z^\circ(U_2) \leq N^\circ(U_1)$, $Z^\circ(U_2) \leq U_p(N^\circ(U_1)) = U_1$, and $U_1 = U_2$ by Theorem 4.1. This shows that U_1 and U_2 are conjugate and completes our proof. □

In case (1) of Lemma 5.7, we have stated the equality $N^\circ(U) = UC^\circ(U)$ between parentheses in order to depreciate this aspect not true in the algebraic case. A conclusion closer to the algebraic case would be case (2) combined with case (1) without this aspect. But even in the well described context of [CJ04] there are potentially Borel subgroups as in case (2) but not as in case (1) without this aspect (in sets of Borel subgroups usually denoted by $\mathfrak{B}$ in [CJ04]).

If the ambient group G is $*$-locally$^\circ$ solvable in Lemma 5.7, then one sees by the same argument, and using the results of Section 4.1 adapted to the $*$-locally$^\circ$ solvable case, that Borel subgroups as in cases (1) and (3) have trivial intersections indeed, and are all generous. In particular a maximal p-unipotent subgroup U as in case (3) must satisfy $N^\circ(U)$ generous, and must be conjugate to one as in case (1) if it exists. But in this case one also has $N^\circ(U)$ of unbounded exponent, and thus cases (1) and (3) are mutually exclusive. It follows also that cases (2) and (3) are mutually exclusive, and as cases (1) and (2) are obviously mutually exclusive all cases are pairwise mutually exclusive, and with a generous Carter subgroup in any case. One can summarize this as follows.

Lemma 5.8. *Let G be a $*$-locally$^\circ$ solvable group of finite Morley rank such that $U_p(G)$ is nontrivial for some prime p. Then exactly one of the following two cases occurs.*

(1) *Maximal p-unipotent subgroups are conjugate in G° and Borel subgroups of the form $N^\circ(U)$, for U a maximal p-unipotent subgroup, are pairwise*

disjoint, generous, of the form $UC^\circ(U)$, and either of unbounded exponent or nilpotent of bounded exponent.

(2) *There is a maximal p-unipotent subgroup U normalized but not centralized by a nontrivial q-torus T for some (and in fact infinitely many) prime(s) $q \neq p$. Moreover T is contained in a generous Carter subgroup of G.*

As in Section 5.2 one may wish to consider the (p, ∞)-homogeneous case for some prime p, or more generally the case in which all Borel subgroups are nilpotent but now of bounded exponent. In this case any Borel subgroup is a Carter subgroup of bounded exponent, and cases (1) and (2) of Lemma 5.7 cannot occur (recall that in case (1) $N^\circ(U)$ has unbounded exponent). One can also see in this case that any two distinct Borel subgroups have a finite intersection, using Corollary 4.3.

We continue with the mere presence of a nontrivial p-unipotent subgroup for some prime p.

Lemma 5.9. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, p and q two primes (possibly the same). Assume that some nontrivial p-unipotent subgroup U of G commutes with a nontrivial q-torus T of G. Then there is a Borel subgroup B of G containing: U, a maximal p-unipotent subgroup of G, T, and a Carter subgroup of G (and also of B) which contains T and is generous in G.*

Proof. Let Q be a Carter subgroup of G containing T, which exists and is generous in G by Fact 3.32. We have Q and U in $N^\circ(T)$, and $N^\circ(T) \leq B$ for some Borel subgroup by $*$-local$^\circ_\circ$ solvability of G. Now B is the unique Borel subgroup of G containing U by the Uniqueness Theorem, here Corollary 4.3 or Corollary 4.4, and our claim follows. □

Definition 5.10. *If M is a proper definable subgroup of a group G of finite Morley rank and p a prime, we say that*

(1) *M is* p-weakly embedded *in G if M has infinite p-subgroups and $M \cap M^g$ has no infinite p-subgroups for any g in $G \setminus M$.*

(2) *M is* p-strongly embedded *in G if M has nontrivial p-subgroups and $M \cap M^g$ has no nontrivial p-subgroups for any g in $G \setminus M$.*

Again the following remarks were obviously made around [CJ04], but were not explicitely stated there in order to keep that paper not too long.

Lemma 5.11. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank, p a prime, U_1 and U_2 two distinct (nontrivial) maximal p-unipotent subgroups of G. Then p-subgroups of $N(U_1) \cap N(U_2)$ are exceptional and have order at most $e(G)$.*

Proof. By assumption, $N^\circ(U_1)$ and $N^\circ(U_2)$ are two distinct Borel subgroups of G.

Assume toward a contradiction that $N(U_1) \cap N(U_2)$ contains a p-subgroup X with $C^\circ(X)$ solvable. We then have $C^\circ(X) \leq B$ for some Borel subgroup B. Notice that $C^\circ_{U_1}(X)$ and $C^\circ_{U_2}(X)$ are both nontrivial by Fact 2.13 (2). Now Lemma 4.6 implies $N^\circ(U_1) = N^\circ(U_2)$, a contradiction. □

Corollary 5.12. *Let G be a $*$-locally$^\circ_\circ$ solvable group of finite Morley rank with G° nonsolvable. Assume that for some prime p maximal p-unipotent subgroups of G are nontrivial. Then $N(U)$ is p-weakly embedded in G for any maximal p-unipotent subgroup U of G, and p-strongly embedded whenever G is $*$-locally$^\circ$ solvable.*

Proof. Assume $N(U) \cap N(U^g)$ has an infinite p-subgroup S for some g in G. We have $S^\circ \leq N^\circ(U) \cap N^\circ(U^g)$, and as S is infinite S° is infinite as well. By Fact 2.13 (1), S° is a central product of a p-unipotent subgroup V and a p-torus T, and one of the two factors is nontrivial by assumption. Now Lemma 4.6 or Lemma 5.11 gives in any case $N^\circ(U) = N^\circ(U^g)$. Thus $g \in N(U)$.

When G is $*$-locally$^\circ$ solvable one proceeds similarly, but now the only exceptional p-element is the identity. □

We also observe that when a nontrivial p-unipotent subgroup commutes with a nontrivial p-torus, then a maximal p-torus commutes with a maximal p-unipotent subgroup by Lemma 5.9. One can then build a p-weakly embedded subgroup as for the elimination of 2-mixed type simple groups [ABC08]. If U is a definable p-unipotent subgroup of G, we denote by $U^\perp$ the definable connected subgroup $T_p(C(U))$, the subgroup of $C(U)$ generated by the definable hulls of its p-tori. By $*$-local$^\circ_\circ$ solvability, this group is solvable (for U nontrivial). One easily sees that if $[U_1, U_2] = 1$, then $U_1^\perp = U_2^\perp$. Then one observes that the graph on the set of nontrivial p-unipotent subgroups, where adjacency is commutation, is not connected, as otherwise $U^\perp$ is independent of the choice of U, hence normal in G, and as it is nontrivial connected and solvable, G° is solvable by $*$-local$^\circ_\circ$ solvability, a contradiction to the assumption. The group G acts naturally on this graph. Let M be the normalizer in G of a connected component $\mathcal{C}$. If $U \in \mathcal{C}$, then $M \leq N(U^\perp)$. In particular M° is solvable, and M has a unique maximal p-unipotent subgroup U. Hence $M = N(U) = N(U^\perp)$. Notice that $B = M^\circ$ is a Borel subgroup of G. But in any case one shows that M is p-weakly embedded in G.

With $p = 2$ these notions suffice to eliminate connected non-solvable mixed type $*$-locally$^\circ_\circ$ solvable groups in [DJ10], by methods and/or results from the simple case. For $p \neq 2$ Configuration 3.15 stands around. Prüfer ranks are controlled via the notion of a strongly embedded subgroup in [DJ07b].

If one is not interested in conjugacy in Lemma 5.7 but merely in genericity, then one can notice that a connected $*$-locally$^\circ_\circ$ solvable group with $U_p(G)$ nontrivial but no generous Borel subgroup must satisfy that $N^\circ(U)$ is a Carter subgroup of bounded exponent for each maximal p-unipotent subgroup U; otherwise $N^\circ(U)$ has unbounded exponent and one gets as in the proof of Lemma 5.7 either a nontrivial decent torus or $N^\circ(U)$ generous, a contradiction to the assumption.

In particular, if the generic element of a connected $*$-locally$^\circ_\circ$ solvable group G of finite Morley rank is not in a connected nilpotent subgroup, then G contains no decent tori (Fact 3.32), contains nontrivial p-unipotent subgroups (Facts 5.1 and 5.6), and $N^\circ(U)$ is a Carter subgroup of bounded exponent for each such maximal p-unipotent subgroup U, generically composed of exceptional elements by Lemma 4.11.

5.4 The toral homogeneous case

We shall now consider the case in which there is no bounded exponent subgroup, and more specifically the toral homogeneous case. Before studying this specific case precisely, we note that Carter subgroups are conjugate in any $*$-locally$^\circ_\circ$ solvable group G of finite Morley rank such that $d(G) < \infty$, by the same proof as in [Fré08].

As in Section 5.1, it is natural to call a $*$-locally$^\circ_\circ$ solvable group in which Borel subgroups are all abelian a *$*$-locally$^\circ_\circ$ abelian* group. Requiring in addition that Borel subgroups are divisible is equivalent to requiring that $d(G) < \infty$.

Theorem 5.13. *Let G be a $*$-locally$^\circ_\circ$ abelian group of finite Morley rank satisfying $d(G) < \infty$. Assume furthermore that nontrivial toral elements are not exceptional and that G contains no involution. Then, either*

(1) *G° is abelian, or*

(2) *G° is a full Frobenius group, with malnormal subgroup some (any) Borel subgroup T.*

We will use the following fact.

Fact 5.14 (Ali's Lemma)**.** *Let G be a group, T_1 and T_2 two disjoint subgroups, $x_1 \in T_1 \cap (N(T_2) \setminus T_2)$ and $x_2 \in T_2 \cap (N(T_1) \setminus T_1)$ satisfying $x_1 T_2 = {x_1}^{T_2}$, $x_2 T_1 = {x_2}^{T_1}$, and $(x_1^2)T_2 = (x_1^2)^{T_2}$. Then x_1 and x_1^2 are conjugate in G. In particular, if x_1 has prime order $p \neq 2$ and is central in T_1, $N(T_1)$ controls fusion in T_1, and $N(T_1)/T_1$ is finite, then some nontrivial prime divisor of $N(T_1)/T_1$ divides $p-1$.*

Proof. This is one of the essential contents of [Nes89], already re-employed through the scope of [CJ04, Lemma 7.23]. By the fusion assumptions one can conjugate x_1 to $x_1 x_2$ in $x_1 T_2$, $x_1 x_2$ to $x_1^2 x_2$ in $x_2 T_1$, and $x_1^2 x_2$ to x_1^2 in $x_1^2 T_2$, which yields the G-conjugacy of x_1 and x_1^2.

For the second point we now have a nontrivial induced automorphism of $\langle x_1 \rangle$ in $N(T_1)/T_1$, and the cyclic group $\langle x_1 \rangle$ of prime order p has an automorphism group of order $p-1$. □

We now proceed to the proof of Theorem 5.13.

Proof. In order to prove Theorem 5.13 we now consider G a connected $*$-locally$^\circ_\circ$ abelian group, and fix a Borel subgroup T, which is divisible abelian by assumption. If G is solvable, then $G = T$ and we are in case (1). So we may assume G not solvable.

As in Lemma 5.2, any two Borel subgroups T_1 and T_2 must have a finite intersection E, and being a finite subgroup of a divisible abelian group E must be toral if it is nontrivial. If $E \neq 1$, then T_1, $T_2 \leq C^\circ(E)$, and one gets either $T_1 = T_2$ when $C^\circ(E)$ is solvable, or a nontrivial exceptional toral subgroup otherwise, which is excluded by assumption. Hence distinct Borel subgroups are pairwise disjoint. As usual, each is generous, and they are all conjugate.

As any element of G also has an infinite centralizer, any such element must centralize an infinite abelian subgroup by Lemma 2.6 (1), and in particular normalizes the unique conjugate of T containing it.

This shows that $G = N(T)^G$. If $N(T) = T$, then T is malnormal in G by disjointness of pairwise distinct Borel subgroups, and $G = T^G$, and thus G is a full Frobenius group with malnormal subgroup T, as desired.

Hence the analysis boils down to showing that T is selfnormalizing. Assume on the contrary $T < N(T)$, and let x be an element of order p modulo T for some prime $p > 2$ (as G contains no involutions), which may be assumed to be itself a p-element of G by Fact 5.6, and in fact inside a p-torus.

By conjugacy, one concludes that T contains a maximal p-torus T_p which is nontrivial. Now x is in a conjugate T_p^g of T_p and $x^p \in T_p$. As $T^g \cap T = 1$, as otherwise $T = T^g$ and $x \in T^g = T$, $x^p \in T \cap T^g = 1$. For any element y in xT, the definable hull $H(y)$ of y contains also a p-element y_1 by Fact 5.6, which similarly belongs to a maximal torus T_1 distinct from T. Now $C_T^\circ(y) \leq C_T^\circ(y_1) \leq T \cap T_1$, and thus any element y in xT has a finite centralizer in T. Hence y^T is generic in xT for any y in xT, and as the Morley degree is one one gets $xT = x^T$. Now x normalizes T and centralizes a nontrivial element z in the elementary abelian p-subgroup of T. We have z normalizing T_x, the torus containing x, without being inside, and similarly $zT_x = z^{T_x}$ (this is typical of [Nes89]. See also [CJ04, Lemma 7.19]). We are now in situation to apply Fact 5.14. Noticing that $N(T)$ controls fusion in the torsion subgroup of T by Corollary 2.12, this gives a contradiction by choosing for p the smallest prime divisor of $N(T)/T$. □

We note similarities between groups as in Theorem 5.13 (2) with those of [JOH04]. These are far from being stable by [JMN08], but there are some hints for the existence of (at least partially) stable such groups, as envisioned in [Jal08a, §1].

We also note that a reduction to Fact 5.14 yields involutions or the triviality of the Weyl group in any group of finite Morley rank without non-trivial p-unipotent subgroups [BC07]. In particular for the last paragraph of the proof of Theorem 5.13 we could have refered to this.

Using the triviality of Weyl groups in connected groups without involutions and without p-unipotent subgroups, one can give a general decomposition as in Theorem 5.13 without non-exceptionality assumption, with the Galois connection of Section 3.4.

Theorem 5.15. *Let G be a $*$-locally$^\circ_\circ$ abelian group of finite Morley rank satisfying $d(G) < \infty$, and without involutions. Then G° has an abelian generous selfnormalizing Carter subgroup T such that $G^\circ = T^{G^\circ}$.*

Proof. This is similar to the proof of Theorem 5.13, using Lemma 5.2 for the generosity of (divisible abelian) Borel subgroups T. Notice also that such (conjugate) Borel subgroups T are selfnormalizing by the above mentioned result of [BC07], or a more direct reduction to Fact 5.14 here, and cover G° by the same argument as in the proof of Theorem 5.13 again. □

In general one cannot say much more in Theorem 5.15, except describing the full group G° by the graph of finite exceptional closed subsets of the divisible abelian Borel subgroup T introduced at the end of Section 3.4 and delineated in Lemma 3.31. In fact, exceptional subsets of T are in the divisible torsion subgroup of T, and in a finite subset of it by Lemma 3.28. One easily sees that closed exceptional subsets of T correspond exactly to intersections of T with distinct conjugates of T. If $(X_0, \cdots, X_k)$ is a maximal chain of exceptional closed subsets of T in the graph of exceptional closed subsets of T (i.e., with (X_i, X_{i+1}) a minimal extension for each i), then $X_0 = Z(G^\circ)$, $C^\circ(X_i) = C_{G^\circ}(X_i)$ for each i (by a Frattini Argument following the conjugacy of generous Carter subgroups and by triviality of the Weyl group $N(T)/T$ in G°) and the center of this group is X_i, and each group $C^\circ(X_i)/X_i$ also satisfies the assumptions of Theorem 5.15, with decreasing exception indices as i increases. The last factor $C^\circ(X_k)/X_k$ is as in Theorem 5.13 by Lemma 3.29.

Hence any group as in Theorem 5.15 is entirely described as above by the finite graph of exceptional closed subsets of T. In particular the picture in Theorem 5.15 looks like Configuration 3.15, where all Borel subgroups involved are decent tori but potentially with more complexity involved in the finite graph of exceptional subsets of T. As for Theorem 5.13, constructions of such abstract groups can be obtained as in [JOH04] with *any* finite graph for T (compatible with the conditions of Lemma 3.31), similarly with a bad control on the complexity of their model theory by the general construction but perhaps with some stability if more care is taken.

The case of groups as in Theorems 5.13 and 5.15 but with involutions will be considered in [DJ07a], and eventually disappear by the analysis of this paper and the contents of [Nes89].

References

[ABC08] T. Altınel, A. V. Borovik, and G. Cherlin. *Simple groups of finite Morley rank*, volume 145 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[Bau96] A. Baudisch. A new uncountably categorical group. *Trans. Amer. Math. Soc.*, 348(10):3889–3940, 1996.

[BBC07] A. Borovik, J. Burdges, and G. Cherlin. Involutions in groups of finite Morley rank of degenerate type. *Selecta Math. (N.S.)*, 13(1):1–22, 2007.

[BC02] J. Burdges and G. Cherlin. Borovik-Poizat rank and stability. *J. Symbolic Logic*, 67(4):1570–1578, 2002.

[BC07] J. Burdges and G. Cherlin. Semisimple torsion in groups of finite Morley rank. Preprint: arXiv:0801.3953, 2007.

[BCJ07] J. Burdges, G. Cherlin, and E. Jaligot. Minimal connected simple groups of finite Morley rank with strongly embedded subgroups. *J. Algebra*, 314(2):581–612, 2007.

[Ben70a] H. Bender. On groups with abelian Sylow 2-subgroups. *Math. Z.*, 117:164–176, 1970.

[Ben70b] H. Bender. On the uniqueness theorem. *Illinois J. Math.*, 14:376–384, 1970.

[BG94] H. Bender and G. Glauberman. *Local analysis for the odd order theorem*. Cambridge University Press, Cambridge, 1994. With the assistance of Walter Carlip.

[Bir67] G. Birkhoff. *Lattice theory*. American Mathematical Society, Providence, R.I., 1967. Third edition. American Mathematical Society Colloquium Publications, Vol. XXV.

[BN92] A. V. Borovik and A. Nesin. On the Schur-Zassenhaus theorem for groups of finite Morley rank. *J. Symbolic Logic*, 57(4):1469–1477, 1992.

[BN94] A. Borovik and A. Nesin. *Groups of finite Morley rank*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[BP90] A. V. Borovik and B. P. Poizat. Tores et p-groupes. *J. Symbolic Logic*, 55(2):478–491, 1990.

[Bur04a] J. Burdges. Odd and degenerate types groups of finite Morley rank. Doctoral Dissertation, Rutgers University, 2004.

[Bur04b] J. Burdges. A signalizer functor theorem for groups of finite Morley rank. *J. Algebra*, 274(1):215–229, 2004.

[Bur06] J. Burdges. Sylow theory for $p = 0$ in solvable groups of finite Morley rank. *J. Group Theory*, 9(4):467–481, 2006.

[Bur07] J. Burdges. The Bender method in groups of finite Morley rank. *J. Algebra*, 312(1):33–55, 2007.

[Che79] G. Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17(1-2):1–28, 1979.

[Che05] G. Cherlin. Good tori in groups of finite Morley rank. *J. Group Theory*, 8(5):613–622, 2005.

[CJ04] G. Cherlin and E. Jaligot. Tame minimal simple groups of finite Morley rank. *J. Algebra*, 276(1):13–79, 2004.

[Del07a] A. Deloro. Groupes simples connexes minimaux algébriques de type impair. *J. Algebra*, 317(2):877–923, 2007.

[Del07b] A. Deloro. Groupes simples connexes minimaux de type impair. Thèse de doctorat, Université de Paris 7, 2007.

[Del08] A. Deloro. Groupes simples connexes minimaux non-algébriques de type impair. *J. Algebra*, 319(4):1636–1684, 2008.

[DJ07a] A. Deloro and E. Jaligot. Groups of finite Morley rank with solvable local subgroups and of odd type. In preparation, 2007.

[DJ07b] A. Deloro and E. Jaligot. Lie rank in groups of finite Morley rank with solvable local subgroups. Preprint, arXiv:0802.1394, Submitted, 2007.

[DJ10] A. Deloro and E. Jaligot. Small groups of finite Morley rank with involutions. *J. Reine Angew. Math.*, 644:23–45, 2010.

[FJ05] O. Frécon and E. Jaligot. The existence of Carter subgroups in groups of finite Morley rank. *J. Group Theory*, 8(5):623–644, 2005.

[FJ08] O. Frécon and E. Jaligot. Conjugacy in groups of finite Morley rank. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 1–58. Cambridge Univ. Press, Cambridge, 2008.

[Fré06a] O. Frécon. Around unipotence in groups of finite Morley rank. *J. Group Theory*, 9(3):341–359, 2006.

[Fré06b] O. Frécon. Carter subgroups in tame groups of finite Morley rank. *J. Group Theory*, 9(3):361–367, 2006.

[Fré08] O. Frécon. Conjugacy of Carter subgroups in groups of finite Morley rank. *J. Math. Log.*, 8(1):41–92, 2008.

[FT63] W. Feit and J. G. Thompson. Solvability of groups of odd order. *Pacific J. Math.*, 13:775–1029, 1963.

[Gag76] T. M. Gagen. *Topics in finite groups.* Cambridge University Press, Cambridge, 1976. London Mathematical Society Lecture Notes Series, No. 16.

[GLS94] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups.* American Mathematical Society, Providence, RI, 1994.

[Jal99] E. Jaligot. Groupes de type mixte. *J. Algebra*, 212(2):753–768, 1999.

[Jal00] E. Jaligot. FT-groupes. Prépublication de l'Institut Girard Desargues 33, CNRS UPRESA 5028, Lyon, Janvier 2000, `http://math.univ-lyon1.fr/~jaligot/publ.html`, 2000.

[Jal01a] E. Jaligot. Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem. *Bull. Symbolic Logic*, 7(3):315–328, 2001.

[Jal01b] E. Jaligot. Groupes de rang de Morley fini de type pair avec un sous-groupe faiblement inclus. *J. Algebra*, 240(2):413–444, 2001.

[Jal06] E. Jaligot. Generix never gives up. *J. Symbolic Logic*, 71(2):599–610, 2006.

[Jal08a] E. Jaligot. Groups of finite dimension in model theory. In C. Glymour, W. Wang, and D. Westerstahl, editors, *Proceedings from the 13th International Congress of Logic, Methodology, and Philosophy of Sciences, Beijing, august 2007*. Studies in Logic and the Foundations of Mathematics, King's College Publications, London, 2008.

[Jal08b] E. Jaligot. Simple groups of finite Morley rank. In *Logic Colloquium 2005*, volume 28 of *Lect. Notes Log.*, pages 57–67. Assoc. Symbol. Logic, Urbana, IL, 2008.

[Jal09] E. Jaligot. Cosets, genericity, and the Weyl group. *J. Algebra*, 322(4):1060–1071, 2009.

[JMN08] E. Jaligot, A. Muranov, and A. Neman. Independence property and hyperbolic groups. *Bull. Symbolic Logic*, 14(1):88–98, 2008.

[JOH04] E. Jaligot and A. Ould Houcine. Existentially closed CSA-groups. *J. Algebra*, 280(2):772–796, 2004.

[Nes89] A. Nesin. Nonsolvable groups of Morley rank 3. *J. Algebra*, 124(1):199–218, 1989.

[Nes90] A. Nesin. On solvable groups of finite Morley rank. *Trans. Amer. Math. Soc.*, 321(2):659–690, 1990.

[Nes91] A. Nesin. Poly-separated and ω-stable nilpotent groups. *J. Symbolic Logic*, 56(2):694–699, 1991.

[Tho68] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Amer. Math. Soc.*, 74:383–437, 1968.

[Tho70] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. II. *Pacific J. Math.*, 33:451–536, 1970.

[Tho71] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. III. *Pacific J. Math.*, 39:483–534, 1971.

[Tho73] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. IV, V, VI. *Pacific J. Math.*, 48:511–592, 50 (1974) 215–297, 51 (1974) 573–630, 1973.

[Wag01] F. Wagner. Fields of finite Morley rank. *J. Symbolic Logic*, 66(2):703–706, 2001.

[Wag03] F. O. Wagner. Bad fields in positive characteristic. *Bull. London Math. Soc.*, 35(4):499–502, 2003.

[Zil77] B. I. Zil′ber. Groups and rings whose theory is categorical. *Fund. Math.*, 95(3):173–188, 1977.