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▶ **To cite this version:**

Nadine Badr, Emmanuel Russ. Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs. Publicacions Matemàtiques, 2009, 53 (2), pp.273-328. hal-00243816

HAL Id: hal-00243816
https://hal.science/hal-00243816

Submitted on 7 Feb 2008

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Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs

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February 7, 2008

hal-00243816, version 1 - 7 Feb 2008

Abstract. Let Γ be a graph endowed with a reversible Markov kernel p, and P the associated operator, defined by $Pf(x) = \sum_y p(x,y)f(y)$. Denote by ∇ the discrete gradient. We give necessary and/or sufficient conditions on Γ in order to compare $\|\nabla f\|_p$ and $\left\|(I-P)^{1/2}f\right\|_p$ uniformly in f for $1 < p < +\infty$. These conditions are different for $p < 2$ and $p > 2$. The proofs rely on recent techniques developed to handle operators beyond the class of Calderón-Zygmund operators. For our purpose, we also prove Littlewood-Paley inequalities and interpolation results for Sobolev spaces in this context, which are of independent interest.

AMS numbers 2000: Primary: 60J10. Secondary: 42B20, 42B25.

Keywords: Graphs, discrete Laplacian, Riesz transforms, Littlewood-Paley inequalities, Sobolev spaces, interpolation.

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1 Introduction and results

It is well-known that, if $n \geq 1$, $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ and $\left\|(-\Delta)^{1/2} f\right\|_{L^p(\mathbb{R}^n)}$ are comparable uniformly in f for all $1 < p < +\infty$. This fact means that the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ defined by means of the gradient coincides with the Sobolev space defined through the Laplace operator. This is interesting in particular because ∇ is a local operator, while $(-\Delta)^{1/2}$ is not.

Generalizations of this result to geometric contexts can be given. On a Riemannian manifold M, it was asked by Strichartz in [47] whether, if $1 < p < +\infty$, there exists $C_p > 0$ such that, for all function $f \in C_0^\infty(M)$,

$$C_p^{-1} \left\|\Delta^{1/2} f\right\|_p \leq \left\||df|\right\|_p \leq C_p \left\|\Delta^{1/2} f\right\|_p, \tag{1.1}$$

where Δ stands for the Laplace-Beltrami operator on M and d for the exterior differential. Under suitable assumptions on M, which can be formulated, for instance, in terms of the volume growth of balls in M, uniform L^2 Poincaré inequalities on balls of M, estimates on the heat semigroup (*i.e.* the semigroup generated by Δ) or the Ricci curvature, each of the two inequalities contained in (1.1) holds for a range of p's (which is, in general, different for the two inequalities). The second inequality in (1.1) means that the Riesz transform $d\Delta^{-1/2}$ is L^p-bounded. We refer to ([3, 5, 9, 22]) and the references therein.

In the present paper, we consider a graph equipped with a discrete gradient and a discrete Laplacian and investigate the corresponding counterpart of (1.1). To that purpose, we prove, among other things, an interpolation result for Sobolev spaces defined via the differential, similar to those already considered in [42], as well as L^p bounds for Littlewood-Paley functionals.

1.1 Presentation of the discrete framework

Let us give precise definitions of our framework. The following presentation is partly borrowed from [27]. Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. We call (Γ, μ) a weighted graph. In the sequel, we write most of the time Γ instead of (Γ, μ), somewhat abusively. If $x, y \in \Gamma$, say that $x \sim y$ if and only if $\mu_{xy} > 0$. Denote by E the set of edges in Γ, *i.e.*

$$E = \{(x, y) \in \Gamma \times \Gamma;\ x \sim y\},$$

and notice that, due to the symmetry of μ, $(x, y) \in E$ if and only if $(y, x) \in E$.

For $x, y \in \Gamma$, a path joining x to y is a finite sequence of edges $x_0 = x, ..., x_N = y$ such that, for all $0 \leq i \leq N-1$, $x_i \sim x_{i+1}$. By definition, the length of such a path is N. Assume that Γ is connected, which means that, for all $x, y \in \Gamma$, there exists a path joining x to y. For all $x, y \in \Gamma$, the distance between x and y, denoted by $d(x, y)$, is the shortest length of a path joining x and y. For all $x \in \Gamma$ and all $r \geq 0$, let $B(x, r) = \{y \in \Gamma,\ d(y, x) \leq r\}$. In the sequel, we always assume that Γ is locally uniformly finite, which means that there exists $N \in \mathbb{N}^*$ such that, for all $x \in \Gamma$, $\sharp B(x, 1) \leq N$(here and after, $\sharp A$ denotes the cardinal of any subset A of Γ). If $B = B(x, r)$ is a ball, set $\alpha B = B(x, \alpha r)$ for all $\alpha > 0$, and write $C_1(B) = 4B$ and $C_j(B) = 2^{j+1}B \setminus 2^j B$ for all integer $j \geq 2$.
For any subset $A \subset \Gamma$, set

$$\partial A = \{x \in A;\ \exists y \sim x,\ y \notin A\}.$$

For all $x \in \Gamma$, set $m(x) = \sum_{y \sim x} \mu_{xy}$. We always assume in the sequel that $m(x) > 0$ for all $x \in \Gamma$. If $A \subset \Gamma$, define $m(A) = \sum_{x \in A} m(x)$. For all $x \in \Gamma$ and $r > 0$, write $V(x, r)$ instead of $m(B(x, r))$ and, if B is a ball, $m(B)$ will be denoted by $V(B)$.

For all $1 \leq p < +\infty$, say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p \, m(x) \right)^{1/p} < +\infty.$$

Say that $f \in L^\infty(\Gamma, m)$ (or $L^\infty(\Gamma)$) if

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

Define $p(x, y) = \mu_{xy}/m(x)$ for all $x, y \in \Gamma$. Observe that $p(x, y) = 0$ if $d(x, y) \geq 2$. Set also

$$p_0(x, y) = \delta(x, y)$$

and, for all $k \in \mathbb{N}$ and all $x, y \in \Gamma$,

$$p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z) p_k(z, y).$$

The p_k's are called the iterates of p. Notice that, for all $x \in \Gamma$, there are at most N non-zero terms in this sum. Observe also that, for all $x \in \Gamma$,

$$\sum_{y \in \Gamma} p(x, y) = 1 \tag{1.2}$$

and, for all $x, y \in \Gamma$,

$$p(x,y)m(x) = p(y,x)m(y). \tag{1.3}$$

For all function f on Γ and all $x \in \Gamma$, define

$$Pf(x) = \sum_{y \in \Gamma} p(x,y) f(y)$$

(again, this sum has at most N non-zero terms). Since $p(x,y) \geq 0$ for all $x, y \in \Gamma$ and (1.2) holds, one has, for all $p \in [1, +\infty]$ and all $f \in L^p(\Gamma)$,

$$\|Pf\|_{L^p(\Gamma)} \leq \|f\|_{L^p(\Gamma)}. \tag{1.4}$$

We make use of the operator P to define a Laplacian on Γ. Consider a function $f \in L^2(\Gamma)$. By (1.4), $(I-P)f \in L^2(\Gamma)$ and

$$\begin{aligned} \langle (I-P)f, f \rangle_{L^2(\Gamma)} &= \sum_{x,y} p(x,y)(f(x) - f(y)) f(x) m(x) \\ &= \frac{1}{2} \sum_{x,y} p(x,y) \left| f(x) - f(y) \right|^2 m(x), \end{aligned} \tag{1.5}$$

where we use (1.2) in the first equality and (1.3) in the second one. If we define now the operator "length of the gradient" by

$$\nabla f(x) = \left(\frac{1}{2} \sum_{y \in \Gamma} p(x,y) \left| f(y) - f(x) \right|^2 \right)^{1/2}$$

for all function f on Γ and all $x \in \Gamma$ (this definition is taken from [23]), (1.5) shows that

$$\langle (I-P)f, f \rangle_{L^2(\Gamma)} = \|\nabla f\|^2_{L^2(\Gamma)}. \tag{1.6}$$

Because of (1.3), the operator P is self-adjoint on $L^2(\Gamma)$ and $I - P$, which, by (1.6) , can be considered as a discrete "Laplace" operator, is non-negative and self-adjoint on $L^2(\Gamma)$. By means of spectral theory, one defines its square root $(I-P)^{1/2}$. The equality (1.6) exactly means that

$$\left\| (I-P)^{1/2} f \right\|_{L^2(\Gamma)} = \|\nabla f\|_{L^2(\Gamma)}. \tag{1.7}$$

This equality has an interpretation in terms of Sobolev spaces defined through ∇. Let $1 \leq p \leq +\infty$. Say that a scalar-valued function f on Γ belongs to the (inhomogeneous) Sobolev space $W^{1,p}(\Gamma)$ (see also [42], [34]) if and only if

$$\|f\|_{W^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \|\nabla f\|_{L^p(\Gamma)} < +\infty.$$

If B is any ball in Γ and $1 \leq p \leq +\infty$, denote by $W^{1,p}_0(B)$ the subspace of $W^{1,p}(\Gamma)$ made of functions supported in B.

We will also consider the homogeneous versions of Sobolev spaces. For $1 \leq p \leq +\infty$, define $\dot{E}^{1,p}(\Gamma)$ as the space of all scalar-valued functions f on Γ such that $\nabla f \in L^p(\Gamma)$, equipped with the semi-norm

$$\|f\|_{\dot{E}^{1,p}(\Gamma)} := \|\nabla f\|_{L^p(\Gamma)}.$$

Then $\dot{W}^{1,p}(\Gamma)$ is the quotient space $\dot{E}^{1,p}(\Gamma)/\mathbb{R}$, equipped with the corresponding norm. It is then routine to check that both inhomogeneous and homogeneous Sobolev spaces on Γ are Banach spaces.

The equality (1.7) means that $\left\|(I-P)^{1/2}f\right\|_{L^2(\Gamma)} = \|f\|_{\dot{E}^{1,2}(\Gamma)}$. In other words, for $p=2$, the Sobolev spaces defined by ∇ and by the Laplacian coincide. In the sequel, we address the analogous question for $p \neq 2$.

1.2 Statement of the problem

To that purpose, we consider separately two inequalities, the validity of which will be discussed in the sequel. Let $1 < p < +\infty$. The first inequality we look at says that there exists $C_p > 0$ such that, for all function f on Γ such that $(I-P)^{1/2}f \in L^p(\Gamma)$,

$$\|\nabla f\|_p \leq C_p \left\|(I-P)^{1/2}f\right\|_p. \tag{R_p}$$

This inequality means that the operator $\nabla(I-P)^{-1/2}$, which is nothing but the *Riesz transform* associated with $(I-P)$, is $L^p(\Gamma)$-bounded. Here and after, say that a (sub)linear operator T is L^p-bounded, or is of strong type (p,p), if there exists $C>0$ such that $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(\Gamma)$. Say that it is of weak type (p,p) if there exists $C>0$ such that $m(\{x \in \Gamma,\ |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda^p}\|f\|_p^p$ for all $f \in L^p(\Gamma)$ and all $\lambda > 0$. Notice that he functions f will be defined on Γ, whereas Tf may be defined on Γ or on E.
The second inequality under consideration says that there exists $C_p > 0$ such that, for all function $f \in \dot{E}^{1,p}(\Gamma)$,

$$\left\|(I-P)^{1/2}f\right\|_p \leq C_p \|\nabla f\|_p. \tag{RR_p}$$

(The notations (R_p) and (RR_p) are borrowed from [3].) We have just seen, by (1.7), that (R_2) and (RR_2) always hold. A well-known fact (see [43] for a proof in this context) is that, if (R_p) holds for some $1 < p < +\infty$, then $(RR_{p'})$ holds with p' such that $1/p + 1/p' = 1$, while the converse is unclear in this discrete situation (it is false in the case of Riemannian manifolds, see [3]). As we will see, we have to consider four distinct issues: (R_p) for $p<2$, (R_p) for $p>2$, (RR_p) for $p<2$, (RR_p) for $p>2$.

1.3 The L^p-boundedness of the Riesz transform

1.3.1 The case when $p < 2$

Let us first consider (R_p) when $p<2$. This problem was dealt with in [43], and we just recall the result proved therein, which involves some further assumptions on Γ. The first one is of geometric nature. Say that (Γ, μ) satisfies the doubling property if there exists $C>0$ such that, for all $x \in \Gamma$ and all $r>0$,

$$V(x,2r) \leq CV(x,r). \tag{D}$$

Note that this assumption implies that there exist $C, D > 0$ such that, for all $x \in \Gamma$, all $r > 0$ and all $\theta > 1$,

$$V(x, \theta r) \leq C\theta^D V(x, r). \tag{1.8}$$

Remark 1.1 *Observe also that, since (Γ, μ) is infinite, it is also unbounded (since it is locally uniformly finite) so that, if (D) holds, then $m(\Gamma) = +\infty$ (see [40]).*

The second assumption on (Γ, μ) is a uniform lower bound for $p(x, y)$ when $x \sim y$, *i.e.* when $p(x, y) > 0$. For $\alpha > 0$, say that (Γ, μ) satisfies the condition $\Delta(\alpha)$ if, for all $x, y \in \Gamma$,

$$(x \sim y \Leftrightarrow \mu_{xy} \geq \alpha m(x)) \text{ and } x \sim x. \tag{$\Delta(\alpha)$}$$

The next two assumptions on (Γ, μ) are pointwise upper bounds for the iterates of p. Say that (Γ, μ) satisfies (DUE) (a on-diagonal upper estimate for the iterates of p) if there exists $C > 0$ such that, for all $x \in \Gamma$ and all $k \in \mathbb{N}^*$,

$$p_k(x, x) \leq \frac{Cm(x)}{V(x, \sqrt{k})}. \tag{DUE}$$

Say that (Γ, μ) satisfies (UE) (an upper estimate for the iterates of p) if there exist $C, c > 0$ such that, for all $x, y \in \Gamma$ and all $k \in \mathbb{N}^*$,

$$p_k(x, y) \leq \frac{Cm(x)}{V(x, \sqrt{k})} e^{-c\frac{d^2(x,y)}{k}}. \tag{UE}$$

Recall that, under assumption (D), estimates (DUE) and (UE) are equivalent (and the conjunction of (D) and (DUE) is also equivalent to a Faber-Krahn inequality, [23], Theorem 1.1). The following result holds:

Theorem 1.2 *([43]) Under assumptions (D), ($\Delta(\alpha)$) and (DUE), (R_p) holds for all $1 < p \leq 2$. Moreover, the Riesz transform is of weak $(1, 1)$ type, which means that there exists $C > 0$ such that, for all $\lambda > 0$ and all function $f \in L^1(\Gamma)$,*

$$m\left(\left\{x \in \Gamma;\ \nabla(I - P)^{-1/2} f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda} \|f\|_1 .$$

As a consequence, under the same assumptions, (RR_p) holds for all $2 \leq p < +\infty$.

Notice that, according to [37], the assumptions of Theorem 1.2 hold, for instance, when Γ is the Cayley graph of a group with polynomial volume growth and $p(x, y) = h(y^{-1}x)$, where h is a symmetric bounded probability density supported in a ball and bounded from below by a positive constant on an open generating neighborhood of e, the identity element of G, and actually Theorem 1.2 had already been proved in [37].

1.3.2 The case when $p > 2$

When $p > 2$, assumptions (D), (UE) and $(\Delta(\alpha))$ are not sufficient to ensure the validity of (R_p), as the example of two copies of $\mathbb{Z}^2$ linked between with an edge shows (see [43], Section 4). More precisely, in this example, as explained in Section 4 of [43], the validity of (R_p) for $p > 2$ would imply an L^2 Poincaré inequality on balls.
Say that (Γ, μ) satisfies a scaled L^2 Poincaré inequality on balls (this inequality will be denoted by (P_2) in the sequel) if there exists $C > 0$ such that, for any $x \in \Gamma$, any $r > 0$ and any function f locally square integrable on Γ such that ∇f is locally square integrable on E,

$$\sum_{y \in B(x,r)} |f(y) - f_B|^2 \, m(y) \leq C r^2 \sum_{y \in B(x,r)} |\nabla f(y)|^2 \, m(y), \tag{P_2}$$

where

$$f_B = \frac{1}{V(B)} \sum_{x \in B} f(x) m(x)$$

is the mean value of f on B. Under assumptions (D), (P_2) and $(\Delta(\alpha))$, not only does (UE) hold, but the iterates of p also satisfy a pointwise Gaussian lower bound. Namely, there exist $c_1, C_1, c_2, C_2 > 0$ such that, for all $n \geq 1$ and all $x, y \in \Gamma$ with $d(x, y) \leq n$,

$$\frac{c_1 m(x)}{V(x, \sqrt{n})} e^{-C_1 \frac{d^2(x,y)}{n}} \leq p_n(x, y) \leq \frac{C_2 m(x)}{V(x, \sqrt{n})} e^{-c_2 \frac{d^2(x,y)}{n}}. \tag{LUE}$$

Actually, (LUE) is equivalent to the conjunction of (D), (P_2) and $(\Delta(\alpha))$, and also to a discrete parabolic Harnack inequality, see [27] (see also [4] for another approach of (LUE)).

Let $p > 2$ and assume that (R_p) holds. Then, if $f \in L^p(\Gamma)$ and $n \geq 1$,

$$\|\nabla P^n f\|_p \leq \frac{C_p}{\sqrt{n}} \|f\|_p . \tag{G_p}$$

Indeed, (R_p) implies that

$$\|\nabla P^n f\|_p \leq C_p \left\|(I - P)^{1/2} P^n f\right\|_p ,$$

and, due to the analyticity of P on $L^p(\Gamma)$, one also has

$$\left\|(I - P)^{1/2} P^n f\right\|_p \leq \frac{C'_p}{\sqrt{n}} \|f\|_p .$$

More precisely, as was explained in [43], assumption $\Delta(\alpha)$ implies that -1 does not belong to the spectrum of P on $L^2(\Gamma)$. As a consequence, P is analytic on $L^2(\Gamma)$ (see [25], Proposition 3), and since P is submarkovian, P is also analytic on $L^p(\Gamma)$ (see [25], p. 426). Proposition 2 in [25] therefore yields the second inequality in (G_p). Thus, condition (G_p) is necessary for (R_p) to hold. Our first result is that, under assumptions (D), (P_2) and $(\Delta(\alpha))$, for all $q > 2$, condition (G_q) is also sufficient for (R_p) to hold for all $2 < p < q$:

Theorem 1.3 *Let $p_0 \in (2, +\infty]$. Assume that (Γ, μ) satisfies (D), (P_2), $(\Delta(\alpha))$ and (G_{p_0}). Then, for all $2 \leq p < p_0$, (R_p) holds. As a consequence, if p'_0 is such that $1/p_0 + 1/p'_0 = 1$, (RR_p) holds for all $p'_0 < p \leq 2$.*

An immediate consequence of Theorem 1.3 and the previous discussion is the following result:

Theorem 1.4 *Assume that (Γ, μ) satisfies (D), (P_2) and $(\Delta(\alpha))$. Let $p_0 \in (2, +\infty]$. Then, the following two assertions are equivalent:*

(i) for all $p \in (2, p_0)$, (G_p) holds,

(ii) for all $p \in (2, p_0)$, (R_p) holds.

Remark 1.5 *In the recent work [29], property (G_p) is shown to be true for all $p \in (1, 2]$ under the sole assumption that Γ satisfies a local doubling property for the volume of balls.*

Remark 1.6 *On Riemannian manifolds, the L^2 Poincaré inequality on balls is neither necessary, nor sufficient to ensure that the Riesz transform is L^p-bounded for all $p \in (2, \infty)$, see [3] and the references therein. We do not know if the corresponding assertion holds in the context of graphs.*

1.3.3 Riesz transforms and harmonic functions

We also obtain another characterization of the validity of (R_p) for $p > 2$ in terms of **reverse Hölder inequalities for the gradient of harmonic functions**, in the spirit of [45] (in the Euclidean context for second order elliptic operators in divergence form) and [3] (on Riemannian manifolds). If B is any ball in Γ and u a function on B, say that u is harmonic on B if, for all $x \in B \setminus \partial B$,

$$(I - P)u(x) = 0. \tag{1.9}$$

We will prove the following result:

Theorem 1.7 *Assume that (D), $(\Delta(\alpha))$ and (P_2) hold. Then, there exists $p_0 \in (2, +\infty]$ such that, for all $q \in (2, p_0)$, the following two conditions are equivalent:*

1. *(R_p) holds for all $p \in (2, q)$,*
2. *for all $p \in (2, q)$, there exists $C_p > 0$ such that, for all ball $B \subset \Gamma$, all function u harmonic in $32B$,*

$$\left(\frac{1}{V(B)} \sum_{x \in B} |\nabla u(x)|^p \, m(x) \right)^{\frac{1}{p}} \leq C_p \left(\frac{1}{V(16B)} \sum_{x \in 16B} |\nabla u(x)|^2 \, m(x) \right)^{\frac{1}{2}}. \tag{RH_p}$$

Assertion 3. says that the gradient of any harmonic function in $32B$ satisfies a reverse Hölder inequality. Remember that such an inequality always holds for solutions of $\mathrm{div}(A\nabla u) = 0$ on any ball $B \subset \mathbb{R}^n$, if u is assumed to be in $H^1(B)$ and A is bounded and uniformly elliptic (see [41]). In the present context, a similar self-improvement result can be shown:

Proposition 1.8 *Assume that (D), $(\Delta(\alpha))$ and (P_2) hold. Then there exists $p_0 > 2$ such that (RH_p) holds for any $p \in (2, p_0)$. As a consequence, (R_p) holds for any $p \in (2, p_0)$.*

As a corollary of Theorem 1.2 and Proposition 1.8, we get:

Corollary 1.9 *Assume that (D), $(\Delta(\alpha))$ and (P_2) hold. Then, there exists $\varepsilon > 0$ such that, for all $2 - \varepsilon < p < 2 + \varepsilon$, $\|\nabla f\|_p \sim \left\|(I - P)^{1/2} f\right\|_p$.*

1.4 The reverse inequality

Let us now focus on (RR_p). As already seen, (RR_p) holds for all $p > 2$ under (D), $(\Delta(\alpha))$ and (DUE), and for all $p_0' < p < 2$ under (D), (P_2), $(\Delta(\alpha))$ and (G_{p_0}) if $p_0 > 2$ and $1/p_0 + 1/p_0' = 1$. However, we can also give a sufficient condition for (RR_p) to hold for all $p \in (q_0, 2)$ (for some $q_0 < 2$) which does not involve any assumption such that (G_{p_0}). For $1 \leq p < +\infty$, say that (Γ, μ) satisfies a scaled L^p Poincaré inequality on balls (this inequality will be denoted by (P_p) in the sequel) if there exists $C > 0$ such that, for any $x \in \Gamma$, any $r > 0$ and any function f on Γ such that $|f|^p$ and $|\nabla f|^p$ are locally integrable on Γ,

$$\sum_{y \in B(x,r)} |f(y) - f_B|^p \, m(y) \leq C r^p \sum_{y \in B(x,r)} |\nabla f(y)|^p \, m(y). \tag{P_p}$$

If $1 \leq p < q < +\infty$, then (P_p) implies (P_q) (this is a very general statement on spaces of homogeneous type, *i.e.* on metric measured spaces where (D) holds, see [36]). The converse implication does not hold but an L^p Poincaré inequality still has a self-improvement in the following sense:

Proposition 1.10 *Let (Γ, μ) satisfy (D). Then, for all $p \in (1, +\infty)$, if (P_p) holds, there exists $\varepsilon > 0$ such that $(P_{p-\varepsilon})$ holds.*

This deep result actually holds in the general context of spaces of homogeneous type, *i.e.* when (D) holds, see [39].

Assuming that (P_q) holds for some $q < 2$, we establish (RR_p) for $q < p < 2$:

Theorem 1.11 *Let $1 \leq q < 2$. Assume that (D), $(\Delta(\alpha))$ and (P_q) hold. Then, for all $q < p < 2$, (RR_p) holds. Moreover, there exists $C > 0$ such that, for all $\lambda > 0$,*

$$m\left(\left\{x \in \Gamma;\ \left|(I-P)^{1/2} f(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \|\nabla f\|_q^q. \tag{1.10}$$

As a corollary of Theorem 1.2, Proposition 1.10 and Theorem 1.11, we get the following consequence:

Corollary 1.12 *Assume that (D), $(\Delta(\alpha))$ and (P_p) hold for some $p \in (1, 2)$. Then, there exists $\varepsilon > 0$ such that, for all $p - \varepsilon < q < +\infty$, (RR_q) holds. In particular, (RR_p) holds.*

1.5 An overview of the method

Let us briefly describe the proofs of our results. Let us first consider Theorem 1.3. The operator $T = \nabla(I-P)^{-1/2}$ can be written as

$$T = \nabla \left(\sum_{k=0}^{+\infty} a_k P^k \right),$$

where the a_k's are defined by the expansion

$$(1-x)^{-1/2} = \sum_{k=0}^{+\infty} a_k x^k \tag{1.11}$$

for $-1 < x < 1$. The kernel of T is therefore given by

$$\nabla_x \left(\sum_{k=0}^{+\infty} a_k p_k(x,y) \right).$$

It was proved in [44] that, under (D) and (P_2), this kernel satisfies the Hörmander integral condition, which implies the $H^1(\Gamma) - L^1(\Gamma)$ boundedness of T and therefore its $L^p(\Gamma)$-boundedness for all $1 < p < 2$, where $H^1(\Gamma)$ denotes the Hardy space on Γ defined in the sense of Coifman and Weiss ([18]). However, the Hörmander integral condition does not yield any information on the L^p-boundedness of T for $p > 2$. The proof of Theorem 1.3 actually relies on a theorem due to Auscher, Coulhon, Duong and Hofmann ([5]), which, given some $p_0 \in (2, +\infty]$, provides sufficient conditions for an L^2-bounded sublinear operator to be L^p-bounded for $2 < p < p_0$. Let us recall this theorem here in the form to be used in the sequel for the sake of completeness (see [5], Theorem 2.1, [2], Theorem 2.2):

Theorem 1.13 *Let $p_0 \in (2, +\infty]$. Assume that Γ satisfies the doubling property (D) and let T be a sublinear operator acting on $L^2(\Gamma)$. For any ball B, let A_B be a linear operator acting on $L^2(\Gamma)$, and assume that there exists $C > 0$ such that, for all $f \in L^2(\Gamma)$, all $x \in \Gamma$ and all ball $B \ni x$,*

$$\frac{1}{V^{1/2}(B)} \|T(I - A_B)f\|_{L^2(B)} \leq C \left(\mathcal{M}(|f|^2) \right)^{1/2}(x) \tag{1.12}$$

and

$$\frac{1}{V^{1/p_0}(B)} \|TA_B f\|_{L^{p_0}(B)} \leq C \left(\mathcal{M}(|Tf|^2) \right)^{1/2}(x). \tag{1.13}$$

If $2 < p < p_0$ and if, for all $f \in L^p(\Gamma)$, $Tf \in L^p(\Gamma)$, then there exists $C_p > 0$ such that, for all $f \in L^2(\Gamma) \cap L^p(\Gamma)$,

$$\|Tf\|_{L^p(\Gamma)} \leq C_p \|f\|_{L^p(\Gamma)}.$$

Notice that, to simplify the notations in our foregoing proofs, the formulation of Theorem 1.13 is slightly different from the one given in [2] and in [5], since the family of operators $(A_r)_{r>0}$ used in these papers is replaced by a family (A_B) indexed by the balls $B \subset \Gamma$, see Remark 5 after Theorem 2.2 in [2]. Observe also that this theorem extends to vector-valued functions (this will be used in Section 3). Finally, here and after, $\mathcal{M}$ denotes the Hardy-Littlewood maximal function: for any locally integrable function f on Γ and any $x \in \Gamma$,

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y),$$

where the supremum is taken over all balls B containing x. Recall that, by the Hardy-Littlewood maximal theorem, since (D) holds, $\mathcal{M}$ is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p \leq +\infty$.

Following the proof of Theorem 2.1 in [5], we will obtain Theorem 1.3 by applying Theorem 1.13 with $A_B = I - (I - P^{k^2})^n$ where k is the radius of B and n is an integer only depending from the constant D in (1.8).

As far as Theorem 1.11 is concerned, note first that (RR_p) cannot be derived from $(R_{p'})$ in this situation (where $1/p + 1/p' = 1$), since we do not know whether $(R_{p'})$ holds or not under these assumptions. Following [3], we first prove (1.10). The proof relies on a Calderón-Zygmund decomposition for Sobolev functions, which is the adaptation to our context of Proposition 1.1 in [3] (see also [1] in the Euclidean case and [6] for the extension to a weighted Lebesgue measure):

Proposition 1.14 *Assume that (D) and (P_q) hold for some $q \in [1, \infty)$ and let $p \in [q, +\infty)$. Let $f \in \dot{E}^{1,p}(\Gamma)$ and $\alpha > 0$. Then one can find a collection of balls $(B_i)_{i\in I}$, functions $(b_i)_{i\in I} \in \dot{E}^{1,q}(\Gamma)$ and a function $g \in \dot{E}^{1,\infty}$ such that the following properties hold:*

$$f = g + \sum_{i\in I} b_i, \tag{1.14}$$

$$\|\nabla g\|_\infty \leq C\alpha, \tag{1.15}$$

$$\operatorname{supp} b_i \subset B_i, \ \sum_{x\in B_i} |\nabla b_i|^q(x)m(x) \leq C\alpha^q V(B_i), \tag{1.16}$$

$$\sum_{i\in I} V(B_i) \leq C\alpha^{-p} \sum_{x\in\Gamma} |\nabla f|^p(x)m(x), \tag{1.17}$$

$$\sum_{i\in I} \chi_{B_i} \leq N, \tag{1.18}$$

where C and N only depend on q, p and on the constants in (D) and (P_q).

As in [3], we rely on this Calderón-Zygmund decomposition to establish (1.10). The argument also uses the $L^p(\Gamma)$-boundedness, for all $2 < p < +\infty$, of a discrete version of the Littlewood-Paley-Stein g-function (see [46]), which does not seem to have been stated before in this context and is interesting in itself. For all function f on Γ and all $x \in \Gamma$, define

$$g(f)(x) = \left(\sum_{l\geq 1} l \left| (I-P)P^l f(x) \right|^2 \right)^{1/2}.$$

Observe that this is indeed a discrete analogue of the g-function introduced by Stein in [46], since $(I - P)P^l = P^l - P^{l+1}$ can be seen as a discrete time derivative of P^l and P is a Markovian operator.

It is easy to check that the sublinear operator g is bounded in $L^2(\Gamma)$. Indeed, as already said, the assumption $(\Delta(\alpha))$ implies that the spectrum of P is contained in $[a, 1]$ for some $a > -1$. As a consequence, P can be written as

$$P = \int_a^1 \lambda dE(\lambda),$$

so that, for all integer $l \geq 1$,

$$(I - P)P^l = \int_a^1 (1 - \lambda)\lambda^l dE(\lambda)$$

and, for all $f \in L^2(\Gamma)$,

$$\left\|(I-P)P^l f\right\|_2^2 = \int_a^1 (1-\lambda)^2 \lambda^{2l} dE_{f,f}(\lambda).$$

It follows that, for all $f \in L^2(\Gamma)$,

$$\begin{aligned} \|g(f)\|_2^2 &= \sum_{l\geq 1} l \left\|(I-P)P^l f\right\|_2^2 \\ &= \int_a^1 (1-\lambda)^2 \sum_{l\geq 1} l\lambda^{2l} dE_{f,f}(\lambda) \\ &= \int_a^1 \left(\frac{\lambda}{1+\lambda}\right)^2 dE_{f,f}(\lambda) \\ &\leq \|f\|_2^2. \end{aligned}$$

It turns out that, as in the Littlewood-Paley-Stein semigroup theory, g is also L^p-bounded for $1 < p < +\infty$:

Theorem 1.15 *Assume that (D), (DUE) and ($\Delta(\alpha)$) hold. Let $1 < p < +\infty$. There exists $C_p > 0$ such that, for all $f \in L^p(\Gamma)$,*

$$\|g(f)\|_p \leq C_p \|f\|_p .$$

Actually, this inequality will only be used for $p > 2$ in the sequel, but the result, which is interesting in itself, does hold and will be proved for all $1 < p < +\infty$.
Before going further, let us mention that, in [29], N. Dungey establishes, under a local doubling property for the volume of balls, the L^p-boundedness for all $p \in (1, 2]$ of another version of the Littlewood-Paley-Stein functional, involving the gradient instead of the "time derivative" and the (continuous time) semigroup generated by $I - P$. Although we do not use Dungey's result here, it may prove useful to study the boundedness of Riesz transforms on graphs.
The proof of Theorem 1.15 for $p > 2$ relies on the vector-valued version of Theorem 1.13, while, for $p < 2$, we use the vector-valued version of the following result (see [2], Theorem 2.1 and also [13] for an earlier version):

Theorem 1.16 *Let $p_0 \in [1, 2)$. Assume that Γ satisfies the doubling property (D) and let T be a sublinear operator of strong type $(2, 2)$. For any ball B, let A_B be a linear operator acting on $L^2(\Gamma)$. Assume that, for all $j \geq 1$, there exists $g(j) > 0$ such that, for all ball $B \subset \Gamma$ and all function f supported in B,*

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|T(I-A_B)f\|_{L^2(C_j(B))} \leq g(j) \frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{1.19}$$

for all $j \geq 2$ and

$$\frac{1}{V^{1/2}(2^{j+1}B)} \|A_B f\|_{L^2(C_j(B))} \leq g(j) \frac{1}{V^{1/p_0}(B)} \|f\|_{L^{p_0}} \tag{1.20}$$

for all $j \geq 1$. If $\sum_{j\geq 1} g(j) 2^{Dj} < +\infty$ where D is given by (1.8), then T is of weak type (p_0, p_0), and is therefore of strong type (p, p) for all $p_0 < p < 2$.

Going back to Theorem 1.11, once (1.10) is established, we conclude by applying real interpolation theorems for Sobolev spaces, which are also new in this context. More precisely, we prove:

Theorem 1.17 *Let $q \in [1, +\infty)$ and assume that (D), (P_q) and $(\Delta(\alpha))$ hold. Then, for all $q < p < +\infty$, $\dot{W}^{1,p}(\Gamma) = \left(\dot{W}^{1,q}(\Gamma), \dot{W}^{1,\infty}(\Gamma)\right)_{1-\frac{q}{p},p}$.*

As an immediate corollary, we obtain:

Corollary 1.18 (The reiteration theorem) *Assume that Γ satisfies (D), (P_q) for some $1 \leq q < +\infty$ and $(\Delta(\alpha))$. Define $q_0 = \inf\{q \in [1, \infty) : (P_q) \text{ holds}\}$. For $q_0 < p_1 < p < p_2 \leq +\infty$, if $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, then $\dot{W}^{1,p}(\Gamma) = \left(\dot{W}^{1,p_1}(\Gamma), \dot{W}^{1,p_2}(\Gamma)\right)_{\theta,p}$.*

Corollary 1.18, in conjunction with (1.10), conclude the proof of Theorem 1.11. Notice that, since we know that Sobolev spaces interpolate by the real method, we do not need any argument as the one in Section 1.3 of [3].

For the proof of Theorem 1.7, we introduce a *discrete differential* and go through a property analogous to (Π_p) in [3], see Section 8 for detailed definitions. Proposition 1.8 follows essentially from Gehring's self-improvement of reverse Hölder inequalities ([32]).
The plan of the paper is as follows. After recalling some well-known estimates for the iterates of p and deriving some consequences (Section 2), we first prove Theorem 1.15, which is of independent interest, in Section 3. In Section 4, we prove Theorem 1.3 using Theorem 1.13. Section 5 is devoted to the proof of Proposition 1.14. Theorem 1.17 is established in Section 6 by methods similar to [8] and, in Section 7, we prove Theorem 1.11. Finally, Section 8 contains the proof of Theorem 1.7 and of Proposition 1.8.

2 Kernel bounds

In this section, we gather some estimates for the iterates of p and some straightforward consequences of frequent use in the sequel. We always assume that (D), (P_2) and $(\Delta(\alpha))$ hold. First, as already said, (LUE) holds. Moreover, we also have the following pointwise estimate for the discrete "time derivative" of p_l: there exist $C, c > 0$ such that, for all $x, y \in \Gamma$ and all $l \in \mathbb{N}^*$,

$$|p_l(x,y) - p_{l+1}(x,y)| \leq \frac{Cm(y)}{lV(x,\sqrt{l})} e^{-c\frac{d^2(x,y)}{l}}. \tag{2.1}$$

This "time regularity" estimate, which is a consequence of the L^2 analyticity of P, was first proved by Christ ([16]) by a quite difficult argument. Simpler proofs have been given by Blunck ([12]) and, more recently, by Dungey ([28]).

Thus, if B is a ball in Γ with radius k, f any function supported in B and $i \geq 2$, one has, for all $x \in C_i(B)$ and all $l \geq 1$,

$$\left|P^l f(x)\right| + l\left|(I-P)P^l f(x)\right| \leq \frac{C}{V(B)} e^{-c\frac{4^i k^2}{l}} \|f\|_{L^1}. \tag{2.2}$$

This "off-diagonal" estimate follows from (UE) and (2.1) and the fact that, for all $y \in B$, by (D),

$$V(y,k) \sim V(B) \text{ and } \frac{V(y,k)}{V(y,\sqrt{l})} \leq C \sup\left(1, \left(\frac{k}{\sqrt{l}}\right)^D\right).$$

Similarly, if B is a ball in Γ with radius k, $i \geq 2$ and f any function supported in $C_i(B)$, one has, for all $x \in B$ and all $l \geq 1$,

$$\left|P^l f(x)\right| + l\left|(I-P)P^l f(x)\right| \leq \frac{C}{V(2^i B)} e^{-c\frac{4^i k^2}{l}} \|f\|_{L^1}. \tag{2.3}$$

Finally, for all ball B with radius k, all $i \geq 2$, all function f supported in $C_i(B)$ and all $l \geq 1$,

$$\left\|\nabla P^l f\right\|_{L^2(B)} \leq \frac{C}{\sqrt{l}} e^{-c\frac{4^i k^2}{l}} \|f\|_{L^2(C_i(B))}. \tag{2.4}$$

See Lemma 2 in [43]. If one furthermore assumes that (G_{p_0}) holds for some $p_0 > 2$, then, by interpolation between (2.4) and (G_{p_0}), one obtains, for all $p \in (2, p_0)$, all f supported in $C_i(B)$ and all $l \geq 1$,

$$\left\|\nabla P^l f\right\|_{L^p(B)} \leq \frac{C_p}{\sqrt{l}} e^{-c\frac{4^i k^2}{l}} \|f\|_{L^p(C_i(B))}. \tag{2.5}$$

Inequalities (2.4) and (2.5) may be regarded as "Gaffney" type inequalities, in the spirit of [31].

3 Littlewood-Paley inequalities

In this section, we establish Theorem 1.15.

The case $1 < p < 2$: We apply the vector-valued version of Theorem 1.16 with $T = g$ and $p_0 = 1$ and, for all ball B with radius k, A_B defined by

$$A_B = I - (I - P^{k^2})^n,$$

where n is a positive integer, to be chosen in the proof. More precisely, we consider, for $f \in L^2(\Gamma)$ and $x \in \Gamma$,

$$Tf(x) = \left(\sqrt{l}(I-P)P^l f(x)\right)_{l \geq 1},$$

so that T maps $L^2(\Gamma)$ into $L^2(\Gamma, l^2)$.

Let B be a ball and f supported in B. Let us first check (1.19). Using the expansion

$$(I - P^{k^2})^n = \sum_{p=0}^{n} C_n^p (-1)^p P^{pk^2},$$

we obtain

$$T(I - A_B)f = \left(\alpha_l (I-P)P^l f\right)_{l \geq 1}$$

where

$$\alpha_l := \sum_{0\le p\le n;\ l\ge pk^2} C_n^p (-1)^p \sqrt{l-pk^2}.$$

Since it follows from (2.2) that

$$\frac{1}{V(2^{j+1}B)} \left\| (I-P)P^l f \right\|^2_{L^2(C_j(B))} \le \frac{C}{l^2 V^2(B)} e^{-c\frac{4^j k^2}{l}} \|f\|^2_{L^1}, \tag{3.1}$$

we will be able to go on thanks to the following estimate:

Lemma 3.1 *There exists $C>0$ only depending on n such that, for all $j\ge 2$,*

$$\sum_{l\ge 1} \frac{|\alpha_l|^2}{l^2} e^{-c\frac{4^j k^2}{l}} \le C 4^{-2nj}.$$

Proof of Lemma 3.1: If $mk^2 \le l < (m+1)k^2$ for some integer $0\le m\le n$, one obviously has

$$|\alpha_l| \le Ck\sqrt{m+1}, \tag{3.2}$$

where $C>0$ only depends on n, while, if $l>(n+1)k^2$, one has

$$|\alpha_l| \le C l^{-\frac{2n-1}{2}} k^{2n}. \tag{3.3}$$

This estimate follows from the following inequality, valid for any C^n function φ on $(0,+\infty)$:

$$\left| \sum_{p=0}^n C_n^p (-1)^p \varphi(t-pk^2) \right| \le C \sup_{u\ge \frac{t}{n+1}} \left|\varphi^{(n)}(u)\right| k^{2n}, \tag{3.4}$$

where $C>0$ only depends on n (see [30], problem 16, p. 65). It follows from (3.2) that, for all $0\le m\le n$,

$$\begin{aligned}
\sum_{mk^2<l\le (m+1)k^2} \frac{|\alpha_l|^2}{l^2} e^{-c\frac{4^j k^2}{l}} &\le C \sum_{mk^2<l\le (m+1)k^2} \frac{(m+1)k^2}{l^2} e^{-c\frac{4^j k^2}{l}} \\
&\le C \int_{mk^2}^{(m+1)k^2} \frac{(m+1)k^2}{t^2} e^{-c\frac{4^j k^2}{t}} dt \\
&\le C e^{-c4^j}
\end{aligned}$$

where $C,c>0$ only depend on n. Similarly, thanks to (3.3),

$$\begin{aligned}
\sum_{l>(n+1)k^2} \frac{|\alpha_l|^2}{l^2} e^{-c\frac{4^j k^2}{l}} &\le C \sum_{l>(n+1)k^2} \frac{l^{-(2n-1)} k^{4n}}{l^2} e^{-c\frac{4^j k^2}{l}} \\
&\le C \int_{(n+1)k^2}^{+\infty} \frac{k^{4n}}{t^{2n+1}} e^{-c\frac{4^j k^2}{t}} dt \\
&\le C 4^{-2nj} \int_0^{+\infty} \frac{e^{-c/w}}{w^{2n+1}} dw, \\
&= C 4^{-2nj},
\end{aligned}$$

which concludes the proof of Lemma 3.1. □
Finally, one obtains

$$\frac{1}{V^{1/2}(2^{j+1}B)}\left\|T(I-A_B)f\right\|_{L^2((2^{j+1}B\setminus 2^jB),l^2)} \;\leq\; C\frac{4^{-nj}}{V(B)}\left\|f\right\|_{L^1},$$

which means that (1.19) holds with $g(j)=4^{-nj}$, and one just has to choose $n>\dfrac{D}{2}$ in order to have $\displaystyle\sum_j g(j)2^{Dj}<+\infty$.
Let us now check (1.20). Since

$$A_B=\sum_{p=1}^{n}C_n^p(-1)^pP^{pk^2},$$

it is enough to prove that, for all $j\geq 1$ and all $1\leq p\leq n$,

$$\frac{1}{V^{1/2}(2^{j+1}B)}\left\|P^{pk^2}f\right\|_{L^2(C_j(B))}\leq g(j)\frac{1}{V(B)}\left\|f\right\|_{L^1(B)}. \tag{3.5}$$

For all $x\in C_j(B)$, (2.2) yields

$$\left|P^{pk^2}f(x)\right|\leq C\frac{e^{-c'\frac{4^j}{p}}}{V(B)}\left\|f\right\|_{L^1(B)}$$

if $j\geq 2$, and

$$\left|P^{pk^2}f(x)\right|\leq \frac{C}{V(B)}\left\|f\right\|_{L^1(B)}$$

for $j=1$, just by (UE). As a consequence,

$$\left\|P^{pk^2}f\right\|_{L^2(C_j(B))}\leq C\frac{e^{-c'\frac{4^j}{p}}}{V(B)}V^{1/2}(2^{j+1}B)\left\|f\right\|_{L^1(B)},$$

so that (3.5) holds. This ends the proof of Theorem 1.15 when $1<p<2$.

The case $2<p<+\infty$: This time, we apply the vector-valued version of Theorem 1.13 with the same choices of T and A_B. Let us first check (1.12), which reads in this situation as

$$\frac{1}{V^{1/2}(B)}\left\|T(I-A_B)f\right\|_{L^2(B,l^2)}\leq C\left(\mathcal{M}\left(|f|^2\right)\right)^{1/2}(y)$$

for all $f\in L^2(\Gamma)$, all ball $B\subset\Gamma$ and all $y\in B$. Fix such an f, such a ball B and $y\in B$. Write

$$f=\sum_{j\geq 1}f\chi_{C_j(B)}:=\sum_{j\geq 1}f_j.$$

The L^2-boundedness of g and A_B and the doubling property (D) yield

$$\frac{1}{V^{1/2}(B)}\left\|T(I-A_B)f_1\right\|_{L^2(B,l^2)}\leq\frac{C}{V^{1/2}(B)}\left\|f\right\|_{L^2(4B)}\leq C\left(\mathcal{M}\left(|f|^2\right)\right)^{1/2}(y).$$

Let $j \geq 2$. Using the same notations as for the case $1 < p < 2$, one has

$$\|T(I - A_B)f_j\|^2_{L^2(B,l^2)} = \sum_{l\geq 1} |\alpha_l|^2 \sum_{x\in B} \left|(I-P)P^l f_j(x)\right|^2 m(x).$$

For all $x \in B$, it follows from (2.3) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\left|(I-P)P^l f_j(x)\right| &\leq \frac{C}{l} e^{-c'\frac{4^j k^2}{l}} \frac{1}{V(2^j B)} \sum_{z\in 2^{j+1}B} |f_j(z)|\, m(z) \\
&\leq \frac{C}{l} e^{-c'\frac{4^j k^2}{l}} \frac{1}{V^{1/2}(2^{j+1}B)} \left(\sum_{z\in 2^{j+1}B} |f_j(z)|^2 m(z) \right)^{1/2} \\
&\leq \frac{C}{l} e^{-c'\frac{4^j k^2}{l}} \left(\mathcal{M}\left(|f|^2\right)(y)\right)^{1/2}.
\end{aligned}$$

As a consequence, by Lemma 3.1,

$$\begin{aligned}
\|T(I - A_B)f_j\|^2_{L^2(B,l^2)} &\leq C \left(\sum_{l\geq 1} \frac{|\alpha_l|^2}{l^2} e^{-c\frac{4^j k^2}{l}} \right) \mathcal{M}\left(|f|^2\right)(y) V(B) \\
&\leq CV(B) 4^{-2nj} \mathcal{M}\left(|f|^2\right)(y),
\end{aligned}$$

which yields (1.12) by summing up on $j \geq 1$.
To prove (1.13), it suffices to establish that, for all $1 \leq j \leq n$, all ball $B \subset \Gamma$ and all $y \in B$,

$$\left\| TP^{jk^2} f \right\|_{L^\infty(B,l^2)} \leq C \left(\mathcal{M}\, \|Tf\|^2_{l^2}(y)\right)^{1/2}.$$

Let $x \in B$. By Cauchy-Schwarz and the fact that

$$\sum_{y\in\Gamma} p_{jk^2}(x,y) = 1$$

for all $x \in \Gamma$, one has, for any function $h \in L^2(\Gamma)$,

$$\left|P^{jk^2}h(x)\right| \leq \left(P^{jk^2}|h|^2(x)\right)^{1/2}.$$

It follows that, for all $l \geq 1$,

$$\left|P^{jk^2}(\sqrt{l}(I-P)P^l f)(x)\right|^2 \leq P^{jk^2}\left(l\left|(I-P)P^l f\right|^2\right)(x),$$

so that

$$\begin{aligned}
\sum_{l\geq 1} \left|P^{jk^2}(\sqrt{l}(I-P)P^l f)(x)\right|^2 &\leq P^{jk^2}\left(\sum_{l\geq 1} l\left|(I-P)P^l f\right|^2 \right)(x) \\
&= P^{jk^2}\left(\|Tf\|^2_{l^2}\right)(x) \\
&\leq C\mathcal{M}\left(\|Tf\|^2_{l^2}\right)(y),
\end{aligned}$$

which is the desired estimate (note that the last inequality follows easily from (UE)). Thus, (1.13) holds and the proof of Theorem 1.15 is therefore complete. □

4 Riesz transforms for $p > 2$

In the present section, we establish Theorem 1.3, applying Theorem 1.13 with the same choice of A_B as in Section 3. One has $\|A_B\|_{2,2} = 1$. In view of Theorem 1.13, it suffices to show that

$$\frac{1}{V^{1/2}(B)} \left\| T(I - P^{k^2})^n f \right\|_{L^2(B)} \leq C \left(\mathcal{M}(|f|^2) \right)^{1/2} (x) \tag{4.1}$$

and

$$\frac{1}{V^{1/p_0}(B)} \left\| T \left(I - (I - P^{k^2})^n \right) f \right\|_{L^{p_0}(B)} \leq C \left(\mathcal{M}(|Tf|^2) \right)^{1/2} (x) \tag{4.2}$$

for all $f \in L^2(\Gamma)$, all $x \in \Gamma$ and all ball $B \subset \Gamma$ containing x. Fix such data f, x and B.

Proof of (4.1): Set $f_i = f\chi_{C_i(B)}$ for all $i \geq 1$. The L^2-boundedness of $T(I - P^{k^2})^n$ yields

$$\frac{1}{V^{1/2}(B)} \left\| T(I - P^{k^2})^n f_1 \right\|_{L^2(B)} \leq \frac{C}{V^{1/2}(B)} \|f_1\|_{L^2(\Gamma)} \leq C \left(\mathcal{M}(|f|^2) \right)^{1/2} (x). \tag{4.3}$$

Fix now $i \geq 2$. In order to estimate the left-hand side of (4.1) with f replaced by f_i, we use the expansion

$$(I - P)^{-1/2} = \sum_{l=0}^{+\infty} a_l P^l,$$

where the a_l's are defined by (1.11) (observe that, for all $l \geq 0$, $a_l > 0$). Therefore, one has

$$\begin{aligned} (I - P)^{-1/2} (I - P^{k^2})^n f_i &= \sum_{l=0}^{+\infty} a_l P^l (I - P^{k^2})^n f_i \\ &= \sum_{l=0}^{+\infty} a_l \sum_{j=0}^{n} C_n^j (-1)^j P^{l+jk^2} f_i \\ &= \sum_{l=0}^{+\infty} d_l P^l f_i, \end{aligned}$$

where

$$d_l = \sum_{0 \leq j \leq n,\ jk^2 \leq l} (-1)^j C_n^j a_{l-jk^2}.$$

It follows that

$$\left| T(I - P^{k^2})^n f_i(x) \right| \leq \sum_{l=1}^{+\infty} |d_l| \, \nabla P^l f_i(x)$$

for all $x \in B$. Indeed, if $x \in B$ and $l = 0$, $\nabla P^l f_i(x) = \nabla f_i(x) = 0$ because f_i is supported in $C_i(B)$. Thus, one has

$$\left\| T(I - P^{k^2})^n f_i \right\|_{L^2(B)} \leq \sum_{l=1}^{+\infty} |d_l| \, \left\| |\nabla P^l f_i| \right\|_{L^2(B)}.$$

According to (2.4), one has

$$\left\|T(I-P^{k^2})^n f_i\right\|_{L^2(B)} \le C\sum_{l=1}^{+\infty} |d_l|\,\frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}}\,\|f\|_{L^2(2^{i+1}B\setminus 2^iB)}\,. \tag{4.4}$$

We claim that the following estimates hold for the d_l's:

Lemma 4.1 *There exists $C>0$ only depending on n with the following properties: for all integer $l\ge 1$,*

(i) if there exists an integer $0\le m\le n$ such that $mk^2<l<(m+1)k^2$, $|d_l|\le \frac{C}{\sqrt{l-mk^2}}$,

(ii) if there exists an integer $0\le m\le n$ such that $l=(m+1)k^2$, $|d_l|\le C$,

(iii) if $l>(n+1)k^2$, $|d_l|\le Ck^{2n}l^{-n-\frac{1}{2}}$.

We postpone the proof of this lemma to the Appendix and end the proof of (4.1). According to (4.4), one has

$$\begin{aligned}\left\|T(I-P^{k^2})^n f_i\right\|_{L^2(B)} &\le C\sum_{m=0}^{n}\sum_{mk^2<l<(m+1)k^2} |d_l|\,\frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}}\,\|f\|_{L^2(2^{i+1}B\setminus 2^iB)}\\ &+ C\sum_{m=0}^{n}\left|d_{(m+1)k^2}\right|\,\frac{e^{-c\frac{4^i}{m+1}}}{k\sqrt{m+1}}\,\|f\|_{L^2(2^{i+1}B\setminus 2^iB)}\\ &+ C\sum_{l>(n+1)k^2} |d_l|\,\frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}}\,\|f\|_{L^2(2^{i+1}B\setminus 2^iB)}\\ &:= S_1+S_2+S_3.\end{aligned} \tag{4.5}$$

For S_1, Lemma 4.1 yields

$$|S_1|\le C\sum_{m=0}^{n}\sum_{mk^2<l<(m+1)k^2}\frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}\sqrt{l-mk^2}}\,\|f\|_{L^2(2^{i+1}B\setminus 2^iB)}\,.$$

But, for each $1\le m\le n$,

$$\begin{aligned}\sum_{mk^2<l<(m+1)k^2}\frac{e^{-c\frac{4^i k^2}{l}}}{\sqrt{l}\sqrt{l-mk^2}} &\le C\int_{mk^2}^{(m+1)k^2}\frac{e^{-c\frac{4^i k^2}{t}}}{\sqrt{t-mk^2}\sqrt{t}}dt\\ &\le C\int_0^1\frac{e^{-c\frac{4^i}{n(1+w)}}}{\sqrt{w(w+1)}}dw\\ &\le Ce^{-c4^i},\end{aligned}$$

where $C,c>0$ only depend on n. For $m=0$,

$$\sum_{0<l<k^2}\frac{e^{-c\frac{4^i k^2}{l}}}{l}\le\int_0^1 e^{-c\frac{4^i}{u}}\frac{du}{u}\le Ce^{-c4^i}.$$

Therefore,

$$|S_1| \leq Ce^{-c4^i} \|f\|_{L^2(2^{i+1}B\setminus 2^iB)}. \tag{4.6}$$

As for S_2, Lemma 4.1 gives at once

$$|S_2| \leq Ce^{-c4^i} \|f\|_{L^2(2^{i+1}B\setminus 2^iB)}, \tag{4.7}$$

where $C, c > 0$ only depend on n once more. Finally, for S_3, Lemma 4.1 provides

$$|S_3| \leq Ck^{2n} \sum_{l>(n+1)k^2} l^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^ik^2}{l}}}{\sqrt{l}} \|f\|_{L^2(2^{i+1}B\setminus 2^iB)}.$$

But one clearly has

$$\begin{aligned}
\sum_{l>(n+1)k^2} l^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^ik^2}{l}}}{\sqrt{l}} &\leq \int_{(n+1)k^2}^{+\infty} t^{-n-\frac{1}{2}} \frac{e^{-c\frac{4^ik^2}{t}}}{\sqrt{t}} dt \\
&= (4^ik^2)^{-n} \int_{\frac{n+1}{4^i}}^{+\infty} u^{-n} e^{-\frac{c}{u}} \frac{du}{u} \\
&\leq Ck^{-2n}4^{-in} \int_0^{+\infty} u^{-n} e^{-\frac{c}{u}} \frac{du}{u} \leq C4^{-in},
\end{aligned}$$

so that, since $k \geq 1$,

$$|S_3| \leq C4^{-in} \|f\|_{L^2(2^{i+1}B\setminus 2^iB)}. \tag{4.8}$$

Summing up the upper estimates (4.6), (4.7) and (4.8) and using (4.5), one obtains

$$\left\| T(I - P^{k^2})^n f_i \right\|_{L^2(B)} \leq C4^{-in} \|f\|_{L^2(2^{i+1}B\setminus 2^iB)}. \tag{4.9}$$

The definition of the maximal function and property (1.8) yield

$$\|f\|_{L^2(2^{i+1}B\setminus 2^iB)} \leq V^{1/2}(2^{i+1}B) \left(\mathcal{M}(|f|^2)(x)\right)^{1/2} \leq C2^{(i+1)D/2} V(B)^{1/2} \left(\mathcal{M}(|f|^2)(x)\right)^{1/2}.$$

Choosing now $n > \dfrac{D}{4}$ and summing up over $i \geq 1$, one concludes from (4.3) and (4.9) that

$$\left\| T(I - P^{k^2})^n f \right\|_{L^2(B)} \leq C \left(\sum_{i=0}^{+\infty} 2^{i\left(\frac{D}{2}-2n\right)} \right) V(B)^{1/2} \left(\mathcal{M}(|f|^2)(x)\right)^{1/2},$$

which ends the proof of (4.1). □

Proof of (4.2): We use the following lemma:

Lemma 4.2 *For all $p \in (2, p_0)$, there exists $C, \alpha > 0$ such that, for all ball $B \subset \Gamma$ with radius k, all integer $i \geq 1$ and all function $f \in L^2(\Gamma)$ supported in $C_i(B)$, and for all $j \in \{1, ..., n\}$ (where n is chosen as above), one has*

$$\left(\frac{1}{V(B)^{1/p}}\right) \left\| \nabla P^{jk^2} f \right\|_{L^p(B)} \leq \frac{Ce^{-\alpha 4^i}}{k} \frac{1}{V(2^{i+1}B)^{1/2}} \|f\|_{L^2(\Gamma)}.$$

Proof of Lemma 4.2: This proof is very similar to the one of Lemma 3.2 in [5], and we will therefore only indicate the main steps. Consider first the case when $i = 1$. If $j = 2m$ for some integer $m \geq 0$, (2.5) yields

$$\left\| \nabla P^{jk^2} f \right\|_{L^p(B)} \leq \frac{C}{k} \left\| P^{mk^2} f \right\|_{L^p(\Gamma)} . \tag{4.10}$$

Using (UE), and noticing that, by (D), for $y \in B$, $V(y, k\sqrt{m}) \sim V(B)$, one has, for all $x \in \Gamma$ and all $y \in B$,

$$p_{mk^2}(x,y) \leq \frac{C}{V(B)} \exp\left(-c \frac{d^2(x,y)}{mk^2} \right) m(y).$$

As a consequence, for all $x \in \Gamma$,

$$\left| P^{mk^2} f(x) \right| \leq \frac{C}{V^{1/2}(B)} \left\| f \right\|_{L^2(4B)} . \tag{4.11}$$

The L^2 contractivity of P shows that

$$\left\| P^{mk^2} f \right\|_{L^2(\Gamma)} \leq C \left\| f \right\|_{L^2(4B)} , \tag{4.12}$$

so that, gathering (4.11) and (4.12),

$$\left\| P^{mk^2} f \right\|_{L^p(\Gamma)} \leq C V(B)^{\frac{1}{p} - \frac{1}{2}} \left\| f \right\|_{L^2(\Gamma)} . \tag{4.13}$$

Finally, (4.13) and (4.10) yield the conclusion of Lemma 4.2 when $i = 1$ and $j = 2m$. If $j = 2m + 1$, argue similarly, writing $j = m + (m + 1)$.

Consider now the case when $i \geq 2$ and assume that $j = 2m$ (one argues similarly if $j = 2m + 1 = m + (m + 1)$). Let χ_l the characteristic function of $C_l(B)$ for all $l \geq 1$. One has, for all $x \in \Gamma$,

$$\nabla P^{jk^2} f(x) \leq \sum_{l \geq 1} \nabla P^{mk^2} \chi_l P^{mk^2} f(x) =: \sum_{l \geq 1} g_l(x).$$

By (2.5) and (1.8),

$$\begin{aligned} \frac{1}{V^{1/p}(B)} \left\| g_l \right\|_{L^p(B)} &\leq C \left(\frac{V(2^{l+1}B)}{V(B)} \right)^{1/p} \frac{e^{-c4^l}}{k} \frac{1}{V^{1/p}(2^{l+1}B)} \left\| P^{mk^2} f \right\|_{L^p(2^{l+1}B \setminus 2^l B)} \\ &\leq C 2^{(l+1)D/p} \frac{e^{-c4^l}}{k} \frac{1}{V^{1/p}(2^{l+1}B)} \left\| P^{mk^2} f \right\|_{L^p(2^{l+1}B \setminus 2^l B)} . \end{aligned}$$

Using (UE) and arguing as in the proof of Lemma 3.2 in [5], one obtains

$$\frac{1}{V(2^{l+1}B)} \left\| P^{mk^2} f \right\|^2_{L^2(C_l)} \leq K_{il} \frac{1}{V(2^{i+1}B)} \left\| f \right\|^2_{L^2(C_i)} \tag{4.14}$$

and, for all $x \in 2^{l+1}B \setminus 2^l B$,

$$\left|P^{mk^2} f(x)\right| \leq K_{il} 2^{(i+2)D} \frac{1}{V^{1/2}(2^{i+1}B)} \|f\|_{L^2(2^{i+1}B \setminus 2^i B)}, \tag{4.15}$$

where

$$K_{il} = \begin{cases} Ce^{-c4^i} & \text{if } l \leq i-2, \\ C & \text{if } i-1 \leq l \leq i+1, \\ Ce^{-c4^l} & \text{if } l \geq i+2. \end{cases}$$

Interpolating between (4.14) and (4.15) therefore yields

$$\frac{1}{V^{1/p}(2^{l+1}B)} \left\|P^{mk^2} f\right\|_{L^p(C_l)} \leq K_{il} 2^{(i+2)D\left(1-\frac{2}{p}\right)} \frac{1}{V^{1/2}(2^{i+1}B)} \|f\|_{L^2(C_i)}.$$

Summing up in l, one ends the proof of Lemma 4.2 as in [5]. □

To prove (4.2), it is enough to show that, if $p \in (2, p_0)$, there exists $C_p > 0$ such that, for all $j \in \{1, ..., n\}$, all function $f \in L^2_{loc}(\Gamma)$ with $\nabla f \in L^2_{loc}(\Gamma)$, all ball $B \subset \Gamma$ with radius k and any point $x \in B$,

$$\frac{1}{V^{1/p}(B)} \left\|\nabla P^{jk^2} f\right\|_{L^p(B)} \leq C \left(\mathcal{M}(|\nabla f|^2)\right)^{1/2}(x).$$

But, since for all $l \geq 0$, $P^l 1 = 1$, one has

$$\nabla P^l f = \nabla P^l (f - f_{4B}),$$

so that

$$\nabla P^{jk^2} f = \sum_{l \geq 1} \nabla P^{jk^2} (\chi_l (f - f_{4B})).$$

One concludes the proof of (4.2) as in [5], using the Poincaré inequality and Lemma 4.2. □

5 The Calderón-Zygmund decomposition for functions in Sobolev spaces

The present section is devoted to the proof of Proposition 1.14, for which we adapt the proof of Proposition 1.1 in [3] to the discrete setting. Let $f \in \dot{E}^{1,p}(\Gamma)$, $\alpha > 0$. Consider $\Omega = \{x \in \Gamma : \mathcal{M}(|\nabla f|^q)(x) > \alpha^q\}$. If $\Omega = \emptyset$, then set

$$g = f, \quad b_i = 0 \text{ for all } i \in I$$

so that (1.15) is satisfied thanks to the Lebesgue differentiation theorem and the other properties in Proposition 1.14 obviously hold. Otherwise the Hardy-Littlewood maximal theorem gives

$$\begin{aligned} m(\Omega) &\leq C\alpha^{-p} \|(\nabla f)^q\|_{\frac{p}{q}}^{\frac{p}{q}} \\ &= C\alpha^{-p} \Big(\sum_x |\nabla f|^p(x) m(x) \Big) \\ &< +\infty. \end{aligned} \tag{5.1}$$

In particular, Ω is a proper open subset of Γ, as $m(\Gamma) = +\infty$ (see Remark 1.1). Let $(\underline{B_i})_{i\in I}$ be a Whitney decomposition of Ω ([18]). That is, Ω is the union of the $\underline{B_i}$'s, the $\underline{B_i}$'s being pairwise disjoint open balls, and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that, if $F = \Gamma \setminus \Omega$,

1. the balls $B_i = C_1\underline{B_i}$ are contained in Ω and have the bounded overlap property;
2. for each $i \in I$, $r_i = r(B_i) = \frac{1}{2}d(x_i, F)$ where x_i is the center of B_i;
3. for each $i \in I$, if $\overline{B_i} = C_2\underline{B_i}$, $\overline{B_i} \cap F \neq \emptyset$ ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i \in I;\ x \in B_i\}$. By the bounded overlap property of the balls B_i, there exists an integer N such that $\sharp I_x \leq N$ for all $x \in \Omega$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3}r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (1.18) is nothing but the bounded overlap property of the B_i's and (1.17) follows from (1.18) and (5.1). The doubling property and the fact that $\overline{B_i} \cap F \neq \emptyset$ yield:

$$\sum_{x\in B_i} |\nabla f|^q(x)m(x) \leq \sum_{x\in\overline{B_i}} |\nabla f|^q(x)m(x) \leq \alpha^q V(\overline{B_i}) \leq C\alpha^q V(B_i). \tag{5.2}$$

Let us now define the functions b_i's. Let $(\chi_i)_{i\in I}$ be a partition of unity of Ω subordinated to the covering $(\underline{B_i})_{i\in I}$, which means that, for all $i \in I$, χ_i is a Lipschitz function supported in B_i with $\|\nabla\chi_i\|_\infty \leq \dfrac{C}{r_i}$ and $\sum_{i\in I}\chi_i(x) = 1$ for all $x \in \Gamma$ (it is enough to choose $\chi_i(x) = \psi\left(\dfrac{C_1 d(x_i,x)}{r_i}\right)\left(\sum_k \psi\left(\dfrac{C_1 d(x_k,x)}{r_k}\right)\right)^{-1}$, where $\psi \in \mathcal{D}(\mathbb{R})$, $\psi = 1$ on $[0,1]$, $\psi = 0$ on $[\frac{1+C_1}{2}, +\infty)$ and $0 \leq \psi \leq 1$). Note that $\nabla\chi_i$ is supported in $2B_i \subset \Omega$. We set $b_i = (f - f_{B_i})\chi_i$. It is clear that $\operatorname{supp} b_i \subset B_i$. Let us estimate $\sum_{x\in B_i} |\nabla b_i|^q(x)m(x)$. Since

$$\nabla b_i(x) = \nabla((f - f_{B_i})\chi_i)(x) \leq \max_{y\sim x} \chi_i(y)\nabla f(x) + |f(x) - f_{B_i}|\nabla\chi_i(x)$$

and since $\chi_i(y) \leq 1$ for all $y \in \Gamma$, we get by (P_q) and (5.2) that

$$\begin{aligned}
\sum_{x\in B_i} |\nabla b_i|^q m(x) &\leq C\left(\sum_{x\in B_i} |\nabla f|^q(x)m(x) + \sum_{x\in B_i} |f - f_{B_i}|^q(x)|\nabla\chi_i|^q(x)m(x)\right) \\
&\leq C\alpha^q V(B_i) + C\frac{C^q}{r_i^q} r_i^q \sum_{x\in B_i} |\nabla f|^q(x)m(x) \\
&\leq C'\alpha^q V(B_i).
\end{aligned}$$

Thus (1.16) is proved.

Set $g = f - \sum_{i\in I} b_i$. Since the sum is locally finite on Ω, g is defined everywhere on Γ and $g = f$ on F.

It remains to prove (1.15). Since $\sum_{i\in I}\chi_i(x)=1$ for all $x\in\Omega$, one has

$$g = f\chi_F + \sum_{i\in I} f_{B_i}\chi_i$$

where χ_F denotes the characteristic function of F. We will need the following lemma:

Lemma 5.1 *There exists $C>0$ such that, for all $j\in I$, all $u\in F\cap 4B_j$ and all $v\in B_j$,*

$$|g(u)-g(v)| \le C\alpha d(u,v).$$

Proof: Since $\sum_{i\in I}\chi_i = 1$ on Γ, one has

$$\begin{aligned} g(u)-g(v) &= f(u) - \sum_{i\in I} f_{B_i}\chi_i(v) \\ &= \sum_{i\in I} \left(f(u)-f_{B_i}\right)\chi_i(v). \end{aligned} \tag{5.3}$$

For all $i\in I$ such that $v\in B_i$,

$$|f(u)-f_{B_i}| \le \sum_{k=0}^{+\infty} \left|f_{B(u,2^{-k}r_i)} - f_{B(u,2^{-k-1}r_i)}\right| + \left|f_{B(u,r_i)} - f_{B_i}\right|.$$

For all $k\ge 0$, (P_q) yields

$$\begin{aligned} \left|f_{B(u,2^{-k}r_i)} - f_{B(u,2^{-k-1}r_i)}\right| &= \frac{1}{V(u,2^{-k-1}r_i)}\left|\sum_{z\in B(u,2^{-k-1}r_i)} \left(f(z)-f_{B(u,2^{-k}r_i)}\right)m(z)\right| \\ &\le \frac{C}{V(u,2^{-k}r_i)} \sum_{z\in B(u,2^{-k}r_i)} \left|f(z)-f_{B(u,2^{-k}r_i)}\right| m(z) \\ &\le \left(\frac{C}{V(u,2^{-k}r_i)} \sum_{z\in B(u,2^{-k}r_i)} \left|f(z)-f_{B(u,2^{-k}r_i)}\right|^q m(z)\right)^{\frac{1}{q}} \\ &\le C2^{-k}r_i \left(\frac{1}{V(u,2^{-k}r_i)} \sum_{z\in B(u,2^{-k}r_i)} |\nabla f(z)|^q\, m(z)\right)^{\frac{1}{q}} \\ &\le C2^{-k}r_i \left(\mathcal{M}\left(\nabla f\right)^q\right)^{\frac{1}{q}}(u) \\ &\le C2^{-k}\alpha r_i \le C2^{-k}\alpha r_j, \end{aligned} \tag{5.4}$$

where the penultimate inequality relies on the fact that $u\in F$ and the last one from the fact that $B_i\cap B_j\neq\emptyset$. Moreover, since $u\in 4B_j$,

$$\begin{aligned} B(u,r_i) &\subset B(x_j, r_i + d(u,x_j)) \\ &\subset B(x_j, r_i+4r_j) \subset 7B_j. \end{aligned}$$

Since one also has $B_i \subset 7B_j$, one obtains, arguing as before,

$$\begin{aligned} \left|f_{B(u,r_i)} - f_{B_i}\right| &\leq \left|f_{B(u,r_i)} - f_{7B_j}\right| + \left|f_{7B_j} - f_{B_i}\right| \\ &\leq \frac{C}{V(7B_j)} \sum_{z\in 7B_j} \left|f(z) - f_{7B_j}\right| m(z) \\ &\leq C\alpha r_j. \end{aligned} \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$|f(u) - f_{B_i}| \leq C\alpha r_j \leq C\alpha d(u,v),$$

since

$$\begin{aligned} r_j &= \frac{1}{2}d(x_j, F) \leq \frac{1}{2}d(x_j,u) \leq \frac{1}{2}d(x_j,v) + \frac{1}{2}d(v,u) \\ &\leq \frac{1}{2}r_j + \frac{1}{2}d(v,u). \end{aligned}$$

This ends the proof of Lemma 5.1 because of (5.3). □

To prove (1.15), it is clearly enough to check that $|g(x) - g(y)| \leq C\alpha$ for all $x \sim y \in \Gamma$. Let us now prove this fact, distinguishing between three cases:

1. Assume that $x \in \Omega$. Then, $x \in B_j$ for some $j \in I$, and for all $y \sim x$, $y \in 2B_j \subset \Omega$, so that $\chi_F(x) = \chi_F(y) = 0$. It follows that

$$g(y) - g(x) = \sum_{i\in I} \left(f_{B_i} - f_{B_j}\right)\left(\chi_i(y) - \chi_i(x)\right),$$

so that $|g(y) - g(x)| \leq C\sum_{i\in I} \left|f_{B_i} - f_{B_j}\right| \nabla\chi_i(x) := h(x)$. We claim that $|h(x)| \leq C\alpha$. To see this, note that, for all $i \in I$ such that $\nabla\chi_i(x) \neq 0$, we have $|f_{B_i} - f_{B_j}| \leq Cr_j\alpha$. Indeed, $d(x, B_i) \leq 1$, which easily implies that $r_i \leq 3r_j + 1 \leq 4r_j$, hence $B_i \subset 10B_j$. As a consequence, we have, arguing as before again,

$$\begin{aligned} |f_{B_i} - f_{10B_j}| &\leq \frac{1}{V(B_i)} \sum_{y\in B_i} |f(y) - f_{10B_j}| m(y) \\ &\leq \frac{C}{V(B_j)} \sum_{y\in 10B_j} |f(y) - f_{10B_j}| m(y) \\ &\leq Cr_j \left(\frac{1}{V(10B_j)} \sum_{y\in 10B_j} |\nabla f|^q(y) m(y)\right)^{\frac{1}{q}} \\ &\leq Cr_j\alpha \end{aligned} \tag{5.6}$$

where we used Hölder inequality, (D), (P_q) and the fact that $(|\nabla f|^q)_{10B_j} \leq \mathcal{M}(|\nabla f|)^q(z)$ for some $z \in F \cap \overline{B_j}$. Analogously $|f_{10B_j} - f_{B_j}| \leq Cr_j\alpha$. Hence

$$\begin{aligned} |h(x)| &= \left|\sum_{i\in I;\ x\in 2B_i} (f_{B_i} - f_{B_j})\nabla\chi_i(x)\right| \\ &\leq C \sum_{i\in I;\ x\in 2B_i} |f_{B_i} - f_{B_j}| r_i^{-1} \\ &\leq CN\alpha. \end{aligned}$$

2. Assume now that $x \in F \setminus \partial F$. In this case $|g(y) - g(x)| = |f(y) - f(x)| \leq C\alpha$ by the definition of F.

3. Assume finally that $x \in \partial F$.

   i. If $y \in F$, we have $|g(y) - g(x)| = |f(x) - f(y)| \leq C\nabla f(x) \leq C\alpha$.

   ii. Consider now the case when $y \in \Omega$. There exists $j \in I$ such that $y \in B_j$. Since $x \sim y$, one has $x \in 4B_j$, Lemma 5.1 therefore yields

$$|g(x) - g(y)| \leq C\alpha d(x, y) \leq C\alpha.$$

Thus the proof of Proposition 1.14 is complete. □

Remark 5.2 *It is easy to get the following estimate for the b_i's: for all $i \in I$,*

$$\frac{1}{V(B_i)} \|b_i\|_1 \leq \frac{1}{V(B_i)^{1/q}} \|b_i\|_q \leq C\alpha r_i.$$

Indeed, the first inequality follows from Hölder and the fact that b_i is supported in B_i. Moreover, by (P_q) and (5.2),

$$\frac{1}{V(B_i)^{1/q}} \|b_i\|_q = \frac{1}{V(B_i)^{1/q}} \|f - f_{B_i}\|_{L^q(B_i)} \leq Cr_i \frac{1}{V(B_i)^{1/q}} \|\nabla f\|_{L^q(B_i)} \leq C\alpha r_i.$$

6 An interpolation result for Sobolev spaces

To prove Theorem 1.17, we will characterize the K functional of interpolation for homogeneous Sobolev spaces in the following theorem.

Theorem 6.1 *Under the same hypotheses as Theorem 1.17 we have that*

1. *there exists C_1 such that for every $f \in \dot{W}^{1,q}(\Gamma) + \dot{W}^{1,\infty}(\Gamma)$ and all $t > 0$*

$$K(f, t^{\frac{1}{q}}, \dot{W}^{1,q}, \dot{W}^{1,\infty}) \geq C_1 t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t);$$

2. *for $q \leq p < \infty$, there exists C_2 such that for every $f \in \dot{W}^{1,p}(\Gamma)$ and every $t > 0$*

$$K(f, t^{\frac{1}{q}}, \dot{W}^{1,q}, \dot{W}^{1,\infty}) \leq C_2 t^{\frac{1}{q}} \left(|\nabla f|^{q**}\right)^{\frac{1}{q}}(t).$$

Proof: We first prove item 1. Assume that $f = h + g$ with $h \in \dot{W}^{1,q}$, $g \in \dot{W}^{1,\infty}$, we then have

$$\begin{aligned} \|h\|_{\dot{W}^{1,q}} + t^{\frac{1}{q}} \|g\|_{\dot{W}^{1,\infty}} &\geq \|\nabla h\|_q + t^{\frac{1}{q}} \|\nabla g\|_\infty \\ &\geq K(\nabla f, t^{\frac{1}{q}}, L^q, L^\infty) \\ &\geq Ct^{\frac{1}{q}} (|\nabla f|^{q**})^{\frac{1}{q}}(t). \end{aligned}$$

Hence we conclude that $K(f,t^{\frac{1}{q}},\dot{W}^{1,q},\dot{W}^{1,\infty})\geq C_1 t^{\frac{1}{q}}(|\nabla f|^{q**})^{\frac{1}{q}}(t)$.
We prove now item 2. Let $f\in\dot{W}^{1,p}$, $q\leq p<\infty$. Let $t>0$, we consider the Calderón-Zygmund decomposition of f given by Proposition 1.14 with $\alpha=\alpha(t)=\left(\mathcal{M}(|\nabla f|)^q\right)^{*\frac{1}{q}}(t)$. Thus we have $f=\sum_{i\in I} b_i+g=b+g$ where $(b_i)_{i\in I}$, g satisfy the properties of the proposition. We have the estimate

$$\begin{aligned}\|\nabla b\|_q^q &\leq \sum_{x\in\Gamma}\left(\sum_{i\in I}|\nabla b_i|\right)^q(x)m(x)\\ &\leq CN\sum_{i\in I}\sum_{x\in B_i}|\nabla b_i|^q(x)m(x)\\ &\leq C\alpha^q(t)\sum_{i\in I}V(B_i)\\ &\leq C\alpha^q(t)m(\Omega),\end{aligned}$$

where the B_i's are given by Proposition 1.14 and Ω is defined as in the proof of Proposition 1.14. The last inequality follows from the fact that $\sum_{i\in I}\chi_{B_i}\leq N$ and $\Omega=\bigcup_i B_i$. Hence $\|\nabla b\|_q\leq C\alpha(t)m(\Omega)^{\frac{1}{q}}$. Moreover, since $(\mathcal{M}f)^*\sim f^{**}$ (see [11], Chapter 3, Theorem 3.8), we obtain

$$\alpha(t)=(\mathcal{M}(|\nabla f|)^q)^{*\frac{1}{q}}(t)\leq C\,(|\nabla f|^{q**})^{\frac{1}{q}}(t).$$

Hence, also noting that $m(\Omega)\leq t$ (see [11], Chapter 2, Proposition 1.7), we get $K(f,t^{\frac{1}{q}},\dot{W}^{1,q},\dot{W}^{1,\infty})\leq Ct^{\frac{1}{q}}|\nabla f|^{q**\frac{1}{q}}(t)$ for all $t>0$ and obtain the desired inequality. □

Proof of Theorem 1.17: The proof follows directly from Theorem 6.1. Indeed, item 1. of Theorem 6.1 gives us that $(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}\subset\dot{W}^{1,p}$ and $\|f\|_{\dot{W}^{1,p}}\leq C\|f\|_{1-\frac{q}{p},p}$, while item 2. gives us that $\dot{W}^{1,p}\subset(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}$ and $\|f\|_{1-\frac{q}{p},p}\leq C\|f\|_{\dot{W}^{1,p}}$. Hence $\dot{W}^{1,p}=(\dot{W}^{1,q},\dot{W}^{1,\infty})_{1-\frac{q}{p},p}$ with equivalent norms. □

7 The proof of (RR_p) for $p<2$

In view of Theorem 1.17 and since (RR_2) holds, it is enough, for the proof of Theorem 1.11, to establish (1.10).

Proof of (1.10): We follow the proof of (1.9) in [3]. Consider such an f and fix $\lambda>0$. Perform the Calderon-Zygmund decomposition of f given by Proposition 1.14. We also use the following expansion of $(I-P)^{1/2}$:

$$(I-P)^{1/2}=\sum_{k=0}^{+\infty}a_k(I-P)P^k \tag{7.1}$$

where the (a_k)'s were already considered in Section 4. For each $i \in I$, pick the integer $k \in \mathbb{Z}$ such that $2^k \leq r(B_i) < 2^{k+1}$ and define $r_i = 2^k$. We split the expansion (7.1) into two parts:

$$(I-P)^{1/2} = \sum_{k=0}^{r_i^2} a_k (I-P)P^k + \sum_{k=r_i^2+1}^{+\infty} a_k (I-P)P^k := T_i + U_i.$$

We first claim that

$$m\left(\left\{x \in \Gamma;\ \left|(I-P)^{1/2} g(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \left\|\nabla f\right\|_q^q. \tag{7.2}$$

Indeed, one has

$$\begin{aligned} m\left(\left\{x \in \Gamma;\ \left|(I-P)^{1/2} g(x)\right| > \lambda\right\}\right) &\leq \frac{C}{\lambda^2} \left\|(I-P)^{1/2} g\right\|_2^2 \\ &= \frac{C}{\lambda^2} \left\|\nabla g\right\|_2^2, \end{aligned}$$

and since $\nabla g \leq C\lambda$ on Γ and $\left\|\nabla g\right\|_q \leq C \left\|\nabla f\right\|_q$, we obtain

$$\left\|\nabla g\right\|_2^2 \leq C\lambda^{2-q} \left\|\nabla g\right\|_q^q \leq C\lambda^{2-q} \left\|\nabla f\right\|_q^q,$$

which ends the proof of (7.2).

We now claim that, for some constant $C > 0$,

$$m\left(\left\{x \in \Gamma;\ \left|\sum_{i \in I} T_i b_i(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \left\|\nabla f\right\|_q^q. \tag{7.3}$$

To prove (7.3), write

$$m\left(\left\{x \in \Gamma;\ \left|\sum_{i \in I} T_i b_i(x)\right| > \lambda\right\}\right) \leq m\left(\bigcup_i 4B_i\right) + m\left(\left\{x \notin \bigcup_i 4B_i;\ \left|\sum_{i \in I} T_i b_i(x)\right| > \lambda\right\}\right). \tag{7.4}$$

Observe first that, by (D) and Proposition 1.14,

$$m\left(\bigcup_i 4B_i\right) \leq C \sum_{i \in I} V(4B_i) \leq \frac{C}{\lambda^q} \left\|\nabla f\right\|_q^q.$$

As far as the second term in the right-hand side of (7.4) is concerned, it can be estimated by

$$m\left(\left\{x \notin \bigcup_i 4B_i;\ \left|\sum_{i \in I} T_i b_i(x)\right| > \lambda\right\}\right) \leq \frac{1}{\lambda^2} \sum_{x \in \Gamma} \left|\sum_{i \in I} \chi_{\Gamma \setminus 4B_i}(x) T_i b_i(x)\right|^2 m(x).$$

Arguing as in [3, 13, 38], we estimate this last quantity by duality. Fix a function $u \in L^2(\Gamma, m)$ with $\left\|u\right\|_2 = 1$. One has

$$\left|\sum_{x \in \Gamma} \sum_{i \in I} \chi_{\Gamma \setminus 4B_i}(x) T_i b_i(x) u(x) m(x)\right| \leq \sum_{i \in I} \sum_{j=2}^{+\infty} A_{i,j}$$

where, for all $i \in I$ and all $j \geq 2$,

$$A_{i,j} := \sum_{x \in 2^{j+1}B_i \setminus 2^j B_i} |T_i b_i(x)| \, |u(x)| \, m(x).$$

If i, j are fixed, since $(I-P)b_i$ is supported in $2B_i$,

$$\begin{aligned} \|T_i b_i\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} &\leq \sum_{k=0}^{r_i^2} |a_k| \left\|(I-P)P^k b_i\right\|_{L^2(2^{j+1}B_i)\setminus 2^j B_i)} \\ &= \sum_{k=1}^{r_i^2} |a_k| \left\|(I-P)P^k b_i\right\|_{L^2(2^{j+1}B_i)\setminus 2^j B_i)} \end{aligned}$$

Given $1 \leq k \leq r_i^2$, one has, for all $x \in 2^{j+1}B_i \setminus 2^j B_i$, using (2.1),

$$\left|(I-P)P^k b_i(x)\right| \leq \sum_{y \in B_i} |p_k(x,y) - p_{k+1}(x,y)| \, |b_i(y)| \leq \sum_{y \in B_i} \frac{C}{kV(y,\sqrt{k})} e^{-c\frac{d^2(x,y)}{k}} \, |b_i(y)| \, m(y).$$

Using (1.8) and arguing exactly as in [3] (relying, in particular, on Remark 5.2), we obtain

$$\left\|(I-P)P^k b_i\right\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} \leq C \frac{r_i}{k} \left(\frac{r_i}{\sqrt{k}}\right)^{2D} e^{-c\frac{4^j r_i^2}{k}} V^{1/2}(2^{j+1}B_i)\lambda.$$

Since

$$a_k \sim \frac{1}{\sqrt{k\pi}}$$

(see Appendix), it follows that

$$\|T_i b_i\|_{L^2(2^{j+1}B_i \setminus 2^j B_i)} \leq C e^{-c4^j} V^{1/2}(2^{j+1}B_i)\lambda.$$

One concludes, as in [3], that (7.3) holds.

What remains to be proved is that

$$m\left(\left\{x \in \Gamma;\ \left|\sum_{i \in I} U_i b_i(x)\right| > \lambda\right\}\right) \leq \frac{C}{\lambda^q} \|\nabla f\|_q^q. \tag{7.5}$$

Define, for all $j \in \mathbb{Z}$,

$$\beta_j = \sum_{i \in I;\ r_i = 2^j} \frac{b_i}{r_i},$$

so that, for all $j \in \mathbb{Z}$,

$$\sum_{i \in I;\ r_i = 2^j} b_i = 2^j \beta_j.$$

One has

$$\begin{aligned}\sum_{i\in I} U_i b_i &= \sum_{i\in I}\sum_{k>r_i^2} a_k (I-P)P^k b_i \\ &= \sum_{k>0} a_k (I-P)P^k \sum_{i\in I;\ r_i^2<k} b_i \\ &= \sum_{k>0} a_k (I-P)P^k \sum_{i\in I;\ r_i^2=2^{2j}<k} b_i \\ &= \sum_{k>0} a_k (I-P)P^k \sum_{j;\ 4^j<k} 2^j \beta_j.\end{aligned}$$

For all $k>0$, define

$$f_k = \sum_{j;\ 4^j<k} \frac{2^j}{\sqrt{k}}\beta_j.$$

It follows from the previous computation and Theorem 1.15 that

$$\left\| \sum_{i\in I} U_i b_i \right\|_q \le C \left\| \left(\sum_{k=1}^{+\infty} \frac{1}{k} |f_k|^2 \right)^{1/2} \right\|_q .$$

To see this, we estimate the left-hand side of this inequality by duality, as in [3] and use the fact that $|a_k| \le \dfrac{C}{\sqrt{k}}$ for all $k \ge 1$. Since, by Cauchy-Schwarz,

$$|f_k|^2 \le 2 \sum_{j;\ 4^j<k} \frac{2^j}{\sqrt{k}} |\beta_j|^2 ,$$

one obtains

$$\left\| \left(\sum_{k=1}^{+\infty} \frac{1}{k} |f_k|^2 \right)^{1/2} \right\|_q \le \left\| \left(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q .$$

By the bounded overlap property,

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \le C \sum_{x\in\Gamma}\sum_{i\in I} \frac{|b_i(x)|^q}{r_i^q} m(x),$$

so that, using Remark 5.2, one obtains

$$\sum_{x\in\Gamma}\sum_{i\in I} \frac{|b_i(x)|^q}{r_i^q} m(x) \le C\lambda^q \sum_{i\in I} V(B_i).$$

As a conclusion,

$$m\left(\left\{ x\in\Gamma;\ \left| \sum_{i\in I} U_i b_i(x) \right| > \lambda \right\} \right) \le C \sum_{i\in I} V(B_i) \le \frac{C}{\lambda^q} \|\nabla f\|_q^q ,$$

which is exactly (7.5). The proof of (1.10) is therefore complete. □

8 Riesz transforms and harmonic functions

Let us now prove Theorem 1.7. The proof goes through a property analogous to (Π_p) in [3], the statement of which requires a notion of discrete differential.

8.1 The discrete differential and its adjoint

To begin with, for any $\gamma = (x, y), \gamma' = (x', y') \in E$ (recall that E denotes the set of edges in Γ), set

$$d(\gamma, \gamma') = \max(d(x, x'), d(y, y')).$$

It is straightforward to check that d is a distance on E. We also define a measure on subsets on E. For any $A \subset E$, set

$$\mu(A) = \sum_{(x,y)\in A} \mu_{xy}.$$

We claim that E, equipped with the metric d and the measure μ, is a space of homogeneous type. Indeed, let $\gamma = (a, b) \in E$ and $r > 0$. Assume first that $r \geq 5$. Then, by (D),

$$\mu(B(\gamma, 2r)) = \sum_{d(x,a)<2r,\ d(y,b)<2r} \mu_{xy} \leq \sum_{d(x,a)<2r} \sum_{y\in\Gamma} \mu_{xy} = V(a, 2r) \leq CV\left(a, \frac{r}{100}\right).$$

But

$$V\left(a, \frac{r}{100}\right) = \sum_{d(x,a)<\frac{r}{100}} \sum_{d(y,x)\leq 1} \mu_{xy} \leq \sum_{d(x,a)<\frac{r}{2}} \sum_{d(y,b)<\frac{r}{2}} \mu_{xy} = \mu\left(B\left(\gamma, \frac{r}{2}\right)\right),$$

since, when $d(x, a) < \frac{r}{100}$ and $d(y, x) \leq 1$, then $d(y, b) < 2 + \frac{r}{100} \leq \frac{r}{2}$.
Assume now that $r < 5$. One has, using (D) again,

$$\mu(B(\gamma, 2r)) \leq V(a, 2r) \leq V(a, 10) \leq CV\left(a, \frac{1}{2}\right) = Cm(a) \leq C'\mu_{ab} \leq C'\mu(B(\gamma, r)),$$

since, whenever $x \sim y$, one has $\alpha m(x) \leq \mu_{xy}$ by $(\Delta(\alpha))$. The claim is therefore proved.
We can then define L^p spaces on E in the following way. For $1 \leq p < +\infty$, say that a function F on E belongs to $L^p(E)$ if and only if F is antisymmetric (which means that $F(x, y) = -F(y, x)$ for all $(x, y) \in E$) and

$$\|F\|^p_{L^p(E)} := \frac{1}{2} \sum_{(x,y)\in E} |F(x, y)|^p \mu_{xy} < +\infty.$$

Observe that the $L^2(E)$-norm derives from the scalar product

$$\langle F, G\rangle_{L^2(E)} = \frac{1}{2} \sum_{x,y\in\Gamma} F(x, y)G(x, y)\mu_{xy}.$$

Finally, say that $F \in L^\infty(E)$ if and only if F is antisymmetric and

$$\|F\|_{L^\infty(E)} := \frac{1}{2} \sup_{(x,y)\in E} |F(x, y)| < +\infty.$$

Our notion of discrete differential is the following one: for any function f on Γ and any $\gamma = (x, y) \in E$, define

$$df(\gamma) = f(y) - f(x).$$

The function df is clearly antisymmetric on E and is related to the length of the gradient of f. More precisely, it is not hard to check that, if $(\Delta(\alpha))$ holds, then for all $p \in [1, +\infty]$ and all function f on Γ,

$$\|df\|_{L^p(E)} \sim \|\nabla f\|_{L^p(\Gamma)}.$$

Indeed, if $1 \leq p < +\infty$, for all function f and all $x \in \Gamma$,

$$\begin{aligned} |\nabla f(x)|^p &\sim \left(\sum_{y \sim x} p(x,y) \, |f(y) - f(x)| \right)^p \\ &\sim \sum_{y \sim x} p^p(x,y) \, |f(y) - f(x)|^p \\ &\sim \sum_{y \sim x} p(x,y) \, |f(y) - f(x)|^p \end{aligned}$$

where the last line is due to $(\Delta(\alpha))$. As a consequence,

$$\begin{aligned} \|\nabla f\|^p_{L^p(\Gamma)} &\sim \sum_{x \in \Gamma} \sum_{y \sim x} p(x,y) \, |f(y) - f(x)|^p \, m(x) \\ &\sim \sum_{x,y \in \Gamma} |df(x,y)|^p \, \mu_{xy} \\ &= \|df\|^p_{L^p(E)}. \end{aligned}$$

The case when $p = +\infty$ is analogous and even easier. We could therefore reformulate properties (R_p) and (RR_p) replacing $\|\nabla f\|_{L^p(\Gamma)}$ by $\|df\|_{L^p(E)}$.
Besides d, we also consider its adjoint in L^2. If $df \in L^2(E)$ and G is any (antisymmetric) function in $L^2(E)$ such that the function $x \mapsto \sum_y p(x,y)G(x,y)$ belongs to $L^2(\Gamma)$, one has

$$\begin{aligned} \langle df, G \rangle_{L^2(E)} &= \frac{1}{2} \sum_{x,y \in \Gamma} df(x,y) G(x,y) \mu_{xy} \\ &= \frac{1}{2} \sum_{x,y \in \Gamma} f(y) G(x,y) \mu_{xy} - \frac{1}{2} \sum_{x,y \in \Gamma} f(x) G(x,y) \mu_{xy} \\ &= -\sum_{x,y \in \Gamma} f(x) G(x,y) \mu_{xy} \\ &= -\sum_{x \in \Gamma} f(x) \left(\sum_{y \in \Gamma} p(x,y) G(x,y) \right) m(x). \end{aligned}$$

Thus, if we define

$$\delta G(x) = \sum_y p(x,y) G(x,y)$$

for all $x \in \Gamma$, it follows that

$$\langle df, G \rangle_{L^2(E)} = -\langle f, \delta G \rangle_{L^2(\Gamma)}$$

whenever $f \in L^2(\Gamma)$, $dF \in L^2(E)$, $G \in L^2(E)$ and $\delta G \in L^2(\Gamma)$. Notice also that $I-P = -\delta d$.

The following lemma, very similar to Lemma 4.2 in [4], holds:

Lemma 8.1 *Assume that (D), ($\Delta(\alpha)$) and (DUE) hold. There exists $C > 0$ such that, for all ball B and all function $f \in L^2(\Gamma)$ supported in B, there exists a unique function $h \in W_0^{1,2}(B)$ such that*

$$(I - P)h = f \text{ in } \Gamma \tag{8.6}$$

and h satisfies

$$\|h\|_{W^{1,2}(\Gamma)} \leq C \|f\|_{L^2(\Gamma)} .$$

Proof: This proof relies on a Sobolev inequality, which will be used again in the proof of Theorem 1.7 and reads as follows: there exist $\nu \in (0,1)$ and $C > 0$ such that, for all ball B with radius $r > \frac{1}{2}$ and all function f supported in B,

$$\|f\|_q \leq CrV(B)^{-\frac{\nu}{2}} \|\nabla f\|_2 \tag{8.7}$$

with $q = \frac{2}{1-\nu}$. This inequality is actually equivalent to a relative Faber-Krahn inequality, which is itself equivalent to the conjunction of (D) and (DUE), see [23, 35, 15, 20, 10, 26]. Let B and f as in the statement of Lemma 8.1. Since $I - P = -\delta d$, (8.6) is equivalent to

$$\langle dh, dv \rangle_{L^2(E)} = \langle f, v \rangle_{L^2(\Gamma)}$$

for all $v \in W_0^{1,2}(B)$. For all $u, v \in W_0^{1,2}(B)$, set $\mathcal{B}(u,v) = \langle du, dv \rangle_{L^2(E)}$. It is obvious that $\mathcal{B}$ is a continuous bilinear form on $W_0^{1,2}(B)$. Moreover, for all $u \in W_0^{1,2}(B)$,

$$\mathcal{B}(u,u) = \|du\|^2_{L^2(E)} \geq c \|u\|^2_{W_0^{1,2}(B)} ,$$

by (8.7) (see also Lemma 4.1 in [4]). The conclusion of Lemma 8.1 follows then from the Lax-Milgram theorem. □

Let $F \in L^2(E)$. It is easy to check that $\delta F \in L^2(\Gamma)$ and

$$\|\delta F\|_{L^2(\Gamma)} \leq \|F\|_{L^2(E)} . \tag{8.8}$$

Indeed, for all $g \in L^2(\Gamma)$,

$$\begin{aligned} \left|\langle \delta F, g \rangle_{L^2(\Gamma)}\right| &= \left| \sum_{x,y \in \Gamma} p(x,y) F(x,y) g(x) m(x) \right| \\ &= \left| \sum_{x,y \in \Gamma} F(x,y) g(x) \mu_{xy} \right| \\ &\leq \left(\sum_{x,y \in \Gamma} |F(x,y)|^2 \mu_{x,y} \right)^{1/2} \left(\sum_{x \in \Gamma} |g(x)|^2 m(x) \right)^{1/2} . \end{aligned}$$

As a consequence of Lemma 8.1, for all $F \in L^2(E)$ with bounded support, there exists a unique function $f \in W^{1,2}(\Gamma)$ such that $(I - P)f = \delta F$. Since functions in $L^2(E)$ with bounded support are dense in $L^2(E)$, we can therefore extend the operator $d(I - P)^{-1}\delta$ to an $L^2(E)$-bounded operator.

8.2 The proof of Theorem 1.7

For all $1 \leq p < +\infty$, say that (Π_p) holds if and only if there exists $C_p > 0$ such that, for all $F \in L^p(E) \cap L^2(E)$,

$$\left\| d(I-P)^{-1}\delta F \right\|_{L^p(E)} \leq C_p \left\| F \right\|_{L^p(E)}. \qquad (\Pi_p)$$

Since $L^2(E) \cap L^p(E)$ is dense in $L^p(E)$, if (Π_p) holds, the operator $d(I-P)^{-1}\delta$ extends to a bounded operator from $L^p(E)$ to itself.

Let us now turn to the proof of Theorem 1.7. Let $p_0 > 2$ and $q \in (2, p_0)$. Denote by $(2')$ the following property:

$$\text{for all } p \in (2,q), (\Pi_p) \text{ holds.} \qquad (2')$$

We show that, for some $p_0 > 2$, if $q \in (2, p_0)$, then $2. \Rightarrow 2'. \Rightarrow 1. \Rightarrow 2.$

Proof of $2. \Rightarrow 2'.$ In order to apply Theorem 2.3 in [3], observe first that E, equipped with the metric d and the measure μ, is a space of homogeneous type. Let $2 < p < \widetilde{p} < q$. Consider $F \in L^2(E) \cap L^p(E)$ with bounded support included in $E \setminus 64B$ where B is a ball in E centered at $\gamma = (a,b)$ and with radius r. Lemma 8.1 and (8.8) therefore yield a function $h \in W^{1,2}(\Gamma)$ such that $(I-P)h = \delta F$ with $\|h\|_{W^{1,2}(\Gamma)} \leq C \|\delta F\|_{L^2(\Gamma)} \leq C \|F\|_{L^2(E)}$.

If $r \geq \frac{1}{16}$, then the function h is harmonic in $B(a, 32r)$. Indeed, if $x \in B(a,32r) \setminus \partial B(a,32r)$,

$$(I-P)h(x) = \delta F(x) = \sum_{y \sim x} p(x,y) F(x,y).$$

When $x \in B(a,32r)$ and $y \sim x$, $d(y,b) \leq d(x,a) + 2 \leq 64r$, so that $F(x,y) = 0$. It follows from $(RH_{\widetilde{p}})$ that

$$\left(\frac{1}{V(B)} \sum_{x \in B} |\nabla h(x)|^{\widetilde{p}}\, m(x) \right)^{\frac{1}{\widetilde{p}}} \leq C \left(\frac{1}{V(16B)} \sum_{x \in 16B} |\nabla h(x)|^2\, m(x) \right)^{\frac{1}{2}}.$$

If $r < \frac{1}{16}$, $B = 16B$ and the same inequality holds. This shows that the operator T defined by $TF = \nabla(I-P)^{-1}\delta F$ for all F with bounded support in E, clearly satisfies the assumptions of Theorem 2.3 in [3], and this theorem therefore yields

$$\|TF\|_{L^p(E)} \leq C_p \|F\|_{L^p(E)} \qquad (8.9)$$

for all F with bounded support in E. Since the space of antisymmetric functions on E with bounded support is dense in $L^p(E)$, (8.9) holds for all $F \in L^p(E)$, which exactly means that (Π_p) holds. □

Proof of $2'. \Rightarrow 1.$ By Theorem 1.11 and Proposition 1.8, there exists $\varepsilon > 0$ such that (RR_q) holds for all $q \in (2-\varepsilon, 2)$. It is therefore enough to check that the conjunction of (Π_p) and $(RR_{p'})$ implies (R_p), with $\frac{1}{p} + \frac{1}{p'} = 1$. But, if $f \in L^p(\Gamma) \cap L^2(\Gamma)$ and $G \in L^{p'}(E) \cap L^2(E)$,

$$\begin{aligned}
\left| \langle d(I-P)^{-1/2} f, G \rangle_{L^2(E)} \right| &= \left| \langle (I-P)^{-1/2} f, \delta G \rangle_{L^2(\Gamma)} \right| \\
&= \left| \langle f, (I-P)^{-1/2} \delta G \rangle_{L^2(\Gamma)} \right| \\
&\leq \|f\|_{L^p(\Gamma)} \left\| (I-P)^{-1/2} \delta G \right\|_{L^{p'}(\Gamma)} \\
&= \|f\|_{L^p(\Gamma)} \left\| (I-P)^{1/2} (I-P)^{-1} \delta G \right\|_{L^{p'}(\Gamma)} \\
&\leq \|f\|_{L^p(\Gamma)} \left\| d(I-P)^{-1} \delta G \right\|_{L^{p'}(\Gamma)} \\
&\leq C \|f\|_{L^p(\Gamma)} \|G\|_{L^{p'}(\Gamma)},
\end{aligned}$$

which ends the proof. □

Proof of 1. ⇒ 3. Assume now that (R_p) holds for some $p \in (2, q)$. Let B be a ball with center x_0 and radius k and u a function harmonic in $32B$, and fix a function φ supported in $3B$, equal to 1 in $2B$ and satisfying $0 \leq \varphi \leq 1$ and $\|\nabla\varphi\|_\infty \leq \frac{C}{k}$. Up to an additive constant, one may assume that the mean value of u in $16B$ is 0. In order to control the left-hand side of (RH_p), it suffices to estimate $\sum_{x\in B} |\nabla(u\varphi)(x)|^p \, m(x)$.

As in [7] p. 35 and [3], Section 2.4, write

$$u\varphi = P^{k^2}(u\varphi) + \sum_{l=0}^{k^2-1} P^l(I-P)(u\varphi),$$

so that

$$\nabla(u\varphi) \leq \nabla\left(P^{k^2}(u\varphi)\right) + \sum_{l=0}^{k^2-1} \nabla\left(P^l(I-P)(u\varphi)\right). \tag{8.10}$$

To treat the first term in the right-hand side of (8.10), fix $\rho \in (p, q)$ and notice that, since (R_ρ) holds by assumption, it follows that $k\left|\nabla P^{k^2}\right|$ is $L^\rho(\Gamma)$-bounded. Then, arguing as in Lemma 4.2, one obtains that

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|\nabla P^l f(x)\right|^p m(x)\right)^{1/p} \leq \frac{Ce^{-\frac{c4^j k^2}{l}}}{\sqrt{l}}\left(\frac{1}{V(2^jB)}\sum_{x\in C_j(B)}|f(x)|^2 m(x)\right)^{1/2} \tag{8.11}$$

for all $j \geq 1$, all $l \in \{1, ..., k^2\}$ and all function f supported in $C_j(B)$. It follows at once from (8.11) applied with $f = u\varphi$, the fact that u has zero integral on $16B$ and the Poincaré inequality (P_2) that

$$\begin{aligned}\left(\frac{1}{V(B)}\sum_{x\in B}\left|\nabla P^{k^2}(u\varphi)(x)\right|^p m(x)\right)^{1/p} &\leq \frac{C}{k}\left(\frac{1}{V(4B)}\sum_{x\in 4B}|u(x)|^2 m(x)\right)^{1/2} \\ &\leq C\left(\frac{1}{V(16B)}\sum_{x\in 16B}|\nabla u(x)|^2 m(x)\right)^{1/2}.\end{aligned} \tag{8.12}$$

Let us now turn to the second term in (8.10). A calculation shows that, for all $x \in \Gamma$,

$$\begin{aligned}(I-P)(u\varphi)(x) &= \sum_{y\in\Gamma} p(x,y)((u\varphi)(x) - (u\varphi)(y)) \\ &= \sum_{y\in\Gamma} p(x,y)u(x)(\varphi(x)-\varphi(y)) + \sum_{y\in\Gamma} p(x,y)(u(x)-u(y))(\varphi(y)-\varphi(x)) \\ &+ \sum_{y\in\Gamma} p(x,y)(u(x)-u(y))\varphi(x) \\ &:= v_1(x) + v_2(x) + v_3(x).\end{aligned} \tag{8.13}$$

For all $x \in \Gamma$, $v_3(x) = 0$ since u is harmonic in $32B$. Because of the support condition on φ, one may apply (8.11) to v_2, and since $\|\nabla\varphi\|_\infty \leq C/k$, one obtains

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|\nabla P^l v_2(x)\right|^p m(x)\right)^{1/p} \leq \frac{C}{k\sqrt{l}}\left(\frac{1}{V(4B)}\sum_{x\in 4B}\left|\nabla u(x)\right|^2 m(x)\right)^{1/2} \tag{8.14}$$

for all $1 \leq l \leq k^2 - 1$.
For v_1, write

$$\begin{aligned} 2v_1(x) &= \sum_y p(x,y)(u(x)+u(y))(\varphi(x)-\varphi(y)) + \sum_y p(x,y)(u(x)-u(y))(\varphi(x)-\varphi(y)) \\ &= \delta F(x) - v_2(x), \end{aligned}$$

where, for all $(x, y) \in E$,

$$F(x,y) = (u(x)+u(y))(\varphi(x)-\varphi(y))$$

is antisymmetric, belongs to $L^2(E)$ and is supported in $B((x_0,x_0),4k)\setminus B((x_0,x_0),2k)$. It is therefore enough to show that, for all $1 \leq l \leq k^2-1$,

$$\left(\sum_{x\in B}\left|\nabla P^l \delta F(x)\right|^p m(x)\right)^{\frac{1}{p}} \leq \frac{Ce^{-c\frac{k^2}{l}}}{l}\left(\frac{1}{V(B(x_0,x_0),4k)}\sum_{(x,y)\in B((x_0,x_0),4k)\setminus B((x_0,x_0),2k)}\left|F(x,y)\right|^2\mu_{xy}\right)^{\frac{1}{2}}. \tag{8.15}$$

To prove this inequality, if $l = 2m$, write $\nabla P^l \delta F = \nabla P^m P^m \delta F$. We establish (8.15) by arguments similar to the proof of Lemma 4.2, combining (2.4) and an inequality analogous to (2.4 and derived by duality (see the proof of (2.6) in [3]). We finally obtain

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|\nabla P^l v_1(x)\right|^p m(x)\right)^{1/p} \leq \frac{C}{k\sqrt{l}}\left(\frac{1}{V(4B)}\sum_{x\in 4B}\left|\nabla u(x)\right|^2 m(x)\right)^{1/2} \tag{8.16}$$

for all $1 \leq l \leq k^2 - 1$. Summing up (8.14) and (8.16) for $1 \leq l \leq k^2-1$, we obtain

$$\left(\frac{1}{V(B)}\sum_{x\in B}\nabla\left(\sum_{l=1}^{k^2-1}\left(P^l(I-P)(u\varphi)\right)\right)(x)^p m(x)\right)^{1/p} \leq C\left(\frac{1}{V(16B)}\sum_{x\in 16B}\left|\nabla u(x)\right|^2 m(x)\right)^{1/2}. \tag{8.17}$$

What remains to be treated in (8.10) is the term $\nabla(I-P)(u\varphi)$. By (8.13),

$$\frac{1}{V(B)^{1/p}}\left\|\nabla(I-P)(u\varphi)\right\|_{L^p(B)} \leq \frac{1}{V(B)^{1/p}}\left\|\nabla v_1\right\|_{L^p(B)} + \frac{1}{V(B)^{1/p}}\left\|\nabla v_2\right\|_{L^p(B)}. \tag{8.18}$$

Let us first deal with v_1. By (R_p),

$$\left\|\nabla v_1\right\|_{L^p(\Gamma)} \leq C\left\|(I-P)^{1/2}v_1\right\|_{L^p(\Gamma)} \leq C\left\|v_1\right\|_{L^p(\Gamma)},$$

where the last inequality follows from the L^p-boundedness of $(I-P)^{1/2}$ (see [25], p. 423 and also [19]). But v_1 is supported in $4B$ and, for all $x \in 4B$,

$$|v_1(x)| \leq \frac{C}{k} |u(x)| .$$

As a consequence,

$$\|v_1\|_{L^p(\Gamma)} \leq \frac{C}{k} \|u\|_{L^p(4B)} \leq \frac{C}{k} \|u\psi\|_{L^p(8B)} ,$$

where ψ is a nonnegative function equal to 1 on $4B$, supported in $8B$ and satisfying $\|\nabla\psi\|_\infty \leq \frac{C}{k}$. Now, (8.7) shows that, if $q_0 = \frac{2}{1-\nu}$ and $p \in (2, q_0)$,

$$\frac{1}{V(B)^{1/p}} \|u\psi\|_{L^p(8B)} \leq \frac{1}{V(8B)^{1/q_0}} \|u\psi\|_{L^{q_0}(8B)} \leq \frac{C}{V(8B)^{1/q_0}} k V(8B)^{-\nu/2} \|\nabla(u\psi)\|_{L^2(8B)} .$$

Using now the fact that $\frac{\nu}{2} = \frac{1}{2} - \frac{1}{q_0}$, we finally conclude

$$\frac{1}{V(B)^{1/p}} \|v_1\|_{L^p(\Gamma)} \leq \frac{C}{V(8B)^{1/2}} \|\nabla(u\psi)\|_{L^2(8B)} \leq \frac{C}{V(16B)^{1/2}} \|\nabla u\|_{L^2(16B)} , \tag{8.19}$$

where the last inequality is due (P_2). All these computations yield

$$\frac{1}{V(B)^{1/p}} \|\nabla v_1\|_{L^p(\Gamma)} \leq \frac{C}{V(16B)^{1/2}} \|\nabla u\|_{L^2(16B)} \tag{8.20}$$

We argue similarly for v_2. We just have to notice that, for all $x \in 4B$,

$$|v_2(x)|^p \leq \frac{C}{k^p} \sum_{y \sim x} \left(|u(y)|^p + |u(x)|^p \right) ,$$

hence

$$\sum_{x \in 4B} |v_2(x)|^p \, m(x) \leq \frac{C}{k^p} \sum_{x \in 4B} \sum_{y \sim x} |u(y)|^p \, m(x) + \frac{C}{k^p} \sum_{x \in 4B} \sum_{y \sim x} |u(x)|^p \, m(x).$$

Since $m(x) \leq C m(y)$ whenever $x \sim y$ (this is a straightforward consequence of (D) and was noticed in [23], Section 4.2) and $\sharp \{y \in \Gamma;\ y \sim x\} \leq N$, we finally obtain that

$$\sum_{x \in 4B} |v_2(x)|^p \, m(x) \leq \frac{C}{k^p} \sum_{x \in 8B} |u(x)|^p \, m(x),$$

and we conclude as for v_1 that

$$\frac{1}{V(B)^{1/p}} \|\nabla v_2\|_{L^p(\Gamma)} \leq \frac{C}{V(16B)^{1/2}} \|\nabla u\|_{L^2(16B)} , \tag{8.21}$$

Summing up (8.12), (8.17, (8.20) and (8.21), we obtain that (RH_p) holds. □

As far as Proposition 1.8 is concerned, its proof is entirely similar to the one of Proposition 2.2 in [3] and will therefore be skipped. Let us just mention that it relies on an elliptic Caccioppoli inequality (analogous to the Euclidean version, see [33]), Proposition 1.10 and Gehring's self-improvement of reverse Hölder inequalities ([32]). □

Appendix

We prove Lemma 4.1. For all $l \geq 0$, $a_l = \dfrac{(2l)!}{4^l (l!)^2}$, and, as already used in Section 7, the Stirling formula shows $a_l \sim \frac{1}{\sqrt{\pi l}}$. Therefore, there exists $C > 0$ such that, for all $l \geq 1$,

$$0 < a_l \leq \frac{C}{\sqrt{l}}.$$

Assume first that $mk^2 < l < (m+1)k^2$ for some integer $0 \leq m \leq n$. For each integer $j \geq 0$ such that $jk^2 \leq l$, one has $l - jk^2 > 0$ and $j \leq m$, so that $|a_{l-jk^2}| \leq \frac{C}{\sqrt{l-jk^2}} \leq \frac{C}{\sqrt{l-mk^2}}$. It follows at once that

$$|d_l| \leq \frac{C}{\sqrt{l-mk^2}}$$

for some $C > 0$ only depending on n.

Assume now that $l = (m+1)k^2$ for some $0 \leq m \leq n$. For each $j \geq 0$ such that $jk^2 \leq l$ and $l - jk^2 > 0$, one has $j \leq m$ again, so that $|a_{l-jk^2}| \leq \frac{C}{\sqrt{l-mk^2}} = \frac{C}{k} \leq C$. Moreover, $a_0 = 1$. One therefore has

$$|d_l| \leq C + C_n^{m+1} \leq C,$$

where, again, C only depends on n.

Finally, assume that $l > (n+1)k^2$. The classical computation of Wallis integrals shows that

$$a_l = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\sin t)^{2l}\, dt = \varphi(l)$$

where, for all $x > 0$, $\varphi(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\frac{\pi}{2}} (\sin t)^{2x}\, dt$. We can then invoke (3.4) and are therefore left with the task of estimating $\varphi^{(n)}$. But, for all $x > 0$,

$$\left|\varphi^{(n)}(x)\right| = \frac{2}{\pi} \left| \int_0^{\frac{\pi}{2}} (2 \log \sin t)^n\, e^{2x \log \sin t} dt \right| \leq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} |2 \log \sin t|^n\, e^{2x \log \sin t} dt := \frac{2}{\pi} I_n(x).$$

We now argue as in the "Laplace" method. For all $\delta \in \left(0, \frac{\pi}{2}\right)$, one clearly has, for all $x > 1$,

$$\begin{aligned} 0 \leq I_n(x) &\leq \int_0^{\frac{\pi}{2}-\delta} |2 \log \sin t|^n\, e^{2x \log \sin t} dt + \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} |2 \log \sin t|^n\, e^{2x \log \sin t} dt \\ &\leq \left(\sin\left(\frac{\pi}{2} - \delta\right)\right)^{2x-2} I_n(1) + J_n(x) = C_{n,\delta} \alpha^{2x-2} + J_n(x) \end{aligned} \tag{8.1}$$

where $C_{n,\delta} > 0$ only depends on n and δ, $0 < \alpha = \sin\left(\frac{\pi}{2} - \delta\right) < 1$ and $J_n(x) := \displaystyle\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} |2 \log \sin t|^n\, e^{2x \log \sin t} dt$.

Observe now that $J_n(x) = \displaystyle\int_0^\delta |2\log\cos u|^n \, e^{2x\log\cos u} du$. Since $\log(\cos u) \sim -\frac{u^2}{2}$ when $u \to 0$, we fix $\delta > 0$ such that, for all $0 < u < \delta$, $-\frac{3}{4}u^2 \leq \log(\cos u) \leq -\frac{1}{4}u^2$, which implies

$$\begin{aligned} J_n(x) &\leq C \int_0^\delta u^{2n} e^{-\frac{1}{2}xu^2} du \\ &\leq C \left(\frac{1}{\sqrt{x}}\right)^{2n+1} \int_0^{+\infty} v^{2n} e^{-v^2} dv \leq C x^{-n-\frac{1}{2}}. \end{aligned} \tag{8.2}$$

It follows from (8.1) and (8.2) that, for all $x > 1$,

$$\left|\varphi^{(n)}(x)\right| \leq C x^{-n-\frac{1}{2}},$$

which, joined with (3.4), yields assertion (*iii*) in Lemma 4.1, the proof of which is now complete. □

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