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**Computing uniform convex approximations
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Juillet 2005

Cahier n° 2005-021

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Computing uniform convex approximations for convex envelopes and convex hulls

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Résumé: Nous établissons une procédure numérique pour approcher uniformément par une suite de fonctions convexes, l'enveloppe convexe d'une fraction rationnelle ayant pour domaine, D , un ensemble de dimension finie supposé compact et semi-algébrique. Calculer la valeur d'une approximation en un point donné de $K=\text{co}(D)$ se résume à résoudre un programme semi-défini. En suite, nous caractérisons $K=\text{co}(D)$ comme projection d'un LMI semi-infini, et en plus nous approximations K par une suite décroissante d'ensembles convexes. Tester si un point donné n'est pas dans K se résume à résoudre un nombre fini de programmes semi-définis.

Abstract: We provide a numerical procedure to compute uniform (convex) approximations $\{f_r\}$ of the convex envelope f of a rational fraction f , on a compact semi-algebraic set D . At each point x in $K=\text{co}(D)$, computing $f_r(x)$ reduces to solving a semidefinite program. We next characterize the convex hull $K=\text{co}(D)$ in terms of the projection of a semi-infinite LMI, and provide outer convex approximations $\{K_r\} \supset K$. Testing whether x is not in K reduces to solving finitely many semidefinite programs.

Mots clés : Enveloppe convexe, programme semi-défini, dualité, ensemble semi-algébrique

Key Words : Convex envelope, semidefinite program, duality, semi-algebraic set

Classification AMS:

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1. INTRODUCTION

Computing the convex envelope $\widehat{f}$ of a given function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a challenging problem, and to the best of our knowledge, there is still no algorithm that approximates $\widehat{f}$ by convex functions (except for the simpler univariate case). For instance, for a function f on a bounded domain Ω , Brighi and Chipot [4] propose triangulation methods and provide piecewise degree-1 polynomial approximations $f_h \geq \widehat{f}$, and derive estimates of $f_h - \widehat{f}$ (where h measures the size of the mesh). Another possibility is to view the problem as a particular instance of the general moment problem, and use geometrical approaches as described in e.g. Anastassiou [1] or Kemperman [7]; but, as acknowledged in [1, 7], this approach is only practical for say, the univariate or bivariate cases.

Concerning convex sets, an important issue raised in Ben-Tal and Nemirovski [3], Parrilo and Sturmfels [12], is to characterize the convex sets that have a *LMI* (or *semidefinite*) *representation*, and called *SDr* sets in e.g. [3]. For instance, the epigraph of a univariate convex polynomial is a SDr set; see [3]. So far, and despite some progress in particular cases (see e.g. the recent proof of the Lax conjecture by Lewis et al [11]), little is known. However, Helton and Vinnikov [6] have proved recently that *rigid convexity* is a necessary condition for a set to be SDr.

In this paper we consider both convex envelopes and convex hulls for certain classes of functions and sets, namely *rational fractions* and compact *semi-algebraic* sets. In both cases we show that one is able to provide relatively simple numerical approximations via semidefinite programming.

Contribution. Our contribution is twofold:

Concerning convex envelopes, we consider the class of *rational fractions* f on a compact semi-algebraic set $\mathbb{D} \subset \mathbb{R}^n$ (and $+\infty$ outside $\mathbb{D}$). We view the problem as a particular instance of the general moment problem, and we provide an algorithm for computing *convex* and *uniform* approximations of its convex envelope $\widehat{f}$. More precisely, with $\mathbb{K} := \text{co}(\mathbb{D})$ being the convex hull of $\mathbb{D}$:

- (a) We provide a sequence of *convex* functions $\{f_r\}$ that converges to $\widehat{f}$, **uniformly** on any compact subset of $\mathbb{K}$ in which $\widehat{f}$ is continuous, as r increases.

- (b) At each point $x \in \mathbb{R}^n$, computing $f_r(x)$ reduces to solving a semidefinite program $\mathbf{Q}_{rx}$.

- (c) For every $x \in \text{int } \mathbb{K}$, the SDP dual $\mathbf{Q}_{rx}^*$ is solvable and any optimal solution provides an element of the subgradient $\partial f_r(x)$ at the point $x \in \text{int } \mathbb{K}$.

Concerning sets, we consider the class of compact semi-algebraic subsets $\mathbb{D} \subset \mathbb{R}^n$, and

- (d) We characterize its *convex hull* $\mathbb{K} := \text{co}(\mathbb{D})$ as the projection of a *semi-infinite* SDr set S_∞ , i.e., a set defined by finitely many LMIs involving matrices of infinite dimension, and countably many variables. Importantly, the LMI representation of S_∞ is simple and given *directly* in terms of the data defining the original set $\mathbb{D}$.

- (e) We provide outer convex approximations of $\mathbb{K}$, namely a monotone nonincreasing sequence of convex sets $\{\mathbb{K}_r\}$, with $\mathbb{K}_r \downarrow \mathbb{K}$. Each set $\mathbb{K}_r$ is the projection of a SDr set S_r , obtained from S_∞ by "finite truncation". Then, checking whether $x \notin \mathbb{K}$ reduces to solving finitely many SDPs based on SDr sets S_r , until one is unfeasible, which eventually happens for some r . Importantly again, the LMI representation of S_r is simple and given *directly* in terms of the data defining the original set $\mathbb{D}$. Other outer approximations are of course possible, like e.g. convex polytopes $\{\Omega_r\}$ containing $\mathbb{K}$, but obtaining such polytopes with $\Omega_r \downarrow \mathbb{K}$ is far from trivial.

2. NOTATION DEFINITIONS AND PRELIMINARY RESULTS

In the sequel, $\mathbb{R}[y](:= \mathbb{R}[y_1, \dots, y_n])$ denotes the ring of real-valued polynomials in the variable $y = (y_1, \dots, y_n)$. Let $\overline{y}_i \in \mathbb{R}[y]$ be the natural projection on the i -variable that is for every $x \in \mathbb{R}^n$, $\overline{y}_i(x) = x_i$. For a real-valued symmetric matrix M , the notation $M \succeq 0$ stands for M is positive semidefinite.

Let $\mathbb{D} \subset \mathbb{R}^n$ be compact, and denote by:

- $\mathbb{K}$, the convex envelope of $\mathbb{D}$. Hence, by a theorem of Caratheodory, $\mathbb{K}$ is convex and compact; see Rockafellar [14].
- $C(\mathbb{D})$, the Banach space of real-valued continuous functions on $\mathbb{D}$, equipped with the sup-norm $\|f\| := \sup_{x \in \mathbb{D}} |f(x)|$, $f \in C(\mathbb{D})$.
- $M(\mathbb{D})$, its topological dual, i.e., the Banach space of finite signed Borel measures on $\mathbb{K}$, equipped with the norm of total variation.
- $M_+(\mathbb{D}) \subset M(\mathbb{D})$, the positive cone, i.e., the set of finite Borel measures on $\mathbb{D}$.
- $\Delta(\mathbb{D}) \subset M_+(\mathbb{D})$, the set of borel probability measures on $\mathbb{D}$.
- for f in $C(\mathbb{D})$, let $\tilde{f}$ be its natural extension to $\mathbb{R}^n$ that is

$$(1) \quad \tilde{f}(x) := \begin{cases} f(x) & \text{on } \mathbb{D} \\ +\infty & \text{on } \mathbb{R}^n \setminus \mathbb{D}. \end{cases}$$

Note that $\tilde{f}$ is lowersemicontinuous (l.s.c.), admits a minimum and its domain is non-empty and compact.

- for f in $C(\mathbb{D})$, let $\hat{f}$ the convex envelope of $\tilde{f}$. This is the greatest convex function smaller than $\tilde{f}$.

Note that the vector spaces $M(\mathbb{D})$ and $C(\mathbb{D})$ are in duality with the duality bracket

$$\langle \sigma, f \rangle := \int_{\mathbb{K}} f d\sigma, \quad \sigma \in M(\mathbb{K}), f \in C(\mathbb{K})$$

Hence, let τ^* denote the associated weak* topology; this is the coarsest topology on $M(\mathbb{D})$ for which $\sigma \rightarrow \langle \sigma, f \rangle$ is continuous for every function f in $C(\mathbb{D})$.

With $f \in C(\mathbb{D})$, and $x \in \mathbb{K} = \text{co}(\mathbb{D})$ fixed, arbitrary, consider the infinite-dimensional linear program (LP):

$$(2) \quad \text{LP}_x : \quad \begin{cases} \inf_{\sigma \in M_+(\mathbb{D})} & \langle \sigma, f \rangle \\ \text{s.t.} & \langle \sigma, \overline{y_i} \rangle = x_i, \quad i = 1, \dots, n \\ & \langle \sigma, 1 \rangle = 1. \end{cases}$$

Its optimal value is denoted by $\inf \text{LP}_x$, and $\min \text{LP}_x$ if the infimum is attained. Notice that LP_x is a particular instance of the general *moment problem*, as described in e.g. Kemperman [7, §2.6]. In particular, the set of $x \in \mathbb{R}^n$ such that LP_x has a feasible solution, is called the *moment space*.

Lemma 2.1. *Let $\mathbb{K} = \text{co}(\mathbb{D})$, $f \in C(\mathbb{D})$ and $\tilde{f}$ be as in (1). Then the convex envelope $\hat{f}$ of $\tilde{f}$ is given by:*

$$(3) \quad \hat{f}(x) = \begin{cases} \min \text{LP}_x, & x \in \mathbb{K}, \\ +\infty, & x \in \mathbb{R}^n \setminus \mathbb{K}, \end{cases}$$

so that the domain of $\hat{f}$ is $\mathbb{K}$.

Proof. For $x \in \mathbb{K}$, let $\Delta_x(\mathbb{D})$ be the set of probability measures σ on $\mathbb{D}$, that are centered at x (that is $\langle \sigma, \overline{y_i} \rangle = x_i$ for $i = 1, \dots, n$). Let $\Delta_x^*(\mathbb{D}) \subset \Delta_x(\mathbb{D})$ be the subset of those probability measures that have a finite support. It is well known that

$$\hat{f}(x) = \inf_{\sigma \in \Delta_x^*(\mathbb{D})} \langle \sigma, f \rangle, \quad \forall x \in \mathbb{K};$$

see Choquet [5] or Laraki [9]. Next, since $\Delta_x^*(\mathbb{D})$ is dense in $\Delta_x(\mathbb{D})$ with respect to the weak* topology, and $\Delta(\mathbb{D})$ is metrizable and compact with respect to the same topology (see Choquet [5]), deduce that for every $x \in \mathbb{K}$

$$\begin{aligned} \hat{f}(x) &= \min_{\sigma \in \Delta_x(\mathbb{D})} \langle \sigma, f \rangle \\ &= \min \text{LP}_x. \end{aligned}$$

If $x \notin \mathbb{K}$, there is no probability measure on $\mathbb{D}$, with finite support, and centered in x ; therefore $\hat{f}(x) = +\infty$. $\square$

Next, let $p, q \in \mathbb{R}[y]$, with $q > 0$ on $\mathbb{D}$, and let $f \in C(\mathbb{D})$ be defined as

$$(4) \quad y \mapsto f(y) = p(y)/q(y), \quad y \in \mathbb{D}.$$

For every $x \in \mathbb{K}$, consider the LP,

$$(5) \quad \mathbf{P}_x : \quad \begin{cases} \inf_{\sigma \in M_+(\mathbb{D})} & \langle \sigma, p \rangle \\ \text{s.t.} & \langle \sigma, \overline{y_i} q \rangle = x_i, \quad i = 1, \dots, n \\ & \langle \sigma, q \rangle = 1. \end{cases}$$

A dual of $\mathbf{P}_x$, is the LP

$$(6) \quad \mathbf{P}_x^* : \quad \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n} \{ \gamma + \langle \lambda, x \rangle : p(y) - q(y) \langle \lambda, y \rangle \geq \gamma q(y), \quad \forall y \in \mathbb{D} \},$$

where $\langle \lambda, y \rangle := \sum_{i=1}^n \lambda_i y_i$ stands for the standard inner product in $\mathbb{R}^n$. The optimal value of $\mathbf{P}_x^*$ is denoted by $\sup \mathbf{P}_x^*$ (and $\max \mathbf{P}_x^*$ if the supremum is attained). Equivalently, as $q > 0$ everywhere on $\mathbb{D}$, and $f = p/q$ on $\mathbb{D}$,

$$(7) \quad \mathbf{P}_x^* : \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^m} \{ \gamma + \langle \lambda, x \rangle : f(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D} \}.$$

In view of the definition (4) of f , notice that

$$f(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D} \quad \Leftrightarrow \quad \tilde{f}(y) - \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{R}^n.$$

Hence $\mathbf{P}_x^*$ in (7) is just the dual LP_x^* of LP_x , for every $x \in \mathbb{K}$, and so $\sup \mathbf{P}_x^* = \sup \text{LP}_x^*$, for every $x \in \mathbb{K}$. In fact we even have the following result:

Theorem 2.2. *Let $p, q \in \mathbb{R}[x]$ with $q > 0$ on $\mathbb{D}$, and let f be as in (4). Let $x \in \mathbb{K} = \text{co}(\mathbb{D})$ be fixed, arbitrary, and let $\mathbf{P}_x$ and $\mathbf{P}_x^*$ be as in (5) and (7), respectively. Then $\mathbf{P}_x$ and LP_x are solvable and there is no duality gap, i.e.,*

$$(8) \quad \sup \mathbf{P}_x^* = \sup \text{LP}_x^* = \min \text{LP}_x = \min \mathbf{P}_x = \hat{f}(x), \quad x \in \mathbb{K}.$$

Proof. Observe that $\mathbf{P}_x$ is equivalent to LP_x . Indeed, with σ an arbitrary feasible solution of $\mathbf{P}_x$, the measure $d\mu := q d\sigma$ is feasible in LP_x , with same value. Similarly, with μ an arbitrary feasible solution of LP_x , the measure $d\sigma := q^{-1} d\mu$, well defined on $\mathbb{D}$ because $q > 0$ on $\mathbb{D}$, is feasible in $\mathbf{P}_x$, and with same value. Finally, it is well known that $\hat{f}$ is the Legendre-Fenchel biconjugate¹ of $\tilde{f}$, and so $\hat{f}(x) = \sup \mathbf{P}_x^*$, for all $x \in \mathbb{K}$. Indeed, let $f^* : \mathbb{R}^n \rightarrow \mathbb{R}$ be the Legendre-Fenchel conjugate of $\tilde{f}$, i.e.,

$$\lambda \mapsto f^*(\lambda) := \sup_{y \in \mathbb{R}^n} \{ \langle \lambda, y \rangle - \tilde{f}(y) \}.$$

In view of the definition of $\tilde{f}$,

$$f^*(\lambda) = \sup_{y \in \mathbb{D}} \{ \langle \lambda, y \rangle - f(y) \},$$

and therefore,

$$\begin{aligned} \sup \mathbf{P}_x^* &= \sup_{\lambda} \{ \langle \lambda, x \rangle + \inf_{y \in \mathbb{D}} \{ f(y) - \langle \lambda, y \rangle \} \} \\ &= \sup_{\lambda} \{ \langle \lambda, x \rangle - \sup_{y \in \mathbb{D}} \{ \langle \lambda, y \rangle - f(y) \} \} \\ &= \sup_{\lambda} \{ \langle \lambda, x \rangle - f^*(\lambda) \} = (f^*)^*(x) = \hat{f}(x). \end{aligned}$$

□

In fact, we have the following.

Corollary 2.3. *Let $x \in \mathbb{K}$ be fixed, arbitrary, and let $\mathbf{P}_x^*$ be as in (7).*

(a) $\mathbf{P}_x^*$ is solvable iff $\partial \hat{f}(x) \neq \emptyset$ ². In that case, any optimal solution (λ^*, γ^*) satisfies:

$$(9) \quad \lambda^* \in \partial \hat{f}(x), \quad \text{and} \quad \gamma^* = -f^*(\lambda^*).$$

(b) If f is a rational fraction on $\mathbb{D}$ as in (4), then $\partial \hat{f}(x) \neq \emptyset$ for every x in $\mathbb{K}$ so that, in this case $\mathbf{P}_x^*$ is solvable and (a) holds for every x in $\mathbb{K}$.

¹See Section 2 in Benoist and Hiriart-Urruty [2]

² $\partial \hat{f}(x) \neq \emptyset$ at least for every x in the relative interior of $\mathbb{K}$ (see Rokafellar 1970, Theorem 23.4)

Proof. Suppose that for some $x \in \mathbb{K}$, $\mathbf{P}_x^*$ is solvable (that is, the supremum is achieved, say at $\lambda^*(x)$ and $\gamma^*(x)$). Then, for every $y \in \mathbb{K}$,

$$\begin{aligned}\widehat{f}(x) &= \langle \lambda^*(x), x \rangle + \gamma^*(x) \\ \widehat{f}(y) &= \sup_{\lambda, \gamma} \{ \langle \lambda, y \rangle + \gamma : f(z) - \langle \lambda, z \rangle \geq \gamma, \quad \forall z \in \mathbb{D} \}, \quad y \in \mathbb{K} \\ &\geq \langle \lambda^*(x), y \rangle + \gamma^*(x),\end{aligned}$$

and in view of (3), the latter inequality also holds for every y in $\mathbb{R}^n$; therefore,

$$\widehat{f}(y) - \widehat{f}(x) \geq \langle \lambda^*(x), y - x \rangle, \quad \forall y \in \mathbb{R}^n.$$

Hence, $\lambda^*(x) \in \partial \widehat{f}(x)$. Finally, from the standard Legendre-Fenchel equality, we deduce that $\gamma^*(x) = -f^*(\lambda^*(x))$ where f^* is the Legendre-Fenchel conjugate of $\widehat{f}$. Conversely, if $\lambda^*(x) \in \partial \widehat{f}(x)$ then, by the Legendre-Fenchel equality we have,

$$\widehat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)).$$

Therefore, we have:

$$(10) \quad \sup \mathbf{P}_x^* = \widehat{f}(x) = \langle \lambda^*(x), x \rangle - f^*(\lambda^*(x)).$$

Next, from

$$f^*(\lambda^*(x)) = \sup_{y \in \mathbb{D}} \langle \lambda^*(x), y \rangle - f(y),$$

we have

$$-f^*(\lambda^*(x)) \leq f(y) - \langle \lambda^*(x), y \rangle, \quad \forall y \in \mathbb{D},$$

which shows that the pair $(\lambda^*(x), -f^*(\lambda^*(x)))$ is a feasible solution of $\mathbf{P}_x^*$, and in view of (10), an optimal solution.

Now, if f is a rational fraction on $\mathbb{D}$, then it is differentiable and Lipschitz on $\mathbb{D}$ so that, from Theorem 3.6 in Benoist and Hiriart-Urruty [2], $\partial \widehat{f}(x)$ is uniformly bounded as x varies on the relative interior of $\mathbb{K}$. Since $\widehat{f}$ is l.s.c. (see below), we deduce that $\partial \widehat{f}(x) \neq \emptyset$ for every x in $\mathbb{K}$. Actually, let $x \in \mathbb{K}$ and let x_n be a sequence in the relative interior of $\mathbb{K}$ that converges to x and let $\lambda_n \in \partial \widehat{f}(x_n)$ such that $\lambda_n \rightarrow \lambda$ (which is possible (passing to a subsequence if needed) since $\partial \widehat{f}(x_n)$ is uniformly bounded). Hence, for every y in $\mathbb{R}^n$,

$$\widehat{f}(y) \geq \widehat{f}(x_n) + \langle \lambda_n, y - x_n \rangle,$$

so that,

$$\begin{aligned}\widehat{f}(y) &\geq \liminf_{n \rightarrow \infty} \widehat{f}(x_n) + \langle \lambda, y - x \rangle \\ &\geq \widehat{f}(x) + \langle \lambda, y - x \rangle\end{aligned}$$

consequently, $\lambda \in \partial \widehat{f}(x)$, the desired result. $\square$

In other words, any optimal solution of $\mathbf{P}_x^*$ provides an element of the subgradient of $\widehat{f}$ at the point x . Corollary 2.3 should be viewed as a refinement for convex envelopes of rational fractions, of Theorem 2.20 in Kemperman [7, p. 28] for the general moment problem, where strong duality results are obtained for the interior of the moment space (here $\text{int } \mathbb{K}$) only.

3. PRESERVATION OF CONTINUITY

Later we will construct a sequence f_r that approximates $\hat{f}$ uniformly at each compact set on which $\hat{f}$ is continuous. Hence it is natural to first define conditions on the data, to ensure that $\hat{f}$ is continuous. The question was solved in Laraki [9] when $\mathbb{D}$ is convex ($\mathbb{K} = \text{co}(\mathbb{D})$). Here we extend this to our general framework.

Definition 3.1. *The compact set $\mathbb{D}$ of $\mathbb{R}^n$ is Splitting-Continuous if and only if $x \mapsto \Delta_x(\mathbb{D})$ is continuous when $\Delta(\mathbb{D})$ is equipped with the weak* topology.*

Examples : this is exactly the Splitting-Continuous condition defined in Laraki [9] when $\mathbb{D}$ is convex. Note that if $\mathbb{D}$ is a polytope or is strictly-convex then it is Splitting-Continuous (see [9][Theorem 1.16]). In fact, if $\mathbb{D}$ is convex then there are equivalence between Splitting-continuous and the more natural condition faces-closed. The latter concept means (when $\mathbb{D}$ is convex) that any Hausdorff converging sequence of faces of $\mathbb{D}$ is also a face of $\mathbb{D}$ (see Laraki 2004, Theorem 1.16).

Lemma 3.2. *Let f be continuous on the compact $\mathbb{D}$ of $\mathbb{R}^n$. Let $\hat{f}$ be as in (3). Then, $\hat{f}$ is always l.s.c. with respect to $\mathbb{K} = \text{co}(\mathbb{D})$ and is continuous on any compact $\bar{\mathbb{K}}$ that is strictly included in the relative interior of $\mathbb{K}$. Moreover, $\mathbb{D}$ is Splitting-Continuous if and only if $\hat{f}$ is continuous on $\mathbb{K}$, for every f which is the restriction on $\mathbb{D}$ of some continuous function on $\mathbb{K}$.*

Proof. From Theorem 10.1 in Rockafellar [14], since $\hat{f}$ is convex, it is always continuous in the relative interior of $\mathbb{K}$. Also, since the correspondence $x \mapsto \Delta_x(\mathbb{D})$ is always uppersemicontinuous (u.s.c.), and since f is continuous, $\hat{f}$ is always l.s.c. ; see e.g. Theorem 6 in Laraki and Sudderth [8].

Again, from Theorem 6 in Laraki and Sudderth [8], $x \mapsto \Delta_x(\mathbb{D})$ is continuous if and only if $\hat{f}$ is continuous on $\mathbb{K}$, for every f which is the restriction on $\mathbb{D}$ of some continuous function on $\mathbb{K}$. $\square$

4. UNIFORM CONVEX APPROXIMATIONS OF $\hat{f}$ BY SDP-RELAXATIONS

In this section, we assume that f is defined as in (4) for some polynomials $p, q \in \mathbb{R}[x]$, with $q > 0$ on $\mathbb{K}$, where $\mathbb{D} \subset \mathbb{R}^n$ is the convex and compact semi-algebraic set defined as

$$(11) \quad \mathbb{K} := \{x \in \mathbb{R}^n : | \quad g_j(x) \geq 0, \quad j = 1, \dots, m\},$$

for some polynomials $\{g_j\} \subset \mathbb{R}[x]$. Depending on its parity, let $2r_j - 1$ or $2r_j$ be the total degree of g_j , for all $j = 1, \dots, m$. Similarly, let $2r_p, 2r_q$ or $2r_p - 1, 2r_q - 1$ be the total degree of p and q respectively.

We next provide a sequence $\{f_r\}_r$ of functions such that for all r

- f_r is convex;
- the domain of f_r is $\mathbb{K}_r \supset \mathbb{K} = \text{co}(\mathbb{D})$;
- $f_r \leq \hat{f}$ and for every $x \in \mathbb{K}$, $f_r(x) \uparrow \hat{f}(x)$ as $r \rightarrow \infty$. In fact, we even have

$$\lim_{r \rightarrow \infty} : \|\hat{f} - f_r\|_{\bar{\mathbb{K}}} \rightarrow 0,$$

that is, f_r converges to $\hat{f}$, *uniformly* on any compact $\bar{\mathbb{K}} \subset \mathbb{K}$ in which $\hat{f}$ is continuous. Consequently, if $\mathbb{D}$ is Splitting-Continuous and if $q > 0$ on $\mathbb{K}$, then we obtain uniform convergence on $\mathbb{K}$. Also, if $\bar{\mathbb{K}}$ is strictly included in the relative interior of $\mathbb{K}$ then we also obtain uniform convergence.

To do this we first introduce some additional notation.

4.1. Notation and definitions. Let $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a sequence indexed in the canonical basis $\{z^\alpha\}$ of $\mathbb{R}[z]$, and let $L_y : \mathbb{R}[z] \rightarrow \mathbb{R}$ be the linear functional defined by

$$h := \sum_{\alpha \in \mathbb{N}^n} h_\alpha z^\alpha \mapsto L_y(h) := \sum_{\alpha \in \mathbb{N}^n} h_\alpha y_\alpha.$$

Let $\mathcal{P}_k \subset \mathbb{R}[z]$ be the space of polynomials of total degree less than k , and let $r_0 := \max[r_p, r_q + 1, r_1, \dots, r_m]$. Then for $r \geq r_0$, consider the optimization problem:

$$(12) \quad \mathbf{Q}_{rx} : \begin{cases} \inf_y L_y(p) \\ \text{s.t. } L_y(z_i q) = x_i, \quad i = 1, \dots, n, \\ L_y(h^2) \geq 0, \quad \forall h \in \mathcal{P}_r, \\ L_y(h^2 g_j) \geq 0, \quad \forall h \in \mathcal{P}_{r-r_j}, j = 1, \dots, m, \\ L_y(q) = 1. \end{cases}$$

Problem $\mathbf{Q}_{rx}$ is a convex optimization problem, in fact, a so-called *semidefinite programming* problem, called a SDP-relaxation of $\mathbf{P}_x$. For more details on semi-definite programming and its applications, the reader is referred to Vandenberghe and Boyd [16].

Indeed, given $y = \{y_\alpha\}$, let $M_r(y)$ be the *moment* matrix associated with y , that is, the rows and columns of $M_r(y)$ are indexed in the canonical basis of $\mathcal{P}_r$, and the entry (α, β) is defined by $M_r(y)(\alpha, \beta) = L_y(z^{\alpha+\beta}) = y_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{N}$, with $|\alpha|, |\beta| \leq r$. Then

$$L_y(h^2) \geq 0, \quad \forall h \in \mathcal{P}_r \quad \Leftrightarrow \quad M_r(y) \succeq 0.$$

Similarly, writing

$$z \mapsto g_j(z) := \sum_{\gamma \in \mathbb{N}^n} (g_j)_\gamma z^\gamma, \quad j = 1, \dots, m,$$

the *localizing* matrix $M_r(g_j y)$ associated with y and $g_j \in \mathbb{R}[z]$, is the matrix also indexed in the canonical basis of $\mathcal{P}_r$, and whose entry (α, β) is defined by

$$M_r(g_j y)(\alpha, \beta) = L_y(g_j(z) z^{\alpha+\beta}) = \sum_{\gamma \in \mathbb{N}^n} y_{\alpha+\beta+\gamma} (g_j)_\gamma,$$

for all $\alpha, \beta \in \mathbb{N}$, with $|\alpha|, |\beta| \leq r$. Then, for every $j = 1, \dots, m$,

$$L_y(g_j h^2) \geq 0, \quad \forall h \in \mathcal{P}_r \quad \Leftrightarrow \quad M_r(g_j y) \succeq 0.$$

Observe that if y has a *representing measure* μ_y , i.e. if

$$y_\alpha = \int x^\alpha d\mu_y, \quad \forall \alpha \in \mathbb{N}^n,$$

then, with $h \in \mathcal{P}_r$, and also denoting by $h = \{h_\alpha\} \in \mathbb{R}^{s(r)}$ its vector of coefficients in the canonical basis,

$$\langle h, M_r(g_j y) h \rangle = \int h^2 g_j d\mu_y.$$

Therefore, if μ_y has its support contained in the level set $\{x \in \mathbb{R}^n : g_j(x) \geq 0\}$, we must have $M_r(g_j y) \succeq 0$.

One also denotes by $M_\infty(y)$ and $M_\infty(g_j y)$ the (obvious) respective "infinite" versions of $M_r(y)$ and $M_r(g_j y)$, i.e., moment and localizing matrices with countably

many rows and columns indexed in the canonical basis $\{z_\alpha\}$, and involving *all* the variables y (as opposed to finitely many in $M_r(y)$ and $M_r(g_j y)$).

For more details on moment and localizing matrices, the reader is referred to Lasserre [10].

4.2. SDP-relaxations. Hence, using the above notation, the optimization problem $\mathbf{Q}_{rx}$ defined in (12) is just the SDP

$$(13) \quad \mathbf{Q}_{rx} : \begin{cases} \inf_y L_y(p) \\ \text{s.t.} & L_y(z_i q) = x_i, \quad i = 1, \dots, n \\ & M_r(y) \succeq 0, \\ & M_{r-r_j}(g_j y) \succeq 0, \quad j = 1, \dots, m, \\ & L_y(q) = 1, \end{cases}$$

with optimal value denoted $\inf \mathbf{Q}_{rx}$, and $\min \mathbf{Q}_{rx}$ if the infimum is attained.

If we write $M_r(y) = \sum_\alpha B_\alpha y_\alpha$, and $M_{r-r_j}(g_j y) = \sum_\alpha C_\alpha^j y_\alpha$, for appropriate symmetric matrices $\{B_\alpha, C_\alpha^j\}$, then the dual of $\mathbf{Q}_{rx}$ is the SDP

$$(14) \quad \mathbf{Q}_{rx}^* : \begin{cases} \sup_{\lambda, \gamma, X, Z_j} \gamma + \langle \lambda, x \rangle \\ \text{s.t.} & \langle B_\alpha, X \rangle + \sum_{j=1}^m \langle C_\alpha^j, Z_j \rangle + \gamma q_\alpha + \sum_{i=1}^n \lambda_i (z_i q)_\alpha = p_\alpha, \quad |\alpha| \leq 2r \\ & X, Z_j \succeq 0. \end{cases}$$

In fact, $\mathbf{Q}_{rx}^*$ is the same as (letting $g_0 \equiv 1$)

$$(15) \quad \mathbf{Q}_{rx}^* : \begin{cases} \sup_{\gamma \in \mathbb{R}, \lambda \in \mathbb{R}^n, u_j \in \mathbb{R}[z]} \gamma + \langle \lambda, x \rangle \\ \text{s.t.} & p - \gamma q - \langle \lambda, z \rangle q = \sum_{j=0}^m u_j g_j \\ & u_j \text{ s.o.s.}, \quad \deg u_j g_j \leq 2r, \quad j = 0, \dots, m \end{cases}$$

(where s.o.s. stands for *sum of squares*).

We next make the following assumption on the polynomials $\{g_j\} \subset \mathbb{R}[z]$ in the definition of the set $\mathbb{D}$ in (11).

Assumption 4.1. *There is a polynomial $u \in \mathbb{R}[z]$, positif on $\mathbb{D}$, which can written*

$$(16) \quad u = u_0 + \sum_{j=1}^m u_j g_j,$$

for a family of s.o.s. polynomials $\{u_j\}_{j=0}^m \subset \mathbb{R}[z]$, and the level set $\{z \in \mathbb{R}^n \mid u(z) \geq 0\}$ is compact.

Assumption 4.1 is not very restrictive. For instance, it is satisfied if:

- all the g_j 's are linear (and so, $\mathbb{D}$ is a polytope), or if
- the level set $\{z \in \mathbb{R}^n \mid g_j(z) \geq 0\}$ is compact, for some $j \in \{1, \dots, m\}$.

Moreover, if one knows that the compact set $\mathbb{D}$ is contained in a ball $\{z \in \mathbb{R}^n \mid \|z\| \leq M\}$, for some $M \in \mathbb{R}$, then it suffices to add the redundant quadratic constraint $M^2 - \|z\|^2 \geq 0$ in the definition (11) of $\mathbb{D}$, and Assumption 4.1 holds true.

Under Assumption 4.1, every polynomial $v \in \mathbb{R}[x]$, strictly positive on $\mathbb{D}$, can be written as

$$v = v_0 + \sum_{j=1}^m v_j g_j,$$

for some family of s.o.s. polynomials $\{v_j\}_{j=0}^m \subset \mathbb{R}[x]$. This is Putinar's Positivstellensatz, an particular version of Schmüdgen's Positivstellensatz (see Putinar [13]).

Then we have the following result:

Theorem 4.2. *Let $\mathbb{D}$ be as in (11), and let Assumption 4.1 hold. Let f be as in (4) with $p, q \in \mathbb{R}[z]$, and with $q > 0$ on $\mathbb{D}$. Let $\hat{f}$ be as in (3), and with $x \in \mathbb{K} = \text{co}(\mathbb{D})$ fixed, consider the SDP-relaxations $\{\mathbf{Q}_{rx}\}$ defined in (12) (or equivalently in (13)). Then:*

(a) *For every $x \in \mathbb{R}^n$,*

$$(17) \quad \inf \mathbf{Q}_{rx} \uparrow \hat{f}(x), \quad \text{as } r \rightarrow \infty.$$

(b) *The function $f_r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by*

$$(18) \quad x \mapsto f_r(x) := \inf \mathbf{Q}_{rx}, \quad x \in \mathbb{R}^n,$$

is convex, and as $r \rightarrow \infty$, $f_r(x) \uparrow \hat{f}(x)$ pointwise, for all $x \in \mathbb{R}^n$.

(c) *If $\mathbb{K}$ has a nonempty interior $\text{int } \mathbb{K}$, then*

$$(19) \quad \sup \mathbf{Q}_{rx}^* = \max \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx} = f_r(x), \quad x \in \text{int } \mathbb{K},$$

and for every optimal solution $(\lambda_r^, \gamma_r^*)$ of $\mathbf{Q}_{rx}^*$,*

$$f_r(y) - f_r(x) \geq \langle \lambda_r^*, y - x \rangle, \quad \forall y \in \mathbb{R}^n,$$

that is, $\lambda_r^ \in \partial f_r(x)$.*

For a proof see §6.1. As a consequence, we also get:

Corollary 4.3. *Let $\mathbb{D}$ be as in (11), and let Assumption 4.1 hold. Let f and $\hat{f}$ be as in (4) and (3) respectively, and let $f_r : \mathbb{K} \rightarrow \mathbb{R}$, be as in Theorem 4.2. Then f_r is always l.s.c. Moreover, for every compact $\bar{\mathbb{K}} \subset \mathbb{K}$ in which $\hat{f}$ is continuous,*

$$(20) \quad \lim_{r \rightarrow \infty} : \sup_{x \in \bar{\mathbb{K}}} : |\hat{f}(x) - f_r(x)| = 0,$$

that is, the monotone nondecreasing sequence $\{f_r\}$ converges to $\hat{f}$, uniformly on every compact in which $\hat{f}$ is continuous. Consequently, if in addition $\mathbb{D}$ is Splitting-continuous and if $q > 0$ on $\mathbb{K}$, then the convergence is uniform on $\mathbb{K}$.

Proof. The lowersemicontinuity of f_r may be obtained using Laraki & Sudderth [8]. This is due to the facts that:

- the objective function of $\mathbf{Q}_{rx}$ does not depend on x , and
 - the feasible set of $\mathbf{Q}_{rx}$ as a function of x , is u.s.c., in the sense of Kuratowski.
- By Theorem 4.2, we already have that $f_r \uparrow \hat{f}$ on $\bar{\mathbb{K}}$.

(1) The convergence $f_r \uparrow \hat{f}$ on $\bar{\mathbb{K}}$, (2) the fact that f_r is l.s.c., (3) that the limit $\hat{f}$ is continuous, and finally (4) that $\bar{\mathbb{K}}$ is compact, imply that the convergence is *uniform* on $\bar{\mathbb{K}}$. Actually, by (1) and (2) the function $\hat{f}(x) - f_r(x)$ is always positive and u.s.c. on $\bar{\mathbb{K}}$, and therefore, by (4), it admits a global maximizer $x_r \in \bar{\mathbb{K}}$. Let $\alpha = \limsup_{r \rightarrow \infty} \hat{f}(x_r) - f_r(x_r)$. Without loss of generality and using (4), suppose

that the limsup is achieved with the sequence $\{x_n\} \subset \overline{\mathbb{K}}$ that converges to some $x_0 \in \overline{\mathbb{K}}$. Let t be some fixed integer. Hence, for any $n \geq t$, using (1) one has

$$f_n(x_n) \geq f_t(x_n), \quad [\text{by monotonicity of } \{f_n\}]$$

so that we obtain

$$\lim_{n \rightarrow \infty} f_n(x_n) \geq f_t(x_0), \quad [\text{by lowersemicontinuity of } f_t].$$

Finally, letting t go to infinity and using (3),

$$\lim_{n \rightarrow \infty} f_n(x_n) \geq \widehat{f}(x_0),$$

and so,

$$\alpha = \lim_{n \rightarrow \infty} \widehat{f}(x_n) - f_n(x_n) = \widehat{f}(x_0) - \lim_{n \rightarrow \infty} f_n(x_n) \leq 0.$$

This implies uniform convergence on $\overline{\mathbb{K}}$. $\square$

Remark 4.4 If Assumption 4.1 does not hold, then in the SDP-relaxation $\mathbf{Q}_{rx}$ in (12), one replaces the m LMI constraints $L_y(g_j h^2) \geq 0$ for all $h \in \mathcal{P}_{r-r_j}$, with the 2^m LMI constraints

$$L_y(g_J h^2) \geq 0, \quad \forall h \in \mathcal{P}_{r-r_J}, \quad \forall J \subseteq \{1, \dots, m\},$$

where $g_J := \prod_{j \in J} g_j$, and $r_J = \deg g_J$, for all $J \subseteq \{1, \dots, m\}$ (and $g_\emptyset \equiv 1$). Indeed, Theorem 4.2 and Corollary 4.3 remain valid with $f_r(x) := \inf \mathbf{Q}_{rx}$, for all $x \in \mathbb{R}^n$ (with the newly defined $\mathbf{Q}_{rx}$). In the proof, one now invokes Schmüdgen's Positivstellensatz [15] (instead of Putinar's Positivstellensatz [13]) which states that every polynomial v , strictly positive on $\mathbb{D}$, can be written as

$$v = \sum_{J \subseteq \{1, \dots, m\}} v_J g_J, \quad [(\text{compare with (16)})]$$

for some family of s.o.s. polynomials $\{v_J\} \subset \mathbb{R}[x]$; see Schmüdgen [15]. $\diamond$

4.3. The univariate case. In the univariate case, simplifications occur. Let $\mathbb{D} \subset \mathbb{R}$ be the interval $[a, b]$, that is, $\mathbb{D}$ has the representation

$$(21) \quad \mathbb{D} := \{x \in \mathbb{R} : g(x) \geq 0\}, \quad \text{with } x \mapsto g(x) := (b-x)(x-a), \quad x \in \mathbb{R}.$$

Theorem 4.5. *Let $\mathbb{D}$ be as in (21), $p, q \in \mathbb{R}[x]$, with $q > 0$ on $\mathbb{D}$, and let $f, \widehat{f}$ be as in (4) and (3) respectively. Then, with $2r \geq \max[\deg p, 1 + \deg q]$, let $\mathbf{Q}_{rx}$ be the SDP-relaxation defined in (12). Then:*

$$(22) \quad \widehat{f}(x) = \inf \mathbf{Q}_{rx}, \quad x \in \mathbb{K}.$$

Proof. Recall that when $\mathbb{D}$ is convex and compact, then for every $x \in \mathbb{D}$, $\widehat{f}(x) = \sup \mathbf{P}_x^*$, with $\mathbf{P}_x^*$ as defined in (6). Next, in the univariate case, a polynomial $h \in \mathbb{R}[x]$ of degree $2r$ or $2r-1$, is nonnegative on $\mathbb{K}$ if and only if $h = h_0 + h_1 g$, for some s.o.s. polynomials $h_0, h_1 \in \mathbb{R}[x]$, and with $\deg h_0, h_1 g \leq 2r$. This is in contrast with the multivariate case, where the degree in Putinar's representation (16) is not known in advance. Therefore, let $2r \geq \max[\deg p, 1 + \deg q]$. The polynomial $p - \gamma q - \langle \lambda, y \rangle q$ (of degree $\leq 2r$) is nonnegative on $\mathbb{D}$ if and only if

$$p - \gamma q - \langle \lambda, y \rangle q = u_0 + u_1 g,$$

for some s.o.s. polynomials $u_0, u_1 \in \mathbf{R}[x]$, with $\deg u_0, u_1 g \leq 2r$. Therefore, as $q > 0$ on $\mathbb{K}$, and recalling the definition of $\mathbf{P}_x^*$ in (6),

$$\begin{aligned} f(y) - \langle \lambda, y \rangle &\geq \gamma, \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle &\geq \gamma q(y), \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle - \gamma q(y) &\geq 0, \quad \forall y \in \mathbb{K} \Leftrightarrow \\ p(y) - q(y)\langle \lambda, y \rangle - \gamma q(y) &= u_0(y) + u_1(y)g(y), \end{aligned}$$

for some s.o.s. polynomials $u_0, u_1 \in \mathbf{R}[x]$, with $\deg u_0, u_1 g \leq 2r$. Therefore, $\mathbf{Q}_{rx}^*$ is identical to $\mathbf{P}_x^*$, from which the result follows. $\square$

So, in the univariate case, the SDP-relaxation $\mathbf{Q}_{rx}$ is exact, that is, the value at $x \in \mathbb{K}$ of the convex envelope $\widehat{f}$, is easily obtained by solving a *single* SDP.

5. THE CONVEX HULL OF A COMPACT SEMI-ALGEBRAIC SET

An important question stated in Ben-Tal and Nemirovski [3, §4.2 and §4.10.2], Parrilo and Sturmfels [12], and not settled yet, is to characterize the convex subsets of $\mathbb{R}^n$ that are *semidefinite representable* (written SDr), or equivalently, have an *LMI representation*; that is, subsets $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega = \{x \in \mathbb{R}^n : M_0 + \sum_{i=1}^n M_i x_i \succeq 0\},$$

for some family $\{M_i\}_{i=0}^n$ of real symmetric matrices. In other words, a SDr set is the *feasible set* of a system of LMI's (Linear Matrix Inequalities), and powerful techniques are now available to solve SDPs. For instance, the epigraph of a univariate convex polynomial is SDr; see [3, p. 292]. Recently, Helton and Vinnikov [6] have proved that *rigid convexity* (as defined in [6]) is a necessary condition for a convex set to be SDr.

In this section, we are concerned with a (large) class of convex sets, namely the convex hull of an arbitrary compact semi-algebraic set, i.e., the convex hull $\mathbb{K} = \text{co}(\mathbb{D})$ of a compact set $\mathbb{D}$ defined by finitely many polynomial inequalities, as in (11). We will show that:

- $\mathbb{K}$ is the *projection* of a *semi-infinite* SDr set S_∞ , that is, S_∞ is defined by finitely many LMIs involving matrices with countably many rows and columns, and involving countably many variables. Importantly, the LMI representation of the set S_∞ is given directly in terms of the data, i.e., in terms of the polynomials g_j 's that define the set $\mathbb{D}$.

- $\mathbb{K}$ can be approximated by a monotone nonincreasing sequence of convex sets $\{\mathbb{K}_r\}$ (with $\mathbb{K}_r \supset \mathbb{K}$ for all r), that are *projections* of SDr sets S_r . Each SDr set S_r is a "finite truncation" of S_∞ , and therefore, also has a specific LMI representation, directly in terms of the data defining the set $\mathbb{D}$. In other words, $\{\mathbb{K}_r\}$ is a converging sequence of *outer convex approximations* of $\mathbb{K}$, i.e. $\mathbb{K}_r \downarrow \mathbb{K}$ as $r \rightarrow \infty$. Detecting whether a point $x \in \mathbb{R}^n$ belongs to $\mathbb{K}_r$ reduces to solving a single SDP that involves the SDr set S_r .

With $\mathbb{D} \subset \mathbb{R}^n$ as in (11), define the 2^m polynomials

$$(23) \quad x \mapsto g_J(x) := \prod_{j \in J} g_j, \quad \forall J \subseteq \{1, \dots, m\},$$

of total degree $2r_J$ or $2r_J - 1$, and with the convention that $g_\emptyset \equiv 1$.

Let $M_r(g_J y) \in \mathbb{R}^{s(r) \times s(r)}$ be the localizing matrix associated with the polynomial g_J , and a sequence y , for all $J \subseteq \{1, \dots, m\}$, and all $r = 0, 1, \dots$; see also §4.1 for the definition of the infinite matrix $M_\infty(g_J y)$.

Define $S_\infty \subset \mathbb{R}^\infty$ by:

$$(24) \quad S_\infty := \{y \in \mathbb{R}^\infty : y_0 = 1; M_\infty(g_J y) \succeq 0, \quad \forall J \subseteq \{1, \dots, m\}\},$$

The set S_∞ is a *semi-infinite* SDr set as it is defined by 2^m LMIs whose matrices have countably many rows and columns, and with countably many variables.

If Assumption 4.1 holds, one may instead use the simpler semi-infinite SDr set

$$S'_\infty := \{y \in \mathbb{R}^\infty : y_0 = 1; M_\infty(y) \succeq 0, M_\infty(g_j y) \succeq 0, \quad \forall j = 1, \dots, m\}.$$

Similarly, define $\mathbb{K}_\infty \subset \mathbb{R}^n$ by:

$$(25) \quad \mathbb{K}_\infty := \{x \in \mathbb{R}^n : \exists y \in S_\infty \text{ s.t. } L_y(z_i) = x_i, i = 1, \dots, n\}.$$

Lemma 5.1. *Let $\mathbb{D} \subset \mathbb{R}^n$ be as in (11) and compact, and let $\mathbb{K}_\infty$ be as in (25). Then $\mathbb{K}_\infty = \mathbb{K} = \text{co}(\mathbb{D})$.*

Proof. If $x \in \text{co}(\mathbb{D}) = \mathbb{K}$, then $x_i = \int z_i d\mu, \forall i = 1, \dots, n$, for some probability measure μ with support contained in $\mathbb{D}$. Let y be the vector of moments of μ , well defined because μ has compact support. Then we necessarily have $y_0 = 1$, and $M_r(g_J y) \succeq 0$ for all r and all $J \subseteq \{1, \dots, m\}$; see §4.1. Equivalently, $M_\infty(g_J y) \succeq 0$, for all $J \subseteq \{1, \dots, m\}$, and so $y \in S_\infty$. From, $\int z_i d\mu = L_y(z_i), \forall i = 1, \dots, n$, we conclude that $x \in \mathbb{K}_\infty$, and so $\mathbb{K} \subseteq \mathbb{K}_\infty$.

Conversely, let $x \in \mathbb{K}_\infty$. Then, there exists $y \in S_\infty$ such that $y_0 = 1$ and $L_y(z_i) = x_i$, for all $i = 1, \dots, n$. As $M_\infty(g_J y) \succeq 0$, for all $J \subseteq \{1, \dots, m\}$, then by Schmüdgen Positivstellensatz [15], y is the vector of moments of some probability measure μ_y , with support contained in $\mathbb{D}$. Next,

$$L_y(z_i) = x_i, \forall i = 1, \dots, n \quad \Leftrightarrow \quad x_i = \int z_i d\mu_y, \forall i = 1, \dots, n,$$

which proves that $x \in \text{co}(\mathbb{D}) = \mathbb{K}$. Therefore, $\mathbb{K}_\infty \subseteq \mathbb{K}$, and the result follows. $\square$

So, Lemma 5.1 states that the convex hull $\mathbb{K}$ of any compact semi-algebraic set $\mathbb{D}$, is the *projection* on the variables y_α with $|\alpha| = 1$, of the semi-infinite SDr set S_∞ (recall that for every $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$). However, the set S_∞ is *not* described by finite-dimensional LMIs, because we have countably many variables y_α , and matrices with infinitely many rows and columns.

We next provide outer approximations $\{\mathbb{K}_r\}$ of $\mathbb{K}$, which are projections of SDr sets $\{S_r\}$, with $S_r \supset S_\infty$, for all r , and $\mathbb{K}_r \downarrow \mathbb{K}$, as $r \rightarrow \infty$.

With $r \geq r_0$, let $S_r \subset \mathbb{R}^{s(2r)}$ be defined as:

$$(26) \quad S_r := \{y \in \mathbb{R}^{s(2r)} : y_0 = 1; M_{r-r_J}(g_J y) \succeq 0, \forall J \subseteq \{1, \dots, m\}\}.$$

Notice that S_r is a SDr set obtained from S_∞ by "finite" truncation. Indeed, S_r contains finitely many variables y_α , namely those with $|\alpha| \leq 2r$. And $M_r(g_J y)$ is a finite truncation of the infinite matrix $M_\infty(g_J y)$; see §4.1.

As for S_∞ , under Assumption 4.1, S_r in (26) may be replaced with the (simpler) SDr set

$$S'_r := \{y \in \mathbb{R}^{s(2r)} : y_0 = 1; M_r(y) \succeq 0, M_{r-r_j}(g_j y) \succeq 0, \quad j = 1, \dots, m\}.$$

Next, let

$$(27) \quad \mathbb{K}_r := \{x \in \mathbb{R}^n : \exists y \in S_r \text{ s.t. } L_y(z_i) = x_i, i = 1, \dots, n\}.$$

Equivalently,

(28)

$$\mathbb{K}_r := \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{s(2r)} \left\{ \begin{array}{ll} L_y(z_i) = x_i & i = 1, \dots, n \\ M_{r-r_J}(g_J y) \succeq 0, & J \subseteq \{1, \dots, m\} \\ y_0 = 1. \end{array} \right. \right\}.$$

In view of the meaning of $L_y(z_i)$, $\mathbb{K}_r$ is the projection on $\mathbb{R}^n$ of the SDr set $S_r \subset \mathbb{R}^{s(2r)}$ defined in (26) (on the n variables y_α , with $|\alpha| = 1$); Obviously, $\{\mathbb{K}_r\}$ forms a *nested* sequence of sets, and we have

$$(29) \quad \mathbb{K}_{r_0} \supset \mathbb{K}_{r_0+1} \dots \supset \mathbb{K}_r \dots \supset \mathbb{K}.$$

Let f be the identity on $\mathbb{D}$ ($f = 1$ on $\mathbb{D}$). Then its convex envelope $\hat{f}$ is given by

$$\hat{f}(x) = \begin{cases} 1, & x \in \mathbb{K}, \\ +\infty & x \in \mathbb{R}^n \setminus \mathbb{K}. \end{cases}$$

Note that $\hat{f}$ is clearly a continuous function on $\mathbb{K}$.

On $\mathbb{D}$, write $f = 1 = p/q$ with $p = q \equiv 1$, so that with $r \geq r_0 := \max_J r_J$, the SDP-relaxation $\mathbf{Q}_{rx}$ defined in (12) and in Remark 4.4, now reads

$$(30) \quad \mathbf{Q}_{rx} : \inf_y \{ y_0 : y \in S_r; \quad L_y(z_i) = x_i, : i = 1, \dots, n \}, \quad x \in \mathbb{R}^n,$$

and so, for all $r \geq r_0$,

$$(31) \quad \inf \mathbf{Q}_{rx} = \begin{cases} 1, & \text{if } x \in \mathbb{K}_r \\ +\infty, & \text{otherwise.} \end{cases}$$

Next, let $f_r : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function $x \mapsto f_r(x) := \inf \mathbf{Q}_{rx}$, with obvious domain $\mathbb{K}_r$.

Corollary 5.2. *Let $\mathbb{D} \subset \mathbb{R}^n$ be compact and defined as in (11), and let $\mathbb{K} := \text{co}(\mathbb{D})$.*

- (a) *If $x \notin \mathbb{K}$, then $f_r(x) = +\infty$ whenever $r \geq r_x$, for some integer r_x .*
- (b) *With $\mathbb{K}_r$ being as in (28), $\mathbb{K}_r \downarrow \mathbb{K}$ as $r \rightarrow \infty$.*

Proof. (a) By Theorem 4.2(b) and Remark 4.4, f_r is convex and $f_r(x) \uparrow \hat{f}(x)$, for all $x \in \mathbb{R}^n$. If $x \in \mathbb{K}$ then $f_r(x) = 1$ for all r . If $x \notin \mathbb{K}$ then $f_r(x) = 1$ if $x \in \mathbb{K}_r$, and $+\infty$ outside $\mathbb{K}_r$. But as $f_r(x) \uparrow \hat{f}(x) = +\infty$, there is some r_x such that $f_r(x) = +\infty$, for all $r \geq r_x$, the desired result.

- (b) As $\{\mathbb{K}_r\}$ is a nonincreasing nested sequence and $\mathbb{K} \subset \mathbb{K}_r$ for all r , one has

$$\mathbb{K}_r \downarrow \mathbb{K}^* := \bigcap_{r=0}^{\infty} \mathbb{K}_r \supset \mathbb{K}.$$

It suffices to show that $\mathbb{K}^* \subseteq \mathbb{K}$, which we prove by contradiction. Let $x \in \mathbb{K}^*$, and suppose that $x \notin \mathbb{K}$. By (a), we must have $f_r(x) = +\infty$ whenever $r \geq r_x$, for some integer r_x . In other words, $x \notin \mathbb{K}_r$ whenever $r \geq r_x$. But then, $x \notin \mathbb{K}^*$, in contradiction with our hypothesis. $\square$

Corollary 5.2 provides us with a means to test whether $x \notin \mathbb{K}$. Indeed, it suffices to solve the SDP-relaxation $\mathbf{Q}_{rx}$ defined in (30), until $\inf \mathbf{Q}_{rx} = +\infty$ for some r (which means that $x \notin \mathbb{K}_r$ for all $r \geq r_x$), which eventually happens if $x \notin \mathbb{K}$.

6. PROOFS

6.1. Proof of Theorem 4.2. (a) By standard weak duality, $\sup \mathbf{Q}_{rx}^* \leq \inf \mathbf{Q}_{rx} \leq \widehat{f}(x)$ for all $r \in \mathbb{N}$, and all $x \in \mathbb{R}^n$. Next, let $x \in \mathbb{K}$ be fixed, arbitrary. From Theorem 2.2 and Corollary 2.3, $\mathbf{P}_x^*$ is solvable, and $\sup \mathbf{P}_x^* = \max \mathbf{P}_x^* = \widehat{f}(x)$ for all $x \in \mathbb{K}$. Therefore, from the definition of $\mathbf{P}_x^*$ in (6), there is some $(\gamma^*, \lambda^*) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$p(y) - q(y)\langle \lambda^*, y \rangle - \gamma^* q(y) \geq 0, \quad y \in \mathbb{D},$$

and $\gamma^* - \langle \lambda^*, x \rangle = \widehat{f}(x)$.

Hence, with $\epsilon > 0$ fixed, arbitrary, $\gamma^* - \epsilon - \langle \lambda^*, x \rangle = \widehat{f}(x) - \epsilon$, and

$$p(y) - q(y)\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q(y) \geq \epsilon q(y) > 0, \quad y \in \mathbb{D}.$$

Therefore, under Assumption 4.1, the polynomial $p - q\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q$, which is strictly positive on $\mathbb{D}$, can be written

$$p(y) - q(y)\langle \lambda^*, y \rangle - (\gamma^* - \epsilon)q(y) = \sum_{j=0}^m u_j g_j,$$

for some s.o.s. polynomials $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$. But then, $(\gamma^* - \epsilon, \lambda^*, \{u_j\})$ is a feasible solution of $\mathbf{Q}_{rx}^*$ as soon as $r \geq r_\epsilon := \max_{j=0,1,\dots,m} \deg(u_j g_j)$, and with value $\gamma^* - \epsilon - \langle \lambda^*, x \rangle = \widehat{f}(x) - \epsilon$. Hence, for every $\epsilon > 0$,

$$\widehat{f}(x) - \epsilon \leq \sup \mathbf{Q}_{rx}^* \leq \inf \mathbf{Q}_{rx} \leq \widehat{f}(x), \quad r \geq r_\epsilon.$$

This concludes the proof of (17) for $x \in \mathbb{K}$.

Next, let $x \notin \mathbb{K}$ so that $\widehat{f}(x) = +\infty$. From the proof of Theorem 2.2, we have seen that $\mathbf{P}_x^* = \widehat{f}(x)$ for all $x \in \mathbb{R}^n$. Therefore, with $M > 0$ fixed, arbitrarily large, one may find $\lambda \in \mathbb{R}^n, \gamma \in \mathbb{R}$ such that

$$M \leq \langle \lambda, x \rangle + \gamma \quad \text{and} \quad f(y) + \langle \lambda, y \rangle \geq \gamma, \quad \forall y \in \mathbb{D}.$$

Hence

$$f(y) + \langle \lambda, y \rangle - \gamma + \epsilon > 0, \quad \forall y \in \mathbb{D}.$$

Therefore, as $q > 0$ on $\mathbb{D}$, the polynomial $g := p + \langle \lambda, y \rangle q - (\gamma - \epsilon)q$ is positive on $\mathbb{D}$. By Putinar Positivstellensatz [13], it may be written

$$g = u_0 + \sum_{j=1}^m u_j g_j$$

for some family of s.o.s. polynomials $\{u_j\}_{j=0}^m \subset \mathbb{R}[x]$. But then, with $2r_M \geq \max[\deg u_0, \deg u_j g_j]$, the 3-uplet $(\lambda, \gamma - \epsilon, \{u_j\})$ is a feasible solution of $\mathbf{Q}_{rx}^*$, whenever $r \geq r_M$, and with value $M - \epsilon$. And so, as M was arbitrarily large, $\sup \mathbf{P}_{rx}^* \rightarrow +\infty = \widehat{f}(x)$, as $r \rightarrow \infty$. This concludes the proof of (17) for $x \notin \mathbb{K}$.

(b) That f_r is convex follows from its definition $f_r(x) = \inf \mathbf{Q}_{rx}$ for all $x \in \mathbb{R}^n$, and the definition (13) of the SDP $\mathbf{Q}_{rx}$. First, observe that for all r sufficiently large, say $r \geq r'_0$, $\inf \mathbf{Q}_{rx} > -\infty$ for all $x \in \mathbb{R}^n$, because $\sup \mathbf{Q}_{rx}^* \geq -1$, for all $x \in \mathbb{R}^n$. Indeed, with $\gamma = -1, \lambda = 0$, the polynomial $p + q = p - q\gamma - q\langle \lambda, y \rangle$ is positive on $\mathbb{D}$, and therefore, by Putinar Positivstellensatz [13], $p + q = u_0 + \sum_j u_j g_j$, for some family of s.o.s. polynomials $\{u_j\}_0^m$. Therefore, $(-1, 0, \{u_j\})$ is feasible for $\mathbf{Q}_{rx}^*$ with

value -1 , whenever $2r'_0 \geq \max[\deg u_0, \deg u_j g_j]$. Next, let $x := \alpha u + (1 - \alpha)v$, with $u, v \in \mathbb{R}^n$, and $0 \leq \alpha \leq 1$. As we want to prove

$$\inf \mathbf{Q}_{r(\alpha u + (1 - \alpha)v)} \leq \alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv}, \quad 0 \leq \alpha \leq 1,$$

we may restrict to $u, v \in \mathbb{R}^n$ such that $\inf \mathbf{Q}_{ru}, \inf \mathbf{Q}_{rv} < +\infty$. So, let y_u (resp. y_v) be feasible for $\mathbf{Q}_{ru}$ (resp. $\mathbf{Q}_{rv}$), and with respective values $\inf \mathbf{Q}_{ru} + \epsilon$, $\inf \mathbf{Q}_{rv} + \epsilon$. As the matrices $M_r(y)$, $M_{r-r_j}(g_j y)$ are all linear in y , and $y \mapsto L_y(\bullet)$ is linear in y as well, $y := \alpha y_u + (1 - \alpha)y_v$ is feasible for $\mathbf{Q}_{rx}$, with value $\alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv} + \epsilon$. Therefore,

$$\inf \mathbf{Q}_{rx} = \inf \mathbf{Q}_{r(\alpha u + (1 - \alpha)v)} \leq \alpha \inf \mathbf{Q}_{ru} + (1 - \alpha) \inf \mathbf{Q}_{rv} + \epsilon, \quad \forall \epsilon > 0,$$

and letting $\epsilon \rightarrow 0$ yields the result. Finally, the pointwise convergence $f_r(x) \uparrow \widehat{f}(x)$ for all $x \in \mathbb{R}^n$, follows from (17).

(c) Let $\mathbb{K}$ be with a nonempty interior $\text{int } \mathbb{K}$, and let $x \in \text{int } \mathbb{K}$. Let μ be the probability measure uniformly distributed on the ball $B_x := \{y \in \mathbb{K} : \|y - x\| \leq \delta\} \subset \mathbb{K}$.

Hence, $\int z_i d\mu = x_i$ for all $i = 1, \dots, n$. Next, define the measure ν to be $d\nu = q^{-1} d\mu$ so that $\int q d\nu = 1$, and $\int z_i d\mu = \int z_i q d\nu = x_i$ for all $i = 1, \dots, n$. Take for $y = \{y_\alpha\}$, the vector of moments of the measure ν . As ν has a density, and is supported on $\mathbb{K}$, it follows that $M_r(y) \succ 0$ and $M_r(q_j y) \succ 0$, $j = 1, \dots, m$, for all r . Therefore, y is a *strictly* feasible solution of $\mathbf{Q}_{rx}$, i.e., Slater's condition holds, which in turn implies the absence of a duality gap between $\mathbf{Q}_{rx}$ and its dual $\mathbf{Q}_{rx}^*$ ($\sup \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx}$). In addition, as $\inf \mathbf{Q}_{rx} > -\infty$, we get $\sup \mathbf{Q}_{rx}^* = \max \mathbf{Q}_{rx}^* = \inf \mathbf{Q}_{rx}$, which is (19).

So, as $\mathbf{Q}_{rx}^*$ is solvable, let $(\gamma_r^*, \lambda_r^*, \{u_j^*\})$ be an optimal solution, that is, $f_r(x) = \gamma_r^* - \langle \lambda_r^*, x \rangle$ and

$$p(z) - \gamma_r^* q(z) - q(z) \langle \lambda_r^*, y \rangle = \sum_{j=0}^m u_j^*(z) g_j(z), \quad \forall z \in \mathbb{R}^n.$$

Therefore, one has

$$\begin{aligned} f_r(x) &= \gamma_r^* - \langle \lambda_r^*, x \rangle \\ f_r(y) &= \sup_{\gamma, \lambda, u} \{ \gamma + \langle \lambda, y \rangle : p(z) - \gamma q(z) - q(z) \langle \lambda, z \rangle = \sum_{j=0}^m u_j(z) g_j(z), \forall z \in \mathbb{R}^n \} \\ &\geq \gamma_r^* - \langle \lambda_r^*, y \rangle, \quad \forall y \in \mathbb{R}^n, \end{aligned}$$

from which we get

$$f_r(y) - f_r(x) \geq \langle \lambda_r^*, y - x \rangle, \quad \forall y \in \mathbb{R}^n,$$

that is, $\lambda_r^* \in \partial f_r(x)$, the desired result. $\square$

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