Analysis of PML formulations for the Maxwell system including pseudo-differential operators : existence and uniqueness results

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Abstract

We consider a formulation of the Maxwell system which involves pseudo-differential operators in time. This formulation has been analyzed in a previous work and it has been proved that it is a Perfectly Matched Layer model for electromagnetic waves propagating into the vacuum. We establish that the system admits a single solution by adapting the Hille-Yosida theory.

1 Introduction

Let us consider the followin formulation:

$$(\mathcal{P})\begin{cases} \varepsilon_0\partial_t\mathbf{E} - \mathrm{rot}\mathbf{H} + [\sigma]\mathbf{E} + [\nu]\mathbf{P} = \mathbf{0}, \\ \mu_0\partial_t\mathbf{H} + \mathrm{rot}\mathbf{E} + [\tau]\mathbf{H} + [\eta]\mathbf{Q} = \mathbf{0}, \\ \varepsilon_0\partial_t\mathbf{P} - [\nu]\mathbf{E} = \mathbf{0}, \\ \mu_0\partial_t\mathbf{Q} - [\eta]\mathbf{H} = \mathbf{0}. \end{cases}$$

Each term $[\sigma]$, $[\nu]$, $[\tau]$ and $[\eta]$ is a tensor with coefficients defined as operators with coefficients depending on the position x only and compactly supported in a given region of the free space. Then, outside the support of the coefficients, the previous system corresponds to the Maxwell system set in the vacuum and the pair $(\mathbf{E}, \mathbf{H})$ denotes the usual electromagnetic field. The vector fields $\mathbf{P}$ and $\mathbf{Q}$ are auxiliary unknowns which can be seen as polarization fields in a bianisotropic medium. The tensors $[\sigma]$ and $[\tau]$ are diagonal matrices with bounded coefficients given as functions of the position vector $\mathbf{x}$. The tensors $[\nu]$ and $[\eta]$ are defined as diagonal matrices whose terms are pseudo-differential operators in time with variable coefficients depending on $\mathbf{x}$. In a previous work [2], we have established sufficient conditions on $[\nu]$ and $[\eta]$ to have a PML model. These conditions allow to define a set of tensors $[\nu]$ and $[\eta]$ in the frequency domain. We begin with briefly recalling this result and then we show that the corresponding PML formulation includes pseudo-differential operators. In the next section, we show how to prove the resulting model is well-posed. In the simplest case where the tensors where functions matrices, we used the Hille-Yosida theory. Herein, we describe how to adapt the theory for including the pseudo-differential terms.

2 PML formulation

To verify the model is PML requires to develop a plane wave analysis which consists in considering the plane wave solutions to the system. Then, the terms of each tensors are matched to ensure that any wave impinging the interface PML is perfectly transmitted into the absorbing layer. We recall the result of the analysis we formerly developed in [2, 4], which will be useful to write the time formulation of the PML model.

2.1 Plane Wave Analysis

Assuming that the solutions to the system are defined as plane waves, one gets the algebraic system :

$$\begin{cases} \mathrm{rot}\mathbf{H} = \left(i\omega\varepsilon_0[1] + [\sigma] + \dfrac{[\nu]^2}{i\varepsilon_0\omega} \right)\mathbf{E}, \\ \\ \mathrm{rot}\mathbf{E} = -\left(i\omega\mu_0[1] + [\tau] + \dfrac{[\eta]^2}{i\mu_0\omega} \right)\mathbf{H}; \end{cases} \tag{1}$$

where [1] denotes the identity. We then have :

$$\begin{cases} \mathrm{rot}\mathbf{H} = i\omega\varepsilon_0\mathcal{M}\mathbf{E} \\ \\ \mathrm{rot}\mathbf{E} = -i\omega\mu_0\mathcal{N}\mathbf{H} \end{cases} \tag{2}$$

with

$$\mathcal{M} = \begin{bmatrix} 1 - i\dfrac{\sigma_x}{\varepsilon_0\omega} - \dfrac{\nu_x^2}{\varepsilon_0^2\omega^2} & 0 & 0 \\ 0 & 1 - i\dfrac{\sigma_y}{\varepsilon_0\omega} - \dfrac{\nu_y^2}{\varepsilon_0^2\omega^2} & 0 \\ 0 & 0 & 1 - i\dfrac{\sigma_z}{\varepsilon_0\omega} - \dfrac{\nu_z^2}{\varepsilon_0^2\omega^2} \end{bmatrix} = \mathrm{diag}(m_x, m_y, m_z)$$

and

$$\mathcal{N} = \begin{bmatrix} 1 - i\dfrac{\tau_x}{\mu_0\omega} - \dfrac{\eta_x^2}{\mu_0^2\omega^2} & 0 & 0 \\ 0 & 1 - i\dfrac{\tau_y}{\mu_0\omega} - \dfrac{\eta_y^2}{\mu_0^2\omega^2} & 0 \\ 0 & 0 & 1 - i\dfrac{\tau_z}{\mu_0\omega} - \dfrac{\eta_z^2}{\mu_0^2\omega^2} \end{bmatrix} = \mathrm{diag}(n_x, n_y, n_z).$$

We suppose the tensors satisfy the compatibility conditions:

$$(\mathrm{H}_1) \qquad \frac{[\sigma]}{\varepsilon_0} = \frac{[\tau]}{\mu_0}\,,$$

$$(\mathrm{H}_2) \qquad \frac{[\nu]^2}{\varepsilon_0^2} = \frac{[\eta]^2}{\mu_0^2}\,.$$

Then we get:

Lemma 2.1 *We suppose that $\sigma_y = \sigma_z$ and $\nu_y = \nu_z = \eta_y = \eta_z = 0$. Then there exist values of ν_x and η_x such that the algebraic system is assiociated to a PML model. For instance, they satisfy :*

$$\nu_x^2 = \frac{\sigma_y^2}{1 + \frac{\sigma_y^2}{\varepsilon_0^2\omega^2}} - i\frac{(\sigma_x + \sigma_y)\varepsilon_0\omega + \sigma_y^2 \frac{\sigma_x}{\varepsilon_0\omega}}{1 + \frac{\sigma_y^2}{\varepsilon_0^2\omega^2}}, \tag{3}$$

$$\eta_x^2 = \frac{\tau_y^2}{1 + \frac{\tau_y^2}{\mu_0^2\omega^2}} - i\frac{(\tau_x + \tau_y)\mu_0\omega + \tau_y^2 \frac{\tau_x}{\mu_0\omega}}{1 + \frac{\tau_y^2}{\mu_0^2\omega^2}}. \tag{4}$$

The previous Lemma gives one example of tensors $[nu]$ and $[eta]$ for which the model is PML. The construction is based on a sufficient condition which must be satisfied by $[nu]$ (and then by $[eta]$, according to the impedance condition they satisfy). We recall that it reads as follows :

$$(1 - i\frac{\sigma_x}{\epsilon_0\omega} - \frac{\nu_x^2}{\epsilon_0^2\omega^2})(1 - i\frac{\sigma_z}{\epsilon_0\omega} - \frac{\nu_z^2}{\epsilon_0^2\omega^2}) = 1 \tag{5}$$

Hence there exist an infinite number of tensors for which the model is PML. But any of the corresponding formulations are equivalent.

3 Time-Formulation of the model

Lemma 2.1 shows that there exists at least a pair of tensors $[\nu]$ and $[\eta]$ such that the plane waves solutions propagating in $\{x < 0\}$ are perfectly transmitted in $\{x > 0\}$ with an exponential attenuation in the absorbing medium. However $[\nu^2]$ and $[\eta^2]$ are defined by functions which depend on x and ω. Hence $[\nu^2]$ and $[\eta^2]$ are the symbols of pseudo-differential operators in time with variables coefficients in x. Then $(\mathbf{E},\mathbf{H},\mathbf{P},\mathbf{Q})$ satisfies the system:

$$\begin{cases} \varepsilon_0\partial_t\mathbf{E} - \mathrm{rot}\mathbf{H} + [\sigma]\mathbf{E} + [A]\mathbf{P} = \mathbf{0}, \\ \mu_0\partial_t\mathbf{H} + \mathrm{rot}\mathbf{E} + [\tau]\mathbf{H} + [B]\mathbf{Q} = \mathbf{0}, \\ \varepsilon_0\partial_t\mathbf{P} - [A]\mathbf{E} = \mathbf{0}, \\ \mu_0\partial_t\mathbf{Q} - [B]\mathbf{H} = \mathbf{0}; \end{cases} \tag{6}$$

where $[A]$ et $[B]$ are diagonal tensors whose terms are pseudo-differential operators in time with variable coefficients depending on x only. Tensors $[\nu]$ and $[\eta]$ in the previous analysis can then be considered as the respective symbols of the operators $[A]$ and $[B]$ providing the following property is satisfied: for any plane-waves field $\boldsymbol{\psi} = \boldsymbol{\psi}_\mathbf{0} e^{i(\omega t - \mathbf{k}.\mathbf{x})}$,

$$[A]\boldsymbol{\psi} = [\nu]\boldsymbol{\psi} \qquad \text{et} \qquad [B]\boldsymbol{\psi} = [\eta]\boldsymbol{\psi}; \tag{7}$$

where $[\nu]$ and $[\eta]$ coincide with the symbols of $[A]$ and $[B]$. Property (7) is direct if $[A]$ and $[B]$ are pseudodifferential tensors. It is also satisfied by classical pseudo-differential operators which is a generalization of differential operators if one adopts the following representation formula. A classical pseudo-differential operator T admits the integral representation:

$$T\phi(\mathbf{y}) = \frac{1}{(2\pi)^N} \int e^{i\mathbf{y}\cdot\boldsymbol{\xi}} t(\mathbf{y}, \boldsymbol{\xi}) \hat{\phi}(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ denotes the dual variable of $\mathbf{y}$ by Fourier transform. The function $t := t(\boldsymbol{y}, \boldsymbol{\xi})$ stnads for the symbol of T. For instance, we refer the reader to as à [9] or [10] for a detailed setting of these operators and their theory. (7) is a direct consequence of the property:

$$t(\mathbf{y}, \boldsymbol{\xi}) = e^{-i\mathbf{y}\cdot\boldsymbol{\xi}} T(e^{i\mathbf{y}\cdot\boldsymbol{\xi}}).$$

Now $[A]$ and $[B]$ should be considered as pseudo-differential operators given by the formula:

$$[A]u(t,x) = (2\pi)^{-1/2} \int_{\mathbb{R}} a(\omega,x) e^{i\omega t} \mathcal{F}_t\, u(\omega,x)\, d\omega,$$
$$[B]u(t,x) = (2\pi)^{-1/2} \int_{\mathbb{R}} b(\omega,x) e^{i\omega t} \mathcal{F}_t\, u(\omega,x)\, d\omega;$$

with $a(x,\omega) = [\nu]$ et $b(x,\omega) = [\eta]$. The notation $\mathcal{F}_t$ denotes the partial Fourier transform in time. Letus notice that the symbols of $[A]$ and $[B]$ do not depend on the time. This propoerty is satisfied by any solution to Eq.(5). Operators $[A]$ and $[B]$ defined by Lemma 2.1 are pseudo-differantial operators in $OPS^{1/2}(\mathbb{R}_t)$, which means that they are pseudo-differential of order 1/2 whose coefficients are regular and are given as functions of the terms of $[\sigma]$ and $[\tau]$ respectively. Lemma 2.1 does not define directly the symbols of $[A]$ and $[B]$. However they are defined completely because the symbols ofe $[A]$ and $[B]$ only depend on x and ω. The chain rule of pseudo-differential operators applies then very easily and we have:

$$s([A])^2 = s([A]^2), \qquad s([B])^2 = s([B]^2);$$

where $s([A])$ and $s([B])$ denote the respective symbols of $[A]$ and $[B]$. In the general case $s([A])^2$ only defines the principal symbol of $[A]^2$. Lemma 2.1 gives an example of $[A]$ and $[B]$ *via* their symbol. Let us remark thatν_x^2 and η_x^2 are fractional functions in ω with a numerator of order 1 and a denominator of order 0. According to [9] or [10], $[A]^2$ and $[B]^2$ are pseudo-differential operators in $OPS^1(\mathbb{R}_t)$ and, thus, $[A]$ and $[B]$ are in $OPS^{1/2}(\mathbb{R}_t)$.
Hence formulation (6) involves pseudo-differential operotors. The use of these operators is sometimes not easy, in particular from a numerical point of view because they do not preserve the local property of differential operators. Nevertheless the coupling of the Maxwell system with pseudo-differential operators can be circumvent. Indeed the formulation involves $[A]$ and $[B]$ both for the construction of the auxiliary unknowns $\mathbf{P}$ and $\mathbf{Q}$, and in the perturbation of the Maxwell equations. However one could proceed differently when ν_x^2 and ν_z^2 (hence ν_y^2) are fractional functions in ω. One can set:

$$[A]^2 = [A_1][A_2],$$

where $[A_1]$ is a pseudo-differential operator whose symbol is given by the numerator of the terms of $[\nu^2]$ and $[A_2]$ is the inverse of a pseudo-differential operator whose symbol is given by the denominator of the terms of $[\nu^2]$. In that case, there exists another equivalent formulation of (6) which is given by:

$$\begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \mathrm{rot}\mathbf{H} + [\sigma]\mathbf{E} + [A_1]\mathbf{P} = \mathbf{0}, \\ \mu_0 \partial_t \mathbf{H} + \mathrm{rot}\mathbf{E} + [\tau]\mathbf{H} + [B_1]\mathbf{Q} = \mathbf{0}, \\ \varepsilon_0 \partial_t [A_2]^{-1}\mathbf{P} - \mathbf{E} = \mathbf{0}, \\ \mu_0 \partial_t [B_2]^{-1}\mathbf{Q} - \mathbf{H} = \mathbf{0}; \end{cases}$$

and only include differential operators and then is more convenient from a numerical point of view. However, this formulation includes higher order differential equations for the auxiliary unknowns. Obviously one can choose other strategy to decompose $[A]^2$ and then propose other formulations of the PML model. These formulations are equivalent because they are associate to the same symbolic writing, as we have seen in the plane wave analysis. We can then formulate the following result:

Theorem 3.1 *Let $[\sigma]$ and $[\tau]$ be two tensors in M_{Σ_+} satisfying the condition of compatibility (H_1) and such that $\sigma_y = \sigma_z$. If $[A]$ and $[B]$ denote the pseudo-differential operators whose symbols $[\nu]$ and $[\eta]$ are defined in Lemma 2.1, then:*
1) Hypotheses (H_2), (H_3) and (H_4) are satisfied,
2) Model $(\mathcal{P})$ is a PML one.

4 Mathematical properties of the PML model

We have seen in 3 that the frequency model (1) gives rise to different formulations which are equivalent. In this section, we choose a formulation and we show that the resulting system admits a single solution in Sobolev spaces.

4.1 Introduction

Let $T > 0$ be given and consider the problem:

$$(\mathcal{P}_1)\begin{cases} \partial_t \mathbf{E} - \mathrm{rot}\mathbf{H} + [\sigma]\mathbf{E} + \mathbf{P} = 0 & \text{dans } \mathbb{R}^3 \times [0,T], \quad (8)\\ \partial_t \mathbf{H} + \mathrm{rot}\mathbf{E} + [\tau]\mathbf{H} + \mathbf{Q} = 0 & \text{dans } \mathbb{R}^3 \times [0,T], \quad (9)\\ \partial_t \mathbf{P} = [\Lambda]\mathbf{E} & \text{dans } \mathbb{R}^3 \times [0,T], \quad (10)\\ \partial_t \mathbf{Q} = [\Gamma]\mathbf{H} & \text{dans } \mathbb{R}^3 \times [0,T], \quad (11)\\ \mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}) & \text{dans } \mathbb{R}^3,\\ \mathbf{P}(\mathbf{x},0) = \mathbf{Q}(\mathbf{x},0) = \mathbf{0} & \text{dans } \mathbb{R}^3. \end{cases}$$

Tensors $[\sigma]$ and $[\tau]$ are in M_W, which is the space of diagonal operators with terms in W, with

$$W = \{w \in L^\infty(\mathbb{R}),\ \forall x < 0,\ w(x) = 0\}.$$

In this section, we do not any asumption on the sign of the terms of $[\sigma]$ and $[\tau]$. Moreover for any $s > 0$, we introduce the space $\mathbb{X}_s$ defined as:

$$\mathbb{X}_s = C^0\left([0,s]; \mathbb{L}^2(\mathbb{R}^3)\right);$$

which is a Banach space for the norm $||\cdot||_{\mathbb{X}_s}$:

$$\forall \mathbf{u} \in \mathbb{X}_s, \quad ||\mathbf{u}||_{\mathbb{X}_s} = \sup_{0\le w\le s} ||\mathbf{u}(\cdot,w)||_0.$$

Operators $[\Lambda]$ and $[\Gamma]$ are pseudo-differential operators in time whose respective symbols are $[\nu^2]$ and $[\eta^2]$. Tensors $[\nu^2]$ and $[\eta^2]$ have been defined in 2.1. They are diagonal and ecah of their terms is a function of ω and x.
Let $S^1_{1,0}(x)$ be the class of symbols such that:

$$\forall s(\omega,x) \in S^1_{1,0}(x), \quad |\partial^\alpha_\omega s(\omega,x)| \le C_\alpha |\omega|^{1-\alpha}, \quad \alpha \in \mathbb{N}.$$

This sybols class has been introduced in [9]. In this section, we will assume that $[\nu^2]$ and $[\eta^2]$ have coefficients in $S^1_{1,0}(x)$. Let us notice that it is the case for the two examples given in 2.1. Then operators $[\Lambda]$ and $[\Gamma]$ belong to OPS^1 in time, with coefficients in W.

4.2 Existence and uniqueness of the PML solution

Since $[\Lambda]$ and $[\Gamma]$ are pseudo-differential in time, we can not apply directly the Hille-Yosida theory, as it was formerly done in [4]. This is why we are going to change our approach to prove the solution to $(\mathcal{P}_1)$ exists and is unique. In the following, we suppose that $[\Lambda]$ and $[\Gamma]$ satisfy the hypotheses:

$$[\Lambda] = [\lambda]\partial_t + [\Lambda_0], \qquad [\Gamma] = [\gamma]\partial_t + [\Gamma_0]; \tag{12}$$

où $[\lambda]$, $[\gamma]$, $[\Lambda_0]$ et $[\Gamma_0]$ sont tels que:

$$(i)\ [\lambda],\ [\gamma] \in M_W \tag{13}$$

$$(ii)\ [\Lambda_0],\ [\Gamma_0] : C^0\left([0,T];\mathbb{L}^2(\mathbb{R}^3)\right) \longrightarrow C^0\left(\mathbb{R}_+;\mathbb{L}^2(\mathbb{R}^3)\right) \tag{14}$$

(iii) there exists a constant $C(T)$ depending on T only such that:

$$\left|\left|\int_0^t [\Lambda_0]\mathbf{u}(.,\xi)\,d\xi\right|\right|_0 \leq C(T)||\mathbf{u}||_{\mathbb{X}_s} \quad \text{et} \quad \left|\left|\int_0^t [\Gamma_0]\mathbf{u}(.,\xi)\,d\xi\right|\right|_0 \leq C(T)||\mathbf{u}||_{\mathbb{X}_s}, \tag{15}$$

for any s in $[0,T]$, $\mathbf{u}$ in $\mathbb{X}_s$ and t in $[0,s]$.
We then obtain the following result.

Theorem 4.1 *Let $\mathbf{E_0},\mathbf{H_0}$ in $\mathbb{L}^2(\mathbb{R}^3)$; $[\sigma],[\tau]$ in M_W; $[\Lambda]$, $[\Gamma]$ be two pseudo-differential operators satisfying (12), (13), (14) and (15). Then $(\mathcal{P}_1)$ admits a single solution*

$$(\mathbf{E},\mathbf{H},\mathbf{P},\mathbf{Q}) \quad in \quad \mathbb{X}_T^4 = C^0\left([0,T],\mathbb{L}^2(\mathbb{R}^3)\right)^4.$$

Moreover there exists a constant C depending on t only such that:

$$\forall t>0, \qquad ||\mathbf{E}(.,t),\mathbf{H}(.,t)||_0 \leq ||\mathbf{E_0},\mathbf{H_0}||_0\, e^{Ct}. \tag{16}$$

Proof. 1) *Existence*. Under assumption (12), we get, by integrating Eqs. (10) and (11) between 0 and t:

$$\mathbf{P}(.,t) = [\lambda]\mathbf{E}(.,t) + \int_0^t [\Lambda_0]\mathbf{E}(.,s)\,ds - [\lambda]\mathbf{E_0} \tag{17}$$

$$\mathbf{Q}(.,t) = [\gamma]\mathbf{H}(.,t) + \int_0^t [\Gamma_0]\mathbf{H}(.,s)\,ds - [\gamma]\mathbf{H_0}. \tag{18}$$

The pair $(\mathbf{P},\mathbf{Q})$ belongs to $\mathbb{X}_T \times \mathbb{X}_T$ according to (14). We then inject relations (17) and (18) in (8) and (9), which leads to:

$$\left\{\begin{array}{l} \partial_t\mathbf{E} - \mathrm{rot}\mathbf{H} + [\sigma]\mathbf{E} + [\lambda]\mathbf{E}(.,t) + \displaystyle\int_0^t [\Lambda_0]\mathbf{E}(.,s)\,ds - [\lambda]\mathbf{E_0} = 0, \\ \\ \partial_t\mathbf{H} + \mathrm{rot}\mathbf{E} + [\tau]\mathbf{H} + [\gamma]\mathbf{H}(.,t) + \displaystyle\int_0^t [\Gamma_0]\mathbf{H}(.,s)\,ds - [\gamma]\mathbf{H_0} = 0; \end{array}\right. \tag{19}$$

and which can be rewritten in the form:

$$\frac{d}{dt}\left[\begin{array}{c} \mathbf{E}(t) \\ \\ \mathbf{H}(t) \end{array}\right] = A\left[\begin{array}{c} \mathbf{E}(t) \\ \\ \mathbf{H}(t) \end{array}\right] + B\left[\begin{array}{c} \mathbf{E}(t) \\ \\ \mathbf{H}(t) \end{array}\right] \tag{20}$$

where A is the maximal monotone operator:

$$A : D(A) = V \times V \to \mathbb{L}^2(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3),$$

defined by:

$$A = \left[\begin{array}{cc} 0 & \mathrm{rot} \\ -\mathrm{rot} & 0 \end{array}\right];$$

et B est l'opérateur défini par:

$$B\left[\begin{array}{c}\mathbf{E}(t)\\ \\ \mathbf{H}(t)\end{array}\right]=\left[\begin{array}{c}-[\lambda+\sigma]\mathbf{E}-\displaystyle\int_0^t[\Lambda_0]\mathbf{E}(.,s)\,ds+[\lambda]\mathbf{E_0}\\ \\ -[\gamma+\tau]\mathbf{H}-\displaystyle\int_0^t[\Gamma_0]\mathbf{H}(.,s)\,ds+[\gamma]\mathbf{H_0}\end{array}\right].$$

Then we use the following estimate:

$$\exists C_1(T)>0,\quad \forall s\in[0,T],\quad \forall \mathbf{u}\in\mathbb{X}_s\times\mathbb{X}_s,\quad ||B\mathbf{u}||_{\mathbb{X}_s}\leq C_1(T)||\mathbf{u}||_{\mathbb{X}_s}. \tag{21}$$

Let us begin with proving this result. Using the Cauchy-Schwarz inequality, we have, for any $\mathbf{u}=(\mathbf{E},\mathbf{H})$ in $\mathbb{X}_s\times\mathbb{X}_s$:

$$\forall w\in[0,s],\quad ||B\mathbf{u}(.,w)||_0^2\leq 2\Bigg(||[\lambda+\sigma]\mathbf{E}(.,w)||_0^2+||[\lambda]\mathbf{E_0}||_0^2+\left|\left|\int_0^w[\Lambda_0]\mathbf{E}(.,\xi)\,d\xi\right|\right|_0^2$$
$$+||[\gamma+\tau]\mathbf{H}(.,w)||_0^2+||[\gamma]\mathbf{H_0}||_0^2+\left|\left|\int_0^w[\Gamma_0]\mathbf{H}(.,\xi)\,d\xi\right|\right|_0^2\Bigg).$$

Now, for $0\leq w\leq s$, $||\mathbf{E}(.,w)||_0\leq||\mathbf{E}||_{\mathbb{X}_s}\leq||\mathbf{u}||_{\mathbb{X}_s}$ and $||\mathbf{H}(.,w)||_0\leq||\mathbf{H}||_{\mathbb{X}_s}\leq||\mathbf{u}||_{\mathbb{X}_s}$, and the previous inequality implies that:

$$\forall w\in[0,s],\quad ||B\mathbf{u}(.,w)||_0^2\leq 2\left(|\lambda+\sigma|^2+|\lambda|^2+|\gamma+\tau|^2+|\gamma|^2\right)||\mathbf{u}||_{\mathbb{X}_s}^2$$
$$+2\left(\left|\left|\int_0^w[\Lambda_0]\mathbf{E}(.,\xi)\,d\xi\right|\right|_0^2+\left|\left|\int_0^w[\Gamma_0]\mathbf{H}(.,\xi)\,d\xi\right|\right|_0^2\right).$$

Hypothesis (15) shows then that:

$$||B\mathbf{u}(.,w)||_0^2\leq 2\left(|\lambda+\sigma|^2+|\lambda|^2+|\gamma+\tau|^2+|\gamma|^2+C(T)^2\right)||\mathbf{u}||_{\mathbb{X}_s}^2,$$

which implies that

$$||B\mathbf{u}||_{\mathbb{X}_s}^2\leq 2\left(|\lambda-\sigma|^2+|\lambda|^2+|\gamma-\tau|^2+|\gamma|^2+C(T)^2\right)||\mathbf{u}||_{\mathbb{X}_s}^2,$$

and completes the proof of (21). Now we define the sequence $(\mathbf{u}^n)_{n\in\mathbb{N}}$ d'éléments of elements in $\mathbb{X}_T$ as:

$$\left\{\begin{array}{l}\mathbf{u}^0(t)=e^{tA}\left[\begin{array}{c}\mathbf{E_0}\\ \mathbf{H_0}\end{array}\right]=e^{tA}\mathbf{u_0},\\ \\ \forall n\geq 0,\quad \mathbf{u}^{n+1}(t)=\displaystyle\int_0^t e^{(t-s)A}B\mathbf{u}^n(s)\,ds.\end{array}\right.$$

We are going to prove that $\sum\limits_{n\geq 0}\mathbf{u}^n$ normally converges on the Banach space $\mathbb{X}_T$ and that the sum is solution to the system (20). First of all, let us notice that $\mathbf{u}^n$ is well-defined. Indeed, $\mathbf{u}^0$ belongs to $\mathbb{X}_T$ and if $\mathbf{u}^n\in\mathbb{X}_T$, then $B\mathbf{u}^n\in\mathbb{X}_T$ (according to (14) and (15)), hence $\mathbf{u}^{n+1}\in\mathbb{X}_T$. In order to establish the normal convergence of the series, we show, by applying a recursive process on n that:

$$\forall n\in\mathbb{N},\quad \forall s\in[0,T],\quad ||\mathbf{u}^n||_{\mathbb{X}_s}\leq\frac{C_1(T)^n s^n}{n!}||\mathbf{u_0}||_0, \tag{22}$$

where $C_1(T)$ is the constant introduced in(21). To begin with, (22) is satisfied for $n=0$ because

$$\forall t\in[0,s],\quad ||\mathbf{u}^0(t)||_0=||e^{tA}\mathbf{u_0}||_0\leq||e^{tA}||\,||\mathbf{u_0}||_0\leq||\mathbf{u_0}||_0,$$

since A generates a contraction semi-group and then, $||e^{tA}|| \leq 1$. Now let us assume that (22) is satisfied for n. We then have:

$$\forall t \in [0,s], \quad ||\mathbf{u}^{n+1}(t)||_0 = \int_0^t ||B\mathbf{u}^n(w)||_0 \, dw. \tag{23}$$

Furthermore

$$\forall w \in [0,t], \quad ||B\mathbf{u}^n(w)||_0 \leq ||B\mathbf{u}^n||_{\mathbb{X}_w},$$

which implies, according to (21),

$$\forall w \in [0,t], \quad ||B\mathbf{u}^n(w)||_0 \leq C_1(T)||\mathbf{u}^n||_{\mathbb{X}_w}.$$

By applying the recursive assumption, we get:

$$\forall w \in [0,t], \quad ||B\mathbf{u}^n(w)||_0 \leq \frac{C_1(T)^{n+1}w^n}{n!}||\mathbf{u_0}||_0.$$

Thanks to (23), we then have:

$$\begin{aligned}\forall t \in [0,s], \quad ||\mathbf{u}^{n+1}(t)||_0 &\leq \frac{C_1(T)^{n+1}}{n!}||\mathbf{u_0}||_0 \int_0^t w^n \, dw \\ &\leq \frac{C_1(T)^{n+1}\, t^{n+1}}{(n+1)!}||\mathbf{u_0}||_0,\end{aligned}$$

which gives $||\mathbf{u}^n||_{\mathbb{X}_s} \leq \frac{C_1(T)^{n+1}\, s^{n+1}}{(n+1)!}||\mathbf{u_0}||_0$, and proves (22) for $n+1$. Estimate 22) shows that we have, in particular:

$$||\mathbf{u}^n||_{\mathbb{X}_T} \leq \frac{C_1(T)^n T^n}{n!}||\mathbf{u_0}||_0;$$

which shows that $\sum_{n\geq 0} \mathbf{u}^n$ normally converges on $\mathbb{X}_T$. Let us notice that $\mathbf{u}$ the limit of this series is:

$$\forall t \in [0,T], \quad \mathbf{u}(t) = \sum_{n=0}^{\infty} \mathbf{u}^n(t).$$

The map $\mathbf{u} \in \mathbb{X}_T \times \mathbb{X}_T$ is solution to (20) because:

$$\frac{d}{dt}\mathbf{u}(t) = Ae^{tA}\mathbf{u_0} + \sum_{n=0}^{\infty}\Big(A\mathbf{u}^{n+1}(t) + B\mathbf{u}^n(t)\Big) = A\mathbf{u}(t) + B\mathbf{u}(t).$$

2) *Unicité.* Let us assume that (20) admits two different solutions $\mathbf{u}$ and $\mathbf{v}$ associated to the same initial datum $\mathbf{u_0} = {}^t[\mathbf{E_0}, \mathbf{H_0}]$. We then have:

$$\begin{aligned}\mathbf{u}(t) &= e^{tA}\mathbf{u_0} + \int_0^t e^{(t-s)A}B\mathbf{u}(s)\, ds, \\ \mathbf{v}(t) &= e^{tA}\mathbf{u_0} + \int_0^t e^{(t-s)A}B\mathbf{v}(s)\, ds;\end{aligned}$$

which implies that:

$$\mathbf{u}(t) - \mathbf{v}(t) = \int_0^t e^{(t-s)A}B\big(\mathbf{u} - \mathbf{v}\big)(s)\, ds. \tag{24}$$

We then show that:

$$\forall n \in \mathbb{N},\ \forall s \in [0,T], \quad ||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s} \leq \frac{C_1(T)^{n+1}s^{n+1}}{(n+1)!}||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s}. \tag{25}$$

Relation (25) is stisfied for 0 since:

$$\begin{aligned}\forall t \in [0,s], \quad ||\mathbf{u}(t)-\mathbf{v}(t)||_0 &\leq \int_0^t ||B\big(\mathbf{u}-\mathbf{v}\big)(w)||_0\, dw \\ &\leq \int_0^t ||B\big(\mathbf{u}-\mathbf{v}\big)||_{\mathbb{X}_w}\, dw \\ &\leq \int_0^t C_1(T)||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_w}\, dw \\ &\leq C_1(T)||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s}\int_0^t dw \\ &\leq C_1(T)s||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s}.\end{aligned}$$

If we suppose that (25) is satisfied for n, then (25) is satisfied for $n+1$ because:

$$\begin{aligned}\forall t \in [0,s], \quad ||\mathbf{u}(t)-\mathbf{v}(t)||_0 &\leq \int_0^t ||B\big(\mathbf{u}-\mathbf{v}\big)(w)||_0\, dw \\ &\leq \int_0^t ||B\big(\mathbf{u}-\mathbf{v}\big)||_{\mathbb{X}_w}\, dw \\ &\leq \int_0^t C_1(T)||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_w}\, dw \\ &\leq \int_0^t C_1(T)\frac{C_1(T)^{n+1}w^{n+1}}{(n+1)!}||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_w}\, dw \\ &\leq \frac{C_1(T)^{n+2}}{(n+1)!}||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s}\int_0^t w^{n+1}\, dw \\ &\leq \frac{C_1(T)^{n+2}s^{n+2}}{(n+2)!}||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_s}.\end{aligned}$$

Relation (25) shows in particular that we have:

$$\forall n \in \mathbb{N}, \quad ||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_T} \leq \frac{C_1(T)^{n+1}T^{n+1}}{(n+1)!}||\mathbf{u}-\mathbf{v}||_{\mathbb{X}_T};$$

which gives that $\mathbf{u}=\mathbf{v}$ by letting n to infinity. Hence there exists a single pair $(\mathbf{E},\mathbf{H})$ in $\mathbb{X}_T \times \mathbb{X}_T$ solution to (20) with $(\mathbf{E_0},\mathbf{H_0})$ as initial condition. The uniqueness of P and Q is then ensured by the formula (17) and (18).

3) *Estimate (16).* We have to prove Estimate (16). The previous work is correct for any $T > 0$.

Consequently we have:

$$
\begin{aligned}
\forall t > 0, \quad ||\mathbf{E}(.,t), \mathbf{H}(.,t)||_0 = ||\mathbf{u}(t)||_0 &\leq \sum_{n=0}^{\infty} ||\mathbf{u}^n(t)||_0 \\
&\leq \sum_{n=0}^{\infty} ||\mathbf{u}^n||_{\mathbb{X}_t} \\
&\leq \sum_{n=0}^{\infty} \frac{C_1(t)^n t^n}{n!} ||\mathbf{u_0}||_0 \\
&\leq e^{C_1(t)t} ||\mathbf{u_0}||_0. \quad \square
\end{aligned}
$$

4.3 Check of the assumptions in the PML framework

We end this section by verifying the hypotheses of (12), (13), (14) and (15) are satisfied when $[\Lambda]$ and $[\Gamma]$ are the operators defined at Lemma 2.1. Let us recall that in that case, $s([\Lambda]) = [\nu^2]$ qnd $s([\Gamma]) = [\eta^2]$ with:

$$
\begin{aligned}
\nu_x^2 &= \frac{\sigma_y^2}{1 + \frac{\sigma_y^2}{\varepsilon_0^2 \omega^2}} - i \frac{(\sigma_x + \sigma_y)\varepsilon_0 \omega + \sigma_y^2 \frac{\sigma_x}{\varepsilon_0 \omega}}{1 + \frac{\sigma_y^2}{\varepsilon_0^2 \omega^2}}, \qquad \nu_y^2 = \nu_z^2 = 0; \\
\eta_x^2 &= \frac{\tau_y^2}{1 + \frac{\tau_y^2}{\mu_0^2 \omega^2}} - i \frac{(\tau_x + \tau_y)\mu_0 \omega + \tau_y^2 \frac{\tau_x}{\mu_0 \omega}}{1 + \frac{\tau_y^2}{\mu_0^2 \omega^2}}, \qquad \eta_y^2 = \eta_z^2 = 0.
\end{aligned}
$$

We begin with proving the following result.

Proposition 4.2 *We assume that* $[\Lambda]$ *and* $[\Gamma]$ *have respectively for symbols* $[\nu^2]$ *and* $[\eta^2]$*; where* $[\nu^2]$ *and* $[\eta^2]$ *have been defined at Lemma 2.1. Then* $[\Lambda]$ *and* $[\Gamma]$ *satisfy (12), (13), (14) and (15).*

Proof. We content ourself to consider the operator $[\Lambda]$; the work is identical for $[\Gamma]$. Using Maple, we apply the inverse Fourier transfrom to ν_x^2 and we get the convolution kernel:

$$
-(\sigma_x + \sigma_y)\delta_t' + \sigma_y^2 \delta_t - \sigma_y^3 e^{-\sigma_y t} Y(t)
$$

où δ_t désigne la distribution de Dirac et Y la fonction de Heaviside. On peut alors écrire l'action de $[\Lambda]$ sur $\mathbf{E}$:

$$
[\Lambda]\mathbf{E} = \begin{bmatrix} -(\sigma_x + \sigma_y)\partial_t E_x + \sigma_y^2 E_x - \sigma_y^3 e^{-\sigma_y t} Y(t) * E_x \\ 0 \\ 0 \end{bmatrix}. \tag{26}
$$

We have:

$$
\begin{aligned}
-\sigma_y^3 e^{-\sigma_y t} Y(t) * E_x(.,t) &= -\int_{-\infty}^{\infty} \sigma_y^3 e^{-(t-s)\sigma_y} Y(t-s) E_x(.,s)\, ds \\
&= -\int_0^{\infty} \sigma_y^3 e^{-(t-s)\sigma_y} Y(t-s) E_x(.,s)\, ds \\
&= -\int_0^t \sigma_y^3 e^{-(t-s)\sigma_y} E_x(.,s)\, ds.
\end{aligned}
$$

As a consequence, we can write
$[\Lambda] = [\lambda] + [\Lambda_0]$ with:

$$[\lambda] = \begin{bmatrix} -(\sigma_x+\sigma_y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad [\Lambda_0] = \begin{bmatrix} \sigma_y^2 - \mathcal{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\mathcal{L}$ is defined by:

$$\mathcal{L}\big(E_x(.,t)\big) = -\int_0^t \sigma_y^3 e^{-(t-s)\sigma_y} E_x(.,s)\,ds.$$

The tensor $[\lambda]$ belongs to M_W and the rank of $\mathbb{X}_T$ by $[\Lambda_0]$ is embedded in $\mathbb{X}_T$, which proves that hypotheses (12), (13) and (14) are satisfied. It remains to show (15). Let $s \in [0,T]$ be fixed and consider $\mathbf{E}$ in $\mathbb{X}_s$. We have, for any t in $[0,s]$:

$$\left(\int_0^t [\Lambda_0]\mathbf{E}(.,w)\,dw\right)_x = \int_0^t \sigma_y^2 E_x(.,w)\,dw - \int_0^t \left(\int_0^w \sigma_y^3 e^{-(w-\xi)\sigma_y} E_x(.,\xi)\,d\xi\right) dw, \tag{27}$$

and $\left(\int_0^t [\Lambda_0]\mathbf{E}(.,w)\,dw\right)_y = \left(\int_0^t [\Lambda_0]\mathbf{E}(.,w)\,dw\right)_z = 0$. Moreover, we have:

$$\begin{aligned}\int_{\mathbb{R}^3} \Big|\int_0^t \sigma_y^2 E_x(\mathbf{x},w)\,dw\Big|^2 d\mathbf{x} &\le \int_{\mathbb{R}^3} \sigma_y^4 \left(\int_0^t |E_x(\mathbf{x},w)|\,dw\right)^2 d\mathbf{x} \\ &\le \int_{\mathbb{R}^3} \sigma_y^4 t \left(\int_0^t |E_x(\mathbf{x},w)|^2\,dw\right) d\mathbf{x} \\ &\le |\sigma|^4 t \int_{\mathbb{R}^3} \left(\int_0^t |E_x(\mathbf{x},w)|^2\,dw\right) d\mathbf{x}.\end{aligned}$$

Since the map $(\mathbf{x},w) \to \mathbf{E}(\mathbf{x},w)$ is integrable on $\mathbb{R}^3\times]0,t[$, we can apply the Fubini Theorem to the last inequality

$$\int_{\mathbb{R}^3} \Big|\int_0^t \sigma_y^2 E_x(\mathbf{x},w)\,dw\Big|^2 d\mathbf{x} \le |\sigma|^4 t \int_0^t ||\mathbf{E}(.,w)||_0^2\,dw \le |\sigma|^4 t ||\mathbf{E}||_{\mathbb{X}_s}^2 \int_0^t dw,$$

which yields:

$$\Big|\Big|\int_0^t \sigma_y^2 E_x(\mathbf{x},w)\,dw\Big|\Big|_0^2 \le |\sigma|^4 T^2 ||\mathbf{E}||_{\mathbb{X}_s}^2. \tag{28}$$

In the same way, we have:

$$\begin{aligned}\Big|\int_0^t \int_0^w \sigma_y^3 e^{-(w-\xi)\sigma_y} E_x(.,\xi)\,d\xi\,dw\Big|^2 &= \sigma_y^6 \Big|\int_0^t \int_0^w e^{-(w-\xi)\sigma_y} E_x(.,\xi)\,d\xi\,dw\Big|^2 \\ &\le |\sigma|^6 e^{2|\sigma|T} \left(\int_0^t \int_0^w |E_x(.,\xi)|\,d\xi\,dw\right)^2 \\ &\le |\sigma|^6 e^{2|\sigma|T} t \int_0^t \left(\int_0^w |E_x(.,\xi)|\,d\xi\right)^2 dw \\ &\le |\sigma|^6 e^{2|\sigma|T} t \int_0^t w \int_0^w |E_x(.,\xi)|^2\,d\xi\,dw \\ &\le |\sigma|^6 e^{2|\sigma|T} T^2 \int_0^t \int_0^w |E_x(.,\xi)|^2\,d\xi\,dw.\end{aligned}$$

We integrate the last inequality on $\mathbb{R}^3$, by using the Fubini theorem again:

$$\begin{aligned}\int_{\mathbb{R}^3}\Big|\int_0^t\int_0^w \sigma_y^3 e^{-(w-\xi)\sigma_y}E_x(\mathbf{x},\xi)\,d\xi\,dw\Big|^2 d\mathbf{x} &\leq |\sigma|^6 e^{2|\sigma|T}T^2\int_0^t\int_0^w ||\mathbf{E}(.,\xi)||_0^2\,d\xi\,dw\\ &\leq |\sigma|^6 e^{2|\sigma|T}T^2\int_0^t\int_0^w ||\mathbf{E}||_{\mathbb{X}_s}^2\,d\xi\,dw\\ &\leq |\sigma|^6 e^{2|\sigma|T}T^4||\mathbf{E}||_{\mathbb{X}_s}^2;\end{aligned}$$

which shows that:

$$\Big|\Big|\int_0^t\int_0^w \sigma_y^3 e^{-(w-\xi)\sigma_y}E_x(\mathbf{x},\xi)\,d\xi\,dw\Big|\Big|_0^2 \leq |\sigma|^6 e^{2|\sigma|T}T^4||\mathbf{E}||_{\mathbb{X}_s}^2. \tag{29}$$

Relations (27), (28) and (29) imply then that we have:

$$\Big|\Big|\int_0^t [\Lambda_0]\mathbf{E}(.,w)\,dw\Big|\Big|_0^2 \leq 2\left(|\sigma|^4T^2 + |\sigma|^6 e^{2|\sigma|T}T^4\right)||\mathbf{E}||_{\mathbb{X}_s}^2;$$

which proves that $[\Lambda_0]$ satisfies (15) and the proof of Proposition 4.2 is then completed. □

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