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Abstract

We investigate some inner bilateral obstacle problems for a class of strongly degenerate parabolic-hyperbolic quasilinear operators associated with homogeneous Dirichlet data in a multidimensional bounded domain. We first introduce the concept of an entropy process solution, more convenient and generalizing the notion of an entropy solution. Moreover, the boundary conditions are expressed by using the background of *Divergence Measure Fields*. We ensure that proposed definition warrants uniqueness. The existence of an entropy process solution is obtained through the vanishing viscosity and penalization methods.

1 Introduction

1.1 Mathematical framework

Obstacle problems for conservation laws in physics and mechanics have been studied by many authors [6, 12, 16],... In this work, we are especially interested in the second-order quasilinear operator stemming from the theory of fluid flows through porous media:

$$\mathbb{P}(t,x,.) : u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i}\varphi_i(t,x,u) + \psi(t,x,u) - \Delta\phi(u),$$

where ϕ is a nondecreasing function; especially ϕ' may vanish on nonempty intervals of $\mathbb{R}$. Let us given two thresholds a and b that are fixed reals such that $a \leq 0 \leq b$. For any positive T, the bilateral obstacle problem for $\mathbb{P}$ on the bounded domain Ω of $\mathbb{R}^p$, $p \geq 1$, may be formally described through the free boundary problem: find a measurable function u on $]0,T[\times\Omega$ such that

$$\left\{\begin{array}{l} a \leq u \leq b \text{ in }]0,T[\times\Omega, \\ \mathbb{P}(t,x,u) = 0 \text{ on } [a < u < b], \\ \mathbb{P}(t,x,u) \leq 0 \text{ on } [a < u = b],\ \mathbb{P}(t,x,u) \geq 0 \text{ on } [a = u < b], \\ u = 0 \text{ on }]0,T[\times\partial\Omega, u(0,.) = u_0 \text{ on } \Omega, \end{array}\right. \tag{1}$$

where u_0 is a measurable function on Ω such that $a \leq u_0 \leq b$ a.e. in Ω.

On the one hand, we emphasize that without any assumption on the sign of the source term for $\mathbb{P}$ a bilateral obstacle condition on initial data does not *a priori* pass on to the associated solution. On the other hand, entropy formulations for the Dirichlet problem to strongly degenerate parabolic-hyperbolic operators have been introduced by J.Carrillo in 1999 ([2]). Since then, numerous works have been achieved on this matter [1, 13, 14, 17, 18]... Here, we first provide in Subsection 1.3 the definition of an entropy solution to (1). However, since we release the smoothness assumptions on the convective term $\boldsymbol{\varphi}$ and the reactive one ψ it is more convenient to introduce the concept of an *entropy process solution* to (1). This notion, highlighted in [18] for the Dirichlet to $\mathbb{P}$, may be considered as an extension to the second order of the framework due to R.Eymard, T.Gallouët & R.Herbin in [7]. The uniqueness of an entropy process solution to (1) is stated in Section 2 and its existence is obtained in Section 3 *via* the vanishing viscosity method and by relaxing the bilateral obstacle condition. These two results provide the existence on an entropy solution to (1) and warrant the strong convergence in $L^q(]0,T[\times\Omega)$, $1 \leq q < +\infty$, of the sequence of approximate solutions.

1.2 Main notations and assumptions on data

The reaction term ψ is a continuous function on $[0,T] \times \bar{\Omega} \times [a,b]$. In addition ψ is Lipschitzian with respect to its third variable with a constant M_ψ, uniformly with respect to (t,x) in $]0,T[\times\Omega$. The flux term $\boldsymbol{\varphi}$ is a $W^{1,+\infty}$-class vector-valued function on $]0,T[\times\Omega\times]a,b[$ such that for all i in $\{1,..,p\}$, $\partial_{x_i}\varphi_i$ is Lipschitzian with respect to its third variable with a constant $M_{\partial_{x_i}\varphi_i}$, uniformly with respect to (t,x) in $]0,T[\times\Omega$. Thus we can set $M_{\boldsymbol{\varphi}} = \max_{i\in\{1,..,p\}} M_{\partial_{x_i}\varphi_i}$ and define for any t of $[0,T]$,

$$M(t) = \max(-a,b)e^{\mathcal{K}_1 t} + \frac{\mathcal{K}_2}{\mathcal{K}_1}\left(e^{\mathcal{K}_2 t} - 1\right), \tag{2}$$

where $\mathcal{K}_1 = M_\psi + M_{\boldsymbol{\varphi}}$ and $\mathcal{K}_2 = \max_{[0,T]\times\bar{\Omega}} |\psi(t,x,0) + Div\boldsymbol{\varphi}(t,x,0)|$.

The diffusive term ϕ is a nondecreasing $W^{1,+\infty}(]a,b[)$-class function such that $\phi(0) = 0$. We note $E = \{l \in \mathbb{R}, \{l\} = \phi^{-1}(\{\phi(l)\})\}$. Hence, if ϕ_0^{-1} denotes a generalized inverse of ϕ, the hypo-inverse for example defined through

$$\forall r \in Im\phi,\ \phi_0^{-1}(r) = Inf\phi^{-1}(\{r\}),$$

then $\phi(E)$ is the set of points where ϕ_0^{-1} is continuous ([2]).

We assume that Ω is a bounded subset of $\mathbb{R}^p$ such that $\Gamma = \partial\Omega$ is Lipschitz-deformable (see [1, 3, 4, 13]...). Then the space of L^2-Divergence Measure fields on Q - denoted $\mathcal{DM}_2(Q)$ - is given by

$$\mathcal{DM}_2(Q) = \left\{V = (v_0, v_1, .., v_p) \in (L^2(Q))^{p+1}, Div_{(t,x)}V \in \mathcal{M}_b(Q)\right\},$$

where $\mathcal{M}_b(Q)$ is the space of bounded Radon measures on Q. For any V in $\mathcal{DM}_2(Q)$, it is useful to define a linear application Λ_V on $H^1(Q)\cap L^\infty(Q)\cap\mathcal{C}(Q)$ through the following generalized Gauss-Green formula coming from [4]:

$$\Lambda_V(\xi) := \langle V, \xi\rangle_\partial = \int_Q V.(\partial_t\xi, \nabla\xi)dxdt + \int_Q \xi\, d[Div_{(t,x)}V], \tag{3}$$

where $d[Div_{(t,x)}V]$ denotes the Borelian measure on Q associated with the bounded Radon measure $Div_{(t,x)}V$. In addition the next property holds (see [13]): let V be an element of $\mathcal{DM}_2(Q)$ such that v_0 is continuous at $t = 0$ and $t = T$ with respect to the $L^1(\Omega)$-norm then, for any ξ in $H^1(Q)\cap L^\infty(Q)\cap\mathcal{C}(Q)$,

$$\begin{aligned}&\lim_{\varrho\to 0^+}\int_Q V\xi.(0,\nabla\rho_\varrho)dxdt\\&=\int_\Omega v_0(T,x)\xi(0,x)dx-\int_\Omega v_0(0,x)\xi(0,x)dx-\langle V,\xi\rangle_\partial\,,\end{aligned}\tag{4}$$

for any boundary-layer sequence $(\rho_\varrho)_{\varrho>0}$, *i.e.* a sequence of $\mathcal{C}^2(\Omega)\cap\mathcal{C}(\overline{\Omega})$-class functions such that $\lim_{\rho\to 0^+}\rho_\varrho = 1$ pointwise in Ω, $0 \le \rho_\varrho \le 1$ in Ω, $\rho_\varrho = 0$ on Γ. In (4), if ξ belongs to $L^2(0,T;H^1_0(\Omega))$ the right-hand side is equal to 0.

We will consider "sgn_λ" the approximation of the function "sgn" given for any positive parameter λ and nonnegative real x by:

$$sgn_\lambda(x)=\min\left(\frac{x}{\lambda},1\right)\text{ and }sgn_\lambda(-x)=-sgn_\lambda(x).$$

To simplify the writing, we refer to the notations

$$\begin{aligned}\mathbf{F}(u,k)&=sgn(u-k)\{\boldsymbol{\varphi}(t,x,u)-\boldsymbol{\varphi}(t,x,k)\},\\G(u,k)&=sgn(u-k)Div_x\boldsymbol{\varphi}(t,x,k)+\psi(t,x,u),\\U_k&=(|u-k|,-\nabla|\phi(u)-\phi(k)|+\mathbf{F}(u,k))\,,\ \overline{\nabla}\zeta=(\partial_t\zeta,\nabla\zeta)\,.\end{aligned}$$

Eventually, for any n in $\mathbb{N}^\star$, $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure and for all s of $]0,T]$, Q_s is the cylinder $]0,s[\times\Omega$, $\Sigma_s=]0,s[\times\Gamma$ with the convention $Q = Q_T$ and $\Sigma = \Sigma_T$. The outer normal of Ω is denoted ν.

1.3 Two concepts of weak solutions

The existence of possible internal and boundary layers leads us to propose a mathematical formulation for (1) through an entropy inequality inside the studied field - using the classical Kruzhkov entropy pairs - and on its boundary; the latter is viewed as an extension to the second order of the F.Otto's formulation provided in [15] for hyperbolic first-order operators and uses - as in [1, 13, 18] - the mathematical framework of *Measure-Divergence Fields*. In addition, to take into account the bilateral obstacle condition we only consider the Kruzhkov pairs for a parameter k that has to belong to the bounded interval $[a,b]$. That is why, by referring to a preliminary study of the positiveness obstacle problem for $\mathbb{P}$ in [11], it will be said that:

Definition 1. *A measurable function u on Q is an entropy solution to (1) if:*

$$a\le u\le b\ a.e.\ in\ Q,\tag{5}$$

$$\begin{gathered}\phi(u)\in L^2(0,T;H^1_0(\Omega)),\ \partial_t\phi(u)\in L^2_{loc}(0,T;L^2(\Omega)),\\\partial_t u\in L^2(0,T;H^{-1}(\Omega)),\end{gathered}\tag{6}$$

$$\forall k\in[a,b],\ U_k\in\mathcal{DM}_2(Q),\tag{7}$$

$\forall k \in [a,b]$, $\forall \xi \in \mathcal{D}(]-\infty,T[\times\Omega)$, $\xi \geq 0$,

$$\int_Q \left(U_k.\overline{\nabla}\xi dxdt - G(u,k)\xi\right) dxdt + \int_\Omega |u_0 - k|\xi(0,.)dx \geq 0, \tag{8}$$

$\forall \xi \in L^\infty(Q) \cap H^1(Q) \cap \mathcal{C}(Q)$, $\xi \geq 0$, $\xi(T,.) = \xi(0,.) = 0$, $\forall k \in [a,b]$,

$$\int_\Sigma \mathbf{F}(k,0).\nu\xi d\mathcal{H}^p \leq \langle U_k, \xi\rangle_\partial + \langle U_0, \xi\rangle_\partial\,. \tag{9}$$

Remark 1. *Considering $k = a$ and ξ in $\mathcal{D}(Q)$, $\xi \geq 0$, in (8) leads to:*

$$\int_Q ((u-a)\partial_t\xi - \nabla\phi(u).\nabla\xi + \mathbf{F}(u,a).\nabla\xi - G(u,a)\xi)dxdt \geq 0.$$

We observe that a.e. in Q, $\mathbf{F}(u,a) = \boldsymbol{\varphi}(t,x,u) - \boldsymbol{\varphi}(t,x,a)$. So thanks to some integrations by parts one gets:

$$-\langle \mathbb{P}(t,x,u), \xi\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} + \int_Q (1 - sgn(u-a))(Div\boldsymbol{\varphi}(t,x,a) + \psi(t,x,u))\xi dxdt \geq 0.$$

Assume now that u is smooth enough so that $\mathbb{P}(t,x,u)$ belongs to $L^1(Q)$ and let us choose $\xi = (u-a)^+\zeta$ with ζ in $\mathcal{D}(Q)$, $\zeta \geq 0$. Since $(1 - sgn(u - a))(Div\boldsymbol{\varphi}(t,x,a) + \psi(t,x,u))(u-a)^+ = 0$ a.e. on Q, we obtain

$$\int_Q \mathbb{P}(t,x,u)(u-a)^+\zeta dxdt \leq 0.$$

That means $\mathbb{P}(t,x,u)(u-a)^+ \leq 0$ a.e. on Q. The same reasoning with $k = b$ leads to $\mathbb{P}(t,x,u)(u-b)^- \geq 0$ a.e. in Q that formally provides (1).

However, one of the feature of this work in comparison with [11] is to release the assumptions of regularity for $\boldsymbol{\varphi}$ and ψ. So for the existence property stated in Section 3 through the vanishing viscosity and penalization methods, we can only refer to an $L^\infty(Q)$-estimate of approximate solutions. That is why we lean on the original presentation of R.Eymard, T.Gallouët & R.Herbin in [7] for first-order quasilinear hyperbolic operators that consists - by using the basic tools of Young measure [5] - in introducing a free variable α living on $]0,1[$ and a new measurable and bounded unknown $]0,1[\times Q$, $\pi \equiv \pi(\alpha,t,x)$, that fulfills an entropy formulation on $]0,1[\times Q$ and called an *entropy process solution.* We adapt this concept when dealing with second order quasilinear operators associated with a forced bilateral constraint. From now, to simplify the writing, we set $\mathcal{Q} =]0,1[\times Q$ and $dq = d\alpha dxdt$:

Definition 2. *Let π be in $L^\infty(]0,1[\times Q)$ and $u(t,x) = \int_0^1 \pi(\alpha,t,x)d\alpha$. We say that π is an entropy process solution to (1) if:*

$$a \leq \pi \leq b \text{ a.e. in } \mathcal{Q}, \tag{10}$$

$$\text{for a.e. } (t,x) \text{ in } Q, \phi(\pi(\alpha,t,x)) = \phi(u(t,x)) \text{ for a.e. } \alpha \text{ in }]0,1[, \tag{11}$$

the smoothness properties (6) hold and:

$$\forall k \in [a,b], \Pi_k \in \mathcal{DM}_2(Q), \tag{12}$$

$\forall k \in [a,b]$, $\forall \xi \in \mathcal{D}(]-\infty,T[\times\Omega)$, $\xi \geq 0$,

$$\int_Q \Pi_k.\overline{\nabla}\xi dxdt - \int_{\mathcal{Q}} G(\pi,k)\xi dq + \int_\Omega |u_0 - k|\xi(0,.)dx \geq 0. \tag{13}$$

$\forall \xi \in L^\infty(Q) \cap H^1(Q) \cap \mathcal{C}(Q)$, $\xi(T,.) = \xi(0,.) = 0$, $\xi \geq 0$, $\forall k \in [a,b]$,

$$\int_\Sigma \mathbf{F}(k,0).\nu\xi d\mathcal{H}^p \leq \langle \Pi_k, \xi \rangle_\partial + \langle \Pi_0, \xi \rangle_\partial \,, \tag{14}$$

where

$$\Pi_k = \left(\int_0^1 |\pi(\alpha,.,.) - k|d\alpha, \int_0^1 \left(-\nabla|\phi(\pi)(\alpha,.,.) - \phi(k)| + \mathbf{F}(\pi(\alpha,.,.),k)\right) d\alpha \right).$$

The concept of an entropy process solution generalize that of an entropy solution since an entropy process solution independent from the variable α is an entropy solution. But the first notion is more convenient that the second one, mainly because it requires a few estimates of approximate solutions and so an existence result is easier to obtain. Moreover, the existence and uniqueness of an entropy solution results from the existence and uniqueness of an entropy process solution. First by reasoning as F.Otto in [15], we may announce:

Proposition 1. *An entropy process solution π to (1) fulfills:*

$$ess \lim_{t \to 0^+} \int_{]0,1[\times\Omega} |\pi(\alpha,t,x) - u_0(x)|d\alpha dx = 0. \tag{15}$$

2 The Uniqueness Theorem

The proof basically relies on a inner comparison property which is an extension to second-order operators of the usual hyperbolic method of doubling variables due to S.N.Kuzhkov [9]; the contribution of diffusive terms being controlled thanks to an *energy inequality* in the same spirit as in the original paper of J.Carrillo [2]. To do so, we need some preliminary lemma. The first one takes into account that ϕ' may vanish on a nonnegligible subset of $\mathbb{R}$:

Lemma 1.
(i) $\forall k \in E$ and a.e. on Q,

$$sgn(u-k) = sgn(\phi(u) - \phi(k)) = sgn(\pi(\alpha,.,.) - k) \text{ for a.e. } \alpha \text{ in }]0,1[.$$

(ii) $\nabla\phi(\pi) = 0$ a.e. on $\mathcal{Q}_0^\pi \equiv \{(\alpha,t,x) \in \mathcal{Q}, \pi(\alpha,t,x) \notin E\}$

Proof.
For the first equality in (i) we remark that when k belongs to E, if $u(t,x) > k$ then $\phi(u(t,x)) > \phi(k)$. This way the second equality is a consequence of (11) and uses the same reasoning as for the first equality with π in the place of u. The point (ii) is already emphasized ([2]) with the setting of entropy solutions. It uses the monotony of ϕ_0^{-1} so that $Im\phi \setminus \phi(E)$ - the set of points where ϕ_0^{-1} is discontinuous - is a countable (and thus a Lebesgue-negligible) subset of $\mathbb{R}$. ■

The second lemma may be viewed as an *inequality version* of the energy equality stated by J.Carrillo in [2]. It permits to determine the sign of diffusive terms appearing in the method of doubling variables. We prove that this inequality is fulfilled by any entropy process solution to (1) but only for k in E. We emphasize that in [11] the forthcoming relation (16) results from an underlying formulation of the unilateral obstacle problem for $\mathbb{P}$ through a strong variational inequality (in the sense of J.L.Lions in [12]). But here, in the context of the bilateral obstacle problem, we have not been able to establish such a formulation and so we directly start from (13). Indeed:

Lemma 2. *Let π be an entropy process solution to (1). Then, for any real k of $[a,b] \cap E$, for any nonnegative function ζ of $\mathcal{D}(Q)$,*

$$\int_{\mathcal{Q}} \Pi_k.\overline{\nabla}\zeta dxdt - \int_{\mathcal{Q}} G(\pi,k)\zeta dq \geq \limsup_{\lambda\to 0^+} \int_{\mathcal{Q}} sgn'_\lambda(\phi(\pi)-\phi(k))[\nabla\phi(\pi)]^2\zeta dq.$$

Proof. We consider (13) for any nonnegative ξ in $\mathcal{D}(Q)$ and, thanks to a density argument, for any nonnegative ξ in $H_0^1(Q)$ so as to choose the test-function $\xi = \zeta|sgn_\lambda(\phi(u)-\phi(k))|$ with ζ in $\mathcal{D}(Q)$, $\zeta \geq 0$. Let us perform the following transformations (by setting $\mathcal{S}_\lambda(v) = sgn_\lambda(\phi(v)-\phi(k))$):
• For the evolution term,

$$\int_{\mathcal{Q}} |\pi - k|\partial_t\xi dq = I_1 + I_2 \text{ where,}$$

$$\begin{aligned} I_1 &= \int_{\mathcal{Q}} |\pi-k||\mathcal{S}_\lambda(u)|\partial_t\zeta dq, \\ I_2 &= \int_{\mathcal{Q}} |\pi-k|\zeta sgn'_\lambda(\phi(u)-\phi(k))sgn(\phi(u)-\phi(k))\partial_t\phi(u)dq. \end{aligned}$$

In I_2 we have taken into account that $sgn(\mathcal{S}_\lambda(u)) = sgn(\phi(u)-\phi(k))$ a.e. on Q when k belongs to E. But due to Lemma 1 (i), $|\pi - k| = sgn(u-k)(\pi-k)$ a.e. on $\mathcal{Q}$ and $sgn(\phi(u)-\phi(k)) = sgn(u-k)$ a.e. on Q. So the integration with respect to α may be calculated. It comes:

$$\begin{aligned} I_2 &= \int_Q \zeta sgn'_\lambda(\phi(u)-\phi(k))(u-k)\partial_t\phi(u)dxdt \\ &= \int_Q \partial_t(\zeta\mathcal{S}_\lambda(u))(u-k)dxdt - \int_{\mathcal{Q}} \partial_t\zeta\mathcal{S}_\lambda(u)(u-k)dxdt. \end{aligned}$$

Since $\partial_t u$ is an element of $L^2(0,T;H^{-1}(\Omega))$,

$$\int_Q \partial_t(\zeta\mathcal{S}_\lambda(u))(u-k)dxdt = -\int_0^T \langle\partial_t u, \mathcal{S}_\lambda(u)\zeta\rangle_{H^{-1},H_0^1}dt.$$

An integration by parts in time using the Mignot-Bamberger formula ([8]) gives

$$-\int_0^T \langle\partial_t u, \mathcal{S}_\lambda(u)\zeta\rangle_{H^{-1},H_0^1}dt = \int_Q \left(\int_k^u \mathcal{S}_\lambda(\tau)d\tau\right)\partial_t\zeta dxdt.$$

We now take the λ-limit through the Lebesgue dominated convergence Theorem in I_1 and I_2. Since k belongs to E, I_2 goes to 0^+ (Lemma 1 (i)).

• For the diffusive term, considering that $sgn(\mathcal{S}_\lambda(\pi)) = sgn(\phi(\pi)-\phi(k))$ a.e. on $\mathcal{Q}$ provides

$$\begin{aligned} & -\int_{\mathcal{Q}} \nabla|\phi(\pi)-\phi(k)|.\nabla\xi dq \\ = & -\int_{\mathcal{Q}} \left[\nabla|\phi(\pi)-\phi(k)|.\nabla\zeta\right]|\mathcal{S}_\lambda(\pi)|dq - \int_{\mathcal{Q}} sgn'_\lambda(\phi(\pi)-\phi(k))[\nabla\phi(\pi)]^2\zeta dq. \end{aligned}$$

• Concerning the convective term, as a consequence of (11), we write:

$$\int_{\mathcal{Q}} \mathbf{F}(\pi,k).\nabla\xi dq = \int_{\mathcal{Q}} \left[\mathbf{F}(\pi,k).\nabla\zeta\right]|\mathcal{S}_\lambda(\pi)|dq + \int_{\mathcal{Q}} \zeta\mathbf{F}(\pi,k).\nabla|\mathcal{S}_\lambda(\pi)|dq.$$

To take the λ-limit, we transform the second term in the right-hand side by by taking into account that due to Lemma 1 (i), $sgn(\mathcal{S}_\lambda(\pi)) = sgn(\phi(\pi)-\phi(k)) = sgn(\pi-k)$ a.e. on $\mathcal{Q}$. It comes

$$\begin{aligned} & \int_{\mathcal{Q}} \zeta\mathbf{F}(\pi,k).\nabla|\mathcal{S}_\lambda(\pi)|dq \\ = & \int_{\mathcal{Q}} \zeta sgn'_\lambda(\phi(\pi)-\phi(k))\{\boldsymbol{\varphi}(t,x,\pi)-\boldsymbol{\varphi}(t,x,k)\}.\nabla\phi(\pi)dq. \end{aligned}$$

Now, we introduce the vector-valued function

$$\mathbf{H}_\lambda(t,x,r) = \int_{\phi(k)}^r [\boldsymbol{\varphi}(t,x,\phi_0^{-1}(\tau)) - \boldsymbol{\varphi}(t,x,k)]sgn'_\lambda(\tau-\phi(k))d\tau,$$

so that thanks to Lemma 1 (ii),

$$\begin{aligned} & \int_{\mathcal{Q}} \zeta sgn'_\lambda(\phi(\pi)-\phi(k))\{\boldsymbol{\varphi}(t,x,\pi)-\boldsymbol{\varphi}(t,x,k)\}\nabla\phi(\pi)dq \\ = & \int_{\mathcal{Q}} \zeta Div\mathbf{H}_\lambda(t,x,\phi(\pi))dq \\ - & \int_{\mathcal{Q}} \zeta \int_{\phi(k)}^{\phi(\pi)} \left(Div_x\boldsymbol{\varphi}(t,x,\phi_0^{-1}(\tau)) - Div_x\boldsymbol{\varphi}(t,x,k)\right) sgn'_\lambda(\tau-\phi(k))d\tau dq, \end{aligned}$$

The first term is integrated by parts and for the second one we come back to the definition of sgn'_λ to obtain:

$$\begin{aligned}
&\int_{\mathcal{Q}} \zeta \mathbf{F}(\pi,k).\nabla |sgn_\lambda(\phi(\pi)-\phi(k))| dq \\
= \quad & -\int_{\mathcal{Q}} \frac{1}{\lambda}\int_{\mathcal{I}} \left([\boldsymbol{\varphi}(t,x,\phi_0^{-1}(\tau)) - \boldsymbol{\varphi}(t,x,k)].\nabla\zeta d\tau \right) dq \\
& -\int_{\mathcal{Q}} \frac{1}{\lambda}\int_{\mathcal{I}} \left(Div_x[\boldsymbol{\varphi}(t,x,\phi_0^{-1}(\tau)) - \boldsymbol{\varphi}(t,x,k)]\zeta d\tau \right) dq,
\end{aligned}$$

where $\mathcal{I} = I(\phi(\pi),\phi(k)) \cap [\phi(k)-\lambda, \phi(k)+\lambda]$ and $I(r,p) = [min(r,p), max(r,p)]$ for all (r,p) of $\mathbb{R}^2$. Since k belongs to E, the generalized function ϕ_0^{-1} is continuous in $\phi(k)$ ([2]). This way, the previous integral goes to 0 when λ tends to 0^+. The study of the reaction term does not present any difficulty. ■

This way the next Kuzhkov-type relation holds:

Proposition 2. *Let π_1 and π_2 satisfying (6,10,11,13). Then for any nonnegative Ψ of $\mathcal{D}(Q \times Q)$:*

$$\begin{aligned}
&- \int_{\mathcal{Q}\times\mathcal{Q}} |\pi_1 - \tilde{\pi}_2|(\partial_t + \partial_{\tilde{t}})\Psi - (\nabla_x + \nabla_{\tilde{x}})|\phi(u_1) - \phi(\tilde{u}_2)|.(\nabla_x + \nabla_{\tilde{x}})\Psi dq d\tilde{q} \\
&- \int_{\mathcal{Q}\times\mathcal{Q}} \left(\mathbf{F}(\pi_1,\tilde{\pi}_2).\nabla_x\Psi + \tilde{\mathbf{F}}(\tilde{\pi}_2,\pi_1).\nabla_{\tilde{x}}\Psi \right) dq d\tilde{q} \\
&+ \int_{\mathcal{Q}\times\mathcal{Q}} (G(\pi_1,\tilde{\pi}_2) + \tilde{G}(\tilde{\pi}_2,\pi_1))\Psi dq d\tilde{q} \leq 0.
\end{aligned}$$

Proof. To simplify we set $d\tilde{q} = d\tilde{\alpha} d\tilde{x} d\tilde{t}$ and we add a "tilde" superscript to any function in "tilde" variables. On the one hand in (16) written in variables (α,t,x) for π_1 we take $k(\tilde{\alpha},\tilde{t},\tilde{x}) = \tilde{\pi}_2$ for a.e. $(\tilde{\alpha},\tilde{t},\tilde{x})$ in $\mathcal{Q} \setminus \mathcal{Q}_0^{\tilde{\pi}_2}$, so that $k(\tilde{\alpha},\tilde{t},\tilde{x})$ belongs to E. On the other hand in (13) written in variables (α,t,x) for π_1, we take $k(\tilde{\alpha},\tilde{t},\tilde{x}) = \tilde{\pi}_2$ for a.e. $(\tilde{\alpha},\tilde{t},\tilde{x})$ in $\mathcal{Q}_0^{\tilde{u}_2}$. Each inequality obtained is integrated with respect to the variables $\tilde{\alpha}$, $\tilde{t}$ and $\tilde{x}$ on the corresponding domain. We add and use a version of the Fatou's Lemma to deal with the "lim sup" in the right-hand side. Indeed we observe by coming back to the proof of Lemma 2 for λ fixed, that the function

$$k \mapsto \int_{\mathcal{Q}} sgn'_\lambda(\phi(\pi) - \phi(k))[\nabla\phi(\pi)]^2 \zeta dq$$

is uniformly bounded with respect to λ and k in $[a,b]$. Finally it comes for π_1:

$$\begin{aligned}
&- \int_{\mathcal{Q}\times\mathcal{Q}} |\pi_1 - \tilde{\pi}_2|\Psi_t \, dq d\tilde{q} + \int_{\mathcal{Q}\times\mathcal{Q}} \left[\nabla_x|\phi(\pi_1) - \phi(\tilde{\pi}_2)| - \mathbf{F}(\pi_1,\tilde{\pi}_2)\right].\nabla_x\Psi \, dq d\tilde{q} \\
&+ \int_{\mathcal{Q}\times\mathcal{Q}} G(\pi_1,\tilde{\pi}_2)\Psi \, dq d\tilde{q} \\
&\leq \; - \limsup_{\lambda\to 0^+} \int_{\mathbb{Q}} sgn'_\lambda(\phi(\pi_1) - \phi(\tilde{\pi}_2))[\nabla\phi(\pi_1)]^2 \Psi \, dq d\tilde{q},
\end{aligned}$$

where in the right-had side we have used the lemma 1 (ii) to rewrite the integration field under the form $\mathbb{Q} = \mathcal{Q}\setminus\mathcal{Q}_0^{\pi_1}\times\mathcal{Q}\setminus\mathcal{Q}_0^{\tilde{\pi}_2}$. Besides, we integrate over $\mathcal{Q}$ the Gauss-Green formula:

$$\int_Q \nabla_x\phi(u_1).\nabla_{\tilde{x}}[sgn_\lambda(\phi(u_1)-\phi(\tilde{u}_2))\Psi]\ d\tilde{t}d\tilde{x} = 0 \text{ a.e. } (t,x) \text{ in } Q.$$

By developing the partial derivatives and taking into account that $\phi(u_2)$ is an element of $L^2(0,T;H_0^1(\Omega))$, the λ-limit provides the next equality:

$$\begin{aligned}
&\int_{\mathcal{Q}\times\mathcal{Q}} sgn(\phi(\pi_1)-\phi(\tilde{\pi}_2))\nabla_x\phi(\pi_1).\nabla_{\tilde{x}}\Psi\ dqd\tilde{q}\\
=&\lim_{\lambda\to0^+}\int_{\mathcal{Q}\times\mathcal{Q}} sgn'_\lambda(\phi(\pi_1)-\phi(\tilde{\pi}_2))\nabla_x\phi(\pi_1).\nabla_{\tilde{x}}\phi(\tilde{\pi}_2)\Psi\ dqd\tilde{q}\\
=&\lim_{\lambda\to0^+}\int_{\mathbb{Q}} sgn'_\lambda(\phi(\pi_1)-\phi(\tilde{\pi}_2))\nabla_x\phi(\pi_1).\nabla_{\tilde{x}}\phi(\tilde{\pi}_2)\Psi\ dqd\tilde{q}.
\end{aligned}$$

We apply the same reasoning with $\tilde{\pi}_2$ and bring the results together to obtain:

$$\begin{aligned}
-&\int_{\mathcal{Q}\times\mathcal{Q}} |\pi_1-\tilde{\pi}_2|(\partial_t+\partial_{\tilde{t}})\Psi-(\nabla_x+\nabla_{\tilde{x}})|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.(\nabla_x+\nabla_{\tilde{x}})\Psi dqd\tilde{q}\\
-&\int_{\mathcal{Q}\times\mathcal{Q}} \Big(\mathbf{F}(\pi_1,\tilde{\pi}_2).\nabla_x\Psi+\tilde{\mathbf{F}}(\tilde{\pi}_2,\pi_1).\nabla_{\tilde{x}}\Psi\Big)\,dqd\tilde{q}\\
+&\int_{\mathcal{Q}\times\mathcal{Q}} (G(\pi_1,\tilde{\pi}_2)+\tilde{G}(\tilde{\pi}_2,\pi_1))\Psi dqd\tilde{q}\\
\le&\ -\limsup_{\lambda\to0^+}\int_{\mathbb{Q}} sgn'_\lambda(\phi(\pi_1)-\phi(\tilde{\pi}_2))\left[\nabla_x\phi(\pi_1)-\nabla_{\tilde{x}}\phi(\tilde{\pi}_2)\right]^2\Psi\ dqd\tilde{q}.
\end{aligned}$$

The desired inequality follows. ■

From Proposition 2 we may state the main result of this section, that is the T-Lipschitzian dependence in L^1 of an entropy process solution to (1) with respect to the corresponding initial data. For the treatment of boundary terms, the proof follows C.Mascia, A.Porretta & A.Terracina's one ([13]) but it needs to be transcribed in the framework of entropy process solutions. It leads to:

Theorem 1. *If π_1 and π_2 are two entropy process solutions to (1) corresponding to initial data $u_{0,1}$ and $u_{0,2}$ respectively then for a.e. t of $[0,T]$,*

$$\int_{]0,1[^2\times\Omega} |\pi_1(\alpha,t,x)-\pi_2(\tilde{\alpha},t,x)|\,dxd\alpha d\tilde{\alpha} \le e^{M_\psi t}\int_\Omega |u_{0,1}-u_{0,2}|\,dx.$$

Proof. In Proposition 2 we choose $\Psi = \xi(t,x,\tilde{t},\tilde{x})\rho_l(x)\rho_m(\tilde{x})$ for any boundary-layer sequences $(\rho_l)_{l>0}$ and $(\rho_m)_{m>0}$ and any nonnegative ξ in $\mathcal{D}((]0,T[\times\overline{\Omega})^2)$.

We develop the partial derivatives and we argue that due to (3) and (4),

$$\lim_{m\to 0^+}\left(\lim_{l\to 0+}\left(\int\limits_{\mathcal{Q}\times\mathcal{Q}}\left(\nabla_x|\phi(\pi_1)-\phi(\tilde{\pi}_2)|-\mathbf{F}(\pi_1,\tilde{\pi}_2)\right)\xi\rho_m\nabla_x\rho_l dqd\tilde{q}\right)\right)$$
$$= \int\limits_{\mathcal{Q}}\langle\Pi^1_{\tilde{\pi}_2},\xi\rangle_\partial d\tilde{q},$$

where $\Pi^1_{\tilde{\pi}_2}$ refers to $\Pi_{\tilde{\pi}_2}$ with $\pi=\pi_1$. Similarly

$$\lim_{m\to 0^+}\left(\lim_{l\to 0+}\left(\int\limits_{\mathcal{Q}\times\mathcal{Q}}\left(\nabla_{\tilde{x}}|\phi(\pi_1)-\phi(\tilde{\pi}_2)|-\tilde{\mathbf{F}}(\tilde{\pi}_2,\pi_1)\right)\xi\rho_l\nabla_{\tilde{x}}\rho_m dqd\tilde{q}\right)\right)$$
$$= \int\limits_{\mathcal{Q}}\langle\tilde{\Pi}^2_{\pi_1},\xi\rangle_\partial dq,$$

and we obtain:

$$\begin{aligned}
& - \int\limits_{Q\times Q}\Pi^1_{\tilde{\pi}_2}.\overline{\nabla}_{(t,x)}\xi dxdtd\tilde{x}d\tilde{t}+\int\limits_{\mathcal{Q}\times\mathcal{Q}}G(\pi_1,\tilde{\pi}_2)\xi dqd\tilde{q}\\
& + \int\limits_{\mathcal{Q}\times\mathcal{Q}}(\nabla_x|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_{\tilde{x}}\xi+\nabla_{\tilde{x}}|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_x\xi)dqd\tilde{q}\\
& - \int\limits_{Q\times Q}\tilde{\Pi}^2_{\pi_1}.\overline{\nabla}_{(\tilde{t},\tilde{x})}\xi dxdtd\tilde{x}d\tilde{t}+\int\limits_{\mathcal{Q}\times\mathcal{Q}}\tilde{G}(\tilde{\pi}_2,\pi_1)\xi dqd\tilde{q}\\
& \le -\int\limits_{\mathcal{Q}}\langle\Pi^1_{\tilde{\pi}_2},\xi\rangle_\partial d\tilde{q}-\int\limits_{\mathcal{Q}}\langle\tilde{\Pi}^2_{\pi_1},\xi\rangle_\partial dq-I_1-I_2, \qquad (16)
\end{aligned}$$

where

$$I_1=\lim_{m\to 0^+}\int\limits_{\mathcal{Q}\times\mathcal{Q}}\xi\nabla_x|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_{\tilde{x}}\rho_m dqd\tilde{q},$$
$$I_2=\lim_{l\to 0^+}\int\limits_{\mathcal{Q}\times\mathcal{Q}}\xi\nabla_{\tilde{x}}|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_x\rho_l dqd\tilde{q}.$$

An integration by parts with respect to x and then with respect to $\tilde{x}$ allows us to express the limit with respect to m in I_1. Indeed:

$$\lim_{m\to 0^+}\int\limits_{\mathcal{Q}\times\mathcal{Q}}\xi\nabla_x|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_{\tilde{x}}\rho_m dqd\tilde{q}=\int\limits_{\mathcal{Q}\times\mathcal{Q}}Div_{\tilde{x}}[|\phi(\pi_1)-\phi(\tilde{\pi}_2)|.\nabla_x\xi]dqd\tilde{q}.$$

An integration by parts with respect to $\tilde{x}$ and then with respect to x provides:

$$I_1=-\int\limits_{\mathcal{Q}\times\Sigma}\nabla_x|\phi(\pi_1)|.\tilde{\nu}\xi dqd\mathcal{H}^p_{(\tilde{t},\tilde{x})}.$$

With the same arguments,

$$I_2 = -\int\limits_{\mathcal{Q}\times\Sigma} \nabla_{\tilde{x}}|\phi(\tilde{\pi}_2)|.\nu\xi d\tilde{q} d\mathcal{H}^p_{(t,x)}.$$

We take now into account (14) for π_1 and $\tilde{\pi}_2$ to have a majoration of the right-hand side of (16) in terms of:

$$\int\limits_{\mathcal{Q}\times\Sigma} \left(\nabla_x|\phi(\pi_1)| - \tilde{\mathbf{F}}(\pi_1,0)\right).\tilde{\nu}\xi dq d\mathcal{H}^p_{(\tilde{t},\tilde{x})} + \int\limits_{\mathcal{Q}} \langle \Pi_0^1, \xi\rangle_\partial d\tilde{q}$$
$$+ \int\limits_{\mathcal{Q}\times\Sigma} \left(\nabla_{\tilde{x}}|\phi(\tilde{\pi}_2)| - \mathbf{F}(\tilde{\pi}_2,0)\right).\nu\xi d\tilde{q} d\mathcal{H}^p_{(t,x)} + \int\limits_{\mathcal{Q}} \langle \tilde{\Pi}_0^2, \xi\rangle_\partial dq. \tag{17}$$

We choose $\xi_n = W_n(x-\tilde{x})w_n(t-\tilde{t})\gamma$ where γ is a nonnegative element of $\mathcal{D}(]0,T[)$, $(W_n)_n$ and $(w_n)_n$ are the standard mollifier sequences on $\mathbb{R}^p$ and $\mathbb{R}$. In addition, n is large enough so that for any $\tilde{t}$ in $]0,T[$, $t\mapsto\gamma(t)w_n(t-\tilde{t})$ and for any t in $]0,T[$, $\tilde{t}\mapsto\gamma(t)w_n(t-\tilde{t})$ belongs to $\mathcal{D}(]0,T[)$. We take the limit with respect to n in (17) by referring to [13] providing that each line goes to 0. For the left-hand side of (16), we use classical techniques and the fact that

$$(\partial_t + \partial_{\tilde{t}})\xi_n = \gamma'(t)W_n(x-\tilde{x})w_n(t-\tilde{t}) \text{ and } (\nabla_x + \nabla_{\tilde{x}})\xi_n = 0.$$

Eventually it comes,

$$-\int\limits_{\mathcal{Q}\times]0,1[} |\pi_1 - \pi_2|\gamma'(t) dq d\tilde{\alpha} \le M_\psi \int\limits_{\mathcal{Q}\times]0,1[} |\pi_1-\pi_2|\gamma dq d\tilde{\alpha},$$

for any nonnegative γ in $\mathcal{D}(]0,T[)$ and so, by density, for any nonnegative γ in $H^1_0(]0,T[)$. Now the conclusion is classical: it uses for γ a Lipschitzian and piecewise linear approximation of $\mathbb{I}_{[0,\tau]}$, for a.e. τ in $]0,T[$, the initial condition (15) for π_1 and π_2 and the Gronwall Lemma. ■

When $u_{0,1} = u_{0,2}$ a.e. on Ω, we deduce (see [7] or [5] in the framework of Young measure solutions) the existence of a measurable function χ on Q, such that $\pi_1(\alpha,t,x) = \pi_2(\tilde{\alpha},t,x) = \chi(t,x)$ for a.e. α and $\tilde{\alpha}$ in $]0,1[$ and for a.e. (t,x) in Q. Thus in Definition 2 the integrations with respect to the Lebesgue measure on $]0,1[$ may be performed. So that $\chi = u$ a.e. on Q and u is namely an entropy solution to (1) in the sense of Definition 1. As a consequence,

Corollary 1. *If (1) has an entropy process solution then it has an entropy solution. In addition, if u_1 and u_2 are two entropy solutions corresponding to initial data $u_{0,1}$ and $u_{0,2}$ respectively, then for a.e. t in $]0,T[$:*

$$\int\limits_\Omega |u_1(t,x) - u_2(t,x)|\,dx \le e^{M_\psi t}\int\limits_\Omega |u_{0,1} - u_{0,2}|\,dx.$$

3 The Viscosity and Penalization Methods

Our aim is to obtain an existence result for (1) by introducing some diffusion on the whole domain and by relaxing the bilateral obstacle condition. This leads us to introduce first, for any function f (or $f(t,x,.)$), the Lipschitz bounded extension $f^\star$ of f outside $[a,b]$ defined through:

$$f^\star(u) = f(u) \text{ if } u \in [a,b], f^\star(u) = f(a) \text{ if } u \le a, f^\star(u) = f(b) \text{ if } u \ge b,$$

and similarly for $f^\star(t,x,.)$ (in this case, observe that $\partial_{x_i} f^\star = (\partial_{x_i} f)^\star$). Now for any positive parameter δ intended to tend to 0^+ - so that it will be supposed less or equal than a fixed value δ_0 - we define $\phi^\star_\delta = \phi^\star + \delta Id_{\mathbb{R}}$ a bilipschitzian function and we consider the nondegenerate penalized parabolic operator

$$\mathbb{P}_\delta(t,x,.) : u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i} \varphi^\star_i(t,x,u) + \psi^\star(t,x,u) - \Delta\phi^\star_\delta(u) + \beta_\delta(u),$$

with $\beta_\delta(z) = \dfrac{1}{\delta}(-(z-a)^- + (z-b)^+)$. We consider the resulting problem: find a measurable and bounded function u_δ on Q satisfying

$$\begin{cases} \mathbb{P}_\delta(t,x,u_\delta) = 0 \text{ on } Q, \\ u_\delta = 0 \text{ on }]0,T[\times\partial\Omega,\ u_\delta(0,.) = u_0^\delta \text{ on } \Omega, \end{cases} \tag{18}$$

where u_0^δ is a regularization of u_0 obtained by the mean of mollifiers, so that:

$$\forall \delta > 0,\ u_0^\delta \in \mathcal{D}(\Omega),\ a \le u_0^\delta \le b \text{ a.e. in } \Omega,\ \lim_{\delta\to 0^+} u_0^\delta = u_0 \text{ in } L^1(\Omega).$$

3.1 Some *a priori* estimates

We know (see for example [10]) that for a fixed positive δ, the problem (18) has a unique weak solution u_δ in $L^\infty(Q) \cap H^1(Q) \cap L^\infty(0,T;H^1_0(\Omega))$ with $\phi^\star_\delta(u_\delta)$ in $L^2(0,T;H^2(\Omega))$. In this section, we are mainly interested in the δ-uniform *a priori* estimations for the sequence $(u_\delta)_{\delta>0}$. Indeed,

Proposition 3. *There exist some constants C_1, C_2 and C_3 independent from δ such that*

$$\forall t \in [0,T],\ |u_\delta(t,.)| \le M(t) \text{ a.e. in } \Omega, \tag{19}$$

$$\frac{1}{\delta}\|(u_\delta - a)^-\|_{L^2(Q)} + \frac{1}{\delta}\|(u_\delta - b)^+\|_{L^2(Q)} \le C_1 \tag{20}$$

$$\|\widehat{\phi^\star_\delta}(u_\delta)\|_{L^2(0,T;H^1_0(\Omega))} + \|\partial_t u_\delta\|_{L^2(0,T;H^{-1}(\Omega))} \le C_2, \tag{21}$$

$$\|\sqrt{t}\,\partial_t \widehat{\phi^\star_\delta}(u_\delta)\|_{L^2(Q)} \le C_3, \tag{22}$$

where $M(t)$ is defined by (2) and $\widehat{\phi^\star}_\delta(x) = \displaystyle\int_0^x (\phi^{\star\prime}_\delta(\tau))^{1/2}\,d\tau$.

Remark 2. *We have not been able to establish any δ-uniform estimates for $(u_\delta)_{\delta>0}$ in $W^{1,1}(Q)$ or even in $BV(Q)$ as in [11] for the positiveness obstacle problem for $\mathbb{P}$ when $\boldsymbol{\varphi}$ is a $W^{2,+\infty}$-class vector-valued function and ψ a $W^{1,+\infty}$-class function. Furthermore even if we have enough compactness for $(\phi^\star_\delta(u_\delta))_{\delta>0}$, we cannot take advantage of it since $(\phi^\star)^{-1}$ may not exist. So to describe the behavior of $(u_\delta)_{\delta>0}$ when δ goes to 0^+ we can only refer to (19), which leads to consider the notion of an entropy process solution to (1).*

Proof of Proposition 3 - With the notations of Section 1.2, $M_{\psi^\star} = M_\psi$ and $M_{\partial_{x_i}\varphi_i^\star} = M_{\partial_{x_i}\varphi_i}$. So the standard maximum principle for u_δ ensures (19), the independence with respect to δ resulting from the monotonicity of the penalized operator β_δ. For (20), we treat each term separately: to estimate $\frac{1}{\delta}(u_\delta - b)^+$ we take the $L^2(Q)$-scalar product between (18) and $\frac{1}{\delta}(u_\delta - b)^+$. We observe that

$$\frac{1}{\delta}\int_Q \partial_t u_\delta (u_\delta - b)^+ dxdt = \frac{1}{2\delta}\int_Q \partial_t[(u_\delta - b)^+]^2 dxdt = \frac{1}{2\delta}\int_\Omega [(u_\delta - b)^+]^2(T,x)dx,$$

since $a \le u_0^\delta \le b$ a.e. on Ω. So the contribution of the evolution term is nonnegative and it is the same for the diffusive one, thanks to the Green formula. For the convective and reactive integrals, due to the definition of $\boldsymbol{\varphi}^\star$ and $\psi^\star$,

$$\begin{aligned} & \frac{1}{\delta}\int_Q (Div\boldsymbol{\varphi}^\star(t,x,u_\delta) + \psi(t,x,u_\delta))(u_\delta - b)^+ dxdt \\ = \ & \frac{1}{\delta}\int_Q (Div\boldsymbol{\varphi}(t,x,b) + \psi(t,x,b))(u_\delta - b)^+ dxdt \\ \le \ & \frac{1}{2}\|\boldsymbol{\varphi}(t,x,b) + \psi(t,x,b)\|^2_{L^2(Q)} + \frac{1}{2}\|\frac{1}{\delta}(u_\delta - b)^+\|^2_{L^2(Q)} dxdt. \end{aligned}$$

Eventually, we remark that

$$\frac{1}{\delta}\int_Q \beta_\delta(u_\delta)(u_\delta - b)^+ dxdt = \|\frac{1}{\delta}(u_\delta - b)^+\|^2_{L^2(Q)}.$$

So we conclude the existence of a constant C, independent from δ, such that The previous techniques with $-\frac{1}{\delta}(u_\delta - a)^-$ provide (20).

The energy estimate (21) results from the $L^2(Q)$-scalar product between (18) and u_δ. Since $\beta_\delta(u_\delta)u_\delta$ is nonnegative a.e. in Q, it warrants thanks to (19) a δ-uniform bound for $\nabla\widehat{\phi_\delta^\star}(u_\delta)$ in $L^2(Q)^p$. Then by coming back to the definition of the norm in $L^2(0,T;H^{-1}(\Omega))$ and referring to (20) we derive the estimation of $\partial_t u_\delta$. Let us focus on (22). We take the $L^2(Q)$-scalar product between (18) and $t\partial_t\phi_\delta^\star(u_\delta)$. This way,

$$\int_Q t\partial_t u_\delta \partial_t \phi_\delta^\star(u_\delta)dxdt = \frac{1}{2}\|\sqrt{t}\partial_t\widehat{\phi_\delta^\star}(u_\delta)\|^2_{L^2(Q)}.$$

For the diffusive integral, since $w = \phi_\delta^\star(u_\delta)$ belongs to the functional space $\mathcal{W}(0,T) \equiv \{v \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)); \partial_t v \in L^2(Q)\}$, we use the density of $\mathcal{D}([0,T];H_0^1(\Omega) \cap H^2(\Omega))$ into $\mathcal{W}(0,T)$ to carry out calculations with a sequence $(w_k)_k$ of mollified functions with respect to the time variable. It comes:

$$\begin{aligned} & -\int_Q t\Delta w_k \partial_t w_k dxdt = \int_Q t\nabla w_k.\partial_t \nabla w_k dxdt = \int_Q \frac{t}{2}\partial_t\left[\nabla w_k\right]^2 dxdt \\ & = -\frac{1}{2}\int_Q \left[\nabla w_k\right]^2 dxdt + \int_\Omega \frac{T}{2}\left[\nabla w_k\right]^2(T,x)dx. \end{aligned}$$

Then, we pass to the limit with k to obtain

$$\begin{aligned}\int_Q t\Delta\phi_\delta^\star(u_\delta)\partial_t\phi_\delta^\star(u_\delta)dxdt &= \frac{1}{2}\int_Q \left[\nabla\phi_\delta^\star(u_\delta)\right]^2 dxdt - \int_\Omega \frac{T}{2}\left[\nabla\phi_\delta^\star(u_\delta)\right]^2(T,x)dx \\ &\le \frac{1}{2}\int_Q \left[\nabla\phi_\delta^\star(u_\delta)\right]^2 dxdt \le C,\end{aligned}$$

where C is a constant independent from δ thanks to (19) and (21).

We develop the partial derivatives in the convective term to write:

$$\begin{aligned}&\int_Q tDiv_x\varphi^\star(t,x,u_\delta)\partial_t\phi_\delta^\star(u_\delta)dxdt \\ =&\int_Q\sum_{i=1}^p\left(\partial_u\varphi_i^\star(t,x,u_\delta)\sqrt{t}\partial_{x_i}\widehat{\phi_\delta^\star}(u_\delta)\sqrt{t}\partial_t\widehat{\phi_\delta^\star}(u_\delta)\right)dxdt \\ &+\int_Q\sum_{i=1}^p\left(\partial_{x_i}\varphi_i^\star(t,x,u_\delta)\sqrt{t}\sqrt{(\phi_\delta^\star)'(u_\delta)}\sqrt{t}\partial_t\widehat{\phi_\delta^\star}(u_\delta)\right)dxdt.\end{aligned}$$

The Young inequality with $p=2$ and (19) prove that:

$$\int_Q tDiv_x\varphi_i^\star(t,x,u_\delta)\partial_t\phi_\delta^\star(u_\delta)dxdt \le C\|\nabla\widehat{\phi_\delta^\star}(u_\delta)\|^2_{L^2(Q)^p}+\frac{1}{4}\|\sqrt{t}\partial_t\widehat{\phi_\delta^\star}(u_\delta)\|^2_{L^2(Q)}.$$

Again, with the Young inequality and (19):

$$\int_Q t\psi^\star(t,x,u_\delta)\partial_t\phi_\delta^\star(u_\delta)dxdt \le C+\frac{1}{4}\|\sqrt{t}\partial_t\widehat{\phi_\delta^\star}(u_\delta)\|^2_{L^2(Q)},$$

where C doest not depend on δ thanks to (19). It is the same for the penalized term since, due to (20), $\frac{1}{\delta}(u_\delta-a)^-$ and $\frac{1}{\delta}(u_\delta-b)^+$ are bounded in $L^2(Q)$ uniformly with respect to δ, that completes the proof of Proposition 3. ■

We give now a formulation of boundary conditions for the solutions to the nondegenerate relaxed problem that will be the starting point to derive the formulation of boundary conditions for the solution to (1) by taking the limit with respect to δ. In what follows $\mathbf{F}^\star$, $G^\star$ and $U_{k,\delta}^\star$ refer to $\mathbf{F}$, G and U_k with $\boldsymbol{\varphi}^\star$, $\psi^\star$ and $\phi_\delta^\star$ in the place of $\boldsymbol{\varphi}$, ψ and ϕ_δ respectively. This way, by arguing as in [11, 18], we prove that:

Proposition 4. *For any δ, the next compatibility condition holds on Σ:*

$$\int_\Sigma \mathbf{F}(0,k).\nu\xi d\mathcal{H}^p \le \left\langle U_{k,\delta}^\star,\xi\right\rangle_\partial+\left\langle U_{0,\delta}^\star,\xi\right\rangle_\partial, \tag{23}$$

for all nonnegative ξ in $L^\infty(Q)\cap H^1(Q)\cap\mathcal{C}(Q)$, $\xi(T,.)=\xi(0,.)=0$ and any real k. Moreover there exists a constant C_4, independent from δ such that

$$\|Div_{(t,x)}U_{k,\delta}^\star\|_{\mathcal{M}_b(Q)} \le C_4. \tag{24}$$

Proof.
First point - We observe that $\Delta|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|$ is a bounded Radon measure on Q, for any real k. As in [17], this assertion comes from the Kato's Inequality and uses the fact that $\phi_\delta^\star(u_\delta)$ is an element of $L^2(0,T;H^2(\Omega))$. As a consequence, $U_{k,\delta}^\star$ and $(0,\nabla|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|)$ belong to $\mathcal{DM}_2(Q)$ for any real k. Thus, by using the boundary condition for u_δ and the relation (4) with $\xi(T,.)=\xi(0,.)=0$, we may successively write:

$$
\begin{aligned}
&\left\langle U_{k,\delta}^\star,\xi\right\rangle_\partial\\
&=-\lim_{\varrho\to 0^+}\int_Q U_{k,\delta}^\star\xi.(0,\nabla\rho_\varrho)\,dxdt\\
&=\lim_{\varrho\to 0^+}\int_Q \xi\nabla|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|.\nabla\rho_\varrho dxdt-\lim_{\varrho\to 0^+}\int_Q \xi\mathbf{F}^\star(u_\delta,k).\nabla\rho_\varrho\,dxdt\\
&=\lim_{\varrho\to 0^+}\int_Q \xi(0,\nabla|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|).(0,\nabla\rho_\varrho)\,dxdt+\int_\Sigma \mathbf{F}(0,k).\nu\xi\,d\mathcal{H}^p\\
&=-\left\langle(0,\nabla|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|),\xi\right\rangle_\partial+\int_\Sigma \mathbf{F}(0,k).\nu\xi\,d\mathcal{H}^p.
\end{aligned}
$$

Let us conclude through the technical property:

Claim 1. *([18]) Let w be in $L^2(0,T;H_0^1(\Omega))$ such that $w\geq 0$ a.e. on Q and Δw belongs to $\mathcal{M}_b(Q)$. Then, for any nonnegative ξ of $L^\infty(Q)\cap H^1(Q)\cap\mathcal{C}(Q)$*

$$\langle(0,\nabla w),\xi\rangle_\partial\leq 0.$$

With $w=|\phi_\delta^\star(u_\delta)-\phi_\delta^\star(k)|+|\phi_\delta^\star(u_\delta)|-|\phi_\delta^\star(k)|$ provides (23) since, due to (4),

$$-\left\langle(0,\nabla|\phi_\delta^\star(u_\delta))|,\xi\right\rangle_\partial=\left\langle U_{0,\delta}^\star,\xi\right\rangle_\partial$$

We remark now that u_δ fulfills (8) with $\phi_\delta^\star$ in the place of ϕ. Indeed for any nonnegative function ξ of $\mathcal{D}(]-\infty,T[\times\Omega)$, we take the $L^2(Q)$-scalar product between (18) and the test-function $sgn_\lambda(u_\delta-k)\xi$. As soon as k belongs to $[a,b]$, $\beta_\delta(u_\delta)sgn_\lambda(u_\delta-k)\xi$ is nonnegative. After some integration by parts it comes (with $S_\lambda(v)=sgn_\lambda(v-k)$ and $I_\lambda(v)=\int_k^v S_\lambda(\tau)d\tau$):

$$
\begin{aligned}
&\int_Q I_\lambda(u_\delta)\partial_t\xi dxdt+\int_Q\left(\int_k^{u_\delta}\partial_u\boldsymbol{\varphi}^\star(t,x,\tau)S_\lambda(\tau)d\tau\right).\nabla\xi dxdt\\
+&\int_\Omega I_\lambda(u_0^\delta)\xi(0,.)dx-\int_Q\left(Div\boldsymbol{\varphi}^\star(t,x,k)+\psi^\star(t,x,u_\delta)\right)S_\lambda(u_\delta)\xi dxdt\\
-&\int_Q\left(\int_k^{u_\delta}[Div\boldsymbol{\varphi}^\star(t,x,\tau)-Div\boldsymbol{\varphi}^\star(t,x,k)]S_\lambda'(\tau)d\tau\right)\xi dxdt\\
\geq&\int_Q S_\lambda(u_\delta)\nabla\phi_\delta^\star(u_\delta).\nabla\xi dxdt.
\end{aligned}\tag{25}
$$

For a.e. x, $sgn'_\lambda(x) = \dfrac{1}{\lambda}\mathbb{I}_{]-\lambda,\lambda[}(x)$ and since the function $z \mapsto Div\boldsymbol{\varphi}^\star(t,x,z)$ is continuous uniformly with respect to (t,x) in Q, the third term in the left-hand side vanishes when λ goes to 0^+. Thus (8) for u_δ is obtained at the $\lambda-$limit thanks to the Lebesgue dominated convergence Theorem. We deduce that for any real k of $[a,b]$,

$$\theta_{k,\delta} \equiv -Div_{(t,x)}U^\star_{k,\delta} - G^\star(u_\delta,k) \tag{26}$$

is a nonnegative Radon measure on Q such that

$$\|\theta_{k,\delta}\|_{\mathcal{M}_b(Q)} = \int_Q d\theta_{k,\delta} = -\int_Q d[Div_{(t,x)}U^\star_{k,\delta}] - \int_Q G^\star(u_\delta,k)\,dxdt.$$

But by using (3) and (4):

$$\begin{aligned} &-\int_Q d[Div_{(t,x)}U^\star_{k,\delta}] \\ =\ &\langle(0,\nabla|\phi^\star_\delta(u_\delta)-\phi^\star_\delta(k)|),1\rangle_\partial - \int_\Sigma \mathbf{F}(0,k).\nu d\mathcal{H}^p + \int_\Omega |u_0^\delta - k|dx \\ -\ &\int_\Omega |u_\delta(T,x)-k|dx. \end{aligned}$$

Then as a consequence of (19) and since u_0^δ is uniformly bounded with respect to δ, we claim the existence of a constant C independent from δ such that:

$$-\int_Q d[Div_{(t,x)}U^\star_{k,\delta}] \le \langle(0,\nabla|\phi^\star_\delta(u_\delta)-\phi^\star_\delta(k)|),1\rangle_\partial + C.$$

On the one hand, the previous inequality with $k=0$ and Claim 1 with $w = |\phi^\star_\delta(u_\delta)|$ provide

$$-\int_Q d[Div_{(t,x)}U^\star_{0,\delta}] \le C,$$

that means that $(\theta_{0,\delta})_{\delta>0}$ given by (26) is a bounded sequence in $\mathcal{M}_b(Q)$ and $(Div_{(t,x)}U^\star_{0,\delta})_{\delta>0}$ too. On the other hand for any k in $[a,b]$, Claim 1 with $w = |\phi^\star_\delta(u_\delta)-\phi^\star_\delta(k)| + |\phi^\star_\delta(u_\delta)| - |\phi^\star_\delta(k)|$ gives:

$$-\int_Q d[Div_{(t,x)}U^\star_{k,\delta}] \le \langle(0,-\nabla|\phi^\star_\delta(u_\delta)|),1\rangle_\partial + C.$$

But by referring to (3) and (4) and using the homogeneous boundary condition for u_δ, we observe that:

$$\langle(0,-\nabla|\phi^\star_\delta(u_\delta)|),1\rangle_\partial = \left\langle U^\star_{0,\delta},1\right\rangle_\partial + \int_\Omega |u_0^\delta|dx - \int_\Omega |u_\delta(T,x)|dx.$$

Thus, due to the uniform boundedness for u_0^δ, there exists a constant C independent from δ such that:

$$|\langle(0,-\nabla|\phi^\star_\delta(u_\delta)|),1\rangle_\partial| \quad\le\quad |\left\langle U^\star_{0,\delta},1\right\rangle_\partial| + C \le \|Div_{(t,x)}U^\star_{0,\delta}\|_{\mathcal{M}_b(Q)} + C.$$

Eventually the estimate of $Div_{(t,x)}U^{\star}_{0,\delta}$ highlighted previously ensures that the sequence $(\theta_{k,\delta})_{\delta>0}$ is a uniformly bounded in $\mathcal{M}_b(Q)$ and so $(Div_{(t,x)}U^{\star}_{k,\delta})_{\delta>0}$ too. Relation (24) follows, which completes the proof of Proposition 4. ■

3.2 Convergence toward an entropy process solution

Due to Proposition 3, there exists a measurable function u on Q such that, up to a subsequence when δ goes to 0^+, $(u_\delta)_{\delta>0}$ converges toward u in $L^\infty(Q)$ weak $\star$ and $(\partial_t u_\delta)_{\delta>0}$ weakly converges toward $\partial_t u$ in $L^2(0,T;H^{-1}(\Omega))$. Besides there exists a function Φ such that, up to a subsequence, $(\phi^\star_\delta(u_\delta))_{\delta>0}$ goes to Φ in $L^2(0,T;H^1_0(\Omega))-$weak. But since $(t\phi^\star_\delta(u_\delta))_{\delta>0}$ is uniformly bounded in $H^1(Q)$ with respect to δ, we may be sure that $(\phi^\star_\delta(u_\delta))_{\delta>0}$ strongly converges (up to a subsequence) toward Φ in $L^q(Q)$ for any q in $[1,+\infty[$. In order to connect $\phi(u)$ and Φ, we first remind some properties of bounded sequences in L^∞:

Claim 2. *([7]) Let $(u_n)_{n>0}$ be a sequence of measurable functions on an open bounded subset $\mathcal{O}$ such that $(u_n)_n$ is uniformly bounded in the $L^\infty(Q)$-norm the by a constant M. Then, there exists a subsequence $(u_{\varphi(n)})_{n>0}$ and a measurable and bounded function π on $]0,1[\times\mathcal{O}$ such that for all continuous and bounded functions h on $\mathcal{O}\times]-M,M[$,*

$$\forall\xi\in L^1(\mathcal{O}),\ \lim_{n\to+\infty}\int_{\mathcal{O}} h(\omega,u_{\varphi(n)})\xi dx=\int_{]0,1[\times\mathcal{O}} h(\omega,\pi(\alpha,\omega))d\alpha\xi d\omega.$$

Such a result has found its first application in the approximation through the artificial viscosity method of the Cauchy problem in $\mathbb{R}^p$ for a scalar conservation law, as one can establish a uniform L^∞-control of approximate solutions. It has also been applied to the numerical analysis of transport equations since "Finite-Volume" schemes mainly give an L^∞-estimate uniformly with respect to the mesh length of the numerical solution. Here, we refer to this concept when the approximating sequence is $(u_\delta)_{\delta>0}$ and so there exists a function π in $L^\infty(]0,1[\times Q)$ such that thanks to (20), $a\leq\pi\leq b$ a.e. in $]0,1[\times Q$. Of course, we also have for a.e. (t,x) in Q,

$$u(t,x)=\int_0^1\pi(\alpha,t,x)d\alpha \text{ and } \Phi(t,x)=\int_0^1\phi(\pi(t,x,\alpha))d\alpha.$$

Furthermore we remark that

$$\int_{]0,1[\times Q}|\phi(\pi(\alpha,t,x))-\Phi(t,x)|d\alpha dxdt=\lim_{\delta\to0^+}\int_Q|\phi^\star_\delta(u_\delta)-\Phi(t,x)|dxdt=0.$$

Thus we even have, for a.e. (α,t,x) of $]0,1[\times Q$, $\phi(\pi(\alpha,t,x))=\Phi(t,x)$. But since ϕ is a nondecreasing function, for a.e. (t,x) in Q, $\phi^{-1}(\{\Phi(t,x)\})=[i_1,i_2]$, where $i_1\leq\pi(.,t,x)\leq i_2$ a.e. in $]0,1[$. This way, by integrating from 0 to 1 it comes that $\phi(i_1)=\phi(u(t,x))=\phi(i_2)=\Phi(t,x)$ that is namely (11). The previous developments guide us toward the next statement:

Theorem 2. - *The obstacle problem (1) admits an entropy process solution.*

Proof. We have highlighted a function u and a process π such that (10,11,6) hold. Moreover due to (24) there exists an element of $\mathcal{M}_b(Q)$ - identified in the sense of distributions on Q to $Div_{(t,x)}\Pi_k$ - such that up to a subsequence when δ goes to 0^+, $(Div_{(t,x)}U^\star_{k,\delta})_{\delta>0}$ converges toward $Div_{(t,x)}\Pi_k$ in $\mathcal{M}_b(Q)$ weak $\star$ and (12) follows. To establish (13) we start from (25). For the left hand-side of the δ-limit only refers to the claim 2 while for the right-hand side of (25) we use the smoothness of u_δ and the Green formula:

$$\begin{aligned}\int_Q S_\lambda(u_\delta)\nabla\phi^\star_\delta(u_\delta).\nabla\xi dxdt &= \int_Q \nabla\left(\int_k^{u_\delta} S_\lambda(\tau)(\phi^\star_\delta)'(\tau)d\tau\right).\nabla\xi dxdt\\ &= -\int_Q K_\delta(u_\delta)\Delta\xi dxdt,\end{aligned}$$

where K_δ is the the continuous function $x \to \int_k^x S_\lambda(\tau)(\phi^\star_\delta)'(\tau)d\tau$. We pass to the limit with δ by arguing that $|H_\delta(x) - H(x)| \le \delta|x-k|$ and referring to Claim 2. It comes

$$\begin{aligned}&\int_Q\left(\int_k^\pi S_\lambda(\tau)d\tau\right)\partial_t\xi dq + \int_Q\int_k^\pi \partial_u\boldsymbol{\varphi}(t,x,\tau)S_\lambda(\tau)d\tau.\nabla\xi dq\\ +\;&\int_\Omega\left(\int_k^{u_0} S_\lambda(\tau)d\tau\right)\xi(0,.)dx - \int_Q (Div\boldsymbol{\varphi}(t,x,\pi)+\psi(t,x,\pi))S_\lambda(\pi)d\tau\xi dq\\ -\;&\int_Q\int_k^\pi [Div\boldsymbol{\varphi}(t,x,\tau) - Div\boldsymbol{\varphi}(t,x,k)]S'_\lambda(\tau)d\tau\xi dq\\ \ge\;& -\int_Q\left(\int_k^\pi S_\lambda(\tau)\phi'(\tau)d\tau\right)\Delta\xi dq.\end{aligned}$$

The λ-limit relies on the Lebesgue dominated convergence Theorem and uses the continuity, uniformly with respect to (t,x) in Q, of $z \mapsto Div\boldsymbol{\varphi}(t,x,z)$ to deal with the third line in the left-hand side. Note that for the right-hand side:

$$\lim_{\lambda\to0^+}\int_{Q\times]0,1[}\left(\int_k^\pi S_\lambda(\tau)\phi'(\tau)d\tau\right)\Delta\xi d\alpha dxdt = \int_{Q\times]0,1[}|\phi(\pi)-\phi(k)|\Delta\xi d\alpha dxdt.$$

An integration by parts gives (13).

We establish (14) by passing to the δ-limit in (23) and using that:

Claim 3. - *([18]) - Let $(\mu_\varrho)_{\varrho>0}$ be a sequence of $\mathcal{M}_b(Q)$, $\mu_\varrho \ge 0$, converging toward μ in $\mathcal{M}_b(Q)$ weak $\star$. Then, for any nonnegative ξ of $L^\infty(Q)\cap\mathcal{C}(Q)$,*

$$\int_Q \xi d\mu \le \liminf_{\varrho\to0^+}\int_Q \xi d\mu_\varrho.$$

This claim with the nonnegative Radon measure $\theta_{k,\delta}$ defined by (26) provide that for any nonnegative ξ of $H^1(Q)\cap L^\infty(Q)\cap \mathcal{C}(Q)$, with $\xi(T,.)=\xi(0,.)=0$, up to a subsequence,

$$\int_Q \xi d\theta_k \leq \liminf_{\delta\to 0^+} \int_Q \xi d\theta_{k,\delta},$$

where in the sense of the bounded Radon measures on Q, $\theta_k = Div_{(t,x)}\Pi_k + G$ with $G = \lim_{\delta\to 0+} G^\star(u_\delta, k)$ in $L^\infty(Q)$ weak $\star$ (up to a subsequence). This way,

$$\limsup_{\delta\to 0^+} \int_Q \xi d[Div_{(t,x)} U^\star_{k,\delta}] \leq \int_Q \xi d[Div_{(t,x)}\Pi_k].$$

Eventually, because $(U^\star_{k,\delta})_{\delta>0}$ weakly converges in $L^2(Q)^{p+1}$ toward Π_k,

$$\limsup_{\delta\to 0^+} \langle U^\star_{k,\delta}, \xi\rangle_\partial = \int_Q \Pi_k.\overline{\nabla}\xi dxdt + \limsup_{\delta\to 0^+} \int_Q \xi d[Div_{(t,x)} U^\star_{k,\delta}],$$

and this way, for any real k of $[a,b]$,

$$\limsup_{\delta\to 0^+} \langle U^\star_{k,\delta}, \xi\rangle_\partial \leq \langle \Pi_k, \xi\rangle_\partial.$$

Relation (14) follows, that completes the proof of Theorem 2. ■

As pointed out in R.Diperna's paper [5] within the framework work of Young measure solutions, the strong convergence of approximate solutions occurs if and only if the process π may be identified to a function independent from the variable α, so that the associated Young measure reduces to a Dirac mass at almost all points of the physical domain. So, as a consequence of Theorem 2 and Corollary 1 we claim that:

Theorem 3. - *The bilateral obstacle problem (1) admits a unique entropy solution that is the limit of the whole sequence $(u_\delta)_{\delta>0}$ of solutions to (18)$_{\delta>0}$ when δ goes to 0^+ in $L^q(Q)$, $1\leq q<+\infty$.*

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