On the skeleton method and an application to a quantum scissor

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Abstract. In the spectral analysis of few one dimensional quantum particles interacting through delta potentials it is well known that one can recast the problem into the spectral analysis of an integral operator (the skeleton) living on the submanifold which supports the delta interactions. We shall present several tools which allow direct insight into the spectral structure of this skeleton. We shall illustrate the method on a model of a two dimensional quantum particle interacting with two infinitely long straight wires which cross one another at angle θ: the quantum scissor.

1. Introduction

Let us consider the following one dimensional model of N quantum particles interacting through delta potentials. In suitable units, the corresponding Hamiltonian reads as

$$-\sum_{i=1}^{N}\frac{\Delta_i}{2m_i}+\sum_{1\le i<j\le N} Z_iZ_j\delta(x_i-x_j),\qquad \text{acting in } L^2(\mathbb{R}^N), \tag{1.1}$$

where m_i and Z_i denote respectively the mass and the charge of the i'th particle. When the particles are identical (i.e. all the m_i's and Z_i's are equal), it is a well known fact that this model is indeed exactly solvable [**LL, McG**]; for a quick and fairly complete review, see [**vD**]; see also the introduction of [**AlGH-KH-E**]. However, it is not known whether the model is exactly solvable if the particles are distinct, and we strongly suspect that it is not. We have shown in [**CDR1**] that one can nevertheless expect partial exact results, at least. To explore this eventual solvability we have developed a mathematical tool, that we call the *skeleton method*, which requires to work with a system of integral operators.

The main issue of this article is to give a thorough exposition of this skeleton method, see sections 2 and 3. Finally we shall demonstrate the power of this tool by the spectral analysis of bound states in a model of leaky wires that we call a *quantum scissor*, see [**BEPS**] for this terminology.

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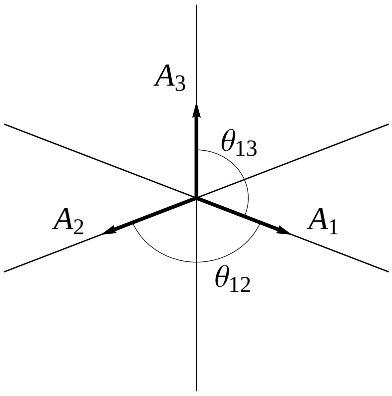

FIGURE 1. The three supports of the δ leaky wires.

1.1. Leaky wires. We shall consider the problem (1.1) only in the case $N = 3$, and

$$m_1 = m_2 > 0, \quad Z_1 = Z_2 < 0 \quad \text{and} \quad Z_3 > 0 \tag{1.2}$$

with the center of mass removed. Then the Hamiltonian expressed in the relative Jacobi coordinates acts in $L^2(\mathbb{R}^2)$. After rescaling (see [**CDR1**] for more details) we have

$$H := -\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y - \delta(A_1^\perp \cdot (x, y)) - \delta(A_2^\perp \cdot (x, y)) + \lambda\delta(A_3^\perp \cdot (x, y)) \tag{1.3}$$

where A_i, $i = 1, 2, 3$ are three normalized vectors as shown in Figure 1 where the angles $\theta_{i,j}$'s and $\lambda \geq 0$ depend on the original parameters m_i's and Z_i's. Here $A_i^\perp$ denotes A_i rotated clockwise by $\pi/2$ and the dot in $A_i^\perp \cdot (x, y)$ stands for the scalar product in $\mathbb{R}^2$. Thus H in (1.3) may be interpreted as the Hamiltonian of a quantum particle confined to a two dimensional plane, which interacts with three straight and infinitely long leaky wires directed by the vectors A_i. The "leaky wire" expression appears probably for the first time in [**EI**] . Another suitable expression for such a quantum model is "leaky graph" which appears in [**EN**].

1.2. Physical applications. Hamiltonians of the type (1.1) are not only convenient mathematical models, but they do also describe physical systems when some physical parameters are pushed to a limit. It has been recognized long time ago, see e.g. [**Spr**], that atoms in a strong homogenous magnetic field can be modelled by (1.1), see [**BaSoY, BD**] for a recent mathematical treatment of this problem. Quasiparticles on carbon nanotubes like excitons can be modelled by a system of charged quantum particles living at the surface of an infinitely long cylinder, see [**P**]. When the radius of the cylinder tends to zero, it has been shown in [**CDP, CDR2**] that a model of the type (1.1) is a good effective Hamiltonian for these quasiparticles. Not only does the quantum world provide us with such models. For example, in classical optics, photonic crystals with a high contrast in the dielectric constant between the (thin) crystal and air, can also be modelled by such a Hamiltonian, see [**KK**, §2] for more details.

2. The skeleton

Most of the content of this section could be obtained as a by-product of [**BEKS**]. However we think it is worth to make public this more "operator theoretical" version. For any normalized vector A in $\mathbb{R}^2$ we introduce $\tau_A : \mathcal{H}^1(\mathbb{R}^2) \to L^2(\mathbb{R})$ as the continuous restriction map

$$\mathcal{H}^1(\mathbb{R}^2) \ni \psi \mapsto \tau_A\psi \in L^2(\mathbb{R}), \quad \tau_A\psi(s) := \psi(sA). \tag{2.1}$$

Let g be a diagonal 3×3 matrix with the diagonal entries $\{g_i\}_{i=1}^3 := (-1,-1,\lambda)$. The Hamiltonian H in (1.3) is properly defined as the unique self-adjoint operator associated to the closed and bounded from below quadratic form:

$$\mathcal{H}^1(\mathbb{R}^2) \ni u \to \frac{1}{2}\|\nabla u\|^2 + \sum_{i=1}^{3} g_i\|\tau_{A_i}u\|^2 \in \mathbb{R}, \tag{2.2}$$

Let us set $\tau_i := \tau_{A_i}$ and $\tau := (\tau_1,\tau_2,\tau_3) : \mathcal{H}^1(\mathbb{R}^2) \to \oplus_{i=1}^3 L^2(\mathbb{R})$. Then H may be rewritten as

$$H = H_0 + \tau^\star g\tau, \quad H_0 := -\frac{1}{2}\Delta, \quad \operatorname{dom} H_0 := \mathcal{H}^2(\mathbb{R}^2). \tag{2.3}$$

Notice that the above sum defining H must be understood in the sense of quadratic forms, and as a matter of fact $\operatorname{dom} H \neq \operatorname{dom} H_0$. Thanks to the particular values of the coupling constant g_i's and by an application of the HVZ theorem one gets

Lemma 2.1. *For all $\lambda \geq -1$, the essential spectrum of H is $[-\frac{1}{2},\infty)$.*

We want to show that the eigenvalue problem $H\Psi = E\Psi$ for $E < -\frac{1}{2}$, i.e. below the essential spectrum can be reduced to a one-dimensional eigenvalue problem involving integral operators. Using Krein's formula with $R(z) := (H-z)^{-1}$, $R_0(z) := (H_0-z)^{-1}$ we get at once:

$$R(z) = R_0(z) - R_0(z)\tau^\star(g^{-1} + \tau R_0(z)\tau^\star)^{-1}\tau R_0(z), \quad z \in \rho(H_0)\cap\rho(H). \tag{2.4}$$

By classical Sobolev trace theorems the following operators are continuous:

$$\tau R_0(z) : L^2(\mathbb{R}^2) \to \bigoplus_{i=1}^{3}\mathcal{H}^{\frac{3}{2}}(\mathbb{R}), \quad \tau R_0(z)\tau^\star : \bigoplus_{i=1}^{3}\mathcal{H}^{s}(\mathbb{R}) \to \bigoplus_{i=1}^{3}\mathcal{H}^{s+1}(\mathbb{R})$$

for all $z \notin \operatorname{spect} H_0$ and all $s \in \mathbb{R}$. This allows to consider $g^{-1} + \tau R_0(z)\tau^\star$ as a bounded operator on $\mathcal{S} := \oplus_{i=1}^3 L^2(\mathbb{R})$ when $z \notin \mathbb{R}_+$.

Definition 2.2. We shall call $S(k) := g^{-1} + \tau R_0(-k^2)\tau^\star$ the *skeleton* of H at energy $-k^2$.

Theorem 2.3. *$E < -\frac{1}{2}$ is an eigenvalue of H iff $\ker(g^{-1} + \tau R_0(E)\tau^\star) \neq \{0\}$. If P is the orthogonal projector on this kernel, then the multiplicity of E is equal to the dimension of P. In addition, the operator $P\tau R_0^2(E)\tau^\star P$ is invertible on the range of P, and the eigenprojector of H associated to E is given by*

$$R_0(E)\tau^\star\left(P\tau R_0^2(E)\tau^\star P\right)^{-1}\tau R_0(E).$$

Proof. 1. We start by showing that $\forall E := k^2 < -\frac{1}{2}$ the essential spectrum of $S(k)$ obeys: for all $\lambda \geq 0$

$$0 \notin \operatorname{spect}_{\mathrm{ess}} S(k) = \operatorname{spect}_{\mathrm{ac}} S(k) = [-1, -1 + \frac{1}{\sqrt{2}k}] \cup [\lambda^{-1}, \lambda^{-1} + \frac{1}{\sqrt{2}k}].$$

Indeed if one sets

$$T_{\theta_{i,j}} := \tau_i R_0(z)\tau_j^\star$$

then

$$S(k) = \begin{pmatrix} -1+T_0 & 0 & 0 \\ 0 & -1+T_0 & 0 \\ 0 & 0 & \lambda^{-1}+T_0 \end{pmatrix} + \begin{pmatrix} 0 & T_{\theta_{1,2}} & T_{\theta_{2,3}} \\ T_{\theta_{1,2}} & 0 & T_{\theta_{2,3}} \\ T_{\theta_{2,3}} & T_{\theta_{2,3}} & 0 \end{pmatrix} \tag{2.5}$$

Since the diagonal of the first matrix consists of multiplication operators (in the Fourier representation see (3.2)) and the entries of the second matrix are all trace class operators (see Theorem 3.3), we are done. That $0 \notin \mathrm{spect}_{\mathrm{ess}} S(k)$ is now obvious.

2. Assume that $E < -\frac{1}{2}$ is an eigenvalue of H, but 0 is not an eigenvalue of $S(k)$. Then $S(k)$ has a bounded inverse (after an easy application of the Fredholm alternative). Since $R_0(E)$ and $\tau R_0(E)$ are bounded operators, it means that $R(z)$ is also bounded at $z = E$. This contradicts the fact that E is an eigenvalue of H. We conclude that $S(k)$ cannot be invertible (injective) if $-k^2$ coincides with an eigenvalue of H.

3. Now let us prove that all singularities of $S(k)^{-1}$ correspond to eigenvalues of H. One has the identity

$$(g^{-1} + \tau R_0(z)\tau^\star)^{-1} = g - g\tau R(z)\tau^\star g \tag{2.6}$$

valid for z where at least one and therefore two members of this identity exists. Now assume that for some $E < -1/2$, the operator $g^{-1} + \tau R_0(E)\tau^\star$ is not invertible (i.e. not injective in our case). Assume also that E is not in the (discrete) spectrum of H. Then (2.6) implies that in a small disc around E we have that $(g^{-1} + \tau R_0(z)\tau^\star)^{-1}$ is uniformly bounded, which means that $g^{-1} + \tau R_0(E)\tau^\star$ is invertible by Neumann series, contradiction.

4. Now let us investigate the dimension of the spectral subspace associated to an eigenvalue. Assume that 0 is an eigenvalue of $g^{-1} + \tau R_0(E)\tau^\star$ and let P be the finite dimensional associated eigenprojector. We have shown that E is also an eigenvalue of H, and denote by $P(E)$ its finite dimensional projection. We want to prove here that $\dim(P) = \dim(P(E))$.

Since $(g^{-1} + \tau R_0(E)\tau^\star)P = 0$ and using the resolvent identity:

$$(g^{-1} + \tau R_0(z)\tau^\star)P = (z - E)\tau R_0(z)R_0(E)\tau^\star P. \tag{2.7}$$

Using (2.6), and knowing that near E we have

$$(z - E)R(z) = -P(E) + \mathcal{O}((z - E)), \tag{2.8}$$

it follows that

$$\begin{aligned} P &= (z - E)(g^{-1} + \tau R_0(z)\tau^\star)^{-1}\tau R_0(z)R_0(E)\tau^\star P \\ &= g\tau P(E)\tau^\star g\tau R_0(z)R_0(E)\tau^\star P + \mathcal{O}((z - E)). \end{aligned} \tag{2.9}$$

Taking the limit $z \to E$ we obtain

$$P = g\tau P(E)\tau^\star g\tau R_0(E)^2\tau^\star P. \tag{2.10}$$

If $\mathrm{Ran}(P(E))$ is spanned by the eigenvectors $\{\psi_j\}_{j=1}^{\dim(P(E))}$, then (2.10) says that $\mathrm{Ran}(P)$ is spanned by $\{g\tau\psi_j\}_{j=1}^{\dim(P(E))}$, therefore

$$\dim(P) \le \dim(P(E)).$$

We now want to prove the reverse inequality. Denote by $Q := \mathrm{id} - P$, so that in matrix notation (use (2.7) and its adjoint)

$$g^{-1}+\tau R_0(z)\tau^\star = (z-E)\begin{pmatrix} P\tau R_0(z)R_0(E)\tau^\star P & P\tau R_0(z)R_0(E)\tau^\star Q \\ Q\tau R_0(z)R_0(E)\tau^\star P & (z-E)^{-1}Q(g^{-1}+\tau R_0(z)\tau^\star)Q \end{pmatrix}.$$

To invert this matrix we use the Feshbach method. One has (i) the operator $Q(g^{-1}+\tau R_0(E)\tau^\star)Q$ has a bounded inverse on the range of Q, thus $(z-E)^{-1}Q(g^{-1}+\tau R_0(z)\tau^\star)Q$ is bounded invertible for z in a neighbourhood of E, except eventually at E, and (ii) the following operator is bounded invertible at least in a neighbourhood of E

$$A(z) := P\tau R_0(z)R_0(E)\tau^\star P - P\tau R_0(z)R_0(E)\tau^\star Q(z-E)$$

$$\cdot(Q(g^{-1}+\tau R_0(z)\tau^\star)Q)^{-1}Q\tau R_0(z)R_0(E)\tau^\star P.$$

Notice that this operator is nothing but a finite dimensional matrix, acting in $\mathrm{Ran}(P)$. The Feshbach formula says that the above operator $A(z)$ is invertible if and only if $(g^{-1}+\tau R_0(z)\tau^\star)/(z-E)$ is invertible. Moreover, for z in a neighborhood of E this formula gives:

$$(z-E)P(g^{-1}+\tau R_0(z)\tau^\star)^{-1}P = A(z)^{-1}, \quad z \neq E.$$

Using again (2.6) and (2.8), we obtain

$$A(z)^{-1} = Pg\tau P(E)\tau^\star gP + \mathcal{O}(z-E), \quad z \neq E.$$

This inverse is bounded near E, and $A(z)$ is continuous at $z = E$, hence $A(E)$ is invertible and

$$A(E)^{-1} = \left\{P\tau R_0(E)^2\tau^\star P\right\}^{-1} = Pg\tau P(E)\tau^\star gP. \tag{2.11}$$

Summarizing, via the Feshbach formula, we obtain that

$$(z-E)(g^{-1}+\tau R_0(z)\tau^\star)^{-1} = A(z)^{-1} + \mathcal{O}((z-E)). \tag{2.12}$$

Multiply (2.6) with $(z-E)$, use (2.8), (2.12), and take the limit $z \to E$. This gives:

$$P(E) = R_0(E)\tau^\star A(E)^{-1}\tau R_0(E) = R_0(E)\tau^\star Pg\tau P(E)\tau^\star gP\tau R_0(E). \tag{2.13}$$

Now assume that $\{\phi_j\}_{j=1}^{\dim(P)}$ are eigenvectors spanning the range of P. Then (2.13) says that the range of $P(E)$ is spanned by $\{R_0(E)\tau^\star\phi_j\}_{j=1}^{\dim(P)}$, which implies

$$\dim(P(E)) \leq \dim(P)$$

and we are done. □

3. The T_θ operators

In this section we shall establish various properties of the $T_{\theta_{i,j}}$ operators.

3.1. Generalities. Let A and B be two normalized vectors of $\mathbb{R}^2$. We shall consider $\tau_A R_0(-k^2)\tau_B^\star$ where τ_A, τ_B are defined by (2.1). We can obtain explicit formulas for their integral kernels using the Fourier transform that we denote by a hat. We summarize the results in the following technical lemma:

LEMMA 3.1. *The operator $\tau_A R_0(-k^2)\tau_B^\star$ depends only on the angle θ between the vectors A and B. When $\det(A, B) \neq 0$, the Fourier transform of $\tau_A R_0(-k^2)\tau_B^\star$ is an integral operator with kernel*

$$\hat{T}_\theta(t, s; k) = \frac{1}{2\pi|\sin(\theta)|} \frac{1}{\frac{t^2 - 2\cos(\theta)ts + s^2}{2\sin^2(\theta)} + k^2}. \tag{3.1}$$

When $A = B$, the Fourier transform of $\tau_A R_0(-k^2)\tau_A^\star$ is the multiplication operator given by the function

$$\hat{T}_0(s; k) := \frac{1}{\sqrt{s^2 + 2k^2}}. \tag{3.2}$$

The proof of this lemma is elementary and left to the reader.

REMARK 3.2. (a) One has:

$$\forall \theta \in (-\pi, \pi), \quad \|T_\theta(k)\| \leq \frac{1}{\sqrt{2}k}.$$

This is clear for the case $\theta = 0$ since then $\hat{T}_0(k)$ is an explicit multiplication operator. For $\theta \neq 0$ we use

$$\|\tau_A R_0 \tau_B^\star\|^2 \leq \|\tau_A R_0^{\frac{1}{2}}\|^2 \|R_0^{\frac{1}{2}}\tau_B^\star\|^2 = \|\tau_A R_0 \tau_A^\star\| \|\tau_B R_0 \tau_B^\star\| = \|T_0\|^2.$$

One can also compute explicitly the Hilbert-Schmidt norm of T_θ:

$$\|T_\theta(k)\|_{\mathrm{HS}}^2 = \frac{1}{2\pi \sin(\theta) k^2}, \quad \theta \neq 0 \bmod \pi.$$

(b) If we perform the scaling $s \to ks$ then clearly $\hat{T}_\theta(k)$ becomes $k^{-1}\hat{T}_\theta(1)$. Since in the sequel we shall use this property and work only with $\hat{T}_\theta(1)$, we denote

$$\hat{T}_\theta := \hat{T}_\theta(1).$$

(c) Let $\Pi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the parity operator $\Pi\varphi(s) = \varphi(-s)$. Then $[\Pi, \hat{T}_\theta] = 0$ so that one can decompose

$$\hat{T}_\theta = \hat{T}_\theta^+ \oplus \hat{T}_\theta^-, \quad \text{with} \quad \hat{T}_\theta^\pm := \frac{1 \pm \Pi}{2}\hat{T}_\theta.$$

(d) By a simple inspection of (3.1) we have the reflection properties:

$$\forall \theta \in (0, \pi), \quad T_{\pi-\theta}^\pm = \pm T_\theta^\pm$$

3.2. Rank one operator decomposition of $\hat{T}_\theta$. Let us first consider $\hat{T}_{\frac{\pi}{2}}$; we have the formula

$$\hat{T}_{\frac{\pi}{2}}(p, q) = \frac{1}{\pi}\frac{1}{p^2 + q^2 + 2} = \frac{1}{\pi}\int_0^\infty e^{-2s} e^{-sp^2} e^{-sq^2} ds \tag{3.3}$$

which shows that $T_{\frac{\pi}{2}}$ is a "sum" of positive rank one operators so that $\hat{T}_{\frac{\pi}{2}} \geq 0$. Since $(p,q) \to T_{\frac{\pi}{2}}(p,q)$ is continuous and

$$\int_{\mathbb{R}} \hat{T}_{\frac{\pi}{2}}(p,p)dp = \frac{1}{2} < \infty$$

this shows in view of [**Si**, th 2.12], that $T_{\frac{\pi}{2}}$ is trace class and that its trace and therefore its trace norm are $1/2$. We are indebted to R. Brummelhuis who showed us the trick (3.3). The above derivation can be generalized to any angle $0 < \theta < \pi$ as follows.

THEOREM 3.3. *For all $\theta \in (0,\pi)$ one has in the trace norm ($\|\cdot\|_1$) topology*

$$\begin{aligned}
(3.4)\,\hat{T}_\theta &= 2^{-\frac{1}{2}} \frac{\sin(\theta)}{\pi} \sum_{n\in\mathbb{N}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \cos^n(\theta) \int_0^\infty ds\, s^{-\frac{1}{2}} e^{-2\sin^2(\theta)s} P_{n,s} \\
\hat{T}_\theta^+ &= 2^{-\frac{1}{2}} \frac{\sin(\theta)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(2n+\frac{1}{2})}{\Gamma(2n+1)} \cos^{2n}(\theta) \int_0^\infty ds\, s^{-\frac{1}{2}} e^{-2\sin^2(\theta)s} P_{2n,s} \\
\hat{T}_\theta^- &= 2^{-\frac{1}{2}} \frac{\sin(\theta)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(2n+\frac{3}{2})}{\Gamma(2n+2)} \cos^{2n+1}(\theta) \int_0^\infty ds s^{-\frac{1}{2}} e^{-2\sin^2(\theta)s} P_{2n+1,s}
\end{aligned}$$

where $P_{n,s}$ denotes the rank one orthogonal projector on the vector $g_{n,s}$ defined by

$$g_{n,s}(p) := \sqrt{\frac{(2s)^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}} p^n e^{-p^2 s}. \tag{3.5}$$

Accordingly one has

$$\forall\theta \in (0, \frac{\pi}{2}], \quad \hat{T}_\theta^\pm \geq 0 \quad \text{and} \quad \forall\theta \in [\frac{\pi}{2}, \pi), \quad \pm\hat{T}_\theta^\pm \geq 0. \tag{3.6}$$

It follows that T_θ and $T_\theta^\pm$ are trace class and

$$\begin{aligned}
\operatorname{tr} T_\theta^+ &= \|T_\theta^+\|_1 = \frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{2\sqrt{2}\sin(\theta)} \\
\operatorname{tr} T_\theta^- &= \frac{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sqrt{2}\sin(\theta)}, \quad \|T_\theta^-\|_1 = \frac{\left|\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right|}{2\sqrt{2}\sin(\theta)} \\
\operatorname{tr} T_\theta &= \frac{1}{2\sqrt{2}\sin\left(\frac{\theta}{2}\right)}, \quad \|T_\theta\|_1 = \frac{\max\left\{\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\right\}}{\sqrt{2}\sin(\theta)}
\end{aligned}$$

PROOF. To find the rank one operator decomposition of $\hat{T}_\theta$ we simply expand its kernel as follows. Let $A := p^2 + q^2 + 2\sin^2\theta$ and $B := 2pq\cos(\theta)$, one can easily check that $A > 0$ and $|B/A| < 1$ for all $0 < \theta < \pi$. Thus one has

$$\begin{aligned}
&\frac{1}{2\pi\sin(\theta)} \frac{1}{\frac{p^2 - 2\cos(\theta)pq + q^2}{2\sin^2(\theta)} + 1} = \frac{\sin\theta}{\pi} \frac{1}{A-B} = \frac{\sin(\theta)}{\pi} \sum_{n=0}^{\infty} A^{-1} \left(\frac{B}{A}\right)^n \\
&= \frac{\sin(\theta)}{\pi} \sum_{n=0}^{\infty} B^n \int_0^\infty ds \frac{s^n e^{-sA}}{n!} \\
&= \frac{\sin(\theta)}{\pi} \sum_{n=0}^{\infty} 2^n \frac{\cos^n(\theta)}{n!} \int_0^\infty ds\, e^{-2s\sin^2(\theta)} s^n e^{-s(p^2+q^2)} (pq)^n.
\end{aligned}$$

To arrive at (3.4) one needs to normalize in $L^2(\mathbb{R})$ the vector $p \to p^n e^{-sp^2}$ which gives the vector $g_{n,s}$ in (3.5). Since $\|P_{n,s}\| = \|P_{n,s}\|_1$ the convergence in the trace norm topology of the r.h.s. of (3.4) is true since the terms in the following sum are all positive and one has explicitly:

$$2^{-\frac{1}{2}} \frac{\sin(\theta)}{\pi} \sum_{n\in\mathbb{N}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} |\cos^n(\theta)| \int_0^\infty ds\, s^{-\frac{1}{2}} e^{-2\sin^2(\theta)s} = \frac{1}{2\sqrt{1-|\cos(\theta)|}}.$$

This shows that $\hat{T}_\theta$ is trace class and that (3.4) is valid in the trace norm topology. Since $g_{n,s}$ has the parity of n, one gets $\hat{T}_\theta^\pm$ by selecting the even and odd values of n in (3.4) resp.. The rest is now obvious up to some tedious explicit computations of sums. □

We shall draw some other useful properties from the above theorem.

Corollary 3.4. *(i) $\theta \to T_\theta$ is a selfadjoint analytic family as a map from $\mathcal{D} := \{\theta \in \mathbb{C},\, |\cos(\theta)| < 1\}$ with values in the ideal of trace class operators.*

(ii) If one labels the eigenvalues of T_θ^+ by descending order: $E_1^+(\theta) \geq E_2^+(\theta) \geq \ldots \geq E_n^+(\theta) \geq \ldots$ then each function $(0,\pi) \ni \theta \to E_n^+(\theta)$ is continuous and decreasing on $(0, \frac{\pi}{2}]$ and increasing on $[\frac{\pi}{2}, \pi)$.

If one labels the eigenvalues of T_θ^- by descending order on $(0, \frac{\pi}{2})$ and ascending order on $(\frac{\pi}{2}, \pi)$, then each function $(0,\pi) \ni \theta \to E_n^-(\theta)$ is continuous and decreasing on $(0,\pi)$.

Proof. (i) is a direct consequence of the convergence of the r.h.s. of (3.4) on $\mathcal{D}$. To prove (ii) we shall consider another s.a. family of operators which is the image of $\{T_\theta\}_\theta$ under the scaling $p \to \sin(\theta)p$, $0 < \theta < \pi$:

$$^\sharp T_\theta(p,q) = \frac{1}{\pi} \frac{1}{p^2 + q^2 - 2pq\cos(\theta) + 2}. \tag{3.7}$$

Then proceeding as in the previous theorem we get

$$\begin{aligned} ^\sharp T_\theta^+ &= \frac{1}{\pi} \sum_{n\in\mathbb{N}} (2\cos\theta)^{2n} B_{2n} \\ ^\sharp T_\theta^- &= \frac{1}{\pi} \sum_{n\in\mathbb{N}} (2\cos\theta)^{2n+1} B_{2n+1} \end{aligned}$$

where B_n denotes the positive operator with kernel

$$B_n(p,q) := \frac{(pq)^n}{(p^2+q^2+2)^{n+1}} = \int_0^\infty \frac{s^n}{n!} e^{-2s} (pq)^n e^{-sp^2} e^{-sq^2}.$$

If we label the eigenvalues of T_θ^+, i.e. the eigenvalues of $^\sharp T_\theta^+$ in descending order they are all continuous in θ and in view of the elementary dependence of $^\sharp T_\theta^+$ on θ, they are decreasing on $(0, \pi/2]$ and increasing on $[\pi/2, \pi)$. We skip the analogous reasoning for T_θ^-. □

3.3. $T_\theta^\pm$ are ergodic. The reader can find the definition of an ergodic operator in [**RS4**, §XIII.12].

Proposition 3.5. *(i) For all $\theta \in (0,\pi)$, T_θ^+ is ergodic and $\sup T_\theta^+$ is a simple eigenvalue of T_θ^+.*

(ii) For all $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, $\mathrm{sign}(\pi/2-\theta)T_\theta^-$ *is ergodic and* $\sup \mathrm{sign}(\pi/2-\theta)T_\theta^-$ *is a simple eigenvalue of* T_θ^-.

PROOF. We have previously seen that $T_\theta^+ \geq 0$. Also T_θ^+ is self adjoint and compact, thus $\|T_\theta^+\| = \sup T_\theta^+$ is an eigenvalue of T_θ^+. Clearly since $T_\theta^+(p,q) > 0$ for all p and q, one has $(T_\theta^+ f, g) > 0$ whenever f and g are positive. Thus T_θ^+ is ergodic. By applying [**RS4**, Th. XIII.43] we get that $\sup T_\theta^+$ is a simple eigenvalue of T_θ^+. The proof for T_θ^- is analogous. □

3.4. T_θ **is injective.** This question was brought to us by T. Dorlas.

LEMMA 3.6. *For all* $0 < \theta < \pi$, *one has* $\ker T_\theta = \{0\}$.

PROOF. We find it more convenient to work with ${}^\sharp T_\theta$, see (3.7), which is unitarily equivalent to T_θ. We recall that Π denotes the parity operator, see Remark 3.2(c). Using the formula derived in the proof of Corollary 3.4, we get with $\varphi \in \mathrm{Ran}\,\Pi^+$, that ${}^\sharp T_\theta^+ \varphi = 0$ implies

$$({}^\sharp T_\theta^+ \varphi, \varphi) = 0 \quad \Rightarrow \quad \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(2\cos(\theta))^{2n}}{(2n)!} \int_0^\infty s^{2n} e^{-2s} (C_{2n,s}\varphi, \varphi) ds = 0$$

where $C_{n,s}$ with kernel $C_{n,s}(p,q) := (pq)^n e^{-sp^2} e^{-sq^2}$ is a positive rank one operator. If $\theta \neq \frac{\pi}{2}$ it follows that: $\forall n \in \mathbb{N}$, $\forall s > 0$, $(C_{2n,s}\varphi, \varphi) = 0$ since $(2\cos(\theta))^{2n}$ and $s^{2n}e^{-s}$ are strictly positive. But

$$(C_{2n,s}\varphi, \varphi) = 0 \iff \int_{\mathbb{R}} |p^{2n} e^{-sp^2} \varphi(p)|^2 dp = 0$$

which shows that $\varphi \perp p^{2n} e^{-\frac{p^2}{2}}$ for all $n \in \mathbb{N}$ by choosing $s = 1/2$. Clearly $\{p^{2n} e^{-\frac{p^2}{2}}, n \in \mathbb{N}\}$ is total in $\mathrm{Ran}\,\Pi^+$ since they generate the even Hermite functions. Thus $\varphi = 0$. A similar argument shows that if $\varphi \in \mathrm{Ran}\,\Pi^-$ and ${}^\sharp T_\theta^- \varphi = 0$ then $\varphi = 0$; notice that it is understood here that $\theta \neq \frac{\pi}{2}$ since otherwise ${}^\sharp T_\theta^- = 0$.

Finally we consider the case $\theta = \pi/2$ and $\varphi \in \mathrm{Ran}\,\Pi^+$. Here we get as above

$$\forall s > 0, \quad (\varphi, e^{-sp^2}) = 0$$

and by differentiating indefinitely this identity with respect to s in $s = \frac{1}{2}$ we find

$$\varphi \perp p^{2n} e^{-\frac{p^2}{2}}, \quad \forall n \in \mathbb{N},$$

which implies as above that $\varphi = 0$. □

3.5. Some properties of $(2^{-\frac{1}{2}} - \hat{T}_0)^{-1/2} \hat{T}_\theta^- (2^{-\frac{1}{2}} - \hat{T}_0)^{-1/2}$**.** From the rank one operator decomposition of T_θ^-, see Theorem 3.3, one gets

$$\begin{aligned}
\widetilde{T}_\theta^- &:= (2^{-\frac{1}{2}} - \hat{T}_0)^{-1/2} \hat{T}_\theta^- (2^{-\frac{1}{2}} - \hat{T}_0)^{-1/2} \\
&= 2^{-\frac{1}{2}} \frac{\sin(\theta)}{\pi} \sum_{n=0}^\infty \frac{\Gamma(2n+\frac{3}{2})}{\Gamma(2n+2)} \cos^{2n+1}(\theta) \int_0^\infty ds s^{-\frac{1}{2}} e^{-2\sin^2(\theta)s} \widetilde{P}_{2n+1,s} \qquad (3.8)
\end{aligned}$$

where

$$\widetilde{P}_{2n+1,s} := (\cdot, \widetilde{g}_{2n+1,s})\widetilde{g}_{2n+1,s} \quad \text{with} \quad \widetilde{g}_{2n+1,s} := (2^{-\frac{1}{2}} - \hat{T}_0)^{-1/2} g_{2n+1,s}.$$

It turns out that $\widetilde{g}_{2n+1,s}$ belongs to $L^2(\mathbb{R})$ and the r.h.s. (3.8) is convergent in the trace norm. More precisely

$$\begin{aligned}\|\widetilde{P}_{2n+1,s}\|_1 &= \|\widetilde{g}_{2n+1,s}\|^2 = \frac{\sqrt{2}\left(4n+8s+4\sqrt{s}\,U\left(-\frac{1}{2},-2n,4s\right)+1\right)}{4n+1} \\ &\leq \frac{8n\sqrt{s}\,\Gamma(2n)+(4n+16s+1)\Gamma\left(2n+\frac{1}{2}\right)}{\sqrt{2}\,\Gamma\left(2n+\frac{3}{2}\right)}\end{aligned}$$

where U denotes the confluent hypergeometric function, see [**AS**, 13.1.3]. The last estimate is obtained by integration of the r.h.s. of the bound

$$|\tilde{g}(2n+1,s)(p)|^2 \leq \frac{(2s)^{2n+\frac{3}{2}}}{\Gamma(2n+\frac{3}{2})}p^{4n}e^{-2sp^2}(\sqrt{2}p^2+2|p|+4\sqrt{2}).$$

which is more convenient in view of the summations over s and n. Then the integration over s gives

$$\begin{aligned}&\sin(\theta)\int_0^\infty \frac{e^{-2s\sin^2(\theta)}}{\sqrt{s}}\frac{\Gamma(2n+\frac{3}{2})}{\Gamma(2n+2)}\|\tilde{g}(2n+1,s)\|^2 ds \leq \\ &\frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{2\Gamma(2n+2)}+\frac{2n\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+2)}+\frac{2\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\sin^2(\theta)\,\Gamma(2n+2)}+\frac{\sqrt{2}\Gamma(2n+1)}{\sin(\theta)\,\Gamma(2n+2)} \\ =:\ & a_1+a_2+a_3+a_4.\end{aligned}$$

In the summation over n of these four terms, only the second one causes problems to arrive at a convenient final formula: (the sums are computed with $0<\theta<\pi/2$)

$$\begin{aligned}\sum_{n=0}^\infty \frac{a_1\cos^{2n+1}(\theta)}{\sqrt{2}\pi} &= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right) \\ \sum_{n=0}^\infty \frac{a_2\cos^{2n+1}(\theta)}{\sqrt{2}\pi} &= \frac{\cos^3(\theta)\,{}_2F_1\left(\frac{5}{4},\frac{7}{4};\frac{5}{2};\cos^2(\theta)\right)}{4\sqrt{2}} \\ \sum_{n=0}^\infty \frac{a_3\cos^{2n+1}(\theta)}{\sqrt{2}\pi} &= \frac{2}{\sin^2(\theta)}\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right) \\ \sum_{n=0}^\infty \frac{a_4\cos^{2n+1}(\theta)}{\sqrt{2}\pi} &= \frac{\tanh^{-1}(\cos(\theta))}{\pi(\sin(\theta)}\end{aligned}$$

One replaces a_2 by the the following bound valid for all $n\in\mathbb{N}$:

$$\frac{2n\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+2)} \leq \frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+1)} = a_2'$$

which gives

$$\sum_{n=0}^\infty \frac{a_2'\cos^{2n+1}(\theta)}{\sqrt{2}\pi} = \frac{1}{2}\frac{\cos(\theta)}{\sin(\theta)}\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right).$$

Summing up gives

Lemma 3.7. *For all $\theta\in(0,\pi)$, $\widetilde{T}_\theta^-$ is trace class and for all $0<\theta<\pi/2$ one has:*

$$\|\tilde{T}_\theta^-\|_1 \leq \frac{\left(4\sin(\theta)\tanh^{-1}(\cos(\theta))+\pi\left(9\cos\left(\frac{\theta}{2}\right)-\cos\left(\frac{5\theta}{2}\right)-9\sin\left(\frac{\theta}{2}\right)+\sin\left(\frac{5\theta}{2}\right)\right)\right)}{4\pi\sin(\theta)}.$$

In particular:

$$\|\widetilde{T}^-_{\frac{2\pi}{3}}\|_1 = \|\widetilde{T}^-_{\frac{\pi}{3}}\|_1 \leq -\frac{4}{3} + \frac{5}{\sqrt{3}} + \frac{\log(3)}{\sqrt{3}\pi} \sim 1.75532 \tag{3.9}$$

REMARK 3.8. (a) If we do not replace a_2 by a_2' we get a better bound:

$$\|\tilde{T}^-_{\frac{2\pi}{3}}\|_1 \leq 1.38929.$$

Moreover a direct numerical evaluation on the Hilbert Schmidt norm gives

$$\|\tilde{T}^-_{\frac{2\pi}{3}}\|_{\mathrm{HS}} \sim 1.01327.$$

(b) We shall see below that -1 is an eigenvalue of $\widetilde{T}_{\frac{2\pi}{3}}$, see (3.13). Since the trace norm of $\widetilde{T}^-_{\frac{2\pi}{3}}$ is less than 2, see (3.9), it follows that this eigenvalue is simple and is the lowest eigenvalue of $\widetilde{T}_{\frac{2\pi}{3}}$. Thus we may conclude that

$$\inf \widetilde{T}_{\frac{2\pi}{3}} = -1. \tag{3.10}$$

(c) The statements in Corollary 3.4(ii) and Proposition 3.5 for eigenvalues of T^-_θ works as well for those of $\widetilde{T}^-_\theta$. In particular the lowest one is simple and monotonically decreasing and pass by -1 for $\theta = 2\pi/3$ in view of (3.10).

3.6. Exact eigenvalues and eigenvectors. We collect here some exact results about these $\hat{T}_\theta$ operators. They can be checked by direct inspection.

$$(\hat{T}_0 + \hat{T}_{\frac{\pi}{2}})\varphi = \varphi, \quad \text{with} \quad \varphi(p) = \sqrt{\frac{2}{\pi}}\,\frac{1}{p^2+1}, \tag{3.11}$$

$$(\hat{T}_0 + 2\hat{T}_{\frac{2\pi}{3}})\varphi = \sqrt{2}\varphi, \quad \text{with} \quad \varphi(p) = \frac{6^{3/4}}{\sqrt{\pi}\,(2p^2+3)} \tag{3.12}$$

and

$$\widetilde{T}_{\frac{2\pi}{3}}\varphi = -\varphi \quad \text{with} \quad \varphi(p) = \frac{\sqrt{T_0(0) - T_0(p)}}{p(2p^2+3)}, \quad \|\varphi(p)\|^2 = \frac{1}{18}\left(6 - \sqrt{3}\pi\right). \tag{3.13}$$

(3.11) and (3.12) are simply obtained by translating known exact eigenfunctions in the skeleton frame work; the first one comes from the exactly solvable quantum scissor, see (4.1), with angle $\pi/2$. The second one comes from the Mc Guire bound state eigenfunction of its three particle system, see [**McG**, §IV.D]. The last one seems to be new. Since $\sqrt{\hat{T}_0(0) - \hat{T}_0(p)} \sim 2^{-\frac{5}{4}}|p|$ as $p \to 0$, this function has a cusp at 0.

4. A quantum scissor

We consider the Hamiltonian (2.3) in the particular case $\lambda = 0$:

$$H_\theta := -\frac{\Delta}{2} - \delta(A_1^\perp\cdot) - \delta(A_2^\perp\cdot) = -\frac{\Delta}{2} - \tau_1\tau_1^\star - \tau_2\tau_2^\star \tag{4.1}$$

which described a two dimensional particle in a scissor-shaped waveguide, a name borrowed from [**BEPS**]. We assume without loss of generality that $\theta := \theta_{1,2}$ belongs to

$$\theta \in [\frac{\pi}{2}, \pi); \tag{4.2}$$

where θ denotes the angle made by the two vectors A_1 and A_2 which generate the supports of the delta interactions, see Figure 1. We note that the case $\theta = \pi$ is exactly solvable, and that the angles $\theta \in (0, \pi/2]$ are covered by the cases (4.2) since H_θ and $H_{\pi-\theta}$ are unitarily equivalent.

Thanks to Lemma 2.1, the essential spectrum of H_θ is $[-1/2, \infty)$. The skeleton associated to H_θ in the Fourier representation is

$$\begin{pmatrix} -1+\hat{T}_0(k) & \hat{T}_\theta(k) \\ \hat{T}_\theta(k) & -1+\hat{T}_0(k) \end{pmatrix} \sim k^{-1} \begin{pmatrix} -k+\hat{T}_0 & \hat{T}_\theta \\ \hat{T}_\theta & -k+\hat{T}_0 \end{pmatrix}$$

where the unitarily equivalent second expression is obtained through the scaling $s \to ks$, see Remark 3.2(b). It acts on $\mathcal{S} := L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Thus, according to Theorem 2.3, $-k^2 < -1/2$ is an eigenvalue of H_θ iff k is an eigenvalue of

$$\mathbf{T}_\theta := \begin{pmatrix} \hat{T}_0 & \hat{T}_\theta \\ \hat{T}_\theta & \hat{T}_0 \end{pmatrix}.$$

Notice that in view of Corollary 3.4, $\{\mathbf{T}_\theta\}_{\theta\in\mathcal{D}}$ is bounded selfadjoint family of analytic operators and since T_θ is trace class (see Theorem 3.3) one has

$$\operatorname{spect}_{\mathrm{ess}} \mathbf{T}_\theta = \operatorname{spect}_{\mathrm{ac}} \mathbf{T}_\theta = \operatorname{spect} T_0 = [0, 2^{-\frac{1}{2}}].$$

4.1. Reduction by symmetries. We use (x, y) for the coordinates in $\mathbb{R}^2$ and we recall that Π stands for the parity operator on $L^2(\mathbb{R})$, see Remark 3.2(c). Let $\Pi_y := \Pi \otimes 1$, $\Pi_x := 1 \otimes \Pi$ acting in $L^2(\mathbb{R}^2)$ denote respectively the reflection with respect to the y and x axis. H_θ fulfills

$$[H_\theta, \Pi_x] = [H_\theta, \Pi_y] = 0.$$

This allows to reduce H_θ as

$$H_\theta = \bigoplus_{\alpha,\beta\in\{\pm1\}} H_\theta^{\alpha,\beta}, \quad \text{where} \quad H_\theta^{\alpha,\beta} := \Pi_x^\alpha \Pi_y^\beta H_\theta$$

and $\Pi_x^\alpha := \frac{1}{2}(\mathrm{id} + \alpha\Pi_x))$, $\Pi_y^\beta := \frac{1}{2}(\mathrm{id} + \beta\Pi_y))$ denote the eigenprojectors of Π_x and Π_y resp.. We also stress that $H_\theta^{\alpha,\beta}$ is unitarily equivalent to the operator acting in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ with same symbol as H_θ but with additional Dirichlet boundary conditions on $x = 0$ if $\alpha = -1$ or on $y = 0$ if $\beta = -1$ and Neumann boundary condition in the opposite case, see Table 1.

Similarly $\mathbf{T}_\theta$ enjoys the following symmetries

$$[\mathbf{T}_\theta, \mathbf{\Pi}] = [\mathbf{T}_\theta, \mathcal{E}] = [\Pi \oplus \Pi, \mathcal{E}] = 0.$$

where $\mathbf{\Pi} := \Pi \oplus \Pi : \mathcal{S} \to \mathcal{S}$ is the parity operator and $\mathcal{E} : \mathcal{S} \to \mathcal{S}$ is the exchange of components operator: $\mathcal{E}(\phi_1 \oplus \phi_2) := \phi_2 \oplus \phi_1$. Thus we may consider separately

$$\mathbf{\Pi}^\alpha \mathcal{E}^\beta \mathbf{T}_\theta, \quad \alpha = \pm1, \quad \beta = \pm1$$

where $\mathbf{\Pi}^\alpha$ and $\mathcal{E}^\beta$ denote the spectral projectors of $\mathbf{\Pi}$ and $\mathcal{E}$ resp.:

$$\mathbf{\Pi}^\alpha := \frac{1}{2}(\mathrm{id} + \alpha\mathbf{\Pi}) \quad \mathcal{E}^\alpha := \frac{1}{2}(\mathrm{id} + \alpha\mathcal{E}).$$

One has the following elementary result the proof of which is left to the reader:

Lemma 4.1. *For all α, β in $\{\pm1\}$, $\mathbf{\Pi}^\alpha \mathcal{E}^\beta \mathbf{T}_\theta$ is unitarily equivalent to*

$$\mathbf{T}_\theta^{\alpha,\beta} := \hat{T}_0 + \beta\hat{T}_\theta^\alpha \quad \textit{acting in } L^2(\mathbb{R}).$$

It is then of practical importance to relate $H_\theta^{\alpha,\beta}$ and $\mathbf{T}_\theta^{\alpha,\beta}$.

LEMMA 4.2. *For all* α, β *in* $\{\pm 1\}$, $-k^2 < -\frac{1}{2}$ *is a discrete eigenvalue of* $H_\theta^{\alpha,\beta}$ *iff* $k > 2^{-\frac{1}{2}}$ *is a discrete eigenvalue of* $\mathbf{T}_\theta^{\alpha\beta,\beta}$.

PROOF. Due to the chosen orientation of the two normalized vectors, see Figure 1, we have the relations between mappings from $\mathcal{H}^1(\mathbb{R}^2)$ to $L^2(\mathbb{R})$

$$\tau\Pi_y = \mathcal{E}\tau, \quad \tau\Pi_x = \mathbf{\Pi}\mathcal{E}\tau$$

so that

$$\begin{aligned} 4\tau\Pi_x^\alpha\Pi_y^\beta &= \tau(\mathrm{id} + \alpha\Pi_x)(\mathrm{id} + \beta\Pi_y) = (\mathrm{id} + \alpha\mathbf{\Pi}\mathcal{E})\tau(\mathrm{id} + \beta\Pi_y) \\ &= (\mathrm{id} + \alpha\mathbf{\Pi}\mathcal{E})(\mathrm{id} + \beta\mathcal{E})\tau = (\mathrm{id} + \alpha\beta\mathbf{\Pi})(\mathrm{id} + \beta\mathcal{E}) = 4\mathbf{\Pi}^{\alpha\beta}\mathcal{E}^\beta\tau. \end{aligned}$$

Let $R(z)^{\alpha,\beta} := \Pi_x^\alpha\Pi_y^\beta(H_\theta - z)^{-1}$ and similarly for $R_0 := (H_0 - z)^{-1}$ then using (2.4) and Definition 2.2 we get

$$\begin{aligned} R(-k^2)^{\alpha,\beta} &= R_0(-k^2)^{\alpha,\beta} - R_0(-k^2)^{\alpha,\beta}\tau^\star S(k)^{-1}\tau R_0(-k^2)^{\alpha,\beta} \\ &= R_0(-k^2)^{\alpha,\beta} - R_0(-k^2)^{\alpha,\beta}\tau^\star \left(S(k)^{\alpha\beta,\beta}\right)^{-1}\tau R_0(-k^2)^{\alpha,\beta} \end{aligned}$$

with $S(k)^{\alpha,\beta} := \mathbf{\Pi}^\alpha\mathcal{E}^\beta S(k)$. The statement of the lemma now follows easily. □

TABLE 1.

$\mathbf{T}_\theta^{\alpha,\beta}$	subspace in $L^2(\mathbb{R}^2)$	B.C. on $\mathbb{R}_+ \times \mathbb{R}_+$ for H_θ
$T_0 + T_\theta^+$	$\mathrm{Ran}\,\Pi_x^+\Pi_y^+$	$\mathrm{N}_{x=0}$ $\mathrm{N}_{y=0}$
$T_0 - T_\theta^+$	$\mathrm{Ran}\,\Pi_x^-\Pi_y^+$	$\mathrm{D}_{x=0}$ $\mathrm{N}_{y=0}$
$T_0 + T_\theta^-$	$\mathrm{Ran}\,\Pi_x^-\Pi_y^-$	$\mathrm{D}_{x=0}$ $\mathrm{D}_{y=0}$
$T_0 - T_\theta^-$	$\mathrm{Ran}\,\Pi_x^+\Pi_y^-$	$\mathrm{N}_{x=0}$ $\mathrm{D}_{y=0}$

4.2. Existence and monotonicity of bound states. First we can quickly fix two cases.

PROPOSITION 4.3. $H_\theta^{-,-}$ *and* $H_\theta^{-,+}$ *have no discrete spectrum for all* $\theta \in [\pi/2, \pi)$.

PROOF. These two cases correspond to $\mathbf{T}_\theta^{\alpha,\beta}$ with $\alpha\beta = -1$, see Lemma 4.2. In view of (3.6) one has $\beta\hat{T}_\theta^\alpha \leq 0$ so that $\mathbf{T}_\theta^{\alpha,\beta} \leq \hat{T}_0 \leq 2^{-\frac{1}{2}}$, and therefore $\mathbf{T}_\theta^{\alpha,\beta}$ cannot have an eigenvalue $k > 2^{-\frac{1}{2}}$. □

PROPOSITION 4.4. *(i)* $H_\theta^{+,-}$ *has no discrete spectrum for* $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. *(ii) It has at least one isolated eigenvalue for* $\theta \in (\frac{2\pi}{3}, \pi)$. *The number of isolated eigenvalues of* $H_\theta^{+,-}$ *is bounded above by* $\|\widetilde{T}_\theta^-\|_1$ *and therefore by the bound given in Lemma 3.7.*

PROOF. Here we have to consider $\mathbf{T}_\theta^{-,-}$. (i) We have using the operator $\widetilde{T}_\theta^-$, see (3.8), that for all $\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$

$$2^{-\frac{1}{2}} - \mathbf{T}_\theta^{-,-} = 2^{-\frac{1}{2}} - \hat{T}_0 + \hat{T}_\theta^- = (2^{-\frac{1}{2}} - \hat{T}_0)^{\frac{1}{2}}(1 + \widetilde{T}_\theta^-)(2^{-\frac{1}{2}} - \hat{T}_0)^{\frac{1}{2}} \geq 0$$

since $\widetilde{T}_\theta^- \geq -1$, see Remark 3.8(b) and (c). Thus $\mathbf{T}_\theta^{-,-} \leq 2^{-\frac{1}{2}}$ which implies that it cannot have an eigenvalue larger than $2^{-\frac{1}{2}}$. To prove (ii) it is sufficient to show that there exists $k > 2^{-\frac{1}{2}}$ so that $\inf \operatorname{spect}(k - \mathbf{T}_\theta^{-,-}) < 0$ for all $\theta > 2\pi/3$. We first establish that the lowest eigenvalue $\widetilde{E}_1^-(\theta)$ of $\widetilde{T}_\theta^-$ is strictly smaller than -1 for all $2\pi/3 < \theta < \pi$. Let φ denote the normalized eigenvector of $\widetilde{T}_{\frac{2\pi}{3}}^-$ associated to the eigenvalue -1, see (3.13). By the Feynman Hellman theorem one has

$$\begin{aligned}
\frac{d}{d\theta}\widetilde{E}_1^-\left(\frac{2\pi}{3}\right) &= \left(\frac{d}{d\theta}\widetilde{T}_\theta^-\varphi, \varphi\right)_{|_{\theta=\frac{2\pi}{3}}} \\
&= \frac{1}{\frac{1}{18}\left(6-\sqrt{3}\pi\right)}\int_{\mathbb{R}} \left(\partial_\theta T_\theta^-(p,q)\right)_{|_{\theta=\frac{2\pi}{3}}} \frac{1}{p\,(2p^2+3)}\frac{1}{q\,(2q^2+3)}dpdq \\
&= -\frac{\pi}{2\left(6-\sqrt{3}\pi\right)} \sim -2.81201.
\end{aligned}$$

This shows that in a right neighbourhood of $2\pi/3$ we have that $\widetilde{E}_1^-(\theta) < -1$. Thanks to Remark 3.8(c), this remains true for all $\theta > 2\pi/3$. Let $A_k := (k - \hat{T}_0)^{\frac{1}{2}}$ then

$$k - \mathbf{T}_\theta^{-,-} = A_k(1 + \widetilde{T}_\theta^-(k))A_k, \quad \text{with} \quad \widetilde{T}_\theta^-(k) := \frac{A_{2^{-\frac{1}{2}}}}{A_k}\widetilde{T}_\theta^-\frac{A_{2^{-\frac{1}{2}}}}{A_k}.$$

Clearly $\widetilde{T}_\theta^-(k)$ converges in norm to $\widetilde{T}_\theta^-$ as $k \to 2^{-\frac{1}{2}}$. Since for all $\theta > 2\pi/3$ there exists $a > 0$ so that $\inf \operatorname{spect}(\widetilde{T}_\theta^-) \leq -1 - a$, one can find k close enough to $2^{-\frac{1}{2}}$ so that $\inf \operatorname{spect}(\widetilde{T}_\theta^-(k)) \leq -1 - a/2$, i.e. there exists $\varphi \in L^2(\mathbb{R})$ such that $((1 + \widetilde{T}_\theta^-(k))\varphi, \varphi) \leq -a/2\|\varphi\|^2$. Let $\psi := A_k^{-1}\varphi$; we finally get that

$$(k - \mathbf{T}_\theta^{-,-}\psi, \psi) \leq -\frac{a}{2}\|A_k\psi\|^2 \leq -\frac{(k - 2^{-\frac{1}{2}})a}{2}\|\psi\|^2$$

which shows that $\inf \operatorname{spect}(k - \mathbf{T}_\theta^{-,-}) < 0$. The bound on the number of isolated eigenvalues is standard, see e.g. the proof of Theorem 3.3 in [**BEKS**]. □

PROPOSITION 4.5. *$H_\theta^{+,+}$ has at least one isolated eigenvalue for all $\theta \in (0, \pi)$ and this eigenvalue is unique in* $[\pi/3, 2\pi/3]$.

PROOF. Here we have to consider $\mathbf{T}_\theta^{+,+}$. Take a smooth even function $j \in C_0^\infty(\mathbb{R}, \mathbb{R}_+^\star)$, such that $\int_{\mathbb{R}} j(x)dx = 1$. For every $\varepsilon > 0$ define $\psi_\varepsilon(x) = (1/\varepsilon)j(x/\varepsilon)$ and

$$\phi_\varepsilon := \frac{\sqrt{\varepsilon}}{||j||}\psi_\varepsilon.$$

We have $||\phi_\varepsilon|| = 1$, while ψ_ε is an approximation of Dirac's distribution. An elementary calculus gives

$$(\hat{T}_0\psi_\varepsilon, \psi_\varepsilon) = \frac{1}{\sqrt{2}} + \mathcal{O}(\varepsilon^2), \quad (\hat{T}_\theta^+\psi_\varepsilon, \psi_\varepsilon) = \varepsilon\hat{T}_\theta^+(0,0) + \mathcal{O}(\varepsilon^2)$$

and since $\hat{T}_\theta^+(0,0) = \hat{T}_\theta(0,0) = \pi|\sin(\theta)|/2 > 0$ it follows that (by taking $\varepsilon > 0$ small enough)

$$\sup_{\psi \in L^2(\mathbb{R}_+)} (\mathbf{T}_\theta^{+,+}\psi, \psi) > \frac{1}{\sqrt{2}} = \sup \operatorname{spect}_{\text{ess}} \mathbf{T}_\theta^{+,+}.$$

Therefore $\mathbf{T}_\theta^{+,+}$ has at least one eigenvalue larger than $\frac{1}{\sqrt{2}}$. Thanks to the monotonicity of ${}^\sharp\mathbf{T}_\theta^{+,+}$ (see below) and its symmetry w.r.t $\pi/2$, it sufficient to show the uniqueness, that ${}^\sharp\mathbf{T}^{+,+}_{2\pi/3}$ or equivalently $H^{+,+}_{2\pi/3}$ has at most one isolated eigenvalue. On the other hand $H_{2\pi/3}$ is bounded below by H of (1.3), with $\lambda = -1$, which is shown to possess a unique bound state by Mc Guire in [**McG**, §IV. D]. □

To prove the monotonicity, we remark that $\mathbf{T}^{\alpha,\alpha}_\theta$ is unitarily equivalent with ${}^\sharp\mathbf{T}^{\alpha,\alpha}_\theta := (2 + p^2\sin^2(\theta))^{-\frac{1}{2}} + \alpha{}^\sharp T^\alpha_\theta$, see (3.7). Since $\pm{}^\sharp T^\pm_\theta \geq 0$ for all $\theta \in [\pi/2, \pi)$, and thanks to the explicit and simple dependence on θ of ${}^\sharp\mathbf{T}^{\alpha,\alpha}_\theta$, we infer that both families, $\alpha = \pm 1$, are monotonously increasing as functions of θ, and therefore their eigenvalues k have the same property. It follows that $-k^2$, the eigenvalues of H_θ are decreasing. Using the bounds $\|T_\theta\| \leq 2^{-\frac{1}{2}}$, see Remark 3.2(a), it follows that $\|\mathbf{T}^{\alpha,\alpha}_\theta\| \leq \sqrt{2}$ and consequently $H_\theta \geq -2$. We gather in a final theorem what we have established so far. Clearly this problem has two mirror symmetries. Following [**BEPS**] we shall call *axis of the scissor* the one which lies in the smaller angle, i.e. the x axis with our notations and *second axis of the scissor* the other one. We warn the reader that our θ is not the one of [**BEPS**].

THEOREM 4.6. *(i) The Hamiltonian H_θ has no isolated bound state which is odd with respect to the axis of the scissor.*

(ii) It has no isolated bound state which is odd with respect to the second axis of the scissor when $\pi/2 \leq \theta \leq 2\pi/3$. It has at least one isolated bound state which is odd with respect to this second axis when $2\pi/3 < \theta < \pi$. The number of such bound states is bounded above by $\|\widetilde{T}^-_\theta\|_1$, and therefore by the bound given in Lemma 3.7.

(iii) It has at least one bound state for all $0 < \theta < \pi$ which is even with respect to both axis of the scissor and unique for θ in $[\pi/3, 2\pi/3]$.

(iv) All bound states of H_θ are bounded below by -2 and monotonously decreasing with respect to θ on $[\pi/2, \pi)$.

5. Concluding remarks and open problems

We are far from having found the answers to all the questions about the quantum scissor of § 4. Let us review these questions; most of them are already in [**BEPS**, §III]:

(1) Every bound state is even w.r.t. the scissor axis: *done, see Th 4.6(i)*.
(2) With respect to the second axis the bound states can have both parities (*done, see Th 4.6(ii) and (iii)*) which are alternating if the bound states are arranged according to their energies: *not done*.
(3) As the angle θ gets larger new bound states emerge from the continuum. Fnd the correponding critical values θ_c of θ: *very partially done*. Thanks to Th 4.6(ii,iii) we know that the first critical value of θ is $2\pi/3$. Compute the asymptotic of the number of bound state as $\theta \to \pi$: *not done*.
(4) All the bound state energies are monotonically decreasing functions of θ: *done*.
(5) Do we have a bound state or a resonance at the threshold, when a bound state emerges from the continuum? *Not done*.
(6) The other way around, when θ decreases, do the bound states become resonances? Can we follow them? *Not done*.
(7) Can one expand the new bound state emerging from the continuum as a function of $\theta - \theta_c$? *Not done*.

Concerning the integral operator T_θ, see § 3,

(1) can we enlarge the list of exact spectral results, see § 3.6?
(2) Are all eigenvalues of $T_\theta^\pm$ simple?
(3) Is it true that $\theta \to T_\theta^\pm$ are monotonous?

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