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Abstract

In this paper, we study the Stokes system in the half-space $\mathbb{R}^N_+$, with $N \geq 2$. We give existence and uniqueness results in weighted Sobolev spaces. After the central case of the generalized solutions, we are interested in strong solutions and symmetricaly in very weak solutions by means of a duality argument.

Key words: Stokes problem, Half-space, Weighted Sobolev spaces
1991 MSC: 35J50, 35J55, 35Q30, 76D07, 76N10

1 Introduction

The purpose of this paper is the resolution of the Stokes system

$$
(S^+)\quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbb{R}^N_+, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma = \mathbb{R}^{N-1}, \end{cases}
$$

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with data and solutions which live in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. We will naturally base on the previously established results on the harmonic and biharmonic operators (see [5], [6], [7], [8]). We will also concentrate on the basic weights because they are the most usual and they avoid the question of the kernel for this operator and symmetricaly the compatibility condition for the data. In a forthcoming work, we will complete these results for the other types of weights in this class of spaces.

Among the first works on the Stokes problem in the half-space, we can cite Cattabriga. In [11], he appeals to the potential theory to explicitly get the velocity and pressure fields. For the homogeneous problem ($\boldsymbol{f} = \mathbf{0}$ and $h = 0$), for instance, he shows that if $\boldsymbol{g} \in \boldsymbol{L}^p(\Gamma)$ and the semi-norm $|\boldsymbol{g}|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} < \infty$, then $\nabla \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^N_+)$ and $\pi \in L^p(\mathbb{R}^N_+)$.

Similar results are given by Farwig-Sohr (see [12]) and Galdi (see [14]), who also have chosen the setting of homogeneous Sobolev spaces. On the other hand, Maz'ya-Plamenevskiĭ-Stupyalis (see [18]), work within the suitable setting of weighted Sobolev spaces and consider different sorts of boundary conditions. However, their results are limited to the dimension 3 and to the Hilbertian framework in which they give generalized and strong solutions. This is also the case of Boulmezaoud (see [10]), who only gives strong solutions. Otherwise, always in dimension 3, by Fourier analysis techniques, Tanaka considers the case of very regular data, corresponding to velocities which belong to $\boldsymbol{W}_2^{m+3,\,2}(\mathbb{R}^3_+)$, with $m \geq 0$ (see [19]).

Let us also quote, for the evolution Stokes or Navier-Stokes problems, Fujigaki-Miyakawa (see [13]), who are interested in the behaviour in $t \to +\infty$; Bochers-Miyakawa (see [9]) and Kozono (see [17]), for the L^N-Decay property; Ukai (see [20]), for the L^p-L^q estimates and Giga (see [15]), for the estimates in Hardy spaces.

This paper is organized as follows. Section 2 is devoted to the notations, functional setting and recalls about the Stokes system in the whole space. In Section 3, we give some results on homogeneous problems with singular boundary conditions and we complete them by Theorem 3.5 with a detailed proof, which is a model for analogous results. In Section 4, we start our study of the Stokes system in the half-space by the central case of generalized solutions which is the pivot of this work. In Section 5, we consider the strong solutions and give regularity results according to the data. In Section 6, we find very weak solutions to the homogeneous problem with singular boundary conditions. The main results of this paper are Theorem 4.2 for the generalized solutions, Theorems 5.2 and 5.6 for the strong solutions, Theorems 6.7 and 6.9 for the very weak solutions.

2 Notations, functional framework and known results

2.1 Notations

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N;\ x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N;\ x_N = 0\} \equiv \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ denote the Euclidean norm of x, we will use two basic weights

$$\varrho = (1 + |x|^2)^{1/2} \quad \text{and} \quad \lg \varrho = \ln(2 + |x|^2).$$

We denote by ∂_i the partial derivative $\dfrac{\partial}{\partial x_i}$, similarly $\partial_i^2 = \partial_i \circ \partial_i = \dfrac{\partial^2}{\partial x_i^2}$, $\partial_{ij}^2 = \partial_i \circ \partial_j = \dfrac{\partial^2}{\partial x_i \partial x_j}$, ... More generally, if $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \cdots \partial_N^{\lambda_N} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_N^{\lambda_N}} \quad , \text{where } |\lambda| = \lambda_1 + \cdots + \lambda_N.$$

In the sequel, for any integer q, we will use the following polynomial spaces:
— $\mathcal{P}_q$ is the space of polynomials of degree smaller than or equal to q;
— $\mathcal{P}_q^{\Delta}$ is the subspace of harmonic polynomials of $\mathcal{P}_q$;
— $\mathcal{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathcal{P}_q$;
— $\mathcal{A}_q^{\Delta}$ is the subspace of polynomials of $\mathcal{P}_q^{\Delta}$, odd with respect to x_N, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
— $\mathcal{N}_q^{\Delta}$ is the subspace of polynomials of $\mathcal{P}_q^{\Delta}$, even with respect to x_N, or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$;
with the convention that these spaces are reduced to $\{0\}$ if $q < 0$.
For any real number s, we denote by $[s]$ the integer part of s.
Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = \{f \in B';\ \forall v \in X,\ \langle f, v \rangle = 0\} = (B/X)'.$$

Lastly, if $k \in \mathbb{Z}$, we will constantly use the notation $\{1, \ldots, k\}$ for the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

For any nonnegative integer m, real numbers $p > 1$, α and β, we define the following space:

$$W^{m,p}_{\alpha,\beta}(\Omega) = \Big\{ u \in \mathcal{D}'(\Omega);\ 0 \le |\lambda| \le k,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta-1}\,\partial^\lambda u \in L^p(\Omega); \\ k+1 \le |\lambda| \le m,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta}\,\partial^\lambda u \in L^p(\Omega) \Big\}, \tag{2.1}$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p}+\alpha \notin \{1,\dots,m\}, \\ m - \dfrac{N}{p} - \alpha & \text{if } \frac{N}{p}+\alpha \in \{1,\dots,m\}. \end{cases}$$

Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{0\le|\lambda|\le k} \|\varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta-1}\,\partial^\lambda u\|^p_{L^p(\Omega)} \\ + \sum_{k+1\le|\lambda|\le m} \|\varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda|=m} \|\varrho^{\alpha}\,(\lg\varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

The weights in the definition (2.1) are chosen so that the corresponding space satisfies two fundamental properties. On the one hand, $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ (see [5], Theorem 1.1): if

$$\frac{N}{p} + \alpha \notin \{1,\dots,m\} \quad \text{or} \quad (\beta-1)p \neq -1, \tag{2.2}$$

then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \quad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}} \le C\,|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}, \tag{2.3}$$

with $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. Now, we define the space

$$\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

which will be characterized in Lemma 2.2 as the subspace of functions with null traces in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. From that, we can introduce the space $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$ as the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. In addition, under the assumption (2.2), $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$. We will now recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+) \ \text{ if } \frac{N}{p} + \alpha \notin \{1,\dots,m\}.$$

When $\frac{N}{p} + \alpha = j \in \{1,\dots,m\}$, then we have:

$$W^{m,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}.$$

Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $\frac{N}{p} + \alpha - \gamma \notin \{1,\dots,m\}$ and $m \in \mathbb{N}$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \varrho^{\gamma} u \in W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \partial^{\lambda} u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - \left(\dfrac{N}{p} + \alpha\right) - 1, & \text{if } \begin{cases} \frac{N}{p} + \alpha \in \{1,\dots,m\} \ \text{ and } \ (\beta-1)p \geq -1, \\ \text{or} \\ \frac{N}{p} + \alpha \in \{j \in \mathbb{Z};\ j \leq 0\} \ \text{ and } \ \beta p \geq -1, \end{cases} \\ \left[m - \left(\dfrac{N}{p} + \alpha\right)\right], & \text{otherwise.} \end{cases} \tag{2.4}$$

Remark 2.1. In the case $\beta = 0$, we simply denote the space $W^{m,p}_{\alpha,0}(\Omega)$ by $W^{m,p}_{\alpha}(\Omega)$. In [16], Hanouzet introduced a class of weighted Sobolev spaces without logarithmic factors, with the same notation. We recall his definition under the notation $H^{m,p}_{\alpha}(\Omega)$:

$$H^{m,p}_{\alpha}(\Omega) = \Big\{u \in \mathcal{D}'(\Omega);\ 0 \leq |\lambda| \leq m,\ \varrho^{\alpha-m+|\lambda|}\, \partial^{\lambda} u \in L^p(\Omega)\Big\}.$$

It is clear that if $\frac{N}{p} + \alpha \notin \{1,\dots,m\}$, we have $W^{m,p}_{\alpha}(\Omega) = H^{m,p}_{\alpha}(\Omega)$. The fundamental difference between these two families of spaces is that the assumption (2.2) and thus the Poincaré-type inequality (2.3), hold for any value of $(N,\, p,\, \alpha)$ in $W^{m,p}_{\alpha}(\Omega)$, but not in $H^{m,p}_{\alpha}(\Omega)$ if $\frac{N}{p} + \alpha \in \{1,\dots,m\}$. □

In order to define the traces of functions of $W_{\alpha}^{m,\,p}(\mathbb{R}_+^N)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in \,]0,1[$, we introduce the space:

$$W_0^{\sigma,\,p}(\mathbb{R}^N) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in L^p(\mathbb{R}^N) \ \text{ and } \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\}, \tag{2.5}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W_0^{\sigma,\,p}(\mathbb{R}^N)} = \left(\left\| \frac{u}{w^{\sigma}} \right\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$W_{\alpha}^{\sigma,\,p}(\mathbb{R}^N) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N),\ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W_{\alpha}^{s,\,p}(\mathbb{R}^N) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ 0 \leq |\lambda| \leq k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg \varrho)^{-1}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N);\ k+1 \leq |\lambda| \leq [s]-1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N);\ |\lambda| = [s],\ \partial^{\lambda} u \in W_{\alpha}^{\sigma,\,p}(\mathbb{R}^N) \Big\},$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W_{\alpha}^{s,\,p}(\mathbb{R}^N)} = \Big(\sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha-s+|\lambda|}\,(\lg \varrho)^{-1}\,\partial^{\lambda} u\|^p_{L^p(\mathbb{R}^N)} + \sum_{k+1 \leq |\lambda| \leq [s]-1} \|\varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda} u\|^p_{L^p(\mathbb{R}^N)} \Big)^{1/p} + \sum_{|\lambda| = [s]} \|\partial^{\lambda} u\|_{W_{\alpha}^{\sigma,\,p}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space:

$$W_{\alpha,\,\beta}^{s,\,p}(\mathbb{R}^N) = \Big\{ v \in \mathcal{D}'(\mathbb{R}^N);\ (\lg \varrho)^{\beta}\, v \in W_{\alpha}^{s,\,p}(\mathbb{R}^N) \Big\}.$$

We can prove some properties of the weighted Sobolev spaces $W_{\alpha,\,\beta}^{s,\,p}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin$

$\{\sigma, \ldots, \sigma + [s] - 1\}$:

$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) \hookrightarrow W^{s-1,\,p}_{\alpha-1,\,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{\sigma,\,p}_{\alpha-[s],\,\beta}(\mathbb{R}^N),$$
$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) \hookrightarrow W^{[s],\,p}_{\alpha+[s]-s,\,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-s,\,\beta}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \ldots, \sigma + [s] - 1\}$, then we have:

$$W^{s,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{s-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta-1},$$
$$W^{s,p}_{\alpha,\beta} \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \hookrightarrow W^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta-1}.$$

If u is a function on $\mathbb{R}^N_+$, we denote its trace of order j on the hyperplane Γ by:

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial^j_N u(x', 0).$$

Let's recall the following trace lemma due to Hanouzet (see [16]) and extended by Amrouche-Nečasová (see [5]) to this class of weighted Sobolev spaces:

Lemma 2.2. *For any integer $m \geq 1$ and real number α, the mapping*

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) : \ \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by γ,

$$\gamma : \ W^{m,\,p}_{\alpha}(\mathbb{R}^N_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,\,p}_{\alpha}(\mathbb{R}^{N-1}).$$

Moreover γ is surjective and $\mathrm{Ker}\gamma = \mathring{W}^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$.

2.4 The Stokes system in the whole space

On the Stokes problem in $\mathbb{R}^N$

$$(S) : \qquad -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = h \ \text{ in } \mathbb{R}^N,$$

let's recall the fundamental result on which we are based in the sequel. First, for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k = \Big\{(\boldsymbol{\lambda},\, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}^{\Delta}_{k-1};\ \operatorname{div} \boldsymbol{\lambda} = 0,\ -\Delta\boldsymbol{\lambda} + \nabla\mu = \mathbf{0}\Big\}.$$

Theorem 2.3 (Alliot-Amrouche [1])**.** *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \ldots, \ell\} \quad \textit{and} \quad N/p \notin \{1, \ldots, -\ell\}.$$

For any $(\boldsymbol{f},\, g) \in \left(\boldsymbol{W}_{\ell}^{-1,\,p}(\mathbb{R}^N) \times W_{\ell}^{0,\,p}(\mathbb{R}^N)\right) \perp \mathcal{S}_{[1+\ell-N/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{\ell}^{1,\,p}(\mathbb{R}^N) \times W_{\ell}^{0,\,p}(\mathbb{R}^N)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-N/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}_{[1-\ell-N/p]}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_{\ell}^{1,\,p}(\mathbb{R}^N)} + \|\pi+\mu\|_{W_{\ell}^{0,\,p}(\mathbb{R}^N)}\right) \\ \leq\; C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_{\ell}^{-1,\,p}(\mathbb{R}^N)} + \|g\|_{W_{\ell}^{0,\,p}(\mathbb{R}^N)}\right).$$

We also have the following result for more regular data:

Theorem 2.4 (Alliot-Amrouche [1])**.** *Let* $\ell \in \mathbb{Z}$ *and* $m \geq 1$ *be two integers and assume that*

$$N/p' \notin \{1, \ldots, \ell+1\} \quad \textit{and} \quad N/p \notin \{1, \ldots, -\ell-m\}.$$

For any $(\boldsymbol{f},\, g) \in \left(\boldsymbol{W}_{m+\ell}^{m-1,\,p}(\mathbb{R}^N) \times W_{m+\ell}^{m,\,p}(\mathbb{R}^N)\right) \perp \mathcal{S}_{[1+\ell-N/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{m+\ell}^{m+1,\,p}(\mathbb{R}^N)\times W_{m+\ell}^{m,\,p}(\mathbb{R}^N)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-N/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}_{[1-\ell-N/p]}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_{m+\ell}^{m+1,\,p}(\mathbb{R}^N)} + \|\pi+\mu\|_{W_{m+\ell}^{m,\,p}(\mathbb{R}^N)}\right) \\ \leq\; C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_{m+\ell}^{m-1,\,p}(\mathbb{R}^N)} + \|g\|_{W_{m+\ell}^{m,\,p}(\mathbb{R}^N)}\right).$$

Note that if we suppose $\ell = 0$, then $\mathcal{S}_{[1-N/p']} = \boldsymbol{\mathcal{P}}_{[1-N/p']} \times \{0\}$ and the orthogonality condition $(\boldsymbol{f},\, g) \perp \mathcal{S}_{[1-N/p']}$ is equivalent to $\boldsymbol{f} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$.

3 Homogeneous problems with singular boundary conditions

The way we will take to solve the Stokes system is based on the existence of very weak solutions to homogeneous problems with singular boundary conditions. The first one is the biharmonic problem: find $u \in W_{\ell-1}^{1,\,p}(\mathbb{R}_+^N)$ solution to the problem

$$(\mathcal{P}): \qquad \Delta^2 u = 0 \;\text{ in } \mathbb{R}_+^N, \quad u = g_0 \;\text{ and }\; \partial_N u = g_1 \;\text{ on } \Gamma,$$

where $g_0 \in W_{\ell-1}^{1-1/p,\,p}(\Gamma)$ and $g_1 \in W_{\ell-1}^{-1/p,\,p}(\Gamma)$ are given. We begin to define for any integer q, the polynomial space $\mathcal{B}_q$ as follows:

$$\mathcal{B}_q = \left\{u \in \mathcal{P}_q^{\Delta^2};\; u = \partial_N u = 0 \text{ on } \Gamma\right\}.$$

Theorem 3.1 (Amrouche-Raudin [8]). *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 1\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell + 1\}. \tag{3.1}$$

For any $g_0 \in W^{1-1/p,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}\ , \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{3.2}$$

problem ($\mathcal{P}$) admits a solution $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q\in\mathcal{B}_{[2-\ell-N/p]}} \|u+q\|_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)} \ \le\ C\left(\|g_0\|_{W^{1-1/p,p}_{\ell-1}(\Gamma)} + \|g_1\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}\right).$$

Remark 3.2. i) In the case where $\ell = 1$, if $1 - N/p' < 0$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and if $1 - N/p' \ge 0$, then $\mathcal{B}_{[3-N/p']} = \mathcal{B}_2 = \mathbb{R}\, x_N^2$.
ii) We also established a result for the lower case, with $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$, but we do not use it in this paper. □

We will also need a result of this type about the Neumann problem for the Laplacian: find $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(\mathcal{Q}): \qquad \Delta u = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{and} \quad \partial_N u = g \ \text{ on } \Gamma,$$

where $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$.

Theorem 3.3 (Amrouche [6]). *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 2\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell + 2\}. \tag{3.3}$$

For any $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}^\Delta_{[\ell-N/p']}, \quad \langle g, \varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0, \tag{3.4}$$

problem ($\mathcal{Q}$) admits a solution $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^\Delta_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q\in\mathcal{N}^\Delta_{[2-\ell-N/p]}} \|u+q\|_{W^{0,p}_{\ell-2}(\mathbb{R}^N_+)} \ \le\ C\,\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}.$$

With the same arguments as for Theorem 3.3, we can prove an intermediate result for this problem:

Theorem 3.4. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.1), for any $g \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition (3.4), problem ($\mathcal{Q}$) admits a solution $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^\Delta_{[2-\ell-N/p]}$, with the estimate*

$$\inf_{q\in\mathcal{N}^\Delta_{[2-\ell-N/p]}} \|u+q\|_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)} \ \le\ C\,\|g\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}.$$

Now, we will establish a similar result about the Dirichlet problem for the Laplacian with very singular boundary conditions: find $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(\mathcal{R}): \qquad \Delta u = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{ and } \quad u = g \ \text{ on } \Gamma,$$

where $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$.

Theorem 3.5. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3), for any $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}, \quad \langle g, \partial_N \varphi \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0, \tag{3.5}$$

problem $(\mathcal{R})$ admits a solution $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, with the estimate

$$\inf_{q \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u + q\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} \ \le \ C \, \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}.$$

Firstly, we must give a meaning to traces for a special class of distributions. We introduce the spaces

$$Y_\ell(\mathbb{R}^N_+) = \left\{ v \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+); \ \Delta v \in W^{0,p}_{\ell+1}(\mathbb{R}^N_+) \right\},$$
$$Y_{\ell,1}(\mathbb{R}^N_+) = \left\{ v \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+); \ \Delta v \in W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+) \right\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|v\|_{Y_\ell(\mathbb{R}^N_+)} = \|v\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,p}_{\ell+1}(\mathbb{R}^N_+)},$$
$$\|v\|_{Y_{\ell,1}(\mathbb{R}^N_+)} = \|v\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+)}.$$

Lemma 3.6. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3), the space $\mathcal{D}\left(\overline{\mathbb{R}^N_+}\right)$ is dense in $Y_\ell(\mathbb{R}^N_+)$ and in $Y_{\ell,1}(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $T \in \left(Y_\ell(\mathbb{R}^N_+)\right)'$, there exists a unique pair $(f, g) \in \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$, such that

$$\forall v \in Y_\ell(\mathbb{R}^N_+), \quad \langle T, v \rangle = \langle f, v \rangle_{\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \int_{\mathbb{R}^N_+} g \, \Delta v \, \mathrm{d}x. \tag{3.6}$$

Thanks to the Hahn-Banach theorem, it suffices to show that any T which vanishes on $\mathcal{D}\left(\overline{\mathbb{R}^N_+}\right)$ is actually zero on $Y_\ell(\mathbb{R}^N_+)$. Let's suppose that $T = 0$ on $\mathcal{D}\left(\overline{\mathbb{R}^N_+}\right)$, thus on $\mathcal{D}(\mathbb{R}^N_+)$. Then we can deduce from (3.6) that

$$f + \Delta g = 0 \quad \text{ in } \mathbb{R}^N_+,$$

hence we have $\Delta g \in \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Let $\tilde{f} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$ and $\tilde{g} \in W^{0,p'}_{-\ell-1}(\mathbb{R}^N)$ be respectively the extensions by 0 of f and g to $\mathbb{R}^N$. Thanks to (3.6), it is clear that $\tilde{f} + \Delta\tilde{g} = 0$ in $\mathbb{R}^N$, and thus $\Delta\tilde{g} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$. Now, thanks to the isomorphism results for the Laplace operator in $\mathbb{R}^N$ (see [4]), we can deduce that $\tilde{g} \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N)$, under hypothesis (3.3). Since $\tilde{g}$ is an extension by 0, it follows that $g \in \overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Then, by density of $\mathcal{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^N_+)$ such that $\varphi_k \to g$ in $\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Thus, for any $v \in Y_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle T, v\rangle &= \langle -\Delta g, v\rangle_{\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle g, \Delta v\rangle_{\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)}\\
&= \lim_{k\to\infty}\left\{\langle -\Delta\varphi_k, v\rangle_{\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle \varphi_k, \Delta v\rangle_{\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)}\right\}\\
&= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+}\varphi_k\,\Delta v\,\mathrm{d}x + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta v\,\mathrm{d}x\right\}\\
&= 0,
\end{aligned}$$

i.e. T is identically zero.

For the density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y_{\ell,1}(\mathbb{R}^N_+)$, the only difference in the proof concerns the logarithmic factors in the weights, with $g \in W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)$. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 3.7. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3), the trace mapping $\gamma_0 : \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathcal{D}(\mathbb{R}^{N-1})$, can be extended to a linear continuous mapping*

$$\begin{aligned}
\gamma_0 : Y_\ell(\mathbb{R}^N_+) &\longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \textit{if } N/p' \notin \{\ell-1,\,\ell,\,\ell+1\},\\
\Big(\textit{resp. } \gamma_0 : Y_{\ell,1}(\mathbb{R}^N_+) &\longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \textit{if } N/p' \in \{\ell-1,\,\ell,\,\ell+1\}\Big).
\end{aligned}$$

Moreover, we have the following Green formula

$$\begin{aligned}
&\forall v \in Y_\ell(\mathbb{R}^N_+),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \textit{ such that } \varphi = \Delta\varphi = 0 \textit{ on } \Gamma,\\
&\quad\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)} - \langle v, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}\\
&\qquad\qquad = \langle v, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}
\end{aligned}\tag{3.7}$$

$\Big($*resp. the Green formula for $v \in Y_{\ell,1}(\mathbb{R}^N_+)$, where the first term of the left-hand side is replaced by $\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)}$*$\Big)$.

Proof. Firstly, let's remark that for any $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, the boundary condition $\varphi = \Delta\varphi = 0$ on Γ is equivalent to $\varphi = \partial^2_N\varphi = 0$ on Γ. Moreover, if

$N/p' \notin \{\ell-1,\, \ell,\, \ell+1\}$, we have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$. So we can write the following Green formula:

$$\begin{gathered}\forall v \in \mathcal{D}\Big(\overline{\mathbb{R}^N_+}\Big),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } \varphi = \Delta\varphi = 0 \text{ on } \Gamma,\\ \int_{\mathbb{R}^N_+} \varphi\, \Delta v \,\mathrm{d}x - \int_{\mathbb{R}^N_+} v\, \Delta \varphi \,\mathrm{d}x = \int_\Gamma v\, \partial_N \varphi \,\mathrm{d}x'.\end{gathered} \tag{3.8}$$

Since $\Delta\varphi = 0$ on Γ, we have the identity

$$\int_{\mathbb{R}^N_+} v\, \Delta\varphi \,\mathrm{d}x = \langle v, \Delta\varphi \rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

This implies

$$\Big| \langle v, \partial_N \varphi \rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma) \times W^{2-1/p',\,p'}_{-\ell+2}(\Gamma)} \Big| \le\ \|v\|_{Y_\ell(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

By Lemma 2.2, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists a lifting function $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi = 0,\ \partial_N\varphi = \mu$ and $\partial^2_N \varphi = 0$ on Γ, satisfying

$$\|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} \le C\ \|\mu\|_{W^{2-1/p',\,p'}_{-\ell+2}(\Gamma)},$$

where C is a constant not depending on φ and μ. Then we can deduce that

$$\|\gamma_0 v\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}\ \le\ C\ \|v\|_{Y_\ell(\mathbb{R}^N_+)}.$$

Thus the linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}\Big(\overline{\mathbb{R}^N_+}\Big)$ is continuous for the norm of $Y_\ell(\mathbb{R}^N_+)$. Since $\mathcal{D}\Big(\overline{\mathbb{R}^N_+}\Big)$ is dense in $Y_\ell(\mathbb{R}^N_+)$, γ_0 can be extended by continuity to a mapping still called $\gamma_0 \in \mathcal{L}\Big(Y_\ell(\mathbb{R}^N_+);\ W^{-1-1/p,\,p}_{\ell-2}(\Gamma)\Big)$. Moreover, we also can deduce the formula (3.7) from (3.8) by density of $\mathcal{D}\Big(\overline{\mathbb{R}^N_+}\Big)$ in $Y_\ell(\mathbb{R}^N_+)$. To finish, note that if $N/p' \in \{\ell-1,\, \ell,\, \ell+1\}$, we only have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)$, hence the necessity to introduce the space $Y_{\ell,\,1}(\mathbb{R}^N_+)$ and the corresponding Green formula with logarithmic factors for these three critical values. □

Proof of Theorem 3.5. We can observe that solve problem $(\mathcal{R})$ is equivalent to find $u \in Y_\ell(\mathbb{R}^N_+)$ if $N/p' \notin \{\ell-1,\, \ell,\, \ell+1\}$ $\Big($resp. $u \in Y_{\ell,\,1}(\mathbb{R}^N_+)$ if $N/p' \in \{\ell-1,\, \ell,\, \ell+1\}\Big)$, satisfying

$$\begin{aligned}&\forall v \in\ W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } v = \Delta v = 0 \text{ on } \Gamma,\\ &\langle u, \Delta v\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}\ =\ -\langle g, \partial_N v\rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma) \times W^{2-1/p',\,p'}_{-\ell+2}(\Gamma)}.\end{aligned} \tag{3.9}$$

Indeed the direct implication is straightforward. Conversely, if u satisfies (3.9) then we have for any $\varphi \in \mathcal{D}(\mathbb{R}^N_+)$,

$$\langle \Delta u, \varphi \rangle_{W^{-3,p}_{\ell-2}(\mathbb{R}^N_+) \times \mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} = \langle u, \Delta\varphi \rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+) \times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)} = 0,$$

thus $\Delta u = 0$ in $\mathbb{R}^N_+$. Moreover, by the Green formula (3.7), we have

$$\begin{aligned} &\forall v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } v = \Delta v = 0 \text{ on } \Gamma, \\ &\langle g, \partial_N v \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = \langle u, \partial_N v \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}. \end{aligned}$$

By Lemma 2.2, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists $v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $v = 0,\ \partial_N v = \mu,\ \partial^2_N v = 0$ on Γ. Consequently,

$$\langle u - g, \mu \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0,$$

i.e. $u - g = 0$ on Γ. Thus u satisfies $(\mathcal{R})$.

Furthermore, for any $f \in \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$, we know that (see [5]) there exists a unique $v \in W^{3,p'}_{-\ell+2}/\mathcal{A}^{\Delta}_{[1+\ell-N/p']}$ such that

$$\Delta v = f \quad \text{in } \mathbb{R}^N_+, \quad v = 0 \quad \text{on } \Gamma,$$

with the estimate

$$\|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[1+\ell-N/p']}} \ \leq\ C\ \|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)},$$

where C denotes a generic constant not depending on v and f. Now, let's consider the linear form $T:\ f \longmapsto -\langle g, \partial_N v \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}$ defined on $\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Thanks to (3.5), we have for any $q \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}$,

$$\begin{aligned} |Tf| = &\ \left| \langle g, \partial_N (v+q) \rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} \right| \\ \leq &\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|v+q\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} \\ \leq &\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[3-N/p']}} \\ \leq &\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}. \end{aligned}$$

Thus we have shown that T is continuous on $\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$ and then, according to Riesz representation theorem, there exists a unique $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)$ such that $Tf = \langle u, f \rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+) \times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}$. So we have (3.9) and u is the unique solution to problem $(\mathcal{R})$. □

Similarly to the Neumann problem, we can give an intermediate result:

Theorem 3.8. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.1), for any $g \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition (3.5), problem $(\mathcal{R})$ admits a solution $u \in W^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, with the estimate*

$$\inf_{q \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u+q\|_{W^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} \;\leq\; C\,\|g\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)}.$$

4 Generalized solutions to the Stokes system in $\mathbb{R}^N_+$

We will establish a first result about the generalized solutions to (S^+) in the homogeneous case. The following proposition is quite natural and we can find similar results in the literature although not expressed in weighted Sobolev spaces (see e.g. Farwig-Sohr [12], Galdi [14], Cattabriga [11]). Moreover, we take up some ideas in [12] and we considerably simplify the proof.

Proposition 4.1. *For any $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,\,p}_0(\Gamma)$, the Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}^N_+, \tag{4.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{4.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma, \tag{4.3}$$

has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,\,p}_0(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, with the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\,p}_0(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \;\leq\; C\;\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,\,p}_0(\Gamma)}. \tag{4.4}$$

Proof. 1) Firstly, we will show that system (4.1)–(4.3) can be reduced to three problems on the fundamental operators Δ^2 and Δ.

Applying the operator div to the first equation (4.1), we obtain

$$\Delta \pi = 0 \quad \text{ in } \mathbb{R}^N_+. \tag{4.5}$$

Now, applying the operator Δ to the same equation (4.1), we deduce

$$\Delta^2 \boldsymbol{u} = \boldsymbol{0} \quad \text{ in } \mathbb{R}^N_+. \tag{4.6}$$

From the boundary condition (4.3), we take out

$$u_N = g_N \quad \text{ on } \Gamma, \tag{4.7}$$

and moreover $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ, where $\operatorname{div}' \boldsymbol{u}' = \sum_{i=1}^{N-1} \partial_i u_i$.
Since $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, we also have $\operatorname{div} \boldsymbol{u} = 0$ on Γ, then we can write

$\partial_N u_N + \mathrm{div}' \boldsymbol{u}' = 0$ on Γ, hence

$$\partial_N u_N = -\mathrm{div}' \boldsymbol{g}' \quad \text{on } \Gamma. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we obtain the following biharmonic problem

$$(\mathcal{P}) : \qquad \Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ and } \ \partial_N u_N = -\mathrm{div}' \boldsymbol{g}' \ \text{ on } \Gamma.$$

Then, combining (4.5) with the trace on Γ of the Nth component in the equations (4.1), we obtain the following Neumann problem

$$(\mathcal{Q}) : \qquad \Delta \pi = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{and} \quad \partial_N \pi = \Delta u_N \ \text{ on } \Gamma.$$

Lastly, if we consider the $N-1$ first components of the equations (4.1) and (4.3), we can write the following Dirichlet problem

$$(\mathcal{R}) : \qquad \Delta \boldsymbol{u}' = \nabla' \pi \ \text{ in } \mathbb{R}^N_+ \quad \text{and} \quad \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

2) Now, we will solve these three problems.

`Step 1`: Problem $(\mathcal{P})$. Since $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, we have $g_N \in W_0^{1-1/p,p}(\Gamma)$ and $\mathrm{div}' \boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)$. So $(\mathcal{P})$ is an homogeneous biharmonic problem with singular boundary conditions, and we can apply Theorem 3.1 provided the compatibility condition (3.2) is fulfilled. If $1 - N/p' < 0$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and the condition vanishes. If $1 - N/p' \geq 0$, then $\mathcal{B}_{[3-N/p']} = \mathbb{R}\, x_N^2$ and this condition is equivalent to

$$\langle \mathrm{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} = 0. \tag{4.9}$$

Since $\mathcal{D}(\mathbb{R}^{N-1})$ is dense in $W_0^{1/p,p'}(\Gamma)$, we know that there exists a sequence $(\varphi_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^{N-1})$ such that $\varphi_k \to 1$ in $W_0^{1/p,p'}(\Gamma)$, hence we can deduce

$$\langle \mathrm{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} = -\lim_{k \to \infty} \int_{\mathbb{R}^{N-1}} \boldsymbol{g}' . \nabla \varphi_k \, \mathrm{d}x' = 0.$$

Thus the orthogonality condition is fulfilled and problem $(\mathcal{P})$ has a unique solution $u_N \in W_0^{1,p}(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned} \|u_N\|_{W_0^{1,p}(\mathbb{R}^N_+)} &\leq C \left(\|g_N\|_{W_0^{1-1/p,p}(\Gamma)} + \|\mathrm{div}' \boldsymbol{g}'\|_{W_0^{-1/p,p}(\Gamma)} \right) \\ &\leq C \, \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}. \end{aligned} \tag{4.10}$$

`Step 2`: Problem $(\mathcal{Q})$. Since $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$, we have $\Delta u_N \in Y_2(\mathbb{R}^N_+)$ and also $\Delta u_N \in Y_{2,1}(\mathbb{R}^N_+)$, hence $\Delta u_N|_\Gamma \in W_0^{-1-1/p,p}(\Gamma)$ by Lemma 3.7. Then we can apply Theorem 3.3, provided the compatibility condition (3.4) is fulfilled, *i.e.*

$$\forall \varphi \in \mathcal{N}^\Delta_{[2-N/p']}, \quad \langle \Delta u_N, \varphi \rangle_{W_0^{-1-1/p,p}(\Gamma) \times W_0^{2-1/p',p'}(\Gamma)} = 0.$$

Knowing that $\mathcal{N}^{\Delta}_{[2-N/p']} \subset \mathcal{P}_1$, an argument similar to that of the condition (4.9) in `step 1` gives us this relation. We can conclude that problem $(\mathcal{Q})$ has a unique solution $\pi \in L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned}
\|\pi\|_{L^p(\mathbb{R}^N_+)} &\leq C \,\|\Delta u_N\|_{W_0^{-1-1/p,\,p}(\Gamma)} \\
&\leq C \,\|\Delta u_N\|_{Y_2(\mathbb{R}^N_+)} \;=\; C \,\|\Delta u_N\|_{W_0^{-1,\,p}(\mathbb{R}^N_+)} \\
&\leq C \,\|u_N\|_{W_0^{1,\,p}(\mathbb{R}^N_+)} \;\leq\; C \,\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)}. \qquad (4.11)
\end{aligned}$$

`Step 3`: Problem $(\mathcal{R})$. By `step 2`, we have $\nabla'\pi \in W_0^{-1,\,p}(\mathbb{R}^N_+)^{N-1}$ and moreover $\boldsymbol{g}' \in W_0^{1-1/p,\,p}(\Gamma)^{N-1}$. Since $\mathcal{A}^{\Delta}_{[1-N/p']} = \{0\}$, we know that problem $(\mathcal{R})$ has an unique solution $\boldsymbol{u}' \in W_0^{1,\,p}(\mathbb{R}^N_+)^{N-1}$ (see [5], Theorem 3.1), satisfying

$$\begin{aligned}
\|\boldsymbol{u}'\|_{W_0^{1,\,p}(\mathbb{R}^N_+)^{N-1}} &\leq C \left(\|\nabla'\pi\|_{W_0^{-1,\,p}(\mathbb{R}^N_+)^{N-1}} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,\,p}(\Gamma)^{N-1}}\right) \\
&\leq C \left(\|\pi\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,\,p}(\Gamma)^{N-1}}\right) \\
&\leq C \,\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)}. \qquad (4.12)
\end{aligned}$$

3) In order, we have found u_N, π and $\boldsymbol{u}'$, which satisfy (4.3) and partially satisfy (4.1), *i.e.*

$$-\Delta\boldsymbol{u}' + \nabla'\pi = \boldsymbol{0} \;\text{ in } \mathbb{R}^N_+.$$

It remains to show they satisfy (4.2) and the Nth component of (4.1), *i.e.*

$$-\Delta u_N + \partial_N\pi = 0 \quad \text{in } \mathbb{R}^N_+.$$

Thanks to (4.5) and (4.6), we obtain

$$\Delta(\Delta u_N - \partial_N\pi) = \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+.$$

With the boundary condition of $(\mathcal{Q})$, we can deduce that the distribution $\Delta u_N - \partial_N\pi \in W_0^{-1,\,p}(\mathbb{R}^N_+)$ satisfies the following Dirichlet problem

$$\Delta(\Delta u_N - \partial_N\pi) = 0 \;\text{ in } \mathbb{R}^N_+, \quad \Delta u_N - \partial_N\pi = 0 \;\text{ on } \Gamma.$$

Thanks to Theorem 3.5, we necessarily have $\Delta u_N - \partial_N\pi = 0$. Thus $(u,\,\pi)$ completely satisfies (4.1).

Now, applying the operator div to (4.1), we have $-\Delta\,\mathrm{div}\,\boldsymbol{u} + \Delta\pi = 0$ in $\mathbb{R}^N_+$, and by the main equation of $(\mathcal{Q})$, *i.e.* (4.5), we obtain $\Delta\,\mathrm{div}\,\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. Moreover, from the boundary condition in $(\mathcal{R})$, we get $\mathrm{div}'\boldsymbol{u}' = \mathrm{div}'\boldsymbol{g}'$ on Γ. Then, with the boundary condition in $(\mathcal{P})$, we can write

$$\mathrm{div}\,\boldsymbol{u} = \mathrm{div}'\boldsymbol{u}' + \partial_N u_N = \mathrm{div}'\boldsymbol{g}' - \mathrm{div}'\boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, we have

$$\Delta\,\mathrm{div}\,\boldsymbol{u} = 0 \;\text{ in } \mathbb{R}^N_+, \quad \mathrm{div}\,\boldsymbol{u} = 0 \;\text{ on } \Gamma,$$

with $\operatorname{div}\boldsymbol{u} \in L^p(\mathbb{R}^N_+)$ and then by Theorem 3.8, we can deduce that $\operatorname{div}\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, *i.e.* (4.2) is satisfied.

4) Finally, let's remark that the uniqueness of $(\boldsymbol{u}, \pi)$ is a consequence of the uniqueness of the solutions to problems $(\mathcal{P})$, $(\mathcal{Q})$ and $(\mathcal{R})$. Moreover, the estimate (4.4) is a consequence of the estimates (4.10), (4.11) and (4.12). □

Now, we can solve the complete problem (S^+). For this, we will show that it can be reduced to an homogeneous problem, solved by Proposition 4.1.

Theorem 4.2. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)$, $h \in L^p(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, and there exists a constant C such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}\right). \quad (4.13)$$

Proof. Firstly, let's write $\boldsymbol{f} = \operatorname{div}\mathbb{F}$, where $\mathbb{F} = (\boldsymbol{F}_i)_{1\leq i\leq N} \in \boldsymbol{L}^p(\mathbb{R}^N_+)^N$, with the estimate

$$\|\mathbb{F}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)^N} \leq C\,\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)};$$

and let's respectively denote by $\tilde{\mathbb{F}} = (\tilde{\boldsymbol{F}}_i)_{1\leq i\leq N} \in \boldsymbol{L}^p(\mathbb{R}^N)^N$ and $\tilde{h} \in L^p(\mathbb{R}^N)$ the extensions by 0 of $\mathbb{F}$ and h to $\mathbb{R}^N$. By Theorem 2.3, we know that there exists $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to the problem

$$(\tilde{S}): \qquad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \operatorname{div}\tilde{\mathbb{F}} \quad\text{and}\quad \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \ \text{ in } \mathbb{R}^N,$$

provided the condition $\operatorname{div}\tilde{\mathbb{F}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled. If $1 - N/p' < 0$, we obviously have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \{\boldsymbol{0}\}$, thus the condition vanishes. If $1 - N/p' \geq 0$, then we have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \mathbb{R}^N$ and this condition is equivalent to

$$\forall i = 1, \dots, N, \quad \left\langle \operatorname{div}\tilde{\boldsymbol{F}}_i, 1\right\rangle_{W_0^{-1,p}(\mathbb{R}^N)\times W_0^{1,p'}(\mathbb{R}^N)} = 0.$$

This is exactly the same argument as for the condition (4.9) in the previous proof. Thus the orthogonality condition is fulfilled, hence the existence of $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to problem $(\tilde{S})$, satisfying

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N)} + \|\tilde{\pi}\|_{L^p(\mathbb{R}^N)} &\leq C\left(\|\operatorname{div}\tilde{\mathbb{F}}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N)} + \|\tilde{h}\|_{L^p(\mathbb{R}^N)}\right)\\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)}\right). \quad (4.14)\end{aligned}$$

Consequently, we can reduce the system (S^+) to the homogeneous problem

$$(S^\sharp): \qquad -\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{0} \quad\text{and}\quad \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \mathbb{R}^N_+, \quad \boldsymbol{v} = \boldsymbol{g}^\sharp \ \text{ on } \Gamma,$$

where we have set $\boldsymbol{g}^\sharp = \boldsymbol{g} - \tilde{\boldsymbol{u}}|_\Gamma \in \boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)$. Now, thanks to Proposition 4.1, we know that $(S^\sharp)$ admits a unique solution $(\boldsymbol{v},\,\vartheta) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+)} + \|\vartheta\|_{L^p(\mathbb{R}^N_+)} &\leq C\ \|\boldsymbol{g}^\sharp\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)}\right). \qquad (4.15)\end{aligned}$$

Then, $(\boldsymbol{u},\,\pi) = (\boldsymbol{v} + \tilde{\boldsymbol{u}}|_{\mathbb{R}^N_+},\, \vartheta + \tilde{\pi}|_{\mathbb{R}^N_+}) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ is solution to (S^+) and the estimate (4.13) is a consequence of the estimates (4.14) and (4.15). Finally, the uniqueness of the solution to (S^+) is a straightforward consequence of Proposition 4.1. □

Remark 4.3. In a forthcoming work, we will show that under hypotheses of Theorem 4.2 and if moreover $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,\,q}(\mathbb{R}^N_+)$, $h \in L^q(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/q,\,q}(\Gamma)$, for any real number $q > 1$, then the solution $(\boldsymbol{u},\,\pi)$ given by Theorem 4.2 verifies, besides, $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_0^{1,\,q}(\mathbb{R}^N_+) \times L^q(\mathbb{R}^N_+)$. □

5 Strong solutions and regularity for the Stokes system in $\mathbb{R}^N_+$

In this section, we are interested in the existence of strong solutions (and then to regular solutions, see Corollaries 5.5 and 5.7), *i.e.* of solutions $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbb{R}^N_+) \times W^{1,p}_{\ell+1}(\mathbb{R}^N_+)$. Here, we limit ourselves to the two cases $\ell = 0$ or $\ell = -1$. Note that in the case $\ell = 0$, we have $W_1^{2,\,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,\,p}(\mathbb{R}^N_+)$ and $W_1^{1,\,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$. The proposition and theorem which follow show that the generalized solution of Theorem 4.2, with a stronger hypothesis on the data, is in fact a strong solution.

Proposition 5.1. *Assume that* $\dfrac{N}{p'} \neq 1$. *For any* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,\,p}(\Gamma)$, *the Stokes problem (4.1)–(4.3) has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} \ \leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,\,p}(\Gamma)}.$$

Proof. The arguments for the estimate are unchanged with respect to the proof of Proposition 4.1. For the surjectivity and the uniqueness, note that we always have the imbedding $W_1^{2-1/p,\,p}(\Gamma) \hookrightarrow W_0^{1-1/p,\,p}(\Gamma)$. By Proposition 4.1, we can deduce that problem (4.1)–(4.3) admits a unique solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, satisfying the estimate (4.4). Then, it suffices to go back to the proof of Proposition 4.1 and to use the established results about problems $(\mathcal{P})$, $(\mathcal{Q})$ and $(\mathcal{R})$, to show that in fact $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$.

In order, for problem $(\mathcal{P})$, we find $u_N \in W_1^{2,p}(\mathbb{R}_+^N)$ (see [7], Lemma 4.9); for problem $(\mathcal{Q})$, thanks to Theorem 3.4, we find $\pi \in W_1^{1,p}(\mathbb{R}_+^N)$; for problem $(\mathcal{R})$, we find $\boldsymbol{u}' \in W_1^{2,p}(\mathbb{R}_+^N)^{N-1}$ (see [5], Theorem 3.3). Note that for these three results, the condition $N/p' \neq 1$ is always necessary. □

Now, we can study the strong solutions for the complete problem (S^+). As for the generalized solutions, we will show that it is equivalent to an homogeneous problem, solved by Proposition 5.1. The following theorem was established in the case $N = 3$, $p = 2$, by Maz'ya-Plamenevskiĭ-Stupyalis (see [18]).

Theorem 5.2. *Assume that* $\dfrac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, $h \in W_1^{1,p}(\mathbb{R}_+^N)$ *and* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$, *problem* (S^+) *admits a unique solution* $(\boldsymbol{u}, \pi)$ *which belongs to* $\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}_+^N)} \leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)} + \|h\|_{W_1^{1,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,p}(\Gamma)}\right).$$

Proof. Here again, the arguments for the estimate are unchanged with respect to the proof of Theorem 4.2. For the surjectivity and the uniqueness, note that the imbedding $W_1^{0,p}(\mathbb{R}_+^N) \hookrightarrow W_0^{-1,p}(\mathbb{R}_+^N)$ holds if $N/p' \neq 1$. Moreover, we have $W_1^{1,p}(\mathbb{R}_+^N) \hookrightarrow L^p(\mathbb{R}_+^N)$ and $W_1^{2-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$. Thus, thanks to Theorem 4.2, we know that problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$, satisfying the estimate (4.13). To show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, we want to find an extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ to $\mathbb{R}^N$, such that the orthogonality condition for the extended problem to the whole space $(\tilde{S})$ holds. To this end, we still can write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Indeed, if $N/p' \neq 1$, for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, the Dirichlet problem

$$\Delta \boldsymbol{w} = \boldsymbol{f} \ \text{ in } \mathbb{R}_+^N, \quad \boldsymbol{w} = \boldsymbol{0} \ \text{ in } \Gamma,$$

admits a unique solution $\boldsymbol{w} \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)$ (see [5], Theorem 3.3). So, if we consider $\mathbb{F} = \nabla \boldsymbol{w} \in \boldsymbol{W}_1^{1,p}(\mathbb{R}_+^N)^N$, we have $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Now, it suffices to go back to the proof of Theorem 4.2. Here again, we know that there exists a continuous linear extension operator from $W_1^{1,p}(\mathbb{R}_+^N)$ to $W_1^{1,p}(\mathbb{R}^N)$, so we get $\tilde{\boldsymbol{f}} = \operatorname{div} \tilde{\mathbb{F}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N)$ and $\tilde{h} \in W_1^{1,p}(\mathbb{R}^N)$, hence the extended problem $(\tilde{S})$, which has, by Theorem 2.4, a solution $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N) \times W_1^{1,p}(\mathbb{R}^N)$. Then, we obtain the equivalent problem $(S^\sharp)$ with $\boldsymbol{g}^\sharp \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$ and this problem is solved by Proposition 5.1. □

Remark 5.3. To give a variant to this proof, we also can consider the extension $\tilde{\boldsymbol{f}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N)$ of $\boldsymbol{f}$ to $\mathbb{R}^N$ defined by:

$$\tilde{\boldsymbol{f}}(x', x_N) = \begin{cases} \boldsymbol{f}(x', x_N) & \text{if } x_N > 0, \\ -\boldsymbol{f}(x', -x_N) & \text{if } x_N < 0, \end{cases}$$

and $\tilde{h} \in W_1^{1,p}(\mathbb{R}^N)$ an extension of h to $\mathbb{R}^N$. Then by Theorem 2.4, there exists $(\tilde{\boldsymbol{u}}, \tilde{\pi})$ solution to the problem

$$(\tilde{S}): \qquad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \tilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \quad \text{in } \mathbb{R}^N,$$

provided the orthogonality condition $\tilde{\boldsymbol{f}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled. Here again, if $1 - N/p' < 0$ this condition vanishes and if $1 - N/p' > 0$, we have

$$\forall i = 1, \ldots, N, \quad \int_{\mathbb{R}^N} \tilde{\boldsymbol{f}}_i(x', x_N)\, \mathrm{d}x \ = \ 0.$$

Thus the orthogonality condition holds. The rest of the proof is identical. □

Remark 5.4. Similarly to Remark 4.3, we could show that under hypotheses of Theorem 5.2 and if moreover $\boldsymbol{f} \in \boldsymbol{W}_1^{0,q}(\mathbb{R}^N_+)$, $h \in W_1^{1,q}(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/q,q}(\Gamma)$, with an arbitrary real number $q > 1$, then the solution $(\boldsymbol{u}, \pi)$ given by Theorem 4.2 verify, besides, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,q}(\mathbb{R}^N_+) \times W_1^{1,q}(\mathbb{R}^N_+)$. □

We will now establish a global regularity result of solutions to the Stokes system (S^+), which includes the case of strong solutions and which rests on Theorem 4.2 and a regularity argument.

Corollary 5.5. *Let $m \in \mathbb{N}$ and assume that $\dfrac{N}{p'} \neq 1$ if $m \geq 1$. For any $\boldsymbol{f} \in \boldsymbol{W}_m^{m-1,p}(\mathbb{R}^N_+)$, $h \in W_m^{m,p}(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_m^{m+1-1/p,p}(\Gamma)$, problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_m^{m+1,p}(\mathbb{R}^N_+) \times W_m^{m,p}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_m^{m+1,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_m^{m,p}(\mathbb{R}^N_+)} \leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_m^{m-1,p}(\mathbb{R}^N_+)} + \|h\|_{W_m^{m,p}(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_m^{m+1-1/p,p}(\Gamma)}\right).$$

Proof. Since we have $W_m^{m-1,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{-1,p}(\mathbb{R}^N_+)$, $W_m^{m,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$ and $W_m^{m+1-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$, thanks to Theorem 4.2, we know that problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$. We will show by induction that

$$\begin{aligned} (\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}_m^{m-1,p}(\mathbb{R}^N_+) \times W_m^{m,p}(\mathbb{R}^N_+) \times \boldsymbol{W}_m^{m+1-1/p,p}(\Gamma) \\ \Rightarrow (\boldsymbol{u}, \pi) \in \boldsymbol{W}_m^{m+1,p}(\mathbb{R}^N_+) \times W_m^{m,p}(\mathbb{R}^N_+). \end{aligned} \tag{5.1}$$

For $m = 0$, (5.1) is true. Assume that (5.1) is true for $0, 1, \ldots, m$ and suppose that $(\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}_{m+1}^{m,p}(\mathbb{R}^N_+) \times W_{m+1}^{m+1,p}(\mathbb{R}^N_+) \times \boldsymbol{W}_{m+1}^{m+2-1/p,p}(\Gamma)$. Let's prove that $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{m+2,\,p}_{m+1}(\mathbb{R}^N_+) \times W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$. Since $W^{m,\,p}_{m+1}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,\,p}_{m}(\mathbb{R}^N_+)$, $W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+) \hookrightarrow W^{m,\,p}_{m}(\mathbb{R}^N_+)$ and $W^{m+2-1/p,\,p}_{m+1}(\Gamma) \hookrightarrow W^{m+1-1/p,\,p}_{m}(\Gamma)$, we know that $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{m+1,\,p}_{m}(\mathbb{R}^N_+) \times W^{m,\,p}_{m}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for any $i \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned}
&-\Delta(\varrho\,\partial_i \boldsymbol{u}) + \nabla(\varrho\,\partial_i \pi) \\
&\qquad = \varrho\,\partial_i \boldsymbol{f} + \frac{2}{\varrho}\,x.\nabla\partial_i \boldsymbol{u} + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right)\partial_i \boldsymbol{u} + \frac{1}{\varrho}\,x\,\partial_i \pi.
\end{aligned}$$

Thus, $-\Delta(\varrho\,\partial_i \boldsymbol{u}) + \nabla(\varrho\,\partial_i \pi) \in \boldsymbol{W}^{m-1,\,p}_{m}(\mathbb{R}^N_+)$. Moreover,

$$\mathrm{div}(\varrho\,\partial_i \boldsymbol{u}) = \frac{1}{\varrho}\,x\,\partial_i \boldsymbol{u} + \varrho\,\partial_i h.$$

Thus, $\mathrm{div}(\varrho\,\partial_i \boldsymbol{u}) \in W^{m,\,p}_{m}(\mathbb{R}^N_+)$. We also have $\gamma_0(\varrho\,\partial_i \boldsymbol{u}) = \varrho'\,\partial_i \gamma_0 \boldsymbol{u} = \varrho'\,\partial_i \boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,\,p}_{m}(\Gamma)$. So, by induction hypothesis, we can deduce that

$$\forall i \in \{1, \ldots, N-1\}, \quad (\partial_i \boldsymbol{u},\, \partial_i \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+) \times W^{m,\,p}_{m+1}(\mathbb{R}^N_+).$$

It remains to prove that $(\partial_N \boldsymbol{u},\, \partial_N \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+) \times W^{m,\,p}_{m+1}(\mathbb{R}^N_+)$. For that, let's observe that for any $i \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned}
\partial_i\partial_N \boldsymbol{u} &= \partial_N \partial_i \boldsymbol{u} && \in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\
\partial^2_N u_i &= -\Delta' u_i + \partial_i \pi - f_i && \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\
\partial^2_N u_N &= \partial_N h - \partial_N\,\mathrm{div}'\,\boldsymbol{u}' && \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\
\partial_N \pi &= f_N + \Delta u_N && \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+).
\end{aligned}$$

Hence, $\nabla(\partial_N \boldsymbol{u}) \in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+)^N$ and knowing that $\partial_N \boldsymbol{u} \in \boldsymbol{W}^{m,\,p}_{m}(\mathbb{R}^N_+)$, we can deduce that $\partial_N \boldsymbol{u} \in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$, according to definition (2.1). Consequently, we have $\nabla \boldsymbol{u} \in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)^N$. Likewise, we have $\nabla \pi \in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+)$. Finally, we can conclude that $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+2,\,p}_{m+1}(\mathbb{R}^N_+) \times W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$. ☐

Now, we examine the basic case $\ell = -1$, corresponding to $f \in \boldsymbol{L}^p(\mathbb{R}^N_+)$. More precisely, we have the following result, corresponding to Theorem 5.2:

Theorem 5.6. *For any $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^N_+)$, $h \in W^{1,\,p}_0(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,\,p}_0(\Gamma)$, problem (S^+) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{2,\,p}_0(\mathbb{R}^N_+) \times W^{1,\,p}_0(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $(\mathbb{R}\,x_N)^{N-1} \times \{0\} \times \mathbb{R}$ if $N \leq p$, with the following estimate if $N \leq p$ (eliminate $(\boldsymbol{\lambda},\, \mu)$ if $N > p$):*

$$\begin{aligned}
&\inf_{(\boldsymbol{\lambda},\,\mu) \in (\mathbb{R}\,x_N)^{N-1} \times \{0\} \times \mathbb{R}} \left(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,\,p}_0(\mathbb{R}^N_+)} + \|\pi + \mu\|_{W^{1,\,p}_0(\mathbb{R}^N_+)} \right) \leq \\
&\qquad\qquad C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|h\|_{W^{1,\,p}_0(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,\,p}_0(\Gamma)} \right).
\end{aligned}$$

Proof. The idea is to go back to the proof of Theorem 4.2 and we will throw light on the modifications. In contrast to Theorem 5.2, the extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ is of no importance because there is no orthogonality condition for the extended problem $(\tilde{S})$ (see Theorem 2.4). Then, we get the reduced problem $(S^\sharp)$. Now, to solve $(S^\sharp)$, this is the proof of Proposition 4.1. Problem $(\mathcal{P})$ yields a unique $u_N \in W_0^{2,\,p}(\mathbb{R}^N_+)$, problem $(\mathcal{Q})$ gives $\pi \in W_0^{1,\,p}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^{\Delta}_{[1-N/p]}$; and $(\mathcal{R})$ yields $\boldsymbol{u}' \in W_0^{2,\,p}(\mathbb{R}^N_+)^{N-1}$ unique up to an element of $(\mathcal{A}^{\Delta}_{[2-N/p]})^{N-1}$. The point 3) of the proof is identical for all N and p (the kernels of the two Dirichlet problems are always reduced to zero). The last point concerns the kernel of the operator associated to this problem. If $N > p$, it is clearly reduced to zero and if $N \leq p$, we have $\mathcal{A}^{\Delta}_{[2-N/p]} = \mathbb{R}\, x_N$ and $\mathcal{N}^{\Delta}_{[1-N/p]} = \mathcal{P}_{[1-N/p]} = \mathbb{R}$. □

Thanks to the corresponding imbeddings, we can give a regularity result with the same proof as Corollary 5.5.

Corollary 5.7. *Let $m \in \mathbb{N}$. For any $\boldsymbol{f} \in \boldsymbol{W}^{m,\,p}_m(\mathbb{R}^N_+)$, $h \in W^{m+1,\,p}_m(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{m+2-1/p,\,p}_m(\Gamma)$, problem (S^+) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+2,\,p}_m(\mathbb{R}^N_+) \times W^{m+1,\,p}_m(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $(\mathbb{R}\, x_N)^{N-1} \times \{0\} \times \mathbb{R}$ if $N \leq p$, with the following estimate if $N \leq p$ (eliminate $(\boldsymbol{\lambda},\, \mu)$ if $N > p$):*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in(\mathbb{R}\,x_N)^{N-1}\times\{0\}\times\mathbb{R}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+2,\,p}_m(\mathbb{R}^N_+)} + \|\pi+\mu\|_{W^{m+1,\,p}_m(\mathbb{R}^N_+)} \right) \leq C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,\,p}_m(\mathbb{R}^N_+)} + \|h\|_{W^{m+1,\,p}_m(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{m+2-1/p,\,p}_m(\Gamma)} \right).$$

6 Very weak solutions for the Stokes system

The aim of this section is to study the Stokes problem with singular data on the boundary. At first, we must give a meaning to singular data for the Stokes problem in the half-space. More precisely, we want to show that a boundary condition of the form $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)$ is meaningful. In mind of this paper, we limit ourselves to the two cases $\ell = 0$ or $\ell = 1$, *i.e.* to $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)$ corresponding to a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$, or $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{0}(\Gamma)$ corresponding to $(\boldsymbol{u},\, \pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W^{-1,\,p}_0(\mathbb{R}^N_+)$. In that way, for every $\ell \in \mathbb{Z}$, we introduce the space

$$\boldsymbol{M}_\ell(\mathbb{R}^N_+) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+);\ \boldsymbol{u} = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 \ \text{ on } \Gamma \right\}.$$

Lemma 6.1. *For any $\ell \in \mathbb{Z}$, we have the identity*

$$\boldsymbol{M}_\ell(\mathbb{R}^N_+) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+);\ \boldsymbol{u} = \boldsymbol{0} \ \text{ and } \ \partial_N u_N = 0 \ \text{ on } \Gamma \right\} \tag{6.1}$$

and the range space of the normal derivative $\gamma_1 : \boldsymbol{M}_\ell(\mathbb{R}^N_+) \to \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)$ *is*

$$\boldsymbol{Z}_\ell(\Gamma) = \left\{\boldsymbol{w} \in \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma);\ w_N = 0 \ \text{ on } \Gamma\right\}. \tag{6.2}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on Γ. Then $\operatorname{div} \boldsymbol{u} = \partial_N u_N$ on Γ and the identity (6.1) holds.

Moreover, it is clear that $\mathcal{I}m\ \gamma_1 \subset \boldsymbol{Z}_\ell(\Gamma)$. Conversely, given $\boldsymbol{w} \in \boldsymbol{Z}_\ell(\Gamma)$, by Lemma 2.2, there exists $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{u} = \boldsymbol{0}$ and $\partial_N \boldsymbol{u} = \boldsymbol{w}$ on Γ. Since $w_N = 0$ on Γ, we have $\boldsymbol{u} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+)$ and $\boldsymbol{w} \in \mathcal{I}m\ \gamma_1$. □

For any open subset Ω of $\mathbb{R}$, we also define the space

$$\boldsymbol{W}^{1,p'}_{-\ell}(\text{div};\, \Omega) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{-\ell}(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,p'}_{-\ell+1}(\Omega)\right\},$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\text{div};\,\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{1,p'}_{-\ell+1}(\Omega)};$$

and the following subspace of $\boldsymbol{W}^{1,p'}_{-\ell}(\text{div};\, \mathbb{R}^N_+)$

$$\boldsymbol{X}_\ell(\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)\right\}.$$

Lemma 6.2. *For any* $\ell \in \mathbb{Z}$*, the space* $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ *is dense in* $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$*.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{X}_\ell(\mathbb{R}^N_+)$ and $\tilde{\boldsymbol{v}}$ the extension by $\boldsymbol{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^N$, then we have $\tilde{\boldsymbol{v}} \in \boldsymbol{W}^{1,p'}_{-\ell}(\text{div};\, \mathbb{R}^N)$.

We begin to apply the cut off functions ϕ_k, defined on $\mathbb{R}^N$ for any $k \in \mathbb{N}$, by

$$\phi_k(x) = \begin{cases} \phi\left(\dfrac{k}{\ln|x|}\right), & \text{if } |x| > 1, \\ 1, & \text{otherwise,} \end{cases}$$

where $\phi \in C^\infty([0,\, \infty[)$ is such that

$$\phi(t) = 0,\ \text{if } t \in [0,\, 1]; \quad 0 \le \phi(t) \le 1,\ \text{if } t \in [1,\, 2]; \quad \phi(t) = 1,\ \text{if } t \ge 2.$$

Note that this truncation process is adapted to the logarithmic weights (see Lemma 7.1 in [3]). Then we have

$$\phi_k\, \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}_k \underset{k\to\infty}{\to} \tilde{\boldsymbol{v}} \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathbb{R}^N)$$

and

$$\operatorname{div}(\phi_k\, \tilde{\boldsymbol{v}}) = \phi_k \operatorname{div} \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{v}} \cdot \nabla\phi_k \underset{k\to\infty}{\to} \operatorname{div} \tilde{\boldsymbol{v}} \quad \text{in } W^{1,p'}_{-\ell+1}(\mathbb{R}^N).$$

Now, for any real number $\theta > 0$ and $x \in \mathbb{R}^N$, we set $\tilde{\boldsymbol{v}}_{k,\theta}(x) = \tilde{\boldsymbol{v}}_k(x - \theta\, e_N)$. Then $\tilde{\boldsymbol{v}}_{k,\theta} \in \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N)$ and $\operatorname{supp} \tilde{\boldsymbol{v}}_{k,\theta}$ is compact in $\mathbb{R}^N_+$, moreover

$$\lim_{\theta \to 0} \tilde{\boldsymbol{v}}_{k,\theta} = \tilde{\boldsymbol{v}}_k \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N).$$

Consequently, for any real number $\varepsilon > 0$ small enough, $\rho_\varepsilon * \tilde{\boldsymbol{v}}_{k,\theta} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ and

$$\lim_{\varepsilon \to 0} \lim_{\theta \to 0} \lim_{k \to \infty} \rho_\varepsilon * \tilde{\boldsymbol{v}}_{k,\theta} = \tilde{\boldsymbol{v}} \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N),$$

where ρ_ε is a mollifier. □

Let $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ be the dual space of $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$, we introduce the spaces:

$$\boldsymbol{T}_\ell(\mathbb{R}^N_+) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \Delta \boldsymbol{v} \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+) \right\},$$
$$\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+) = \left\{ \boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \mathbb{R}^N_+ \right\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\Delta \boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)},$$

where $\|\cdot\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$.

Lemma 6.3. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.1), the space $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $\boldsymbol{z} \in \big(\boldsymbol{T}_\ell(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)$, such that

$$\forall \boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+), \quad \langle \boldsymbol{z}, \boldsymbol{v} \rangle = \int_{\mathbb{R}^N_+} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \langle \Delta \boldsymbol{v}, \boldsymbol{g} \rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)} . \tag{6.3}$$

Thanks to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{z}$ which vanishes on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$. Let's suppose that $\boldsymbol{z} = \boldsymbol{0}$ on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$. Then we can deduce from (6.3) that

$$\boldsymbol{f} + \Delta \boldsymbol{g} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+,$$

hence we have $\Delta \boldsymbol{g} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+)$, $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$ and $\operatorname{div} \boldsymbol{g} \in \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Let $\tilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$ and $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{1,p'}_{-\ell}(\mathbb{R}^N)$ be respectively the extensions by $\boldsymbol{0}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ to $\mathbb{R}^N$. From (6.3), we get $\tilde{\boldsymbol{f}} + \Delta \tilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\mathbb{R}^N$, and thus $\Delta \tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$. Now, according to the isomorphism results for Δ in $\mathbb{R}^N$ (see [4]), we can deduce that $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N)$, under hypothesis (3.1). Since $\tilde{\boldsymbol{g}}$ is an extension by $\mathbf{0}$, it follows that $g \in \mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Then, by density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{g}$ in $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Thus, for any $\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}\langle \boldsymbol{z}, \boldsymbol{v}\rangle &= -\int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot\Delta\boldsymbol{g}\,\mathrm{d}x + \langle \Delta\boldsymbol{v}, \boldsymbol{g}\rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)\times\boldsymbol{X}_\ell(\mathbb{R}^N_+)}\\ &= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot\Delta\boldsymbol{\varphi}_k\,\mathrm{d}x + \langle \Delta\boldsymbol{v}, \boldsymbol{\varphi}_k\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^N_+)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)}\right\}\\ &= 0,\end{aligned}$$

i.e. $\boldsymbol{z}$ is identically zero. □

We also can show that, under hypothesis (3.1), $\left\{\boldsymbol{v}\in\boldsymbol{\mathcal{D}}\left(\overline{\mathbb{R}^N_+}\right);\ \operatorname{div}\boldsymbol{v}=0\right\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$. To study the traces of functions which belong to $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$, we set

$$\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+) = \left\{\boldsymbol{v}\in\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \operatorname{div}\boldsymbol{v}\in W^{0,p}_\ell(\mathbb{R}^N_+)\right\}$$

and their normal trace are described in the following lemma:

Lemma 6.4. *Assume that $\ell\in\mathbb{Z}$ with $N/p'\neq\ell$. The linear mapping*

$$\begin{aligned}\gamma_{e_N} : \boldsymbol{\mathcal{D}}\left(\overline{\mathbb{R}^N_+}\right) &\longrightarrow \mathcal{D}(\mathbb{R}^{N-1})\\ \boldsymbol{v} &\longmapsto v_N|_\Gamma,\end{aligned}$$

can be extended to a linear continuous mapping

$$\gamma_{e_N} : \boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+) \longrightarrow W^{-1/p,p}_{\ell-1}(\Gamma).$$

Moreover, we have the Green formula:

$$\begin{aligned}&\forall\boldsymbol{v}\in\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+),\quad \forall\varphi\in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+),\\ &\int_{\mathbb{R}^N_+}\boldsymbol{v}\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N_+}\varphi\,\operatorname{div}\boldsymbol{v}\ \mathrm{d}x = -\langle v_N,\varphi\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)}.\end{aligned}\tag{6.4}$$

Proof. Note that the assumption $N/p'\neq\ell$ is necessary for the imbedding $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)\hookrightarrow W^{0,p'}_{-\ell}(\mathbb{R}^N_+)$, which is underlying in the Green formula. We will show in remark how to do without.

Here again, we can show by truncation and regularization that $\boldsymbol{\mathcal{D}}\left(\overline{\mathbb{R}^N_+}\right)$ is dense in $\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)$ as in [3].

Let $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}\left(\overline{\mathbb{R}^N_+}\right)$ and $\varphi\in\mathcal{D}\left(\overline{\mathbb{R}^N_+}\right)$, then formula (6.4) obviously holds. Since

$\mathcal{D}\Big(\overline{\mathbb{R}^N_+}\Big)$ is dense in $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ and the mapping

$$\begin{aligned}\gamma_0 : W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) &\longrightarrow W^{1/p,p'}_{-\ell+1}(\Gamma)\\ \varphi &\longmapsto \varphi|_\Gamma\end{aligned}$$

is continuous, formula (6.4) holds for every $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\Big(\overline{\mathbb{R}^N_+}\Big)$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. By Lemma 2.2, for every $\mu \in W^{1/p,p'}_{-\ell+1}(\Gamma)$, there exists $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi = \mu$ on Γ, with $\|\varphi\|_{W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} \leq C\,\|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}$. Consequently,

$$\Big|\langle v_N, \mu\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)}\Big| \leq C\,\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)}\,\|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}.$$

Thus

$$\|v_N\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} \leq C\,\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)}.$$

We can deduce that the linear mapping γ_{e_N} is continuous for the norm of $\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)$. Since $\boldsymbol{\mathcal{D}}\Big(\overline{\mathbb{R}^N_+}\Big)$ is dense in $\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)$, γ_{e_N} can be extended by continuity to $\gamma_{e_N} \in \mathcal{L}\left(\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+);\; W^{-1/p,p}_{\ell-1}(\Gamma)\right)$ and formula (6.4) holds for all $\boldsymbol{v} \in \boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. □

Remark 6.5. If $N/p' = \ell$, the imbedding $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell}(\mathbb{R}^N_+)$ fails, but in that case we have $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell,-1}(\mathbb{R}^N_+)$. Thus, it suffices to introduce the space $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\,\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \mathrm{div}\,\boldsymbol{v} \in W^{0,p}_{\ell,1}(\mathbb{R}^N_+)\right\}$ instead of $\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\,\mathbb{R}^N_+)$. Then, with the same proof, we can show that $\boldsymbol{\mathcal{D}}\Big(\overline{\mathbb{R}^N_+}\Big)$ is dense in the space $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\,\mathbb{R}^N_+)$ and that the mapping γ_{e_N} is continuous from $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\,\mathbb{R}^N_+)$ to $W^{-1/p,p}_{\ell-1}(\Gamma)$, with the corresponding Green formula. □

It follows that the functions $\boldsymbol{v}$ from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ are such their normal trace v_N belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. Furthermore, for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\Big(\overline{\mathbb{R}^N_+}\Big)$ we have the following Green formula:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} \Delta\boldsymbol{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot\Delta\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Gamma \boldsymbol{v}\cdot\partial_N\boldsymbol{\varphi}\,\mathrm{d}x'.$$

Let's now observe that the dual space $\boldsymbol{Z}'_\ell(\Gamma)$ of $\boldsymbol{Z}_\ell(\Gamma)$ can be identified with the space

$$\left\{\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma);\ g_N = 0 \text{ on } \Gamma\right\},$$

and moreover that $\partial_N\boldsymbol{\varphi}$ sweeps $\boldsymbol{Z}_\ell(\Gamma)$ when $\boldsymbol{\varphi}$ sweeps $\boldsymbol{M}_\ell(\mathbb{R}^N_+)$. Thus, thanks to the density of $\boldsymbol{\mathcal{D}}\Big(\overline{\mathbb{R}^N_+}\Big)$ in $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$, we can prove that the tangential trace of functions from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. So, their complete trace belongs to $\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)$ and we have

$$\begin{aligned}&\forall \boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+), \quad \forall \boldsymbol{v} \in \boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+),\\ &\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{X}'_\ell \times \boldsymbol{X}_\ell} = \langle \boldsymbol{v}, \Delta\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}\times \boldsymbol{W}^{0,\,p'}_{-\ell+1}} + \langle \boldsymbol{v}, \partial_N \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}}.\end{aligned} \tag{6.5}$$

We now can solve the homogeneous Stokes problem with singular boundary conditions. We will give separately the results for $\ell = 0$ and $\ell = 1$. The proofs are quite similar and we will just detail the first case. The following proposition and corollary yield the existence of very weak solutions when the data are singular, so extending Proposition 4.1. Note that $W^{1,\,p}_0(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p}_{-1}(\mathbb{R}^N_+)$ and $W^{1-1/p,\,p}_0(\Gamma) \hookrightarrow W^{-1/p,\,p}_{-1}(\Gamma)$ if $N \neq p$.

Proposition 6.6. *Assume that* $\dfrac{N}{p} \neq 1$. *For any* $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)$ *such that* $g_N = 0$, *the Stokes problem (4.1)–(4.3) has a unique solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)} \;\leq\; C\; \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)}.$$

Proof. 1) We will first show that if the pair $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ satisfies (4.1) and (4.2), then we have $\boldsymbol{u} \in \boldsymbol{T}_{0,\sigma}(\mathbb{R}^N_+)$ and thus the boundary condition (4.3) makes sense. With this aim, thanks to Lemma 6.2, observe that if $\pi \in W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$, then we have $\nabla\pi \in \boldsymbol{X}'_0(\mathbb{R}^N_+)$ and

$$\|\nabla \pi\|_{\boldsymbol{X}'_0(\mathbb{R}^N_+)} \leq C\; \|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)},$$

So, we have $\Delta \boldsymbol{u} \in \boldsymbol{X}'_0(\mathbb{R}^N_+)$ and the trace $\gamma_0 \boldsymbol{u} \in \boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)$.

2) Let's show that the problem (4.1)–(4.3) with $g_N = 0$ is equivalent to the variational formulation: Find $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ such that

$$\begin{aligned}&\forall \boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+), \quad \forall \vartheta \in W^{1,\,p'}_1(\mathbb{R}^N_+),\\ &\langle \boldsymbol{u}, -\Delta\boldsymbol{v} + \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} - \langle \pi, \operatorname{div} \boldsymbol{v}\rangle_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{1}(\mathbb{R}^N_+)}\\ &\qquad\qquad = \langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{1}(\Gamma)}.\end{aligned} \tag{6.6}$$

Indeed, let $(\boldsymbol{u},\, \pi)$ be a solution to (4.1)–(4.3) with $g_N = 0$; then the Green formula (6.5) yields for all $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{X}'_0 \times \boldsymbol{X}_0} &= - \langle \boldsymbol{u}, \Delta \boldsymbol{v}\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} -\\ &- \langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{1}(\Gamma)} - \langle \pi, \operatorname{div} \boldsymbol{v}\rangle_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{1}(\mathbb{R}^N_+)} = 0.\end{aligned}$$

Moreover, using the density of the functions of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ with divergence zero in $\boldsymbol{T}_{0,\sigma}(\mathbb{R}^N_+)$, we obtain for all $\vartheta \in W^{1,p'}_1(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)} &= -\langle \operatorname{div}\boldsymbol{u}, \vartheta\rangle_{L^p(\mathbb{R}^N_+)\times L^{p'}(\mathbb{R}^N_+)} - \\ &- \langle u_N, \vartheta\rangle_{W^{-1/p,p}_{-1}(\Gamma)\times W^{1/p,p'}_1(\Gamma)} = 0.\end{aligned}$$

So we show that $(\boldsymbol{u}, \pi)$ satisfies the variational formulation (6.6). Conversely, we can readily prove that if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$ satisfies the variational formulation (6.6), then $(\boldsymbol{u}, \pi)$ is a solution to problem (4.1)–(4.3).

3) Let's solve problem (6.6). According to Theorem 5.2, we know that if $\dfrac{N}{p} \neq 1$, for all $\boldsymbol{f} \in \boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)$ and $\varphi \in \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)$, there exists a unique $(\boldsymbol{v}, \vartheta) \in \boldsymbol{M}_0(\mathbb{R}^N_+) \times W^{1,p'}_1(\mathbb{R}^N_+)$ solution to

$$-\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = \varphi \quad \text{in } \mathbb{R}^N_+, \qquad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,p'}_1(\mathbb{R}^N_+)} + \|\vartheta\|_{W^{1,p'}_1(\mathbb{R}^N_+)} \leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)} + \|\varphi\|_{W^{1,p'}_1(\mathbb{R}^N_+)}\right).$$

Then

$$\begin{aligned}\left|\langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_1(\Gamma)}\right| &\leq C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)}\|\boldsymbol{v}\|_{W^{2,p'}_1(\mathbb{R}^N_+)} \\ &\leq C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{-1}}\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p'}_1} + \|\varphi\|_{W^{1,p'}_1}\right).\end{aligned}$$

In other words, we can say that the linear mapping

$$T:\ (\boldsymbol{f}, \varphi) \longmapsto \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle$$

is continuous on $\boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)$, and according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$ which is the dual space of $\boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)$, such that

$$\begin{aligned}&\forall(\boldsymbol{f}, \varphi) \in \boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+),\\ &T(\boldsymbol{f}, \varphi) = \langle \boldsymbol{u}, \boldsymbol{f}\rangle_{\boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,p'}_1(\mathbb{R}^N_+)} + \langle \pi, -\varphi\rangle_{W^{-1,p}_{-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)},\end{aligned}$$

i.e. the pair $(\boldsymbol{u}, \pi)$ satisfies (6.6). □

We now can drop the hypothesis $g_N = 0$.

Theorem 6.7. *Assume that* $\dfrac{N}{p} \neq 1$. *For any* $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)$, *the Stokes problem (4.1)–(4.3) has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)} \;\leq\; C\;\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)}.$$

Proof. We know that (see Corollary 3.6 in [6]) if $\dfrac{N}{p} \neq 1$, then there exists $\psi \in W^{1,p}_{-1}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^{\Delta}_{[2-N/p]}$ solution to the following Neumann problem:

$$\Delta\psi = 0 \;\text{ in } \mathbb{R}^N_+, \quad \partial_N\psi = g_N \;\text{ on } \Gamma.$$

Let's set $\boldsymbol{w} = \nabla\psi$ and $\boldsymbol{g}^* = \boldsymbol{g} - \gamma_0\boldsymbol{w}$. Then $\boldsymbol{w} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$ and

$$\|\boldsymbol{w}\|_{\boldsymbol{T}_0(\mathbb{R}^N_+)} = \|\boldsymbol{w}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} \leq C\;\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)}.$$

Furthermore, $\boldsymbol{g}^*$ satisfies the hypotheses of Proposition 6.6, hence the existence of a unique pair $(\boldsymbol{z},\,\pi)$ which satisfies

$$-\Delta\boldsymbol{z} + \nabla\pi = \boldsymbol{0} \;\text{ and }\; \operatorname{div}\boldsymbol{z} = 0 \;\text{ in } \mathbb{R}^N_+, \quad \boldsymbol{z} = \boldsymbol{g}^* \;\text{ on } \Gamma.$$

Then the pair $(\boldsymbol{z} + \boldsymbol{w},\,\pi)$ is the required solution. The uniqueness of this solution is a straightforward consequence of Proposition 6.6. □

Here is the corresponding results for the case $\ell = 1$.

Proposition 6.8. *For any $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_0(\Gamma)$ such that $g_N = 0$, and $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N \leq p'$, the Stokes problem (4.1)–(4.3) has a unique solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W^{-1,\,p}_0(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,\,p}_0(\mathbb{R}^N_+)} \;\leq\; C\;\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_0(\Gamma)}.$$

Proof. The two differences from the weight $\ell = 0$ are the absence of critical value (the reason is that here, the dual problem solved by Theorem 5.6 has no critical value), and the orthogonality condition in the case $N \leq p'$ (which corresponds by duality to the non-zero kernel in Theorem 5.6 if $N \leq p$). The rest of the proof is similar. □

Theorem 6.9. *For any $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,\,p}_0(\Gamma)$ such that $\boldsymbol{g} \perp \mathbb{R}^N$ if $N \leq p'$, the Stokes problem (4.1)–(4.3) has a unique solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W^{-1,\,p}_0(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,\,p}_0(\mathbb{R}^N_+)} \;\leq\; C\;\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_0(\Gamma)}.$$

Remark 6.10. Let $p > 1$ be a real number. If $p < N$ and $r = Np/(N-p)$, then we have $W^{1-1/p,p}_0(\Gamma) \hookrightarrow W^{-1/r,\,r}_0(\Gamma)$. Indeed, for every $g \in W^{1-1/p,p}_0(\Gamma)$, there exists $u \in W_0^{2,p}(\mathbb{R}_+^N)$ such that

$$\Delta u = 0 \ \text{ in } \mathbb{R}_+^N, \quad \partial_N u = g \ \text{ on } \Gamma,$$

(see [6], Corollary 3.3). Since we have the imbedding $W_0^{2,p}(\mathbb{R}_+^N) \hookrightarrow W_0^{1,r}(\mathbb{R}_+^N)$, we can deduce that $\boldsymbol{v} = \nabla u \in \boldsymbol{L}^r(\mathbb{R}_+^N)$ and $\operatorname{div} \boldsymbol{v} = 0 \in W_1^{0,r}(\mathbb{R}_+^N)$, *i.e.* $\boldsymbol{v} \in \boldsymbol{W}_1^{0,p}(\operatorname{div}; \mathbb{R}_+^N)$. Moreover, as $r' \neq N$, according to Lemma 6.4, we get $\gamma_{e_N} \boldsymbol{v} = \partial_N u|_\Gamma = g \in W_0^{-1/r,r}(\Gamma)$. Consequently, if $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}_0^{-1/r,r}(\Gamma)$, Proposition 4.1 and Theorem 6.9 respectively yield the unique solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$ and $(\boldsymbol{v}, \vartheta) \in \boldsymbol{L}^r(\mathbb{R}_+^N) \times W_0^{-1,r}(\mathbb{R}_+^N)$, which are identical thanks to the Sobolev imbeddings $W_0^{1,p}(\mathbb{R}_+^N) \hookrightarrow L^r(\mathbb{R}_+^N)$ and $L^p(\mathbb{R}_+^N) \hookrightarrow W_0^{-1,r}(\mathbb{R}_+^N)$. □

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