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A locally compact quantum group of triangular matrices.

Pierre Fima∗ and Leonid Vainerman†

Dedicated to Professor M.L. Gorbachuk on the occasion of his 70-th anniversary.

Abstract
We construct a one parameter deformation of the group of $2 \times 2$ upper triangular matrices with determinant 1 using the twisting construction. An interesting feature of this new example of a locally compact quantum group is that the Haar measure is deformed in a non-trivial way. Also, we give a complete description of the dual $C^*$-algebra and the dual comultiplication.

1 Introduction

In [3, 14], M. Enock and the second author proposed a systematic approach to the construction of non-trivial Kac algebras by twisting. To illustrate it, consider a cocommutative Kac algebra structure on the group von Neumann algebra $M = L(G)$ of a non commutative locally compact (l.c.) group $G$ with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ (here $\lambda_g$ is the left translation by $g \in G$). Let us define on $M$ another, ”twisted”, comultiplication $\Delta_\Omega(\cdot) = \Omega \Delta(\cdot) \Omega^*$, where $\Omega$ is a unitary from $M \otimes M$ verifying certain 2-cocycle condition, and construct in this way new, non cocommutative, Kac algebra structure on $M$. In order to find such an $\Omega$, let us, following to M. Rieffel [10] and M. Landstad [8], take an inclusion $\alpha : L^\infty(\hat{K}) \to M$, where $\hat{K}$ is the dual to some abelian subgroup $K$ of $G$ such that $\delta_{|K} = 1$, where $\delta(\cdot)$ is the module of $G$. Then, one lifts a usual 2-cocycle $\Psi$ of $\hat{K}$: $\Omega = (\alpha \otimes \alpha)\Psi$. The main result of [3, 14] is that the integral by the Haar measure of $G$ gives also the Haar measure of the deformed object. Recently P. Kasprzak studied the deformation of l.c. groups by twisting in [5], and also in this case the Haar measure was not deformed.

In [3], the authors extended the twisting construction in order to cover the case of non-trivial deformation of the Haar measure. The aim of the present paper is to illustrate this construction on a concrete example and to compute

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explicitly all the ingredients of the twisted quantum group including the dual $C^*$-algebra and the dual comultiplication. We twist the group von Neumann algebra $L(G)$ of the group $G$ of $2 \times 2$ upper triangular matrices with determinant 1 using the abelian subgroup $K = \mathbb{C}^*$ of diagonal matrices of $G$ and a one parameter family of bicharacters on $K$. In this case, the subgroup $K$ is not included in the kernel of the modular function of $G$, this is why the Haar measure is deformed. We compute the new Haar measure and show that the dual $C^*$-algebra is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ such that

$$\hat{\alpha} \hat{\beta} = \hat{\beta} \hat{\alpha}, \quad \hat{\alpha} \hat{\beta}^* = q \hat{\beta}^* \hat{\alpha},$$

where $q > 0$. Moreover, the comultiplication $\hat{\Delta}$ is given by

$$\hat{\Delta}_1(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_1(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} \hat{\beta} \hat{\alpha}^{-1},$$

where $\hat{+}$ means the closure of the sum of two operators.

This paper is organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we present in detail our example computing all the ingredients associated. This example is inspired by [5], but an important difference is that in the present example the Haar measure is deformed in a non trivial way. Finally, we collect some useful results in the Appendix.

2 Preliminaries

2.1 Notations

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space $H$, $\otimes$ the tensor product of Hilbert spaces, von Neumann algebras or minimal tensor product of $C^*$-algebras, and $\Sigma$ (resp., $\sigma$) the flip map on it. If $H$, $K$ and $L$ are Hilbert spaces and $X \in B(H \otimes L)$ (resp., $X \in B(H \otimes K)$, $X \in B(K \otimes L)$), we denote by $X_{13}$ (resp., $X_{12}$, $X_{23}$) the operator $(1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)$ (resp., $X \otimes 1$, $1 \otimes X$) defined on $H \otimes K \otimes L$. For any subset $X$ of a Banach space $E$, we denote by $\langle X \rangle$ the vector space generated by $X$ and $[X]$ the closed vector space generated by $X$. All l.c. groups considered in this paper are supposed to be second countable, all Hilbert spaces are separable and all von Neumann algebras have separable preduals.

Given a normal semi-finite faithful (n.s.f.) weight $\theta$ on a von Neumann algebra $M$ (see [12]), we denote: $M_\theta^+ = \{ x \in M^+ \mid \theta(x) < +\infty \}$, $N_\theta = \{ x \in M \mid x^* x \in M_\theta^+ \}$, and $M_\theta = \langle M_\theta^+ \rangle$.

When $A$ and $B$ are $C^*$-algebras, we denote by $M(A)$ the algebra of the multipliers of $A$ and by $\text{Mor}(A,B)$ the set of the morphisms from $A$ to $B$.

2.2 $G$-products and their deformation

For the notions of an action of a l.c. group $G$ on a $C^*$-algebra $A$, a $C^*$ dynamical system $(A, G, \alpha)$, a crossed product $G_\alpha \rtimes A$ of $A$ by $G$ see [1]. The crossed product has the following universal property:
For any $C^*$-covariant representation $(\pi, u, B)$ of $(A, G, \alpha)$ (here $B$ is a $C^*$-algebra, $\pi : A \to B$ a morphism, $u$ is a group morphism from $G$ to the unitaries of $M(B)$, continuous for the strict topology), there is a unique morphism $\rho \in \text{Mor}(G_{\alpha} \ltimes A, B)$ such that
\[
\rho(\lambda_t) = u_t, \quad \rho(\pi_\alpha(x)) = \pi(x) \quad \forall t \in G, x \in A.
\]

**Definition 1** Let $G$ be a l.c. abelian group, $B$ a $C^*$-algebra, $\lambda$ a morphism from $G$ to the unitary group of $M(B)$, continuous in the strict topology of $M(B)$, and $\theta$ a continuous action of $G$ on $B$. The triplet $(B, \lambda, \theta)$ is called a $G$-product if $\theta_\gamma(\lambda_g) = \langle \gamma, g \rangle \lambda_g$ for all $\gamma \in \hat{G}$, $g \in G$.

The unitary representation $\lambda : G \to M(B)$ generates a morphism :
\[
\lambda \in \text{Mor}(C^*(G), B).
\]

Identifying $C^*(G)$ with $C_0(\hat{G})$, one gets a morphism $\lambda \in \text{Mor}(C_0(\hat{G}), B)$ which is defined in a unique way by its values on the characters
\[
u_g = (\gamma \mapsto \langle \gamma, g \rangle) \in C_0(\hat{G}) : \lambda(u_g) = \lambda_g, \quad \text{for all } g \in G.
\]

One can check that $\lambda$ is injective.

The action $\theta$ is done by: $\theta_\gamma(\lambda(u_g)) = \theta_\gamma(\lambda_g) = \langle \gamma, g \rangle \lambda_g = \lambda(u_g(\gamma, g))$. Since the $u_g$ generate $C_b(\hat{G})$, one deduces that:
\[
\theta_\gamma(\lambda(f)) = \lambda(f(\gamma, g)), \quad \text{for all } f \in C_b(\hat{G}).
\]

The following definition is equivalent to the original definition by Landstad [8] (see [3]):

**Definition 2** Let $(B, \lambda, \theta)$ be a $G$-product and $x \in M(B)$. One says that $x$ verifies the Landstad conditions if
\[
\begin{cases}
(i) \quad \theta_\gamma(x) = x, & \text{for any } \gamma \in \hat{G}, \\
(ii) \quad \text{the application } g \mapsto \lambda_g x \lambda_g^* \text{ is continuous}, \\
(iii) \quad \lambda(f)x\lambda(g) \in B, & \text{for any } f, g \in C_0(\hat{G}).
\end{cases}
\]

The set $A \in M(B)$ verifying these conditions is a $C^*$-algebra called the Landstad algebra of the $G$-product $(B, \lambda, \theta)$. Definition 2 implies that if $a \in A$, then $\lambda_g a \lambda_g^* \in A$ and the map $g \mapsto \lambda_g a \lambda_g^*$ is continuous. One gets then an action of $G$ on $A$.

One can show that the inclusion $A \to M(B)$ is a morphism of $C^*$-algebras, so $M(A)$ can be also included into $M(B)$. If $x \in M(B)$, then $x \in M(A)$ if and only if
\[
\begin{cases}
(i) \quad \theta_\gamma(x) = x, & \text{for all } \gamma \in \hat{G}, \\
(ii) \quad \text{for all } a \in A, \text{ the application } g \mapsto \lambda_g x \lambda_g^* a \text{ is continuous}.
\end{cases}
\]

Let us note that two first conditions of (1) imply (2).
The notions of $G$-product and crossed product are closely related. Indeed, if $(A, G, \alpha)$ is a $C^*$-dynamical system with $G$ abelian, let $B = G_\alpha \ltimes A$ be the crossed product and $\lambda$ the canonical morphism from $G$ into the unitary group of $M(B)$, continuous in the strict topology, and $\pi \in \text{Mor}(A, B)$ the canonical morphism of $C^*$-algebras. For $f \in K(G, A)$ and $\gamma \in \hat{G}$, one defines $(\theta_\gamma f)(t) = \langle \gamma, t \rangle f(t)$. One shows that $\theta_\gamma$ can be extended to the automorphisms of $B$ in such a way that $(B, \hat{G}, \theta)$ would be a $C^*$-dynamical system. Moreover, $(B, \lambda, \theta)$ is a $G$-product and the associated Landstad algebra is $\pi(A)$. $\theta$ is called the dual action.

Conversely, if $(B, \lambda, \theta)$ is a $G$-product, then one shows that there exists a $C^*$-dynamical system $(A, G, \alpha)$ such that $B = G_\alpha \ltimes A$. It is unique (up to a covariant isomorphism), $A$ is the Landstad algebra of $(B, \lambda, \theta)$ and $\alpha$ is the action of $G$ on $A$ given by $\alpha_t(x) = \lambda_t x \lambda_t^*$.

**Lemma 1** [5] Let $(B, \lambda, \theta)$ be a $G$-product and $V \subset A$ be a vector subspace of the Landstad algebra such that:

- $\lambda_g \lambda^*_g \subset V$, for any $g \in G$,
- $\lambda(C_0(\hat{G})) V \lambda(C_0(\hat{G}))$ is dense in $B$.

Then $V$ is dense in $A$.

Let $(B, \lambda, \theta)$ be a $G$-product, $A$ its Landstad algebra, and $\Psi$ a continuous bicharacter on $\hat{G}$. For $\gamma \in \hat{G}$, the function on $\hat{G}$ defined by $\Psi_\gamma(\omega) = \Psi(\omega, \gamma)$ generates a family of unitaries $\lambda(\Psi_\gamma) \in M(B)$. The bicharacter condition implies:

$$\theta_\gamma(U_{\gamma_2}) = \lambda(\Psi_{\gamma_2}(\cdot - \gamma_1)) = \Psi(\gamma_1, \gamma_2) U_{\gamma_2}, \quad \forall \gamma_1, \gamma_2 \in \hat{G}.$$ 

One gets then a new action $\theta^\Psi$ of $\hat{G}$ on $B$:

$$\theta^\Psi_\gamma(x) = U_{\gamma} \theta(x) U_{\gamma}^*.$$ 

Note that, by commutativity of $G$, one has:

$$\theta^\Psi_\gamma(\lambda_g) = U_{\gamma} \theta(\lambda_g) U_{\gamma}^* = \overline{\langle \gamma, g \rangle} \lambda_g, \quad \forall \gamma \in \hat{G}, g \in G.$$ 

The triplet $(B, \lambda, \theta^\Psi)$ is then a $G$-product, called a deformed $G$-product.

### 2.3 Locally compact quantum groups [6], [7]

A pair $(M, \Delta)$ is called a (von Neumann algebraic) l.c. quantum group when

- $M$ is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $\ast$-homomorphism which is coassociative: $(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$ (i.e., $(M, \Delta)$ is a Hopf-von Neumann algebra).
- There exist n.s.f. weights $\varphi$ and $\psi$ on $M$ such that
  \begin{itemize}
  \item $\varphi$ is left invariant in the sense that $\varphi((\omega \otimes \text{id}) \Delta(x)) = \varphi(x) \omega(1)$ for all $x \in M_+^\omega$ and $\omega \in M_+^\varphi$,
  \end{itemize}
Here, $\Lambda$ denotes the canonical GNS-map for $\varphi$. Left and right invariant weights are unique up to a positive scalar. We say that $W$ and $\Lambda$ satisfy the pentagonal equation: $W^aW_1W_3W_2 = W_2W_3W_1^a$, and we say that $W$ is a multiplicative unitary. The von Neumann algebra $M$ and the comultiplication on it can be given in terms of $W$ respectively as

$$M = \{(\Lambda \otimes \Lambda)(W) : \omega \in B(H)\}
$$

Left and right invariant weights are unique up to a positive scalar. Let us represent $\hat{M} = \Lambda \Lambda \Lambda$ the canonical GNS-map for $\varphi$ and define the unitary $W$ on $H \otimes H$ by

$$W^a(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)), \quad \text{for all } a, b \in N_\varphi.$$ Here, $\Lambda$ denotes the canonical GNS-map for $\varphi$, $\Lambda \otimes \Lambda$ the similar map for $\varphi \otimes \varphi$. One proves that $W$ satisfies the pentagonal equation: $W_1W_2W_3W_2 = W_2W_3W_1^a$, and we say that $W$ is a multiplicative unitary. The von Neumann algebra $M$ and the comultiplication on it can be given in terms of $W$ respectively as

$$M = \{(\Lambda \otimes \Lambda)(W) : \omega \in B(H)\}$$

and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. Next, the l.c. quantum group $(M, \Delta)$ has an antipode $S$, which is the unique $\sigma$-strongly* closed linear map from $M$ to $M$ satisfying $(\Lambda \otimes \Lambda)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)$, and $S((\Lambda \otimes \Lambda)(W)) = (\Lambda \otimes \Lambda)((W^*)^*)$ such that for all $x \in M$, we take $\psi := \Lambda \Lambda \Lambda$. Hence (see [1]), there is a unique positive, self-adjoint operator $\delta_M$ affiliated to $M$, such that $\sigma_\tau(\delta_M) = \nu \delta_M$ for all $\tau \in \mathbb{R}$ and $\psi = \varphi_{\delta_M}$. It is called the modular element of $(M, \Delta)$. If $\delta_M = 1$ we call $(M, \Delta)$ unimodular. The scaling constant can be characterized as well by the relative invariance $\varphi \varphi = \nu^{-t} \varphi$.

For the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ we have:

$$\hat{M} = \{((\Lambda \otimes \Lambda)(W)) = \Lambda \Lambda \Lambda$$

and $\hat{\Delta}(x) = W^t(1 \otimes x)W^t$ for all $x \in \hat{M}$. A left invariant n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ can be constructed explicitly and the associated multiplicative unitary is $W = \Sigma W^t \Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, let us denote its antipode by $\hat{S}$, its unitary antipode by $\hat{R}$ and its scaling group by $\hat{\tau}$. Then we can construct the dual of $(M, \Delta)$, starting from the left invariant weight $\varphi$. The bidual l.c. quantum group $(\hat{M}, \hat{\Delta})$ is isomorphic to $(M, \Delta)$.

$M$ is commutative if and only if $(M, \Delta)$ is generated by a usual l.c. group $G : M = L^\infty(G), (\Delta_G f)(g, h) = f(gh), (S_G f)(g) = f(g^{-1}), \varphi_G(f) = \int f(g) dg$, where $f \in L^\infty(G)$, $g, h \in G$ and we integrate with respect to the left Haar measure $dg$ on $G$. Then $\psi_G$ is given by $\psi_G(f) = \int f(g^{-1}) dg$ and $\delta_M$ by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

$L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and $(W_G \xi)(g, h) = \xi(g, g^{-1}h)$, for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = \mathcal{L}(G)$ is the group von Neumann
algebra generated by the left translations \((\lambda_g)_{g \in G}\) of \(G\) and \(\Delta_G(\lambda_g) = \lambda_g \otimes \lambda_g\). Clearly, \(\Delta_G^{op} := \sigma \circ \Delta_G = \Delta_G\), so \(\Delta_G\) is cocommutative.

\((M, \Delta)\) is a Kac algebra (see [2]) if \(\tau_t = \text{id}\), for all \(t\), and \(\Delta\) is affiliated with the center of \(M\). In particular, this is the case when \(M = L^\infty(G)\) or \(M = L(G)\).

We can also define the \(C^*\)-algebra of continuous functions vanishing at infinity on \((M, \Delta)\) by

\[
A = [\{(\omega \otimes 1)(W) \mid \omega \in B(H)\}]
\]

and the reduced \(C^*\)-algebra (or dual \(C^*\)-algebra) of \((M, \Delta)\) by

\[
\hat{A} = [\{(1 \otimes \omega)(W) \mid \omega \in B(H)\}].
\]

In the group case we have \(A = C_0(G)\) and \(\hat{A} = C_r(G)\). Moreover, we have \(\Delta \in \text{Mor}(A, A \otimes A)\) and \(\hat{\Delta} \in \text{Mor}(\hat{A}, \hat{A} \otimes \hat{A})\).

A l.c. quantum group is called compact if \(\varphi(1_M) < \infty\) and discrete if its dual is compact.

### 2.4 Twisting of locally compact quantum groups [4]

Let \((M, \Delta)\) be a locally compact quantum group and \(\Omega\) a unitary in \(M \otimes M\).

We say that \(\Omega\) is a 2-cocycle on \((M, \Delta)\) if

\[
(\Omega \otimes 1)(\Delta \otimes \text{id})(\Omega) = (1 \otimes \Omega)(\text{id} \otimes \Delta)(\Omega).
\]

As an example we can consider \(M = L^\infty(G)\), where \(G\) is a l.c. group, with \(\Delta_G\) as above, and \(\Omega = \Psi(\cdot, \cdot) \in L^\infty(G \times G)\) a usual 2-cocycle on \(G\), i.e., a measurable function with values in the unit circle \(T \subset \mathbb{C}\) verifying

\[
\Psi(s_1, s_2)\psi(s_1, s_2, s_3) = \psi(s_2, s_3)\psi(s_1, s_2, s_3), \quad \text{for almost all } s_1, s_2, s_3 \in G.
\]

This is the case for any measurable bicharacter on \(G\).

When \(\Omega\) is a 2-cocycle on \((M, \Delta)\), one can check that \(\Delta_{\Omega}(\cdot) = \Omega \Delta(\cdot) \Omega^*\) is a new coassociative comultiplication on \(M\). If \((M, \Delta)\) is discrete and \(\Omega\) is any 2-cocycle on it, then \((M, \Delta_{\Omega})\) is again a l.c. quantum group (see [1]), finite-dimensional case was treated in [13]). In the general case, one can proceed as follows. Let \(\alpha : (L^\infty(G), \Delta_G) \to (M, \Delta)\) be an inclusion of Hopf-von Neumann algebras, i.e., a faithful unital normal \(*\)-homomorphism such that \((\alpha \otimes \alpha) \circ \Delta_G = \Delta \circ \alpha\). Such an inclusion allows to construct a 2-cocycle of \((M, \Delta)\) by lifting a usual 2-cocycle of \(G : \Omega = (\alpha \otimes \alpha)\psi\). It is shown in [3] that if the image of \(\alpha\) is included into the centralizer of the left invariant weight \(\varphi\), then \(\varphi\) is also left invariant for the new comultiplication \(\Delta_{\Omega}\).

In particular, let \(G\) be a non commutative l.c. group and \(K\) a closed abelian subgroup of \(G\). By Theorem 6 of [1], there exists a faithful unital normal \(*\)-homomorphism \(\hat{\alpha} : \mathcal{L}(K) \to \mathcal{L}(G)\) such that

\[
\hat{\alpha}(\lambda^K_g) = \lambda_g, \quad \text{for all } g \in K, \quad \text{and} \quad \hat{\Delta} \circ \hat{\alpha} = (\hat{\alpha} \otimes \hat{\alpha}) \circ \hat{\Delta}_K,
\]
We define:

\[
\text{continuous bicharacter on } \mathbb{R}^G
\]

Let \( \Psi \) be a continuous bicharacter on the group homomorphism \( t \circ \phi \) of the group, \( S \).

This is why we do the following assumptions. Let \( (M, \delta) \) be a l.c. quantum group, \( G \) an abelian l.c. group and \( \alpha: (L^\infty(G), \Delta_G) \rightarrow (M, \Delta) \) an inclusion of Hopf-von Neumann algebras. Let \( \varphi \) be the left invariant weight, \( \sigma_t \) its modular group. Let \( \tilde{\varphi} \) be the right unitary antipode, \( \tilde{\phi} \) the unitary antipode, \( \tilde{\tau} \) the scaling group. Let \( \psi = \varphi \circ \tilde{\phi} \) be the right invariant weight and \( \sigma_t \) its modular group. We denote by \( \delta \) the modular element of \( (M, \Delta) \).

The 2-cocycle equation implies that \( u_t \) is a \( \sigma_t \)-cocycle and \( v_t \) is a \( \sigma_t' \)-cocycle. The Connes' Theorem gives two n.s.f. weights on \( M, \varphi \Omega \) and \( \psi \Omega \), such that

\[
\begin{align*}
    u_t &= \lambda^\frac{\nu^t}{2} \alpha(\Psi(-, \gamma_t)) \\
    v_t &= \lambda^\frac{-\nu^t}{2} \alpha(\Psi(\cdot, -, \gamma_t)).
\end{align*}
\]

The main result of \( \Omega \) is as follows:

**Theorem 1** \( (M, \Delta_\Omega) \) is a l.c. quantum group with left and right invariant weight \( \varphi \Omega \) and \( \psi \Omega \) respectively. Moreover, denoting by a subscript or a superscript \( \Omega \) the objects associated with \( (M, \Delta_\Omega) \) one has:

- \( \sigma_t^\Omega = \tau_t \),
- \( \nu_\Omega = \nu \) and \( \delta_\Omega = \delta A^{-1} B \),

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• $D(S_\Omega) = D(S)$ and, for all $x \in D(S)$, $S_\Omega(x) = uS(x)u^*$.

Remark that, because $\Psi$ is a bicharacter on $G$, $t \mapsto \alpha(\Psi(\cdot, -\gamma t))$ is a representation of $\mathbb{R}$ in the unitary group of $M$ and there exists a positive self-adjoint operator $A$ affiliated with $M$ such that

$$\alpha(\Psi(\cdot, -\gamma t)) = A^t, \quad \text{for all } t \in \mathbb{R}.$$  

We can also define a positive self-adjoint operator $B$ affiliated with $M$ such that

$$\alpha(\Psi(-\gamma, \cdot)) = B^t.$$  

We obtain:

$$u_t = \lambda^{it^2} A^t, \quad v_t = \lambda^{it^2} B^t.$$  

Thus, we have $\varphi_\Omega = \varphi_A$ and $\psi_\Omega = \psi_B$, where $\varphi_A$ and $\psi_B$ are the weights defined by S. Vaes in [13].

One can also compute the dual $C^*$-algebra and the dual comultiplication. We put:

$$L_\gamma = \alpha(u_\gamma), \quad R_\gamma = JL_\gamma J, \quad \text{for all } \gamma \in \hat{G}.$$  

From the representation $\gamma \mapsto L_\gamma$ we get the unital $*$-homomorphism $\lambda_L : L^\infty(G) \to M$ and from the representation $\gamma \mapsto R_\gamma$ we get the unital normal $*$-homomorphism $\lambda_R : L^\infty(G) \to M'$. Let $\hat{A}$ be the reduced $C^*$-algebra of $(M, \Delta)$. We can define an action of $\hat{G}^2$ on $\hat{A}$ by

$$\alpha_{\gamma_1, \gamma_2}(x) = L_{\gamma_1} R_{\gamma_2} x R^*_{\gamma_2} L^*_{\gamma_1}.$$  

Let us consider the crossed product $C^*$-algebra $B = \hat{G}^2 \rtimes \hat{A}$. We will denote by $\lambda$ the canonical morphism from $\hat{G}^2$ to the unitary group of $M(B)$ continuous in the strict topology on $M(B)$, $\pi \in \text{Mor}(\hat{A}, B)$ the canonical morphism and $\theta$ the dual action of $\hat{G}^2$ on $B$. Recall that the triplet $(\hat{G}^2, \lambda, \theta^\rho)$ is a $\hat{G}^2$-product. Let us denote by $(\hat{G}^2, \lambda, \theta^\Psi)$ the $\hat{G}^2$-product obtained by deformation of the $G^2$-product $(\hat{G}^2, \lambda, \theta)$ by the bicharacter $\omega(g, h, s, t) := \Psi(g, h, s)\Psi(h, t)$ on $G^2$.

The dual deformed action $\theta^\Psi$ is done by

$$\theta^\Psi_{(g_1, g_2)}(x) = U_{g_1} V_{g_1} \theta_{(g_1, g_2)}(x) U^*_{g_1} V^*_{g_2}, \quad \text{for any } g_1, g_2 \in G, \ x \in B,$$

where $U_g = \lambda_L(\Psi^*_g)$, $V_g = \lambda_R(\Psi_g)$, $\Psi_g(h) = \Psi(h, g)$.

Considering $\Psi_g$ as an element of $\hat{G}$, we get a morphism from $G$ to $\hat{G}$, also noted $\Psi$, such that $\Psi(g) = \Psi_g$. With these notations, one has $U_g = u(\psi(-g), 0)$ and $V_g = u(0, \psi(g))$. Then the action $\theta^\Psi$ on $\pi(\hat{A})$ is done by

$$\theta^\Psi_{(g_1, g_2)}(\pi(x)) = \pi(\alpha(\psi(-g_1, \Psi_{(g_2)})(x))). \quad (3)$$

Let us consider the Landstad algebra $A^\Psi$ associated with this $\hat{G}^2$-product. By definition of $\alpha$ and the universality of the crossed product we get a morphism

$$\rho \in \text{Mor}(B, K(H)), \quad \rho(\lambda_{\gamma_1, \gamma_2}) = L_{\gamma_1} R_{\gamma_2} \quad \text{et} \quad \rho(\pi(x)) = x. \quad (4)$$

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It is shown in [4] that \( \rho(A^{\Psi}) = \hat{A}_\Omega \) and that \( \rho \) is injective on \( A^{\Psi} \). This gives a canonical isomorphism \( A^{\Psi} \simeq \hat{A}_\Omega \). In the sequel we identify \( A^{\Psi} \) with \( \hat{A}_\Omega \). The comultiplication can be described in the following way. First, one can show that, using universality of the crossed product, there exists a unique morphism \( \Gamma \in \text{Mor}(B, B \otimes B) \) such that:

\[
\Gamma \circ \pi = (\pi \otimes \pi) \circ \Delta \quad \text{and} \quad \Gamma(\lambda_{\gamma_1, \gamma_2}) = \lambda_{\gamma_1,0} \otimes \lambda_{0,\gamma_2}.
\]

Then we introduce the unitary \( \Upsilon = (\lambda_R \otimes \lambda_L)(\tilde{\Psi}) \in M(B \otimes B) \), where \( \tilde{\Psi}(g, h) = \Psi(g, gh) \). This allows us to define the *-morphism \( \Gamma_{\Omega} \in \text{Mor}(A^{\Psi}, A^{\Psi} \otimes A^{\Psi}) \) is the comultiplication on \( A^{\Psi} \).

Note that if \( M = L(G) \) and \( K \) is an abelian closed subgroup of \( G \), the action \( \alpha \) of \( K^2 \) on \( C_0(G) \) is the left-right action.

### 3 Twisting of the group of 2 × 2 upper triangular matrices with determinant 1

Consider the following subgroup of \( SL_2(\mathbb{C}) \):

\[
G := \left\{ \begin{pmatrix} z & \omega \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^*, \omega \in \mathbb{C} \right\}.
\]

Let \( K \subset G \) be the subgroup of diagonal matrices in \( G \), i.e. \( K = \mathbb{C}^* \). The elements of \( G \) will be denoted by \((z, \omega), z \in \mathbb{C}, \omega \in \mathbb{C}^* \). The modular function of \( G \) is

\[
\delta_G((z, \omega)) = |z|^{-2}.
\]

Thus, the morphism \( (t \mapsto \gamma_t) \) from \( \mathbb{R} \) to \( \mathbb{C}^* \) is given by

\[
\langle \gamma_t, z \rangle = |z|^{2it}, \quad \text{for all} \quad z \in \mathbb{C}^*, t \in \mathbb{R}.
\]

We can identify \( \mathbb{C}^* \) with \( Z \times \mathbb{R}^*_+ \) in the following way:

\[
Z \times \mathbb{R}^*_+ \rightarrow \mathbb{C}^*, \quad (n, \rho) \mapsto \gamma_{n, \rho} = (re^{i\theta} \mapsto e^{i\ln r \ln \rho}e^{i\theta}),
\]

Under this identification, \( \gamma_t \) is the element \((0, e^t)\) of \( Z \times \mathbb{R}^*_+ \). For all \( x \in \mathbb{R} \), we define a bicharacter on \( Z \times \mathbb{R}^*_+ \) by

\[
\Psi_x((n, \rho), (k, r)) = e^{ix(k \ln \rho - n \ln r)}.
\]

We denote by \( (M_x, \Delta_x) \) the twisted l.c. quantum group. We have:

\[
\Psi_x((n, \rho), \gamma_t^{-1}) = e^{ixt} = u_{e^{ixt}}((n, \rho)).
\]

In this way we obtain the operator \( A_x \) deforming the Plancherel weight:

\[
A_x^\ell = \alpha(u_{e^{ixt}}) = \lambda_{e^{ixt},0}^G.
\]
In the same way we compute the operator $B_x$ deforming the Plancherel weight:

$$B_x^{it} = \lambda^{G}_{(e^{-it},0)} = A_x^{-it}.$$  

Thus, we obtain for the modular element:

$$\delta_x^{it} = A_x^{-it} B_x^{it} = \lambda^{G}_{(e^{-2it},0)}.$$  

The antipode is not deformed. The scaling group is trivial but, if $x \neq 0$, $(M_x, \Delta_x)$ is not a Kac algebra because $\delta_x$ is not affiliated with the center of $M$. Let us look if $(M_x, \Delta_x)$ can be isomorphic for different values of $x$. One can remark that, since $\Psi_{-x} = \Psi_x^*$ is antisymmetric and $\Delta$ is cocommutative, we have $\Delta_{-x} = \sigma \Delta_x$, where $\sigma$ is the flip on $\mathcal{L}(G) \otimes \mathcal{L}(G)$. Thus, $(M_{-x}, \Delta_{-x}) \simeq (M_x, \Delta_x)^{op}$, where "op" means the opposite quantum group. So, it suffices to treat only strictly positive values of $x$. The twisting deforms only the comultiplication, the weights and the modular element. The simplest invariant distinguishing the $(M_x, \Delta_x)$ is then the specter of the modular element. Using the Fourier transformation in the first variable, on has immediately $\text{Sp}(\delta_x) = q_x^e \cup \{0\}$, where $q_x = e^{-2x}$. Thus, if $x \neq y$, $x > 0, y > 0$, one has $q_x^e \neq q_y^e$ and, consequently, $(M_x, \Delta_x)$ and $(M_y, \Delta_y)$ are non isomorphic.

We compute now the dual $C^*$-algebra. The action of $K^2$ on $C_0(G)$ can be lifted to its Lie algebra $\mathbb{C}^2$. The lifting does not change the result of the deformation (see [5], Proposition 3.17) but simplify calculations. The action of $\mathbb{C}^2$ on $C_0(G)$ will be denoted by $\rho$. One has

$$\rho_{z_1,z_2}(f)(z,\omega) = f(e^{itz - z_1}z, e^{-(z_1 + z_2)}\omega). \quad (5)$$

The group $\mathbb{C}$ is self-dual, the duality is given by

$$(z_1, z_2) \mapsto \exp(i\text{Im}(z_1z_2)).$$

The generators $u_z, z \in \mathbb{C}$, of $C_0(\mathbb{C})$ are given by

$$u_z(w) = \exp(i\text{Im}(zw)), \quad z, w \in \mathbb{C}.$$  

Let $x \in \mathbb{R}$. We will consider the following bicharacter on $\mathbb{C}$:

$$\Psi_x(z_1, z_2) = \exp(iz\text{Im}(z_1\overline{z_2})).$$

Let $B$ be the crossed product $C^*$-algebra $\mathbb{C}^2 \times C_0(G)$. We denote by $((z_1, z_2) \mapsto \lambda_{z_1,z_2})$ the canonical group homomorphism from $G$ to the unitary group of $M(B)$, continuous for the strict topology, and $\pi \in \text{Mor}(C_0(G), B)$ the canonical homomorphism. Also we denote by $\lambda \in \text{Mor}(C_0(\mathbb{C}^2), B)$ the morphism given by the representation $((z_1, z_2) \mapsto \lambda_{z_1,z_2})$. Let $\theta$ be the dual action of $\mathbb{C}^2$ on $B$. We have, for all $z, w \in \mathbb{C}$, $\Psi_x(w, z) = u_x(z)$. The deformed dual action is given by

$$\theta_{z_1,z_2}^\Psi(b) = \lambda_{z_1,z_2} \theta_{z_1,z_2}(b) \lambda_{z_1,z_2}^*,$$ \quad (6)

Recall that

$$\theta_{z_1,z_2}^\Psi(\lambda(f)) = \theta_{z_1,z_2}(\lambda(f)) = \lambda(f(\cdot - z_1, \cdot - z_2)), \quad \forall f \in C_0(\mathbb{C}^2). \quad (7)$$
Let $\hat{A}_x$ be the associated Landstad algebra. We identify $\hat{A}_x$ with the reduced $C^*$-algebra of $(M_x, \Delta_x)$. We will now construct two normal operators affiliated with $\hat{A}_x$, which generate $A_x$. Let $a$ and $b$ be the coordinate functions on $G$, and $\alpha = \pi(a)$, $\beta = \pi(b)$. Then $\alpha$ and $\beta$ are normal operators, affiliated with $B$, and one can see, using (8), that

$$\lambda_{z_1, z_2}^* \lambda_{z_1, z_2}^* = e^{z_2-z_1} \alpha, \quad \lambda_{z_1, z_2} \lambda_{z_1, z_2}^* = e^{-(z_1+z_2)} \beta. \quad (8)$$

We can deduce, using (8), that

$$\theta_{z_1, z_2}^{\Psi} (\alpha) = e^{x(z_1 - z_2)} \alpha, \quad \theta_{z_1, z_2}^{\Psi} (\beta) = e^{x(z_1 - z_2)} \beta. \quad (9)$$

Let $T_l$ and $T_r$ be the infinitesimal generators of the left and right shift respectively, i.e. $T_l$ and $T_r$ are normal, affiliated with $B$, and

$$\lambda_{z_1, z_2} = \exp (i \text{Im}(z_1 T_l)) \exp (i \text{Im}(z_2 T_r)), \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$ 

Thus, we have:

$$\lambda(f) = f(T_l, T_r), \quad \text{for all } f \in C_b(\mathbb{C}^2).$$

Let $U = \lambda(\Psi_x)$, we define the following normal operators affiliated with $B$:

$$\hat{\alpha} = U^* \alpha U, \quad \hat{\beta} = U \beta U^*.$$ 

**Proposition 1** The operators $\hat{\alpha}$ and $\hat{\beta}$ are affiliated with $\hat{A}_x$ and generate $\hat{A}_x$.

**Proof.** First let us show that $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$. One has, using (8):

$$\theta_{z_1, z_2}^{\Psi}(U) = \lambda(\Psi_x(., -z_1, -z_2)) = \lambda(U e^{ix \text{Im}(-z_1 T_l)} e^{ix \text{Im}(-z_2 T_r)} \Psi_x(z_1, z_2)) = U \lambda_{-z_1, -z_2} \Psi_x(z_1, z_2).$$

Now, using (8) and (9), we obtain:

$$\theta_{z_1, z_2}^{\Psi} (\hat{\alpha}) = \hat{\alpha}, \quad \theta_{z_1, z_2}^{\Psi} (\hat{\beta}) = \hat{\beta}, \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$ 

Thus, for all $f \in C_0(\mathbb{C})$, $f(\hat{\alpha})$ and $f(\hat{\beta})$ are fixed points for the action $\theta^{\Psi_x}$. Let $f \in C_0(\mathbb{C})$. Using (8) we find:

$$\lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* = U^* f(e^{z_2 - z_1} \alpha) U,$$

$$\lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^* = U^* f(e^{-z_1-z_2} \beta) U. \quad (10)$$

Because $f$ is continuous and vanish at infinity, the applications

$$(z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* \quad \text{and} \quad (z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^*$$

are norm-continuous and $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$. 

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Taking in mind Proposition 3 (see Appendix), in order to show that \( \hat{\alpha} \) is affiliated with \( \hat{A}_x \), it suffices to show that the vector space \( \mathcal{I} \) generated by \( f(\hat{\alpha})a \), with \( f \in C_0(\mathbb{C}) \) and \( a \in \hat{A}_x \), is dense in \( \hat{A}_x \). Using (10), we see that \( \mathcal{I} \) is globally invariant under the action implemented by \( \lambda \). Let \( g(z) = (1 + \tau z)^{-1} \). As \( \lambda(C_0(\mathbb{C}^2))U = \lambda(C_0(\mathbb{C}^2)) \), we can deduce that the closure of \( \lambda(C_0(\mathbb{C}^2))g(\hat{\alpha})\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is equal to

\[
\left[ \lambda(C_0(\mathbb{C}^2))(1 + \alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \right].
\]

As the set \( U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \) and \( \alpha \) is affiliated with \( B \), the set \( \lambda(C_0(\mathbb{C}^2))(1 + \alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \). Moreover, it is included in \( \lambda(C_0(\mathbb{C}^2))I\lambda(C_0(\mathbb{C}^2)) \), so \( \lambda(C_0(\mathbb{C}^2))I\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \). We conclude, using Lemma 1, that \( \mathcal{I} \) is dense in \( \hat{A}_x \). One can show in the same way that \( \hat{\beta} \) is affiliated with \( \hat{A}_x \).

Now, let us show that \( \hat{\alpha} \) and \( \hat{\beta} \) generate \( \hat{A}_x \). By Proposition 3, it suffices to show that

\[
\mathcal{V} = \left\{ f(\hat{\alpha})g(\hat{\beta}), \ f, g \in C_0(\mathbb{C}) \right\}
\]

is a dense vector subspace of \( \hat{A}_x \). We have shown above that the elements of \( \mathcal{V} \) satisfy the two first Landstad’s conditions. Let

\[
\mathcal{W} = \left[ \lambda(C_0(\mathbb{C}^2))\mathcal{V}\lambda(C_0(\mathbb{C}^2)) \right].
\]

We will show that \( \mathcal{W} = B \). This proves that the elements of \( \mathcal{V} \) satisfy the third Landstad’s condition, and then \( \mathcal{V} \subseteq \hat{A}_x \). Then (10) shows that \( \mathcal{W} \) is globally invariant under the action implemented by \( \lambda \), so \( \mathcal{V} \) is dense in \( \hat{A}_x \) by Lemma 1.

One has:

\[
\mathcal{W} = \left[ xU^*f(\alpha)U^2g(\beta)U^*y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2)) \right].
\]

Because \( U \) is unitary, we can substitute \( x \) with \( xU \) and \( y \) with \( Uy \) without changing \( \mathcal{W} \):

\[
\mathcal{W} = \left[ xf(\alpha)U^2g(\beta)y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2)) \right].
\]

Using, for all \( f \in C_0(\mathbb{C}) \), the norm-continuity of the application

\[
(z_1, z_2) \mapsto \lambda_{z_1, z_2}f(\alpha)\lambda_{z_1, z_2}^* = e^{z_2^*-z_1^*}\alpha,
\]

one deduces that

\[
[xf(\alpha)x, \ f \in C_0(\mathbb{C}), \ x \in \lambda(C_0(\mathbb{C}^2))]
= [xf(\alpha), \ f \in C_0(\mathbb{C}), \ x \in \lambda(C_0(\mathbb{C}^2))].
\]

In particular,

\[
\mathcal{W} = [xf(\alpha)xU^2g(\beta)y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2))].
\]
Now we can commute $g(\beta)$ and $y$, and we obtain:
\[ W = [f(\alpha)xyg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))] . \]
Substituting $x \mapsto xU^*$, $y \mapsto U^*y$, one has:
\[ W = [f(\alpha)xyg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))] . \]
Commuting back $f(\alpha)$ with $x$ and $g(\beta)$ with $y$, we obtain:
\[ W = [xf(\alpha)g(\beta)y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))] = B. \]
This concludes the proof.

We will now find the commutation relations between $\hat{\alpha}$ and $\hat{\beta}$.

**Proposition 2** One has:

1. $\alpha$ et $T_i^* + T_r^*$ strongly commute and $\hat{\alpha} = e^{x(T_i^* + T_r^*)}\alpha$.
2. $\beta$ et $T_i^* - T_r^*$ strongly commute and $\hat{\beta} = e^{x(T_i^* - T_r^*)}\beta$.

Thus, the polar decompositions are given by:

\[ Ph(\hat{\alpha}) = e^{-ix\text{Im}(T_i + T_r)} Ph(\alpha), \quad |\hat{\alpha}| = e^{x\text{Re}(T_i + T_r)}|\alpha|, \]
\[ Ph(\hat{\beta}) = e^{-ix\text{Im}(T_i - T_r)} Ph(\beta), \quad |\hat{\beta}| = e^{x\text{Re}(T_i - T_r)}|\beta|. \]

Moreover, we have the following relations:

1. $|\hat{\alpha}|$ and $|\hat{\beta}|$ strongly commute,
2. $Ph(\hat{\alpha})Ph(\hat{\beta}) = Ph(\hat{\beta})Ph(\hat{\alpha})$,
3. $Ph(\hat{\alpha})|\hat{\beta}|Ph(\hat{\alpha})^* = e^{2x}|\hat{\beta}|$,
4. $Ph(\hat{\beta})|\hat{\alpha}|Ph(\hat{\beta})^* = e^{2x}|\hat{\alpha}|$.

**Proof.** Using (8), we find, for all $z \in \mathbb{C}$:

\[ e^{ix\text{Im}(T_i + T_r)} \lambda e^{-ix\text{Im}(z(T_i^* + T_r^*))} = \lambda e^{-i\text{Im}(z(T_i^* + T_r^*))}. \]

Thus, $T_i^* + T_r^*$ and $\alpha$ strongly commute. Moreover, because $e^{ix\text{Im}(T_i^* + T_r^*)} = 1$, one has:

\[ \hat{\alpha} = e^{-ix\text{Im}(T_i)T_r^*} \alpha e^{ix\text{Im}(T_i^*T_r)} e^{-ix\text{Im}(T_i^*T_r)} = e^{-ix\text{Im}(T_i^*T_r)} = e^{-ix\text{Im}(T_i^*T_r)} \alpha e^{ix\text{Im}(T_i^*T_r)}. \]

We can now prove the point 1 using the equality $e^{-ix\text{Im}(T_i^*T_r)} \alpha e^{ix\text{Im}(T_i^*T_r)} = e^{-ix\text{Im}(T_i^*T_r)} \alpha$.

The proof of the second assertion is similar and the polar decompositions follows.

From (8), we deduce:
\[ e^{-ix\text{Im}(T_r-T_i)}\alpha e^{ix\text{Im}(T_r-T_i)} = e^{-2x}\alpha, \]
\[ e^{ix\text{Im}(T_l+T_r)}\beta e^{-ix\text{Im}(T_l+T_r)} = e^{-2x}\beta, \]
\[ e^{ix\text{Re}(T_r-T_i)}\alpha e^{-ix\text{Re}(T_r-T_i)} = e^{2ix}\alpha, \]
\[ e^{ix\text{Re}(T_l+T_r)}\beta e^{-ix\text{Re}(T_l+T_r)} = e^{-2ix}\beta. \]

It is now easy to prove the last relations from the preceding equations and the polar decompositions. ■

We can now give a formula for the comultiplication.

**Proposition 3** Let \( \hat{\Delta}_x \) be the comultiplication on \( \hat{A}_x \). One has:
\[
\hat{\Delta}_x(\hat{a}) = \hat{a} \otimes \hat{a}, \quad \hat{\Delta}_x(\hat{b}) = \hat{a} \otimes \hat{b} \hat{b} \hat{a}^{-1}.
\]

**Proof.** Using the Preliminaries, we have that \( \hat{\Delta}_x = \gamma(\cdot)\gamma^* \), where
\[
\gamma = e^{ix\text{Im}T_r \otimes T_l^*}
\]
and \( \gamma \) is given by
- \( \Gamma(T_l) = T_l \otimes 1 \), \( \Gamma(T_r) = 1 \otimes T_r \);
- \( \Gamma \) restricted to \( C_0(G) \) is equal to the comultiplication \( \Delta_G \).

Define \( R = \gamma(\gamma^*) \). One has \( \hat{\Delta}_x(\gamma) = R(\gamma \otimes \gamma)R^* \). Thus, it is sufficient to show that \( (U \otimes U)R \) commute with \( \gamma \otimes \gamma \). Indeed, in this case, one has
\[
\hat{\Delta}_x(\hat{a}) = R(\gamma \otimes \gamma)R^* = (U^* \otimes U^*)(U \otimes U)R(\gamma \otimes \gamma)R^*(U^* \otimes U^*)(U \otimes U) = \gamma \otimes \gamma.
\]

Let us show that \( (U \otimes U)R \) commute with \( \gamma \otimes \gamma \). From the equality \( U = e^{ix\text{Im}T_l T_l^*} \), we deduce that
\[
\Gamma(U^*) = e^{-ix\text{Im}T_l \otimes T_l^*}, \quad U \otimes U = e^{ix\text{Im}(T_l T_l^* \otimes 1 + 1 \otimes T_l T_l^*)}.
\]
Thus, 
\[
R = e^{-ix\text{Im}(T_l^* \otimes T_l^* + T_l \otimes T_l^*)} \quad \text{and} \quad (U \otimes U)R = e^{ix\text{Im}(T_l T_l^* \otimes 1 + 1 \otimes T_l T_l^* - T_l^* T_l - T_l T_l^*)}.
\]

Notice that
\[
T_l T_l^* \otimes 1 + 1 \otimes T_l T_l^* - T_l^* T_l - T_l T_l^* = (T_l \otimes 1 - 1 \otimes T_l)(T_l^* \otimes 1 - 1 \otimes T_l^*).
\]

Thus, it suffices to show that \( T_l \otimes 1 - 1 \otimes T_l \) and \( T_l^* \otimes 1 - 1 \otimes T_l^* \) strongly commute with \( \gamma \otimes \gamma \). This follows from the equations
\[
e^{i\text{Im}(T_l^* \otimes 1 - 1 \otimes T_l^*)}((\gamma \otimes \gamma)e^{-i\text{Im}(T_l^* \otimes 1 - 1 \otimes T_l^*)})
\[
= (\lambda_{0,-\tau} \otimes \lambda_{0,\tau})(\gamma \otimes \gamma)(\lambda_{0,-\tau} \otimes \lambda_{0,\tau})
\[
= e^{-x}e^{\tau}a \otimes a = a \otimes a, \quad \forall z \in \mathbb{C}
\]
and
\[
e^{i\text{Im}z(T_1 \otimes 1 - 1 \otimes T_1)}(\alpha \otimes \alpha)e^{-i\text{Im}z(T_1 \otimes 1 - 1 \otimes T_1)} = (\lambda_{z,0} \otimes \lambda_{-z,0})(\alpha \otimes \alpha)(\lambda_{z,0} \otimes \lambda_{-z,0})^* = e^{-z^2}e^{i\text{Im}z(T_1 \otimes 1 - 1 \otimes T_1)}.
\]

Put \( S = \Upsilon \Gamma(U) \). One has:
\[
\hat{\Delta}_x(\hat{\beta}) = S(\alpha \otimes \beta + \beta \otimes (\alpha^{-1}))S^* = S(\alpha \otimes \beta)S^* + S(\beta \otimes (\alpha^{-1}))S^*. 
\]
As before, we see that it suffices to show that \((U \otimes U^*)S\) commutes with \(\alpha \otimes \beta\) and that \((U^* \otimes U)S\) commutes with \(\beta \otimes (\alpha^{-1})\), and one can check this in the same way.

Let us summarize the preceding results in the following corollary (see \([16, 5]\) for the definition of commutation relation between unbounded operators):

**Corollary 1** Let \( q = e^{8x} \). The \( C^* \)-algebra \( \hat{A}_x \) is generated by 2 normal operators \( \hat{\alpha} \) and \( \hat{\beta} \) affiliated with \( \hat{A}_x \) such that \( \hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \hat{\alpha}^* = q\hat{\beta}^*\hat{\alpha} \).

Moreover, the comultiplication \( \hat{\Delta}_x \) is given by
\[
\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes (\alpha^{-1}).
\]

**Remark.** One can show, using the results of \([4]\), that the application \((q \mapsto W_q)\) which maps the parameter \( q \) to the multiplicative unitary of the twisted l.c. quantum group is continuous in the \( \sigma \)-weak topology.

## 4 Appendix

Let us cite some results on operators affiliated with a \( C^* \)-algebra.

**Proposition 4** Let \( A \subset B(H) \) be a non degenerated \( C^* \)-subalgebra and \( T \) a normal densely defined closed operator on \( H \). Let \( \mathcal{I} \) be the vector space generated by \( f(T)a \), where \( f \in \mathcal{C}_0(\mathbb{C}) \) and \( a \in A \). Then:
\[
(T \eta A) \iff \left( f(T) \in M(A) \text{ for any } f \in \mathcal{C}_0(\mathbb{C}) \right). 
\]

**Proof.** If \( T \) is affiliated with \( A \), then it is clear that \( f(T) \in M(A) \) for any \( f \in \mathcal{C}_0(\mathbb{C}) \), and that \( \mathcal{I} \) is dense in \( A \) (because \( \mathcal{I} \) contains \( (1 + T^*T)^{-\frac{1}{2}}A \)). To show the converse, consider the \( * \)-homomorphism \( \pi_T : \mathcal{C}_0(\mathbb{C}) \to M(A) \) given by \( \pi_T(f) = f(T) \). By hypothesis, \( \pi_T(\mathcal{C}_0(\mathbb{C}))A \) is dense in \( A \). So, \( \pi_T \in \text{Mor}(\mathcal{C}_0(\mathbb{C}), A) \) and \( T = \pi_T(z \mapsto z) \) is then affiliated with \( A \).
Proposition 5 Let $A \subset \mathcal{B}(H)$ be a non degenerated $C^\ast$-subalgebra and $T_1, T_2, \ldots, T_N$ normal operators affiliated with $A$. Let us denote by $V$ the vector space generated by the products of the form $f_1(T_1)f_2(T_2)\ldots f_N(T_N)$, with $f_i \in C_0(\mathbb{C})$. If $V$ is a dense vector subspace of $A$, then $A$ is generated by $T_1, T_2, \ldots, T_N$.

Proof. This follows from Theorem 3.3 in [15].

References


