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A locally compact quantum group of triangular matrices.

Pierre Fima* and Leonid Vainerman†

Dedicated to Professor M.L. Gorbachuk on the occasion of his 70-th anniversary.

Abstract

We construct a one parameter deformation of the group of $2 \times 2$ upper triangular matrices with determinant 1 using the twisting construction. An interesting feature of this new example of a locally compact quantum group is that the Haar measure is deformed in a non-trivial way. Also, we give a complete description of the dual $C^*$-algebra and the dual comultiplication.

1 Introduction

In [3, 14], M. Enock and the second author proposed a systematic approach to the construction of non-trivial Kac algebras by twisting. To illustrate it, consider a cocommutative Kac algebra structure on the group von Neumann algebra $M = L(G)$ of a non commutative locally compact (l.c.) group $G$ with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ (here $\lambda_g$ is the left translation by $g \in G$). Let us define on $M$ another, ”twisted”, comultiplication $\Delta_\Omega(\cdot) = \Omega \Delta(\cdot) \Omega^*$, where $\Omega$ is a unitary from $M \otimes M$ verifying certain 2-cocycle condition, and construct in this way new, non cocommutative, Kac algebra structure on $M$. In order to find such an $\Omega$, let us, following to M. Rieffel [10] and M. Landstad [8], take an inclusion $\alpha : L^\infty(\hat{K}) \to M$, where $\hat{K}$ is the dual to some abelian subgroup $K$ of $G$ such that $\delta|_K = 1$, where $\delta(\cdot)$ is the module of $G$. Then, one lifts a usual 2-cocycle $\Psi$ of $\hat{K}$: $\Omega = (\alpha \otimes \alpha) \Psi$. The main result of [3], [14] is that the integral by the Haar measure of $G$ gives also the Haar measure of the deformed object. Recently P. Kasprzak studied the deformation of l.c. groups by twisting in [5], and also in this case the Haar measure was not deformed.

In [3], the authors extended the twisting construction in order to cover the case of non-trivial deformation of the Haar measure. The aim of the present paper is to illustrate this construction on a concrete example and to compute

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explicitly all the ingredients of the twisted quantum group including the dual $C^*$-algebra and the dual comultiplication. We twist the group von Neumann algebra $L(G)$ of the group $G$ of $2 \times 2$ upper triangular matrices with determinant 1 using the abelian subgroup $K = \mathbb{C}^\ast$ of diagonal matrices of $G$ and a one parameter family of bicharacters on $K$. In this case, the subgroup $K$ is not included in the kernel of the modular function of $G$, this is why the Haar measure is deformed. We compute the new Haar measure and show that the dual $C^*$-algebra is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ such that
\[
\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha}, \quad \hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha},
\]
where $q > 0$. Moreover, the comultiplication $\hat{\Delta}$ is given by
\[
\hat{\Delta}_l(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_r(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\alpha}^{-1},
\]
where $\hat{\beta}$ means the closure of the sum of two operators.

This paper in organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we present in detail our example computing all the ingredients associated. This example is inspired by [5], but an important difference is that in the present example the Haar measure is deformed in a non trivial way. Finally, we collect some useful results in the Appendix.

2 Preliminaries

2.1 Notations

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space $H$, $\otimes$ the tensor product of Hilbert spaces, von Neumann algebras or minimal tensor product of $C^*$-algebras, and $\Sigma$ (resp., $\sigma$) the flip map on it. If $H$, $K$ and $L$ are Hilbert spaces and $X \in B(H \otimes L)$ (resp., $X \in B(H \otimes K)$, $X \in B(K \otimes L)$), we denote by $X_{13}$ (resp., $X_{12}$, $X_{23}$) the operator $(1 \otimes \Sigma)(X \otimes 1)(1 \otimes \Sigma)$ (resp., $X \otimes 1$, $1 \otimes X$) defined on $H \otimes K \otimes L$. For any subset $X$ of a Banach space $E$, we denote by $\langle X \rangle$ the vector space generated by $X$ and $[X]$ the closed vector space generated by $X$. All l.c. groups considered in this paper are supposed to be second countable, all Hilbert spaces are separable and all von Neumann algebras have separable preduals.

Given a normal semi-finite faithful (n.s.f.) weight $\theta$ on a von Neumann algebra $M$ (see [12]), we denote: $M^+_\theta = \{x \in M^+ \mid \theta(x) < +\infty\}$, $N_\theta = \{x \in M \mid x^*x \in M^+\}$, and $M_\theta = \langle M^+_{\theta} \rangle$.

When $A$ and $B$ are $C^*$-algebras, we denote by $M(A)$ the algebra of the multipliers of $A$ and by $\text{Mor}(A,B)$ the set of the morphisms from $A$ to $B$.

2.2 $G$-products and their deformation

For the notions of an action of a l.c. group $G$ on a $C^*$-algebra $A$, a $C^*$ dynamical system $(A, G, \alpha)$, a crossed product $G_\alpha \ltimes A$ of $A$ by $G$ see [1]. The crossed product has the following universal property:
For any $C^*$-covariant representation $(\pi, u, B)$ of $(A, G, \alpha)$ (here $B$ is a $C^*$-algebra, $\pi : A \to B$ a morphism, $u$ is a group morphism from $G$ to the unitaries of $M(B)$, continuous for the strict topology), there is a unique morphism $\rho \in \text{Mor}(G_\alpha \ltimes A, B)$ such that

$$\rho(\lambda_t) = u_t, \quad \rho(\pi_\alpha(x)) = \pi(x) \quad \forall t \in G, x \in A.$$  

**Definition 1** Let $G$ be a l.c. abelian group, $B$ a $C^*$-algebra, $\lambda$ a morphism from $G$ to the unitary group of $M(B)$, continuous in the strict topology of $M(B)$, and $\theta$ a continuous action of $G$ on $B$. The triplet $(B, \lambda, \theta)$ is called a $G$-product if $\theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g$ for all $\gamma \in \hat{G}$, $g \in G$.

The unitary representation $\lambda : G \to M(B)$ generates a morphism:

$$\lambda \in \text{Mor}(C^*(G), B).$$

Identifying $C^*(G)$ with $C_0(\hat{G})$, one gets a morphism $\lambda \in \text{Mor}(C_0(\hat{G}), B)$ which is defined in a unique way by its values on the characters

$$u_g = (\gamma \mapsto \langle \gamma, g \rangle) \in C_0(\hat{G}) : \lambda(u_g) = \lambda_g, \quad \text{for all } g \in G.$$  

One can check that $\lambda$ is injective.

The action $\theta$ is done by: $\theta_\gamma(\lambda(u_g)) = \theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g = \lambda(u_g(\cdot, -\gamma))$. Since the $u_g$ generate $C_b(\hat{G})$, one deduces that:

$$\theta_\gamma(\lambda(f)) = \lambda(f(\cdot, -\gamma)), \quad \text{for all } f \in C_b(\hat{G}).$$

The following definition is equivalent to the original definition by Landstad [8] (see [5]):

**Definition 2** Let $(B, \lambda, \theta)$ be a $G$-product and $x \in M(B)$. One says that $x$ verifies the Landstad conditions if

$$\left\{ \begin{array}{l}
(i) \quad \theta_\gamma(x) = x, \quad \text{for any } \gamma \in \hat{G}, \\
(ii) \quad \text{the application } g \mapsto \lambda_g x \lambda_g^* \text{ is continuous}, \\
(iii) \quad \lambda(f)x \lambda(g) \in B, \quad \text{for any } f, g \in C_0(\hat{G}).
\end{array} \right. \tag{1}$$

The set $A \in M(B)$ verifying these conditions is a $C^*$-algebra called the Landstad algebra of the $G$-product $(B, \lambda, \theta)$. Definition 2 implies that if $a \in A$, then $\lambda_g a \lambda_g^* \in A$ and the map $g \mapsto \lambda_g a \lambda_g^*$ is continuous. One gets then an action of $G$ on $A$.

One can show that the inclusion $A \to M(B)$ is a morphism of $C^*$-algebras, so $M(A)$ can be also included into $M(B)$. If $x \in M(B)$, then $x \in M(A)$ if and only if

$$\left\{ \begin{array}{l}
(i) \quad \theta_\gamma(x) = x, \quad \text{for all } \gamma \in \hat{G}, \\
(ii) \quad \text{for all } a \in A, \text{ the application } g \mapsto \lambda_g x \lambda_g^* a \text{ is continuous.} \tag{2}
\end{array} \right.$$  

Let us note that two first conditions of (1) imply (2).
The notions of $G$-product and crossed product are closely related. Indeed, if $(A, G, \alpha)$ is a $C^\ast$-dynamical system with $G$ abelian, let $B = G_\alpha \ltimes A$ be the crossed product and $\lambda$ the canonical morphism from $G$ into the unitary group of $M(B)$, continuous in the strict topology, and $\pi \in \text{Mor}(A, B)$ the canonical morphism of $C^\ast$-algebras. For $f \in \mathcal{K}(G, A)$ and $\gamma \in \hat{G}$, one defines $(\theta_\gamma f)(t) = \langle \gamma, t \rangle f(t)$. One shows that $\theta_\gamma$ can be extended to the automorphisms of $B$ in such a way that $(B, \hat{G}, \theta)$ would be a $C^\ast$-dynamical system. Moreover, $(B, \lambda, \theta)$ is a $G$-product and the associated Landstad algebra is $\pi(A)$. $\theta$ is called the dual action. Conversely, if $(B, \lambda, \theta)$ is a $G$-product, then one shows that there exists a $C^\ast$-dynamical system $(A, G, \alpha)$ such that $B = G_\alpha \ltimes A$. It is unique (up to a covariant isomorphism), $A$ is the Landstad algebra of $(B, \lambda, \theta)$ and $\alpha$ is the action of $G$ on $A$ given by $\alpha_t(x) = \lambda t x \lambda^\ast t$.

**Lemma 1** [5] Let $(B, \lambda, \theta)$ be a $G$-product and $V \subset A$ be a vector subspace of the Landstad algebra such that:

- $\lambda_g V \lambda_g^\ast \subset V$, for any $g \in G$,
- $\lambda(C_0(\hat{G})) V \lambda(C_0(\hat{G}))$ is dense in $B$.

Then $V$ is dense in $A$.

Let $(B, \lambda, \theta)$ be a $G$-product, $A$ its Landstad algebra, and $\Psi$ a continuous bicharacter on $\hat{G}$. For $\gamma \in \hat{G}$, the function on $\hat{G}$ defined by $\Psi_\gamma(\omega) = \Psi(\omega, \gamma)$ generates a family of unitaries $\lambda(\Psi_\gamma) \in M(B)$. The bicharacter condition implies:

$$\theta_\gamma(U_{\gamma_2}) = \lambda(\Psi_{\gamma_2}(- \gamma_1)) = \overline{\Psi(\gamma_1, \gamma_2) U_{\gamma_2}}, \quad \forall \gamma_1, \gamma_2 \in \hat{G}.$$ 

One gets then a new action $\theta^\Psi$ of $\hat{G}$ on $B$:

$$\theta^\Psi_\gamma(x) = U_{\gamma} \theta(x) U_{\gamma}^\ast.$$ 

Note that, by commutativity of $G$, one has:

$$\theta^\Psi_\gamma(\lambda_g) = U_{\gamma} \theta(\lambda_g) U_{\gamma}^\ast = \overline{\langle \gamma, g \rangle} \lambda_g, \quad \forall \gamma \in \hat{G}, g \in G.$$ 

The triplet $(B, \lambda, \theta^\Psi)$ is then a $G$-product, called a deformed $G$-product.

### 2.3 Locally compact quantum groups [4], [7]

A pair $(M, \Delta)$ is called a (von Neumann algebraic) l.c. quantum group when

- $M$ is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism which is coassociative: $(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$ (i.e., $(M, \Delta)$ is a Hopf-von Neumann algebra).
- There exist n.s.f. weights $\varphi$ and $\psi$ on $M$ such that
  - $\varphi$ is left invariant in the sense that $\varphi((\omega \otimes \text{id}) \Delta(x)) = \varphi(x) \omega(1)$ for all $x \in M_2^+$ and $\omega \in M_1^+$,  
  - $\psi((\omega \otimes \text{id}) \Delta(x)) = \psi(x) \omega(1)$ for all $x \in M_1^+$ and $\omega \in M_2^+$.  

Here, $\Lambda$ denotes the canonical GNS-map for $\varphi$. One proves that and we say that $W$ and the comultiplication on it can be given in terms of $dg$ measure for all $\xi$ and $\Delta(\varphi^\tau) = \varphi^\tau \Delta$.

Let us represent $M$ on the GNS Hilbert space of $\varphi$ and define a unitary $W$ on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)),$$

for all $a, b \in N_\varphi$. Here, $\Lambda$ denotes the canonical GNS-map for $\varphi$, $\Lambda \otimes \Lambda$ the similar map for $\varphi \otimes \varphi$.

One proves that $W$ satisfies the pentagonal equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that $W$ is a multiplicative unitary. The von Neumann algebra $M$ and the comultiplication on it can be given in terms of $W$ respectively as

$$M = \{ (\id \otimes \omega)(W) \mid \omega \in B(H)_+ \}^{\sigma-strong*}$$

and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. Next, the l.c. quantum group $(M, \Delta)$ has an antipode $S$, which is the unique $\sigma$-strongly* closed linear map from $M$ to $M$ satisfying $(\id \otimes \omega)(W) \in D(S)$ for all $\omega \in B(H)$, and $S(\id \otimes \omega)(W) = (\id \otimes \omega)(W^*)$ and such that the elements $(\id \otimes \omega)(W)$ form a $\sigma$-strong* core for $S$. $S$ has a polar decomposition $S = R\tau_{-1/2}$, where $R$ (the unitary antipode) is an anti-automorphism of $M$ and $\tau_1$ (the scaling group of $(M, \Delta)$) is a strongly continuous one-parameter group of automorphisms of $M$. We have $\sigma(R \otimes R) = \Delta R$, so $\varphi R$ is a right invariant weight on $(M, \Delta)$ and we take $\psi := \varphi R$.

Let $\sigma_t$ be the modular automorphism group of $\varphi$. There exist a number $\nu > 0$, called the scaling constant, such that $\psi \sigma_t = \nu^{-t} \psi$ for all $t \in \mathbb{R}$. Hence (see [3]), there is a unique positive, self-adjoint operator $\delta_M$ affiliated to $M$, such that $\sigma_t(\delta_M) = \nu^t \delta_M$ for all $t \in \mathbb{R}$ and $\psi = \varphi^{\delta_M}$. It is called the modular element of $(M, \Delta)$. If $\delta_M = 1$ we call $(M, \Delta)$ unimodular. The scaling constant can be characterized as well by the relative invariance $\varphi \tau_t = \nu^{-t} \varphi$.

For the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ we have:

$$\hat{M} = \{ (\omega \otimes \id)(W) \mid \omega \in B(H)_+ \}^{\sigma-strong*}$$

and $\hat{\Delta}(x) = \Sigma W^*(1 \otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. A left invariant n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ can be constructed explicitly and the associated multiplicative unitary is $W = \Sigma W^*\Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, let us denote its antipode by $\hat{S}$, its unitary antipode by $\hat{R}$ and its scaling group by $\hat{\tau}_1$. Then we can construct the dual of $(M, \Delta)$, starting from the left invariant weight $\varphi$. The bidual l.c. quantum group $(\hat{M}, \hat{\Delta})$ is isomorphic to $(M, \Delta)$.

$M$ is commutative if and only if $(M, \Delta)$ is generated by a usual l.c. group $G : M = L^\infty(G), (\Delta_G f)(g, h) = f(gh), (S_G f)(g) = f(g^{-1}), \varphi_G(f) = \int f(g) \, dg$, where $f \in L^\infty(G)$, $g, h \in G$ and we integrate with respect to the left Haar measure $dg$ on $G$. Then $\psi_G$ is given by $\psi_G(f) = \int f(g^{-1}) \, dg$ and $\delta_M$ by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

$L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and $(W_G \xi)(g, h) = \xi(g, g^{-1}h)$, for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = L(G)$ is the group von Neumann
algebra generated by the left translations $(\lambda_g)_{g \in G}$ of $G$ and $\hat{\Delta}_G(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}_G^{op} := \sigma \circ \hat{\Delta}_G = \hat{\Delta}_G$, so $\hat{\Delta}_G$ is cocommutative.

$(M, \Delta)$ is a Kac algebra (see [2]) if $\tau_t = \text{id}$, for all $t$, and $\delta_M$ is affiliated with the center of $M$. In particular, this is the case when $M = L^\infty(G)$ or $M = \mathcal{L}(G)$.

We can also define the $C^*$-algebra of continuous functions vanishing at infinity on $(M, \Delta)$ by

$$A = \{[\text{id} \otimes \omega](W) \mid \omega \in \mathcal{B}(H)_*\}$$

and the reduced $C^*$-algebra (or dual $C^*$-algebra) of $(M, \Delta)$ by

$$\hat{A} = \{[\omega \otimes \text{id}](W) \mid \omega \in \mathcal{B}(H)_*\}.$$  

In the group case we have $A = C_0(G)$ and $\hat{A} = C^r(G)$. Moreover, we have $\Delta \in \text{Mor}(A, A \otimes A)$ and $\hat{\Delta} \in \text{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$.

A l.c. quantum group is called compact if $\varphi(1_M) < \infty$ and discrete if its dual is compact.

### 2.4 Twisting of locally compact quantum groups [4]

Let $(M, \Delta)$ be a locally compact quantum group and $\Omega$ a unitary in $M \otimes M$. We say that $\Omega$ is a 2-cocycle on $(M, \Delta)$ if

$$(\Omega \otimes 1)(\Delta \otimes \text{id})(\Omega) = (1 \otimes \Omega)(\text{id} \otimes \Delta)(\Omega).$$

As an example we can consider $M = L^\infty(G)$, where $G$ is a l.c. group, with $\Delta_G$ as above, and $\Omega = \Psi(\cdot, \cdot) \in L^\infty(G \times G)$ a usual 2-cocycle on $G$, i.e., a measurable function with values in the unit circle $T \subset \mathbb{C}$ verifying

$$\Psi(s_1, s_2)\Psi(s_1s_2, s_3) = \Psi(s_2, s_3)\Psi(s_1, s_2s_3),$$

for almost all $s_1, s_2, s_3 \in G$.

This is the case for any measurable bicharacter on $G$.

When $\Omega$ is a 2-cocycle on $(M, \Delta)$, one can check that $\Delta_G(\cdot) = \Omega \Delta(\cdot) \Omega^*$ is a new coassociative comultiplication on $M$. If $(M, \Delta)$ is discrete and $\Omega$ is any 2-cocycle on it, then $(M, \Delta_\Omega)$ is again a l.c. quantum group (see [1], finite-dimensional case was treated in [3]). In the general case, one can proceed as follows. Let $\alpha : (L^\infty(G), \Delta_G) \to (M, \Delta)$ be an inclusion of Hopf-von Neumann algebras, i.e., a faithful unital normal $^*$-homomorphism such that $(\alpha \otimes \alpha) \circ \Delta_G = \Delta \circ \alpha$. Such an inclusion allows to construct a 2-cocycle of $(M, \Delta)$ by lifting a usual 2-cocycle of $G : \Omega = (\alpha \otimes \alpha)\Psi$. It is shown in [3] that if the image of $\alpha$ is included into the centralizer of the left invariant weight $\varphi$, then $\varphi$ is also left invariant for the new comultiplication $\Delta_\Omega$.

In particular, let $G$ be a non commutative l.c. group and $K$ a closed abelian subgroup of $G$. By Theorem 6 of [1], there exists a faithful unital normal $^*$-homomorphism $\hat{\alpha} : \mathcal{L}(K) \to \mathcal{L}(G)$ such that

$$\hat{\alpha}(\lambda_g^K) = \lambda_g, \quad \text{for all } g \in K, \quad \text{and} \quad \hat{\Delta} \circ \hat{\alpha} = (\hat{\alpha} \otimes \hat{\alpha}) \circ \hat{\Delta}_K,$$
where $\lambda^K$ and $\lambda$ are the left regular representation of $K$ and $G$ respectively, and $\Delta_K$ and $\Delta$ are the comultiplications on $\mathcal{L}(K)$ and $\mathcal{L}(G)$ respectively. The composition of $\hat{\sigma}$ with the canonical isomorphism $L^\infty(K) \simeq \mathcal{L}(K)$ given by the Fourier transformation, is a faithful unital normal $^*$-homomorphism $\alpha : L^\infty(\hat{K}) \to \mathcal{L}(G)$ such that $\Delta \circ \alpha = (\alpha \circ \alpha) \circ \Delta_K$, where $\Delta_K$ is the comultiplication on $L^\infty(\hat{K})$. The left invariant weight on $\mathcal{L}(G)$ is the Plancherel weight for which

$$\sigma_t(x) = \delta_G^t x \delta_G^{-t}, \quad \text{for all } x \in \mathcal{L}(G),$$

where $\delta_G$ is the modular function of $G$. Thus, $\sigma_t(\lambda_g) = \delta_G^t(g) \lambda_g$ or

$$\sigma_t \circ \alpha(u_g) = \alpha(u_g(\cdot - \gamma_t)),$$

where $u_g(\gamma) = (\gamma, g)$, $g \in G$, $\gamma \in \hat{G}$, $\gamma_t$ is the character $K$ defined by $(\gamma_t, g) = \delta_G^{-t}(g)$. By linearity and density we obtain:

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(\hat{K}).$$

This is why we do the following assumptions. Let $(M, \Delta)$ be a l.c. quantum group, $G$ an abelian l.c. group and $\alpha : (L^\infty(G), \Delta_G) \to (M, \Delta)$ an inclusion of Hopf-von Neumann algebras. Let $\varphi$ be the left invariant weight, $\sigma_t$ its modular group, $S$ the antipode, $R$ the unitary antipode, $\tau_t$ the scaling group. Let $\psi = \varphi \circ R$ be the right invariant weight and $\sigma_{t'}$ its modular group. Also we denote by $\delta$ the modular element of $(M, \Delta)$. Suppose that there exists a continuous group homomorphism $t \mapsto \gamma_t$ from $\mathbb{R}$ to $G$ such that

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(G).$$

Let $\Psi$ be a continuous bicharacter on $G$. Notice that $(t, s) \mapsto \Psi(\gamma_t, \gamma_s)$ is a continuous bicharacter on $\mathbb{R}$, so there exists $\lambda > 0$ such that $\Psi(\gamma_t, \gamma_s) = \lambda^{ts}$. We define:

$$u_t = \lambda^\frac{ts}{2} \alpha(\Psi(\cdot, -\gamma_t)) \quad \text{and} \quad v_t = \lambda^\frac{ts}{2} \alpha(\Psi(-\gamma_t, \cdot)).$$

The 2-cocycle equation implies that $u_t$ is a $\sigma_t$-cocycle and $v_t$ is a $\sigma_{t'}$-cocycle. The Connes’ Theorem gives two n.s.f. weights on $M$, $\varphi_\Omega$ and $\psi_\Omega$, such that

$$u_t = [D\varphi_\Omega : D\varphi_t] \quad \text{and} \quad v_t = [D\psi_\Omega : D\psi_{t'}].$$

The main result of [3] is as follows:

**Theorem 1** $(M, \Delta_\Omega)$ is a l.c. quantum group with left and right invariant weight $\varphi_\Omega$ and $\psi_\Omega$ respectively. Moreover, denoting by a subscript or a superscript $\Omega$ the objects associated with $(M, \Delta_\Omega)$ one has:

- $\sigma_1^\Omega = \tau_t$,
- $\nu_\Omega = \nu$ and $\delta_\Omega = \delta A^{-1} B$,  

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• \( D(S_{\Omega}) = D(S) \) and, for all \( x \in D(S) \), \( S_{\Omega}(x) = uS(x)u^* \).

Remark that, because \( \Psi \) is a bicharacter on \( G \), \( t \mapsto \alpha(\Psi(., -\gamma t)) \) is a representation of \( \mathbb{R} \) in the unitary group of \( M \) and there exists a positive self-adjoint operator \( A \) affiliated with \( M \) such that

\[
\alpha(\Psi(., -\gamma t)) = A^t, \quad \text{for all } t \in \mathbb{R}.
\]

We can also define a positive self-adjoint operator \( B \) affiliated with \( M \) such that

\[
\alpha(\Psi(-\gamma t, .)) = B^t.
\]

We obtain :

\[
u_t = \lambda_t^\frac{it}{2} A^t, \quad v_t = \lambda_t^\frac{it}{2} B^t.
\]

Thus, we have \( \varphi_\Omega = \varphi_A \) and \( \psi_\Omega = \psi_B \), where \( \varphi_A \) and \( \psi_B \) are the weights defined by S. Vaes in [3].

One can also compute the dual \( C^* \)-algebra and the dual comultiplication. We put:

\[
\Lambda_\gamma = \alpha(u_\gamma), \quad R_\gamma = JLCJ, \quad \text{for all } \gamma \in \hat{G}.
\]

From the representation \( \gamma \mapsto \Lambda_\gamma \) we get the unital *-homomorphism \( \lambda_L : L^\infty(G) \to M \) and from the representation \( \gamma \mapsto R_\gamma \) we get the unital normal *-homomorphism \( \lambda_R : L^\infty(G) \to M' \). Let \( \hat{A} \) be the reduced \( C^* \)-algebra of \((M, \Delta)\). We can define an action of \( \hat{G}^2 \) on \( \hat{A} \) by

\[
\alpha_{\gamma_1, \gamma_2}(x) = \Lambda_{\gamma_1} R_{\gamma_2} x R_{\gamma_2}^* \Lambda_{\gamma_1}^*.
\]

Let us consider the crossed product \( C^* \)-algebra \( B = \hat{G}^2 \alpha \ltimes \hat{A} \). We will denote by \( \lambda \) the canonical morphism from \( \hat{G}^2 \) to the unitary group of \( M(B) \) continuous in the strict topology on \( M(B) \), \( \pi \in \text{Mor}(\hat{A}, B) \) the canonical morphism and \( \theta \) the dual action of \( \hat{G}^2 \) on \( B \). Recall that the triplet \((\hat{G}^2, \lambda, \theta)\) is a \( \hat{G}^2 \)-product. Let us denote by \((\hat{G}^2, \lambda, \theta^\Psi)\) the \( \hat{G}^2 \)-product obtained by deformation of the \( \hat{G}^2 \)-product \((\hat{G}^2, \lambda, \theta)\) by the bicharacter \( \omega(g, h, s, t) := \Psi(g, s) \Psi(h, t) \) on \( G^2 \).

The dual deformed action \( \theta^\Psi \) is done by

\[
\theta^\Psi_{(g_1, g_2)}(x) = U_{g_1} V_{g_2} \theta_{(g_1, g_2)}(x) U_{g_1}^* V_{g_2}^*, \quad \text{for any } g_1, g_2 \in G, \ x \in B,
\]

where \( U_g = \lambda_L(\Psi_g), \ V_g = \lambda_R(\Psi_g), \ \Psi_g(h) = \Psi(h, g) \).

Considering \( \Psi_g \) as an element of \( \hat{G} \), we get a morphism from \( G \) to \( \hat{G} \), also noted \( \Psi \), such that \( \Psi(g) = \Psi_g \). With these notations, one has \( U_g = u_{\Psi(-g, 0)} \) and \( V_g = u_{(0, \Psi(g))} \). Then the action \( \theta^\Psi \) on \( \pi(\hat{A}) \) is done by

\[
\theta^\Psi_{(g_1, g_2)}(\pi(x)) = \pi(\alpha(\Psi_{(-g_1, g_2)})(x)). \quad (3)
\]

Let us consider the Landstad algebra \( A^\Psi \) associated with this \( \hat{G}^2 \)-product. By definition of \( \alpha \) and the universality of the crossed product we get a morphism

\[
\rho \in \text{Mor}(B, \mathcal{K}(H)), \quad \rho(\lambda_{\gamma_1, \gamma_2}) = \Lambda_{\gamma_1} R_{\gamma_2} \quad \text{et} \quad \rho(\pi(x)) = x. \quad (4)
\]

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It is shown in [4] that $\rho(A^\Psi) = \hat{A}_\Omega$ and that $\rho$ is injective on $A^\Psi$. This gives a canonical isomorphism $A^\Psi \simeq \hat{A}_\Omega$. In the sequel we identify $A^\Psi$ with $\hat{A}_\Omega$. The comultiplication can be described in the following way. First, one can show that, using universality of the crossed product, there exists a unique morphism $\Gamma \in \text{Mor}(B, B \otimes B)$ such that:

$$\Gamma \circ \pi = (\pi \otimes \pi) \circ \hat{\Delta}$$

and

$$\Gamma(\lambda_{\gamma_1, \gamma_2}) = \lambda_{\gamma_1, 0} \otimes \lambda_{0, \gamma_2}.$$

Then we introduce the unitary $\Upsilon = (\lambda_R \otimes \lambda_L)(\tilde{\Psi}) \in M(B \otimes B)$, where $\tilde{\Psi}(g, h) = \Psi(g, gh)$. This allows us to define the $^*$-morphism $\Gamma_\Omega(x) = \Upsilon \Gamma(x) \Upsilon^*$ from $B$ to $M(B \otimes B)$. One can show that $\Gamma_\Omega \in \text{Mor}(A^\Psi, A^\Psi \otimes A^\Psi)$ is the comultiplication on $A^\Psi$.

Note that if $M = L(G)$ and $K$ is an abelian closed subgroup of $G$, the action $\alpha$ of $K^2$ on $C_0(G)$ is the left-right action.

3 Twisting of the group of $2 \times 2$ upper triangular matrices with determinant 1

Consider the following subgroup of $SL_2(\mathbb{C})$:

$$G := \left\{ \begin{pmatrix} z & \omega \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^*, \omega \in \mathbb{C} \right\}.$$

Let $K \subset G$ be the subgroup of diagonal matrices in $G$, i.e. $K = \mathbb{C}^*$. The elements of $G$ will be denoted by $(z, \omega)$, $z \in \mathbb{C}$, $\omega \in \mathbb{C}^*$. The modular function of $G$ is

$$\delta_G((z, \omega)) = |z|^{-2}.$$

Thus, the morphism $(t \mapsto \gamma_t)$ from $\mathbb{R}$ to $\mathbb{C}^*$ is given by

$$\langle \gamma_t, z \rangle = |z|^{2t}, \text{ for all } z \in \mathbb{C}^*, t \in \mathbb{R}.$$

We can identify $\hat{\mathbb{C}}^*$ with $\mathbb{Z} \times \mathbb{R}_{>0}^*$ in the following way:

$$\mathbb{Z} \times \mathbb{R}_{>0}^* \to \hat{\mathbb{C}}^*, \quad (n, \rho) \mapsto \gamma_{n, \rho} = (r e^{i\theta} \mapsto e^{i n r \ln \rho} e^{i n \theta}).$$

Under this identification, $\gamma_t$ is the element $(0, e^t)$ of $\mathbb{Z} \times \mathbb{R}_{>0}^*$. For all $x \in \mathbb{R}$, we define a bicharacter on $\mathbb{Z} \times \mathbb{R}_{>0}^*$ by

$$\Psi_x((n, \rho), (k, r)) = e^{ix(k \ln \rho - n \ln r)}.$$

We denote by $(M_x, \Delta_x)$ the twisted l.c. quantum group. We have:

$$\Psi_x((n, \rho), \gamma_t^{-1}) = e^{ixtn} = u_{ext}((n, \rho)).$$

In this way we obtain the operator $A_x$ deforming the Plancherel weight:

$$A_x^\Omega = \alpha(u_{ext}) = \lambda_{ext, 0}^G.$$
In the same way we compute the operator $B_x$ deforming the Plancherel weight:

$$B_x^t = \lambda^G_{e^{-itx},0} = A_x^{-it}.$$

Thus, we obtain for the modular element:

$$\delta_x^t = A_x^{-it} B_x^t = \lambda^G_{e^{-2itx},0}.$$

The antipode is not deformed. The scaling group is trivial but, if $x \neq 0$, $(M_x, \Delta_x)$ is not a Kac algebra because $\delta_x$ is not affiliated with the center of $M$. Let us look if $(M_x, \Delta_x)$ can be isomorphic for different values of $x$. One can remark that, since $\Psi_{-x} = \Psi_x^*$ is antisymmetric and $\Delta$ is cocommutative, we have $\Delta_{-x} = \sigma \Delta_x$, where $\sigma$ is the flip on $L(G) \otimes L(G)$. Thus, $(M_{-x}, \Delta_{-x}) \simeq (M_x, \Delta_x)^{\text{op}}$, where "op" means the opposite quantum group. So, it suffices to treat only strictly positive values of $x$. The twisting deforms only the comultiplication, the weights and the modular element. The simplest invariant distinguishing the $(M_x, \Delta_x)$ is then the specter of the modular element. Using the Fourier transformation in the first variable, on has immediately

$$\text{Sp}(\delta_x) = q_x^2 \cup \{0\},$$

where $q_x = e^{-2x}$. Thus, if $x \neq y$, $x > 0, y > 0$, one has $q_x^2 \neq q_y^2$ and, consequently, $(M_x, \Delta_x)$ and $(M_y, \Delta_y)$ are non isomorphic.

We compute now the dual $C^*$-algebra. The action of $K^2$ on $C_0(G)$ can be lifted to its Lie algebra $\mathfrak{c}^2$. The lifting does not change the result of the deformation (see [3], Proposition 3.17) but simplify calculations. The action of $\mathfrak{c}^2$ on $C_0(G)$ will be denoted by $\rho$. One has

$$\rho_{z_1,z_2}(f)(z,\omega) = f(e^{iz_1-z_2}z, e^{-(z_1+z_2)}\omega). \ (5)$$

The group $\mathcal{C}$ is self-dual, the duality is given by

$$(z_1,z_2) \mapsto \exp(i\text{Im}(z_1z_2)).$$

The generators $u_z, z \in \mathbb{C}$, of $C_0(\mathbb{C})$ are given by

$$u_z(w) = \exp(i\text{Im}(zw)), \quad z, w \in \mathbb{C}.$$

Let $x \in \mathbb{R}$. We will consider the following bicharacter on $\mathbb{C}$:

$$\Psi_x(z_1,z_2) = \exp(i z \text{Im}(z_1z_2)).$$

Let $B$ be the crossed product $C^*$-algebra $\mathbb{C}^2 \rtimes C_0(G)$. We denote by $(z_1,z_2) \mapsto \lambda_{z_1,z_2}$ the canonical group homomorphism from $G$ to the unitary group of $M(B)$, continuous for the strict topology, and $\pi \in \text{Mor}(C_0(G), B)$ the canonical homomorphism. Also we denote by $\lambda \in \text{Mor}(C_0(\mathbb{C}^2), B$) the morphism given by the representation $(z_1, z_2) \mapsto \lambda_{z_1,z_2}$. Let $\theta$ be the dual action of $\mathbb{C}^2$ on $B$. We have, for all $z, w \in \mathbb{C}$, $\Psi_x(w,z) = u_x(w)$. The deformed dual action is given by

$$\theta^{\Psi_x}_{z_1,z_2} (b) = \lambda_{-z_1,xz_2} \theta_{z_1,z_2}(b) \lambda_{-xz_1,z_2}^{-1} \lambda_{-z_1,xz_2}.$$ \ (6)

Recall that

$$\theta^{\Psi_x}_{z_1,z_2}(\lambda(f)) = \theta_{z_1,z_2}(\lambda(f)) = \lambda(f(-z_1, -z_2)), \quad \forall f \in C_0(\mathbb{C}^2). \quad (7)$$
Let $\hat{A}_x$ be the associated Landstad algebra. We identify $\hat{A}_x$ with the reduced $C^*$-algebra of $(M_x, \Delta_x)$. We will now construct two normal operators affiliated with $\hat{A}_x$, which generate $A_x$. Let $a$ and $b$ be the coordinate functions on $G$, and $\alpha = \pi(a)$, $\beta = \pi(b)$. Then $\alpha$ and $\beta$ are normal operators, affiliated with $B$, and one can see, using (8), that

$$\lambda_{z_1, z_2} \alpha \lambda_{z_1, z_2}^* = e^{z_2 - z_1} \alpha, \quad \lambda_{z_1, z_2} \beta \lambda_{z_1, z_2}^* = e^{-(z_1 + z_2)} \beta.$$  

(8)

We can deduce, using (9), that

$$\theta_{\Psi_x}^{\alpha}(\alpha) = e^{x(\pi(z_1), \pi(z_2))} \alpha, \quad \theta_{\Psi_x}^{\beta}(\beta) = e^{x(\pi(z_1), \pi(z_2))} \beta.$$  

(9)

Let $T_l$ and $T_r$ be the infinitesimal generators of the left and right shift respectively, i.e. $T_l$ and $T_r$ are normal, affiliated with $B$, and

$$\lambda_{z_1, z_2} = \exp i\text{Im}(z_1 T_l) \exp i\text{Im}(z_2 T_r), \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$  

Thus, we have:

$$\lambda(f) = f(T_l, T_r), \quad \text{for all } f \in C_b(\mathbb{C}^2).$$

Let $U = \lambda(\Psi_x)$, we define the following normal operators affiliated with $B$:

$$\hat{\alpha} = U^* \alpha U, \quad \hat{\beta} = U \beta U^*.$$  

**Proposition 1** The operators $\hat{\alpha}$ and $\hat{\beta}$ are affiliated with $\hat{A}_x$ and generate $\hat{A}_x$.

**Proof.** First let us show that $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$. One has, using (8):

$$\theta_{\Psi_x}^{\Psi_x}(U) = \lambda(\Psi_x(z_1, z_2)) = U e^{i\text{Im}(z_1 T_l)} e^{i\text{Im}(z_2 T_r)} \Psi_x(z_1, z_2) = U \lambda_{z_1, z_2} \Psi_x(z_1, z_2).$$

Now, using (8) and (9), we obtain:

$$\theta_{\Psi_x}^{\alpha}(\hat{\alpha}) = \hat{\alpha}, \quad \theta_{\Psi_x}^{\beta}(\hat{\beta}) = \hat{\beta}, \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$  

Thus, for all $f \in C_0(\mathbb{C})$, $f(\hat{\alpha})$ and $f(\hat{\beta})$ are fixed points for the action $\theta_{\Psi_x}^{\Psi_x}$. Let $f \in C_0(\mathbb{C})$. Using (8) we find:

$$\lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* = U^* f(e^{z_2 - z_1} \alpha) U,$$

$$\lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^* = U^* f(e^{-(z_1 + z_2)} \beta) U.$$  

(10)

Because $f$ is continuous and vanish at infinity, the applications

$$(z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* \quad \text{and} \quad (z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^*$$

are norm-continuous and $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$.  

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Taking in mind Proposition 3 (see Appendix), in order to show that \( \hat{\alpha} \) is affiliated with \( \hat{A}_x \), it suffices to show that the vector space \( \mathcal{I} \) generated by \( f(\hat{\alpha})a \), with \( f \in C_0(\mathbb{C}) \) and \( a \in \hat{A}_x \), is dense in \( \hat{A}_x \). Using (9), we see that \( \mathcal{I} \) is globally invariant under the action implemented by \( \lambda \). Let \( g(z) = (1 + \Xi z)^{-1} \). As \( \lambda(C_0(\mathbb{C}^2))U = \lambda(C_0(\mathbb{C}^2)) \), we can deduce that the closure of \( \lambda(C_0(\mathbb{C}^2))g(\hat{\alpha})\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is equal to

\[
\left[ \lambda(C_0(\mathbb{C}^2))(1 + \alpha^* \alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \right].
\]

As the set \( U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \) and \( \alpha \) is affiliated with \( B \), the set \( \lambda(C_0(\mathbb{C}^2))(1 + \alpha^* \alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \). Moreover, it is included in \( \lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2)) \), so \( \lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2)) \) is dense in \( B \). We conclude, using Lemma 3, that \( \mathcal{I} \) is dense in \( \hat{A}_x \). One can show in the same way that \( \hat{\beta} \) is affiliated with \( \hat{A}_x \).

Now, let us show that \( \hat{\alpha} \) and \( \hat{\beta} \) generate \( \hat{A}_x \). By Proposition 3, it suffices to show that

\[ \mathcal{V} = \left\{ f(\hat{\alpha})g(\hat{\beta}), \ f, g \in C_0(\mathbb{C}) \right\} \]

is a dense vector subspace of \( \hat{A}_x \). We have shown above that the elements of \( \mathcal{V} \) satisfy the two first Landstad’s conditions. Let

\[ \mathcal{W} = \left[ \lambda(C_0(\mathbb{C}^2))\mathcal{V}\lambda(C_0(\mathbb{C}^2)) \right]. \]

We will show that \( \mathcal{W} = B \). This proves that the elements of \( \mathcal{V} \) satisfy the third Landstad’s condition, and then \( \mathcal{V} \subset \hat{A}_x \). Then (9) shows that \( \mathcal{V} \) is globally invariant under the action implemented by \( \lambda \), so \( \mathcal{V} \) is dense in \( \hat{A}_x \) by Lemma 3.

One has:

\[ \mathcal{W} = [xU^*f(\alpha)U^2g(\beta)U^*y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2))] . \]

Because \( U \) is unitary, we can substitute \( x \) with \( xU \) and \( y \) with \( Uy \) without changing \( \mathcal{W} \):

\[ \mathcal{W} = [xf(\alpha)U^2g(\beta)y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2))] . \]

Using, for all \( f \in C_0(\mathbb{C}) \), the norm-continuity of the application

\[ (z_1, z_2) \mapsto \lambda_{z_1, z_2}f(\alpha)\lambda_{z_1, z_2}^* = e^{z_2 - z_1} \alpha, \]

one deduces that

\[ [f(\alpha)x, \ f \in C_0(\mathbb{C}), \ x \in \lambda(C_0(\mathbb{C}^2))] = [xf(\alpha), \ f \in C_0(\mathbb{C}), \ x \in \lambda(C_0(\mathbb{C}^2))] . \]

In particular,

\[ \mathcal{W} = [f(\alpha)xU^2g(\beta)y, \ f, g \in C_0(\mathbb{C}), \ x, y \in \lambda(C_0(\mathbb{C}^2))] . \]
Now we can commute $g(\beta)$ and $y$, and we obtain:

$$W = [f(\alpha)xU^2yg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))].$$

Substituting $x \mapsto xu^\ast$, $y \mapsto u^\ast y$, one has:

$$W = [f(\alpha)xyg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))].$$

Commuting back $f(\alpha)$ with $x$ and $g(\beta)$ with $y$, we obtain:

$$W = [xf(\alpha)g(\beta)y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))] = B.$$  

This concludes the proof.

We will now find the commutation relations between $\hat{\alpha}$ and $\hat{\beta}$.

**Proposition 2** One has:

1. $\alpha$ et $T_1^* + T_r^*$ strongly commute and $\hat{\alpha} = e^{x(T_1^* + T_r^*)}\alpha$.

2. $\beta$ et $T_1^* - T_r^*$ strongly commute and $\hat{\beta} = e^{x(T_1^* - T_r^*)}\beta$.

Thus, the polar decompositions are given by:

$$Ph(\hat{\alpha}) = e^{-ix\text{Im}(T_1 + T_r)}Ph(\alpha), \quad |\hat{\alpha}| = e^{x\text{Re}(T_1 + T_r)}|\alpha|,$$

$$Ph(\hat{\beta}) = e^{-ix\text{Im}(T_1 - T_r)}Ph(\beta), \quad |\hat{\beta}| = e^{x\text{Re}(T_1 - T_r)}|\beta|.$$  

Moreover, we have the following relations:

1. $|\hat{\alpha}|$ and $|\hat{\beta}|$ strongly commute,

2. $Ph(\hat{\alpha})Ph(\hat{\beta}) = Ph(\hat{\beta})Ph(\hat{\alpha})$,

3. $Ph(\hat{\alpha})|\hat{\beta}|Ph(\hat{\alpha})^* = e^{4x}|\hat{\beta}|$,

4. $Ph(\hat{\beta})|\hat{\alpha}|Ph(\hat{\beta})^* = e^{4x}|\hat{\alpha}|$.

**Proof.** Using (8), we find, for all $z \in \mathbb{C}$:

$$e^{i\text{Im}(z(T_1^* + T_r^*))}\alpha e^{-i\text{Im}(z(T_1^* + T_r^*))} = \lambda_{-z^{-1}}\lambda_\omega \lambda_\omega^{-1} = e^{-z^{-1}\omega} = \alpha.$$  

Thus, $T_1^* + T_r^*$ and $\alpha$ strongly commute. Moreover, because $e^{ix\text{Im}(T_1^* + T_r^*)} = 1$, one has:

$$\hat{\alpha} = e^{-ix\text{Im}(T_1^* + T_r^*)}\alpha e^{ix\text{Im}(T_1^* + T_r^*)} = e^{-ix\text{Im}(T_1^* + T_r^*)}\alpha e^{ix\text{Im}(T_1^* + T_r^*)}^* = e^{ix\text{Im}(T_1^* + T_r^*)}^*.$$  

We can now prove the point 1 using the equality $e^{-ix\text{Im}(T_1^* + T_r^*)}\alpha e^{ix\text{Im}(T_1^* + T_r^*)} = e^{ix\text{Im}(T_1^* + T_r^*)}^*$, the preceding equation and the fact that $T_1^* + T_r^*$ and $\alpha$ strongly commute. The proof of the second assertion is similar and the polar decompositions follows. From (8) we deduce:
Notice that

\[
\begin{align*}
e^{-ix\text{Im}(T_r-T_l)}&=e^{-2ix}\alpha, \\
e^{ix\text{Im}(T_l+T_r)}&=e^{-2ix}\beta, \\
e^{ix\text{Re}(T_r-T_l)}&=e^{2ix}\alpha, \\
e^{ix\text{Re}(T_l+T_r)}&=e^{-2ix}\beta.
\end{align*}
\]

It is now easy to prove the last relations from the preceding equations and the polar decompositions.

We can now give a formula for the comultiplication.

**Proposition 3** Let \( \hat{\Delta}_x \) be the comultiplication on \( \hat{A}_x \). One has:

\[
\hat{\Delta}_x(\hat{a}) = \hat{a} \otimes \hat{a}, \quad \hat{\Delta}_x(\hat{b}) = \hat{a} \otimes \hat{b} + \hat{b} \otimes \hat{a}^{-1}.
\]

**Proof.** Using the Preliminaries, we have that \( \hat{\Delta}_x = \Upsilon \Gamma(.)\Upsilon^* \), where

\[
\Upsilon = e^{ix\text{Im}(T_r \otimes T_l^*)}
\]

and \( \Gamma \) is given by

- \( \Gamma(T_l) = T_l \otimes 1 \), \( \Gamma(T_r) = 1 \otimes T_r \);
- \( \Gamma \) restricted to \( C_0(G) \) is equal to the comultiplication \( \Delta_G \).

Define \( R = \Upsilon \Gamma(U^*) \). One has \( \hat{\Delta}_x(\hat{a}) = R(\alpha \otimes \alpha)R^* \). Thus, it is sufficient to show that \( (U \otimes U)R \) commute with \( \alpha \otimes \alpha \). Indeed, in this case, one has

\[
\hat{\Delta}_x(\hat{a}) = R(\alpha \otimes \alpha)R^* = (U^* \otimes U^*)(U \otimes U^*)R(\alpha \otimes \alpha)R^*(U^* \otimes U^*)(U \otimes U) = \hat{a} \otimes \hat{a}.
\]

Let us show that \( (U \otimes U)R \) commute with \( \alpha \otimes \alpha \). From the equality \( U = e^{ix\text{Im}(T_l T_l^*)} \), we deduce that

\[
\Gamma(U^*) = e^{-ix\text{Im}(T_l \otimes T_l^*)}, \quad U \otimes U = e^{ix\text{Im}(T_l T_l^* \otimes 1 + 1 \otimes T_l T_l^*)}.
\]

Thus, \( R = e^{-ix\text{Im}(T_l^* \otimes T_l + T_l \otimes T_l^*)} \) and

\[
(U \otimes U)R = e^{ix\text{Im}(T_l^* T_l^* \otimes 1 + 1 \otimes T_l T_l^* \otimes 1 - T_l \otimes T_l^*)}.
\]

Notice that

\[
T_l T_l^* \otimes 1 + 1 \otimes T_l T_l^* - T_l^* \otimes T_l - T_l \otimes T_l^* = (T_l \otimes 1 - 1 \otimes T_l)(T_l \otimes 1 - 1 \otimes T_l^*).
\]

Thus, it suffices to show that \( T_l \otimes 1 - 1 \otimes T_l \) and \( T_l^* \otimes 1 - 1 \otimes T_l^* \) strongly commute with \( \alpha \otimes \alpha \). This follows from the equations

\[
e^{i\text{Im}(T_l \otimes 1 - 1 \otimes T_l^*)}(\alpha \otimes \alpha)e^{-i\text{Im}(T_l \otimes 1 - 1 \otimes T_l^*)} = (\lambda_{0,\tau} \otimes \lambda_{0,\tau})(\alpha \otimes \alpha)(\lambda_{0,\tau} \otimes \lambda_{0,\tau})^* = e^{-i\tau}e^{\bar{\tau}}(\alpha \otimes \alpha) = \alpha \otimes \alpha, \quad \forall z \in \mathbb{C}
\]
and
\[ e^{i \text{Im}(z)} (T_1 \otimes 1 - 1 \otimes T_1) (\alpha \otimes \alpha) e^{-i \text{Im}(z)} (T_1 \otimes 1 - 1 \otimes T_1) \]
\[ = (\lambda_{x,0} \otimes \lambda_{-z,0}) (\alpha \otimes \alpha) (\lambda_{x,0} \otimes \lambda_{-z,0})^* \]
\[ = e^{-z} e^{i z} \alpha \otimes \alpha = \alpha \otimes \alpha, \quad \forall z \in \mathbb{C}. \]

Put \( S = \Upsilon \Gamma(U) \). One has:
\[ \hat{\Delta}_x(\hat{\beta}) = S(\alpha \otimes \beta + \beta \otimes \alpha^{-1}) S^* = S(\alpha \otimes \beta) S^* + S(\beta \otimes \alpha^{-1}) S^*. \]
As before, we see that it suffices to show that \((U \otimes U^*) S\) commutes with \(\alpha \otimes \beta\) and that \((U^* \otimes U) S\) commutes with \(\beta \otimes \alpha^{-1}\), and one can check this in the same way.

Let us summarize the preceding results in the following corollary (see\[16, 5\] for the definition of commutation relation between unbounded operators):

**Corollary 1** Let \( q = e^{8x} \). The \( C^* \)-algebra \( \hat{A}_x \) is generated by 2 normal operators \( \hat{\alpha} \) and \( \hat{\beta} \) affiliated with \( \hat{A}_x \) such that
\[ \hat{\alpha} \hat{\beta} = \hat{\beta} \hat{\alpha} \quad \hat{\alpha} \hat{\beta}^* = q \hat{\beta}^* \hat{\alpha}. \]
Moreover, the comultiplication \( \hat{\Delta}_x \) is given by
\[ \hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\alpha}^{-1}. \]

**Remark.** One can show, using the results of \[11\], that the application \((q \mapsto W_q)\) which maps the parameter \( q \) to the multiplicative unitary of the twisted l.c. quantum group is continuous in the \( \sigma \)-weak topology.

4 Appendix

Let us cite some results on operators affiliated with a \( C^* \)-algebra.

**Proposition 4** Let \( A \subset B(H) \) be a non degenerated \( C^* \)-subalgebra and \( T \) a normal densely defined closed operator on \( H \). Let \( \mathcal{I} \) be the vector space generated by \( f(T)a \), where \( f \in C_0(\mathbb{C}) \) and \( a \in A \). Then:
\[ (T\eta A) \iff \left( \begin{array}{c} f(T) \in M(A) \text{ for any } f \in C_0(\mathbb{C}) \\ \text{et } \mathcal{I} \text{ is dense in } A \end{array} \right). \]

**Proof.** If \( T \) is affiliated with \( A \), then it is clear that \( f(T) \in M(A) \) for any \( f \in C_0(\mathbb{C}) \), and that \( \mathcal{I} \) is dense in \( A \) (because \( \mathcal{I} \) contains \((1 + T^{-1} T)^{-1} A\)). To show the converse, consider the *-homomorphism \( \pi_T : C_0(\mathbb{C}) \to M(A) \) given by \( \pi_T(f) = f(T) \). By hypothesis, \( \pi_T(C_0(\mathbb{C})) A \) is dense in \( A \). So, \( \pi_T \in \text{Mor}(C_0(\mathbb{C}), A) \) and \( T = \pi_T(z \mapsto z) \) is then affiliated with \( A \).
Proposition 5  Let $A \subset B(H)$ be a non degenerated $C^*$-subalgebra and $T_1, T_2, \ldots, T_N$ normal operators affiliated with $A$. Let us denote by $V$ the vector space generated by the products of the form $f_1(T_1)f_2(T_2)\ldots f_N(T_N)$, with $f_i \in C_0(\mathbb{C})$. If $V$ is a dense vector subspace of $A$, then $A$ is generated by $T_1, T_2, \ldots, T_N$.

Proof. This follows from Theorem 3.3 in [15].

References


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