

In the core of longevity risk: hidden dependence in stochastic mortality models and cut-offs in prices of longevity swaps

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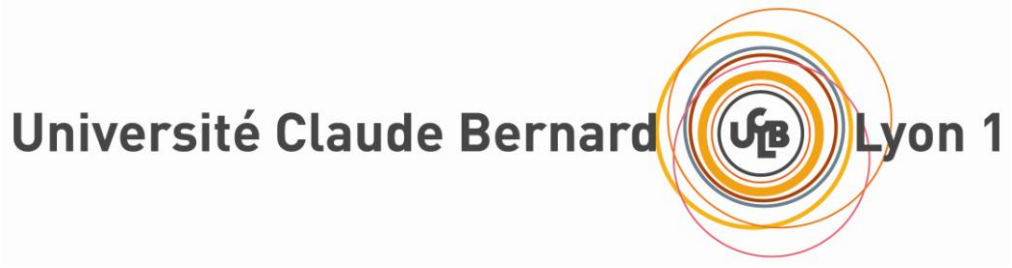
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IN THE CORE OF LONGEVITY RISK : HIDDEN DEPENDENCE IN STOCHASTIC MORTALITY MODELS AND CUT-OFFS IN PRICES OF LONGEVITY SWAPS

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In the core of longevity risk : dependence in stochastic mortality models and cut-offs in prices of longevity swaps.

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Abstract

In most stochastic mortality models, either one stochastic intensity process (for example a jump-diffusion process) or a collection of independent processes is used to model the stochastic evolution of survival probabilities. We propose and calibrate a new model that takes inter-age correlations into account. The so-called stochastic logit's Deltas model is based on the study of the multivariate time series of the differences of logits of yearly mortality rates. These correlations are important and we illustrate our study on a real-life portfolio. We determine their impact on the price of a longevity swap type reinsurance contract, in which most of the financial risk is taken by a third party. The hypotheses of our model are statistically tested and various measures of risk of the present value of liabilities are found to be significantly smaller in our model than in the case of one common underlying stochastic process. [1]

Keywords : Longevity risk, longevity swap, inter-age correlations, stochastic mortality, multivariate process.

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1 Introduction

The evolution of longevity is of remarkable importance to life insurance and reinsurance companies and pension schemes. As pointed out by Millossovich and Biffis (2006), the management of such types of business involves the simultaneous consideration of several classes of policyholders, differing for example by type of policy taken out, age, sex, health status, etc... The resulting longevity risk at aggregate level depends on the kind of stochastic dependence between the different random variables that contribute to the construction of the distribution of the vector of future cash-flows associated with a contract or a collection of contracts. Our goal is to propose and calibrate a stochastic mortality model that takes into account inter-age and inter-period correlations: the so-called stochastic logit's Deltas model.

Classical stochastic mortality models are often based on doubly stochastic processes. The time of death of the policyholder is regarded as the first jump-time of a counting process with random intensity, and the stochastic intensity process is often chosen to belong to the class of affine diffusion processes to simplify computations, in analogy with credit-risk and interest-rate-risk models.

Dahl (2004), Luciano and Vigna (2005a,b), Cairns et al. (2006); Blake et al. (2006a), Lin and Cox (2005), Milevsky and Promislow (2001) among others have recently studied that kind of models. Generally speaking, several problems often arise during the calibration phase : the fit is usually worse with a Feller intensity process than with an Ornstein-Uhlenbeck process, which often leads to tolerate a (very small in practice, but positive) probability that the intensity process becomes negative (see Luciano and Vigna (2005a) for example). The main issue is that the estimated volatility is very often zero or very close to zero for classical data sets. To avoid this, it has been proposed to incorporate a variable mean-reverting speed, while others, like Luciano and Vigna (2005b), observe that the fit is better for non-mean-reverting processes, which according to Blake et al. (2006a) comes from the fact that mortality is less constrained and more volatile than interest rates for which long-term trends are more trustable. Another possibility is to add a jump component to the diffusion in the intensity process, in analogy with what is often done nowadays in finance and credit risk. Those jumps, considered by Lin (2006) in her PhD thesis, might correspond to wars, pandemics or medical progresses. The most interesting extension for our model is the one of Millossovich and Biffis (2006), who use random field theory to model the whole mortality surface. This is a common point with our approach, which is though quite different. These models are well adapted to a financial pricing framework, but there are some reasons for which they do not model always very accurately the pure longevity risk, in particular when the period effect is more important that the cohort effect.

The pricing framework has been also recently widely discussed in the liter-

ature (see e.g. Blake et al. (2006b)), even if no consensus has appeared yet. Some authors assume the completeness of the life-insurance market, or that the market price of mortality risk is zero, which makes it possible for them to develop a valuation framework, on condition that the longevity risk is modeled in a cohort-based manner with a certain kind of stochastic mortality model. It is now widely accepted that the life-insurance market is far from being complete, and that it is hard to rely on traditional financial methods due to the lack of observed market prices. The choice of a particular risk-neutral probability measure is not an easy task in this framework. Hedging is also another important issue of course. This is the reason why some traditional actuarial pricing methods which involve Wang-transforms or two-parameter Wang-transforms are sometimes preferred to classical financial methods, see for example Lin (2006) and Cox et al. (2006). This approach is particularly adapted to the case where the financial risk is kept by the insurer, taken by a third party, or strongly reduced due to the description of the contract, which leaves the insurer or the reinsurer mostly exposed to the sole "pure" longevity risk. In this paper, we illustrate the pricing of the non-financial part of some life-insurance contracts with a one-parameter Wang-transform, but financial methods involving risk-neutral probability measures might be used as well with our model.
Let us first identify the different sources of risk before making some observations on mortality evolutions that will be key motivation to build our model.

1.1 Cartography of risks and impact of their relative importance

For longevity-risk based contracts, four main sources of randomness arise :

- the acceleration of longevity improvements : this corresponds to the (systemic) risk of change of drift of longevity-based processes
- the risk associated with oscillations of longevity improvements around the average drift(part of this risk, like the first one, is not diversifiable)
- financial risks : interest rate risk usually represents a large part of the overall risk if it is not transferred to a third party or strongly reduced by the form of the contract.
- portfolio variations or sampling risk : the smaller the portfolio, the bigger the relative importance of this risk in comparison to others. This risk is diversifiable, a reinsurer can benefits from a diversification effect coming from the heterogeneity between several ceding companies' portfolios.

Past experience on prospective life tables shows that future trends can be extrapolated quite correctly for the next 20-30 years, but that some input from demographers or scenario-based approaches must be used after 30 years from now.

As our goal is to price a pure longevity swap, the discounting factor and the size effect (fewer people are still alive) reduce the impact of long-term prediction errors. *A contrario,* on the short term, fluctuations around the (well-predicted) average drift must be securely controlled. After a certain time, the relative importance of these oscillations becomes more and more neglectable in comparison to uncertainty on the trend, model risk coming mainly from the choice of the long-term average trend, as experts have very diversified views on future longevity improvements. To be short, the average drift is likely to be given by national or experience prospective life tables, and after 25-30 years by a transition from the trend of these tables to long-term scenarios obtained from demographers. Probability weights of these scenarios are likely to become more and more accurate thanks to credibility methods. We thus have two models for the trend to link smoothly. The main conclusion of this cartography of risk is that it is highly desirable to deal with a stochastic process in which we can easily incorporate one or a weighted collection of average trends. Our stochastic mortality process is thus going to be obtained from a multi-dimensional, discrete-time Gaussian process, such that inter-age correlations are carefully modeled and taken into account.

1.2 Things that have to be taken into account

Let us now go into the core of longevity risk, and briefly make some observations that will lead us in section 3 to define a model that better takes them into account than usual stochastic mortality models. Millossovich and Biffis (2006) used random field theory to model the whole mortality surface. They explain that this enables them *"to look simultaneously at mortality profiles (evolution of yearly mortality rates over time for given age), contemporary life tables (yearly mortality rates across all ages at given time) and cohort tables (yearly mortality rates over time for people born in the same year). This is important because there is evidence that the well-documented downward mortality trends in most developed countries are not uniform across ages nor across different policyholders's classes (e.g. Renshaw et al. (1996)).*" This is not the case of most other stochastic mortality models, in which the approach is most-often cohort-based by nature. This is a very important point, because we observed that specific cohort-type oscillations are most often second-order oscillations in comparison to the oscillations of yearly mortality rates over time for given age, except in a few countries as the United Kingdom and Italy. Our first point is thus that it is better to model the joint evolution of yearly mortality rates over time for given ages. For countries where cohort-effect is important, this effect could be taken into account a posteriori.
The second point is that it is most often more consistent with data to model

oscillations of the logit

$$\text{logit } (q_{x,t}) = \ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right)$$

of yearly mortality rates $q_{x,t}$ at age x during calendar year t, than to model directly oscillations of the yearly mortality rates, or oscillations of mortality forces $\mu_{x,t}$ at age x and time t or of the logarithm of mortality forces like in Lee and Carter (1992) and Millossovich and Biffis (2006) or other stochastic mortality models.

It is of course very important to take into account these discrepancies between risk classes, but it is also very important to take into account the dependence structure between the processes that represent the evolution of some longevity-related quantities for different risk classes. In most stochastic mortality models, either one stochastic process (for example a jump-diffusion process) or a collection of independent processes is used to model the stochastic evolution of survival probabilities. Some practitioners even use one single random variable to model the simultaneous decay of yearly mortality rates over time for all ages. These approaches (with one single random variable, or one single process underlying intensity processes for all cohorts) correspond to the case where oscillations are perfectly positively correlated among ages or cohorts, which increases the global risk and thus the price of the contract if one uses a coherent risk measure.

The analog model with independent copies of the intensity process or of the "decay factor" random variable would be quite simple and far less penalizing, but our study confirmed us that it was not realistic at all. Positive correlations do exist, and cannot be neglected. The independence hypothesis would not be cautious. Conversely, the perfectly-correlated case would be too cautious, as we shall see in Section 4. Following the International Accounting Standards Board (IASB), any benefits of diversification and correlation within a book of contracts should be reflected in the measurement of liabilities. This will be confirmed by our study of the impact on the price of contract, which is already expected to be high, and on the solvency capital that is required at aggregation level in Solvency II project (see Section 4). The use of an internal model which better takes diversification effect into account will lead to a gain in price or in solvency required capital, as in any bottom-up approach.

There are also hidden correlations in oscillations of male and female yearly mortality rates over time. If one has prospective life tables for male and female, like the new ones in France that just appeared, aggregation needs to be done carefully. The fact that oscillations around the drift for men and for women are far from being independent and the impact on portfolio risk are illustrated in Devineau et al. (2007).

Our last point is that the model should be tractable consistently with this male/female aggregation problem, but also with inputs from demographers and actuaries on the long-term trends (a trend or a collection of scenarios),

including catastrophic scenarios to perform stress-testing techniques. This is why we built a model in which we can incorporate any drift prescribed by specialists or chosen by the two parties.

1.3 Structure of the paper

Our paper is organized as follows. Section 2 starts by sketching the usual Lee-Carter model, its advantages and drawbacks, and showing the variety of ideas of demographers on future mortality trends. Section 3 describes our model that can be seen as a kind of multi-dimensional extension of the Lee-Carter model. In section 4, we use our model to price longevity swaps, and we compare the obtained price with the ones we would have obtained in the independent case and in the perfectly correlated case.

2 Prospective life tables

2.1 The Lee-Carter Model

Lee and Carter (1992) proposed to model the evolution of the logarithm of mortality force $\mu_{x,t}$ at age x and time t as follows:

$$\ln \mu_{x,t} = \alpha(x) + \beta(x)\kappa(t) + \epsilon_{x,t},$$

where x is the age, t is the calendar year, $\exp(\alpha(x))$ is the general shape of mortality at age x, $\kappa(t)$ represents the time trend, $\beta(x)$ indicates the sensitivity of the logarithm of the force of mortality at age x to variations in $\kappa(t)$, and the error terms $\epsilon_{x,t}$ are a collection of i.i.d. Gaussian random variables. To ensure identifiability of the model, one generally assumes that

$$\sum_t \kappa(t) = 0 \text{ and } \sum_x \beta(x) = 1.$$

This structure is quite restrictive for the evolution of mortality. Indeed, if it exists, the partial derivative can be written as

$$\beta(x).\kappa'(t),$$

which can be interpreted as follows:

(1) for all t_1, t_2, quotients

$$\frac{\frac{\partial(E[\ln \mu_{x,t}])}{\partial t}(x,t_1)}{\frac{\partial(E[\ln \mu_{x,t}])}{\partial t}(x,t_2)} = \frac{\kappa'(t_1)}{\kappa'(t_2)}$$

are unsensitive to x! For example, if the relative variation of mortality hasard rates at age $x = 50$ was equal to 80% of what it was in 1990, this coefficient (80%) would be used for all the other ages!

(2) for each ages x_1, x_2, quotients

$$\frac{\frac{\partial(E[\ln \mu_{x,t}])}{\partial t}(x_1, t)}{\frac{\partial(E[\ln \mu_{x,t}])}{\partial t}(x_2, t)} = \frac{\beta(x_1)}{\beta(x_2)}$$

are unsensitive to t! For example, if the relative variation of mortality hasard rates at time $t = 1950$ for age 80 was twice as large as the one at age 20, this equal to 80% of what it was in 1990, this phenomenon would be assumed to hold for every time t.

A certain number of stochastic mortality models are based on the Lee-Carter model, either on the study of the $\epsilon_{x,t}$ or on the study of the $\kappa(t)$. The stochastic logit's Deltas model we propose in Section 3 is based on the fact that the observed $\kappa(t)$ are not far from being linear in t in the Lee-Carter model, and that a similar phenomenon is observed for the logits of yearly mortality rates. A way to introduce long-term systemic risk is to incorporate stochastic changes in the linear drift of the $\kappa(t)$. This is going to be possible in our model as well.

2.2 Long-term vision of demographers

After a certain amount of time (30-40 years), it seems clear that models that are only based on past trends have poor reliability. One has then to take into account the long-term vision of demographers. The problem is that these experts strongly disagree on the long-term behavior of mortality trends. Some demographers more or less assume that some evolutions are going to continue for a long time as in the past, while some others think that a biological limit makes longevity improvement almost impossible after a certain threshold that would explain rectangularization of mortality curves. Hayflick (2002) even writes: *"Those who predict enormous gains in life expectation in the future based only on mathematically sound predictions of life table data but ignore the biological facts that underlie longevity determination and aging do so at their own peril and the peril of those who make health policy for the future of this country." A contrario* some others assume that scientific progress will enable us to maintain longevity improvements, and that one will be able to replace defaultable organs in the human body as in a car! Vaupel (see Yashin et al. (1985); Vaupel et al. (1988); Vaupel (2003)) points out that the best national life expectancy at birth in the world linearly increases, and thinks that this is likely to continue. Note however that even if this were true, for a fixed country, longevity improvements may decrease and one should be concerned about the difference between the longevity of a fixed country and the one of

the leading country. This is what Keilman (2003), Alho (2003) and Bengtsson (2003) do int their study of longevity improvements in the Scandinavian countries.
The study of long-term longevity trends may use prospective life tables methods, jump models to take into account new cancer treatments, wars and pandemics, or even reliability models (see for example Gavrilov and Gavrilova (2001)) in which the human body is regarded as a machine with components at risk. It is hard to calibrate this select-effect models as causes of death are not so easy to determine and to incorporate in mortality databases (for example in the *Human Mortality Database*).
Given these uncertainties, it seems really useful to have a refined model for the oscillations around the trend for the 30-40 first years that is compliant with the addition of long-term scenarios. This is what we do in the next section with a centered multivariate process. We then illustrate our model with the addition of three weighted long-term average trends:

- a slightly favorable long-term scenario (from the reinsurer's point of view)
- a slightly unfavorable long-term scenario (from the reinsurer's point of view)
- a slightly unfavorable mid-term scenario (from the reinsurer's point of view) that worsens into a quite unfavorable scenario (according to "Vaupel-type" predictions).

Of course, the weight of the last scenario has a strong impact on the price of longevity swaps and generates model risk that has to be taken into account in risk measurement. The long-term trends and the weights of these trends may be easily adjusted in our model with credibility methods.

3 Taking inter-age correlations into account: the stochastic logit's Deltas model

The cohort effect seems to exist in the UK and in Italy (see for example Haberman and Renshaw (2006) and Pitacco and Olivieri (2007)), but seems to be of second order in comparison to period effect in most other countries (France, Belgium, Scandinavian countries, U.S.A., ...). We thus use here a "period-based" model to take into account the main source of oscillations. For countries like the UK, it would be possible to take the cohort effect into account a posteriori in our model.
Our main point is that inter-age correlations are important. We thus use a multivariate approach as Millossovich and Biffis (2006) who use a Brownian sheet (kind of extension of Brownian motion with two indices x and t). Our model, yet quite different and with dependent increments, takes inter-age correlations into account.
In many stochastic mortality models, the intensity $(\mu_{x+t,t})_t$ of the jump pro-

cess whose first jump instant corresponds to time of death is a diffusion process (e.g. Ornstein-Uhlenbeck or Feller process), which implies that $\mu_{x+t,t}$ follows a Gaussian distribution. But as $\mu_{x+t,t}$ is very similar to $q_{x+t,t}$ for small values of $q_{x+t,t}$, in classical models (Lee and Carter (1992), etc...), Gaussian distributions are used for $\ln q_{x,t}$ or logit $(q_{x,t})$, not for $q_{x,t}$. Our discrete-time multivariate process is thus going to describe the evolution of logit $(q_{x,t})$ instead of $q_{x,t}$. The logit function is one-to-one from $]0,1[$ onto $]-\infty,+\infty[$, which gives us freedom to model the logit $(q_{x,t})$ with the guarantee to get back values of $q_{x,t}$ that lie between 0 and 1. This is not the case if $q_{x,t}$ or $\mu_{x+t,t}$ is given by a Gaussian process, which is one of the issues in some stochastic mortality models as in Luciano and Vigna (2005a). Delwaerde and Denuit (2006) have shown that the logit $(q_{x,t})$ are very similar to logarithms of mortality forces, which are used in many stochastic mortality models. Nevertheless, we do not observe mortality forces and are only able to obtain them by some assumptions. To avoid the noise induced by these assumptions and the resulting errors, we directly use the logit $(q_{x,t})$ because we were interested in the oscillations and correlations between these oscillations. Those quantities are very sensitive to errors that would be involved in extrapolated datasets of mortality forces. Our model is based on a Gaussian process in which one may incorporate any trend or stochastic trend process to take into account prospective life tables and then long-term scenarios prescribed by demographers (see Section 2.2). This guarantees a smooth transition from prospective models (based on statistical methods) to long-term stochastic scenarios, and the ability to perform stress-tests in the spirit of Pillar II of Solvency II.

We aim at modeling the discrete-time stochastic multivariate process

$$(q_{x,t})_{(x,t)\in[x_0,x_1]\times[t_0,t_1]}$$

in a way that is consistent with data. Recall that $q_{x,t}$ represents the yearly mortality rate at age x during calendar year t. To illustrate our model, we calibrate it with yearly, bulk French INSEE male and female mortality tables on the period $[t_0,t_1]=[1962,2000]$. The range of ages (in years) has been fixed to $[x_0,x_1]=[60,90]$. We use 60 as a lower bound because we are interested in swaps linked with annuities, and 90 as an upper bound in order to avoid sampling errors for high ages (those fluctuations are present even in national mortality tables). We model mortality profiles (evolution of yearly mortality rates over time for given age), and then deduce cohort tables (yearly mortality rates over time for people born in the same year). Stochastic life tables and correlations were "closed" manually at high ages. More sophisticated methods might be used but the impact of this is limited in comparison to the ones of the phenomena we study here. We also apply our methods to a real-life portfolio.

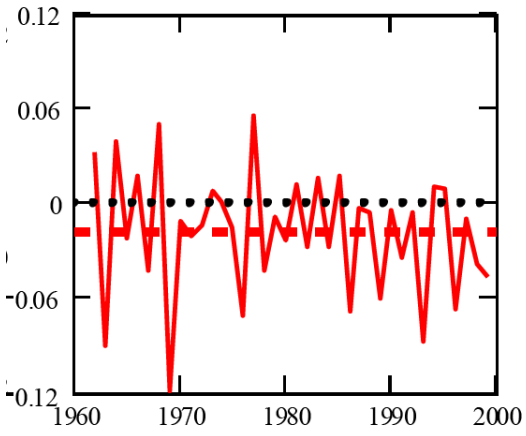


Figure 1. Logit's Deltas for French males (1962-1999) at age $x = 60$.

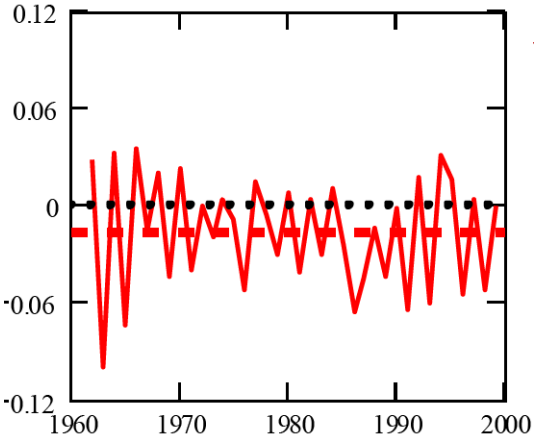


Figure 2. Logit's Deltas for French males (1962-1999) at age $x = 80$.

3.1 Construction of the model : exploring phase

The distribution of the process

$$(q_{x,t})_{[x,t]\in[x_0,x_1]\times[t_0,t_1]}$$

is defined by :

- the distribution of
$$\left(\ln\left[\frac{q_{x,t_0}}{1-q_{x,t_0}}\right]\right)_{x\in[x_0,x_1]},$$
which may be obtained thanks to prospective life tables for example (it can be deterministic and correspond to the last prospective life table that is known (year 2000 in our study),
- and the distribution of the so-called "logit's Deltas" process
$$\left(\ln\left[\frac{q_{x,t+1}}{1-q_{x,t+1}}\right]-\ln\left[\frac{q_{x,t}}{1-q_{x,t}}\right]\right)_{[x,t]\in[x_0,x_1]\times[t_0,t_1-1]}.$$

From the linear structure of the time series $\kappa(t)$ in the Lee-Carter model, a natural candidate for this distribution is the one of a stationary Gaussian multivariate process of Box-Jenkins type. The consistency of the stationarity of the logit's Deltas process is illustrated by Figures 1 and 2 for French males (respectively at ages 60 and 80). This is confirmed by a test of stationarity. In the following Table (see Figure 3), the p-values of the test of non-stationarity in t (alternative hypothesis) of the $L\Delta_{x,t}$ age given for each age x (for males and females).

age	p-value (F)	p-value (M)	age	p-value (F)	p-value (M)
60	0.95	0.33	75	0.79	0.31
61	0.99	0.20	76	0.77	0.44
62	0.97	0.09	77	0.97	0.58
63	0.67	0.32	78	0.76	0.38
64	0.91	0.19	79	0.80	0.45
65	0.88	0.37	80	0.77	0.58
66	0.78	0.48	81	0.70	0.40
67	0.94	0.36	82	0.65	0.77
68	0.97	0.29	83	0.93	0.87
69	0.83	0.35	84	0.95	0.80
70	0.90	0.44	85	0.92	0.87
71	0.96	0.21	86	1.00	0.91
72	0.85	0.34	87	0.99	0.93
73	0.77	0.33	88	0.95	0.84
74	0.79	0.39	89	0.87	0.92
			90	0.97	0.90

Figure 3. Table of p-values of the test of non-stationarity in t (alternative hypothesis) of the $L\Delta_{x,t}$ for each age x for males (M) and females (F).

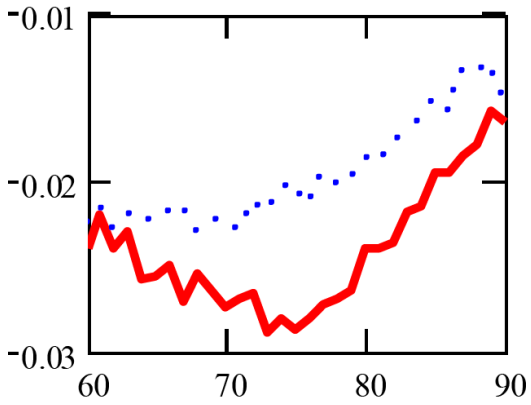


Figure 4. Average level of the $L\Delta_{x,t}$ for age x on period 1969-1999 for males (dashed) and females (solid).

Figure 4 presents the average level of the $L\Delta_{x,t}$ for age x on period 1969-1999. Before 1969, the drift was smaller and was ignored in order not to take into account unappropriate, old trends. For males and females, the longevity drift gets weakened for ages greater than 80. Note also that the maximal drift is reached at age 75 for French females.

Figure 5 illustrates the consistency of the Gaussian assumption: the Deltas of logits have been standardized (age by age) and the empirical p.d.f. is plotted. This is confirmed by a Kolmogorov-Smirnov normality test: normality is far from being rejected by the test, as the p-value is 0.15 for French males and 0.14 for French males.

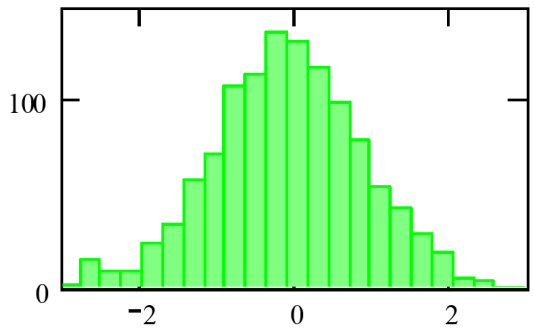


Figure 5. Empirical pdf of the standardized logit's Deltas for French males (1962-1999).

For $x \in [x_0, x_1]$ and $t \in [t_0, t_1 - 1]$, let us define the logit's Delta $L\Delta(x,t)$ at age x during calendar year t by

$$L\Delta(x,t) = \ln\left[\frac{q_{x,t+1}}{1-q_{x,t+1}}\right] - \ln\left[\frac{q_{x,t}}{1-q_{x,t}}\right].$$

For $t \in [t_0, t_1 - 1]$, let us define the multivariate logit's Delta $L\Delta_t$ during calendar year t by

$$L\Delta_t = \left(\ln\left[\frac{q_{x,t+1}}{1-q_{x,t+1}}\right] - \ln\left[\frac{q_{x,t}}{1-q_{x,t}}\right]\right)_{x\in[x_0,x_1]}.$$

This vectorial notation enables us to write the Box-Jenkins modeling assumption as :

$$L\Delta_t = \mu + \sum_{i=1}^{p} AR_i.L\Delta_{t-i} + U_t + \sum_{i=1}^{q} AR_i.U_{t-i},$$

where

- μ is a constant column vector of size $N = x_1 - x_0 + 1$,
- autoregressive (AR_i 's) and mobile average (MA_i 's) terms are $N \times N$ square matrices,
- the $(U_t)_{t\in[t_0+1,t_1]}$ is a sequence of i.i.d., centered Gaussian vectors with covariance matrix Γ.
- The autoregressive and mobile average matrices AR_i and MA_j, and covariance matrix Γ determine inter-period and inter-age correlations of the $L\Delta_{x,t}$.
- Vector μ and autoregressive matrices AR_i determine the level of the process: as the process $(L\Delta_t)_{t\in[t_0,t_1]}$ is stationary, the average level m of this process satisfies

$$m = \mu + \left[\sum_{i=1}^{p} AR_i\right] m. \tag{1}$$

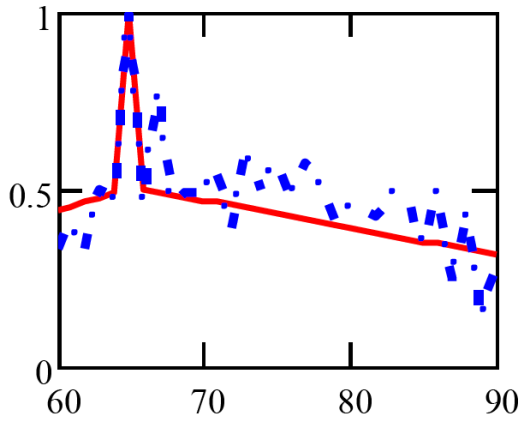


Figure 6. Empirical inter-age correlation coefficients $\rho(65, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(65, y)$ for French males (1969-1999).

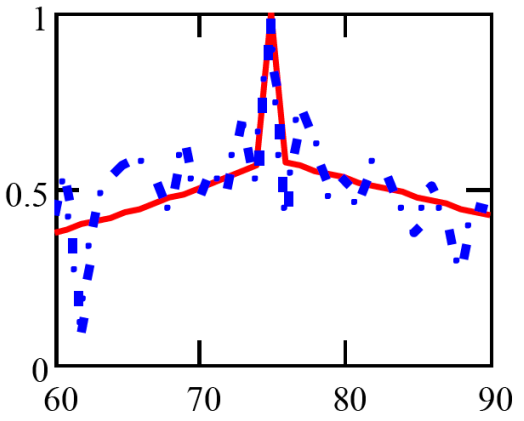


Figure 7. Empirical inter-age correlation coefficients $\rho(75, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(75, y)$ for French males (1969-1999).

To introduce systemic long-term risk, m could depend on t and become a stochastic process with sample paths with constant-value parts, in the spirit of some stochastic extensions of the Lee-Carter model.

3.2 Exploring time and space correlations

3.2.1 Inter-age correlations

In Figures 6 to 11, empirical inter-age correlation coefficients $\rho(x, y)$ for ages x and y (computed on period 1969-1999) are represented, as well as their smooth fits $\tilde{\rho}(x, y)$ of the form:

$$\tilde{\rho}(x,y) = \mathbf{1}_{\{x\}}(y) + \left[1 - \mathbf{1}_{\{x\}}(y)\right] \left(s_0 + \left(s_1 + s_2 \frac{x+y}{2} \right) e^{\left(s_3 + s_4 \frac{x+y}{2}\right)|x-y|} \right). \quad (2)$$

Correlation seems to be increasing in the age, as shown in Figures 12 and 13. It seems also quite symmetrical around each age x (i.e. $\rho(x, x+h) \sim \rho(x, x-h)$ for h such that $x-h$ and $x+h$ are in $[60, 90]$). Besides, even for quite different ages x and y ($x - y = 15$, say), correlation remains significant (around 50%), which shows the importance of the period effect. Some localized effects may also be noted at high ages.

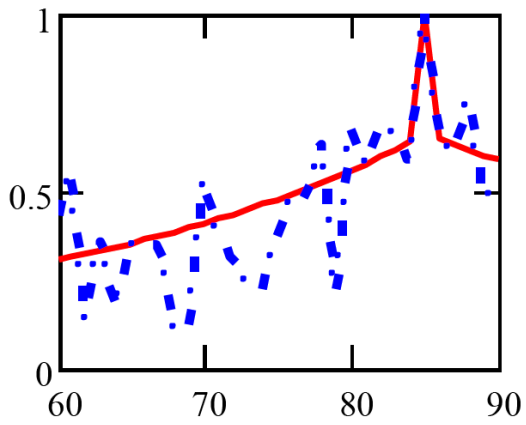


Figure 8. Empirical inter-age correlation coefficients $\rho(85, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(65, y)$ for French males (1969-1999).

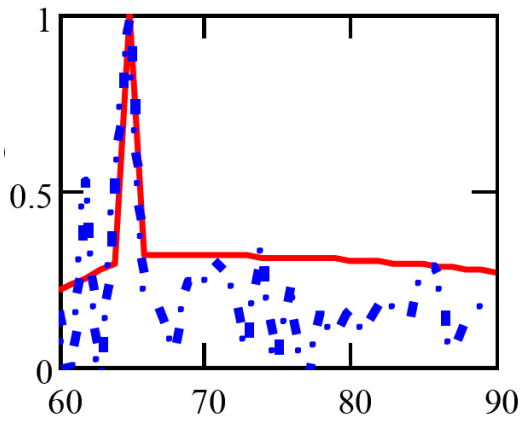


Figure 9. Empirical inter-age correlation coefficients $\rho(65, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(65, y)$ for French females (1969-1999).

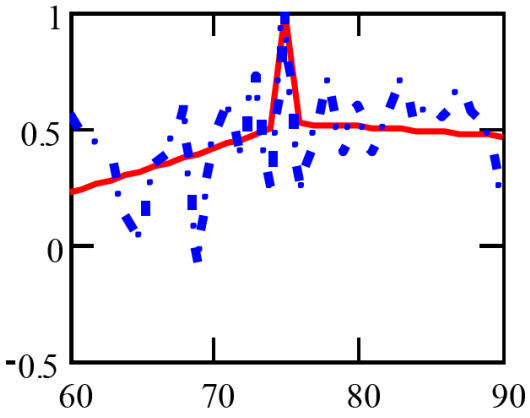


Figure 10. Empirical inter-age correlation coefficients $\rho(75, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(75, y)$ for French females (1969-1999).

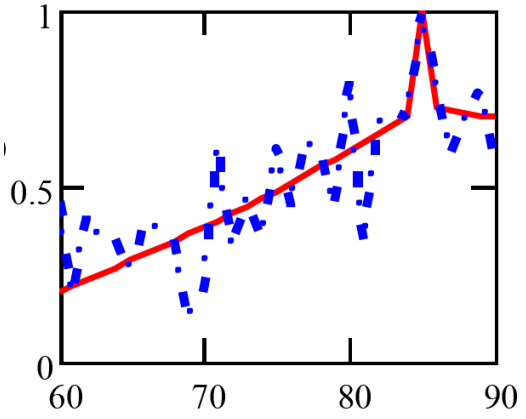


Figure 11. Empirical inter-age correlation coefficients $\rho(85, y)$ for age y (dashed) and smooth fit $\tilde{\rho}(85, y)$ for French females (1969-1999).

3.2.2 Inter-period correlations

To empirically detect and model inter-period correlation is not an easy task because for a fixed period t, the series $x \to L\Delta_{x,t}$ are not observations of i.i.d. random variables. Let us consider the series of the "average logit's Deltas", defined by

$$t \to \bar{L\Delta}_t = \frac{1}{31} \sum_{x=60}^{90} L\Delta_{x,t}$$

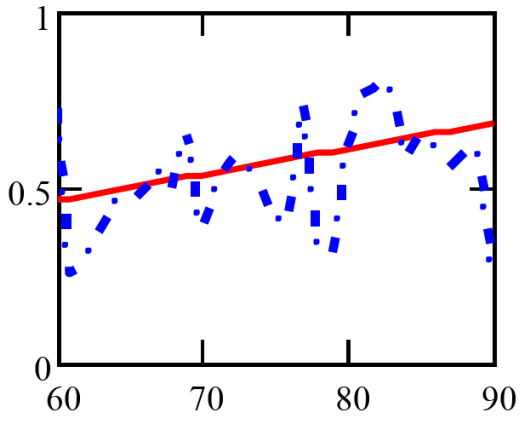


Figure 12. Empirical inter-age correlation coefficients $\rho(x, x+1)$ for age x (dashed) and smooth fit $\tilde{\rho}(x, x+1)$ for French males (1969-1999).

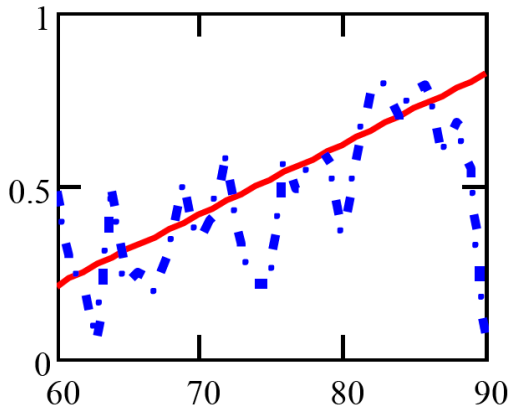


Figure 13. Empirical inter-age correlation coefficients $\rho(x, x+1)$ for age y (dashed) and smooth fit $\tilde{\rho}(x, x+1)$ for French females (1969-1999).

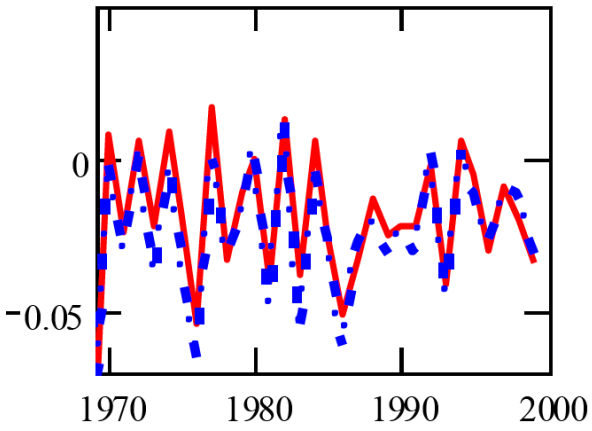


Figure 14. Average time series $L\bar{\Delta}_t$ for French females (dashed) and French males (solid).

and shown in Figure 14. The empirical study of this time series leads to a first-order autoregressive process with autocorrelation -0.56 for males and -0.57 for females. This corresponds to a short memory process, with a negative first-order autocorrelation as one could imagine from Figures 1, 2 and 14. The other phenomenon that strikingly appears in Figure 14 is the strong correlation between the time series for males and females. This is confirmed by the linear correlation coefficient which equals 96%. This correlation must be taken into account while aggregating longevity risk for males and females. Its detailed characteristics and its impacts are studied in another paper, see Devineau et al. (2007).

In order to be able to identify the process and to estimate its parameters by least squared errors or maximum likelihood methods, one must have a number of periods greater than the number of ages simultaneously considered. Besides, the estimation process is reliable if the number of periods is greater than 1.5 times the number of ages. This condition is difficult to respect as a too large number of periods would lead us to take into account old trends that are unappropriate and would generate significative bias (see Delwaerde and Denuit (2003)). To avoid this, we propose an alternative method: first decorrelate the time series, treat each univariate corresponding time series separately, and then get back to the initial time series. The passage matrices that are introduced are estimated ones and contribute to the volatility of estimations. To quantify more precisely this volatility would require further analysis and is left for future research.
We have also chosen to study separately the average trend of the process and the oscillations around this trend. The advantage of this approach is that it enables us to estimate the average trend either internally from the model (the trend is then estimated by the empirical average computed age by age on the chosen period) or from an exogenous longevity trend given by prospective life tables and demographers' long-term vision (see Section 2.2).

Let us describe the method we use more precisely. Let $\hat{\Gamma}$ be the empirical covariance matrix of the vector time series $(L\Delta_t)_{t\in t_0,t_1}$. If the process $(L\Delta_t)_t$ is stationary, the matrix $\hat{\Gamma}$ is an unbiased estimator of the covariance matrix of the $L\Delta_t$. Using diagonalization with a orthonormal basis of eigenvectors, one may write

$$\hat{\Gamma} = \hat{P}\hat{D}\hat{P}^T$$

with D diagonal and P orthonormal. Define

$$t \to w_t = \hat{P}^T \tilde{L\Delta_t},$$

where $\tilde{L\Delta_t}$ corresponds to the "centered" version of $L\Delta_t$ (obtained after substraction of the empirical average). The vectorial time series $(w_t)_t$ has uncorrelated components (and thus independent under the Gaussian assumption).

We now study each series $t \to w_t^x$ for each age x. After identification and estimation, the series $t \to w_t^x$ may be written under the form

$$w_t^x = \sum_{i=1}^{p_x} AR_i^x w_{t-i}^x + \epsilon_t^x + \sum_{j=1}^{r_x} MA_j^x \epsilon_{t-j}^x, \quad 1 \leq x \leq x_1 - x_0 + 1,$$

where ϵ_t^x is a Gaussian random variable with variance v_i.

From

$$w_t = \hat{P}^T \tilde{L\Delta}_t,$$

we then get back the process $(L\Delta_t)$ with

$$L\Delta_t = \sum_{i=1}^{\max_x p_x} AR_i L\Delta_{t-i} + U_t + \sum_{j=1}^{\max_x r_x} MA_j U_{t-j},$$

where $p_x \geq 1$ and $q_x \geq 1$ for all x, and where AR_i is the autoregressive matrix of order i defined by

$$AR_i = \hat{P} Diag\left(ar_i^1, \ldots, ar_i^{x_1 - x_0 + 1}\right) \hat{P}^T,$$

MA_j is the mobile average matrix of order j defined by

$$MA_j = \hat{P} Diag\left(ma_j^1, \ldots, ma_j^{x_1 - x_0 + 1}\right) \left[\hat{P}^T\right]^2,$$

and (U_t) is a centered Gaussian vector with covariance matrix estimated by

$$K = \hat{P} Diag\left(v_1, \ldots, v_{x_1 - x_0 + 1}\right) \hat{P}^T.$$

Note that in all the examples we have considered, the p_x and the q_x are never greater than 3, which means that long memory is not really present, and makes calibration possible in practice.

4 Impact of hidden dependence on actuarial pricing of longevity swaps

In this Section, we compare three models: the so-called independent model in which the $(L\Delta_{x,t})_{x,t}$ are independent and the $L\Delta_t$ are i.i.d. (but the variance of each component depends on the age of course), our so-called stochastic logit's Deltas model, and the so-called comonotone model, in which the the $L\Delta_t$ are i.i.d., and for a fixed calendar year t, the $(L\Delta_{x,t})_x$ are comonotone. The latter model is commonly used by some practitioners and this is why we compare it with ours. For the sake of brevity we omit results in the model where independent copies of univariate time series (for each age x) are used as it gives results that are very similar to the ones obtained with the so-called independent model. We first investigate the impact of correlations on the residual life expectation at age 60 in 10 years and in 20 years (see Figures 16 and 15, and on prospective residual life expectation at age 60 (see Figure 4 and Table 2). Unsurprisingly, the more positive the correlation is, the more dispersed those quantities are. The most important thing we observe from Figures 16 to 4 and Table 2 is that the impact of correlation is important, and that one can expect significant cut-offs in prices of longevity swaps with our model in comparison to the so-called comonotone model.

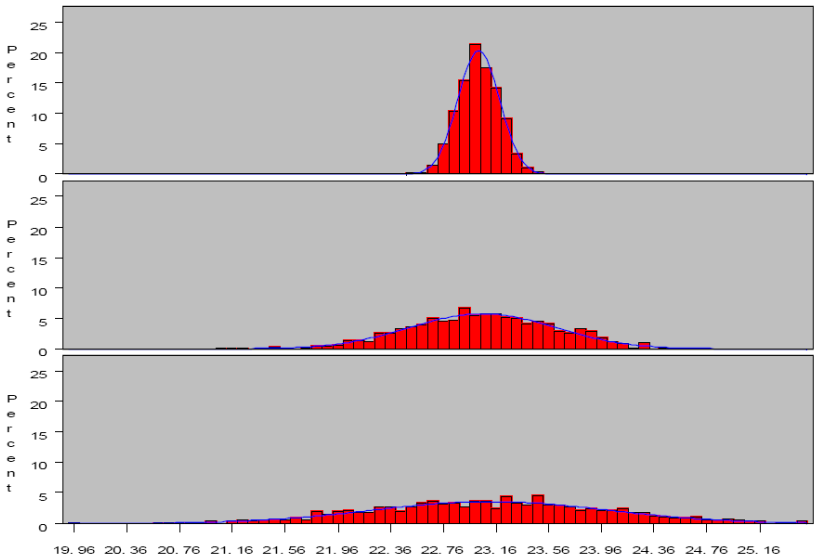


Figure 15. Empirical p.d.f. of residual life expectation at age $x = 60$ in 20 years (percentiles bar diagram, independent model (top), stochastic logit's Deltas model (middle) and comonotone model (down)).

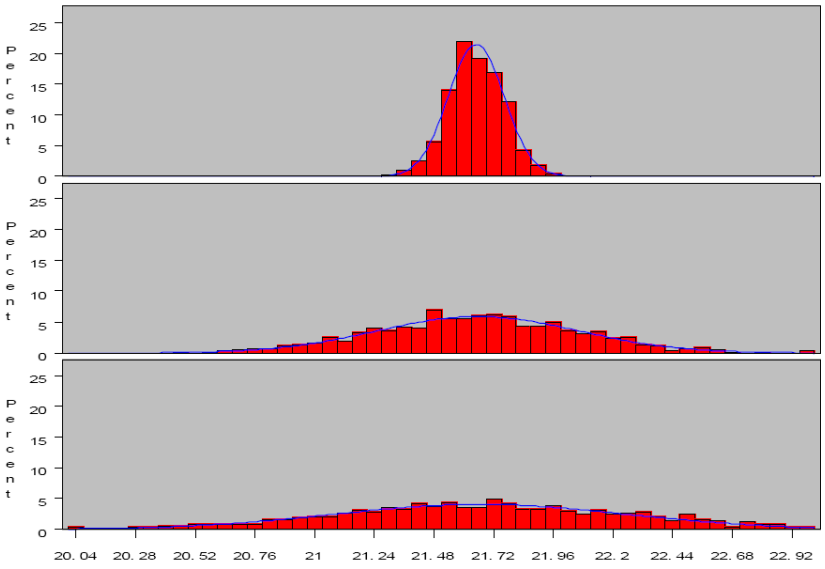


Figure 16. Empirical p.d.f. of residual life expectation at age $x = 60$ in 10 years (percentiles bar diagram, independent model (top), stochastic logit's Deltas model (middle) and comonotone model (down)).

We now consider a pure longevity swap: each year, the reinsurer pays the algebraic difference between the loaded expected cash-flows and the real ones. This corresponds to the risk that is kept by the reinsurer in a treaty where the financial part of risk is transferred to a third party (usually a financial institution). We first plot the expected discounted cashflow for each future year in Figure 21 (it is of course the same in the three models). Figure 22 confirms that coefficients of variation of the discounted cashflows of each future year in

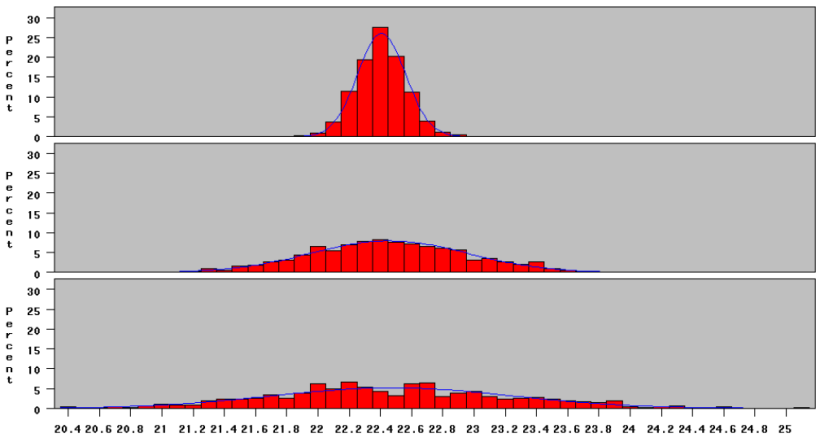


Figure 17. Empirical p.d.f. of prospective residual life expectation at age $x = 60$ (percentiles bar diagram, independent model (top), stochastic logit's Deltas model (middle) and comonotone model (down)).

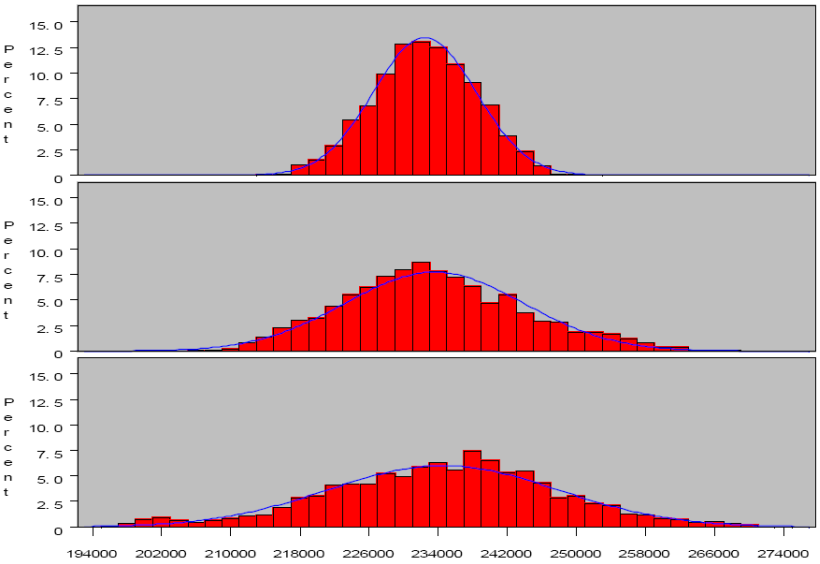


Figure 18. Empirical p.d.f. of sum of expected future cashflows (percentiles bar diagram, independent model (top), Logit's Deltas model (middle) and comonotone model (down)) without discounting effect.

th three models are ordered as one could imagine from the study of residual life expectations at age 60.

We compare the price of such a contract on a real-life (slightly modified) portfolio (with approximately 3500 individuals) and Solvency Capital Requirements (respectively with a Wang-transform and with a Value-at-Risk) in the three above models. The calibration of the parameter of the Wang-transform has been made through an internal model of SCOR and is thus not exposed here. We just illustrate here our example with results for three values of this parameter : 0.8, 1.5 and 2.1. The coherence of the Wang-transform takes the

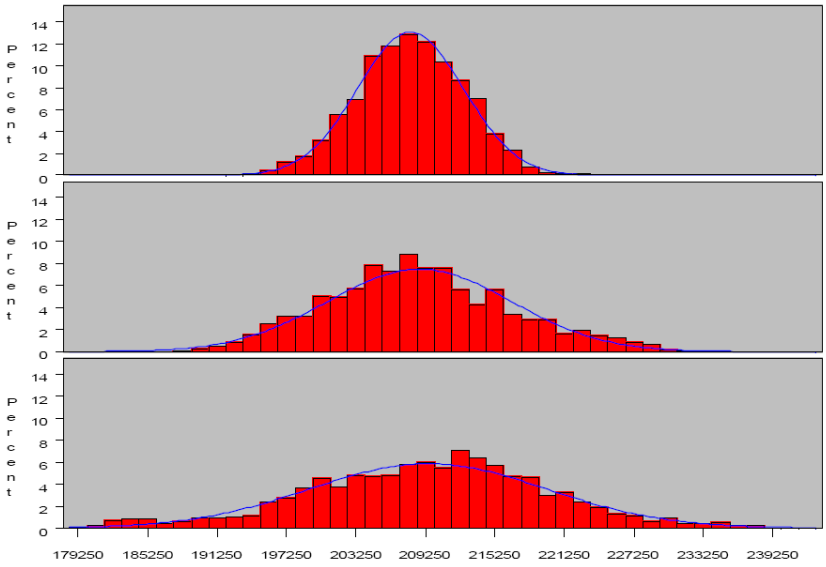


Figure 19. Empirical p.d.f. of sum of discounted expected future cashflows (percentiles bar diagram, independent model (top), stochastic logit's Deltas model (middle) and comonotone model (down)).

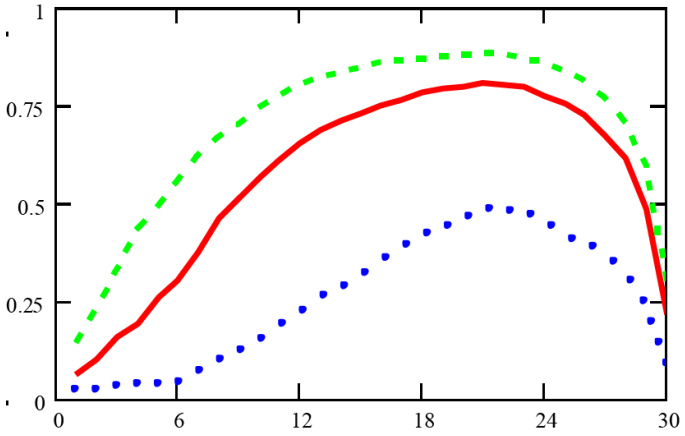


Figure 20. Part of variance of cash-flows of each year due to systemic risk (real-life portfolio), independent model (solid), stochastic logit's Deltas model (dashed), comonotone model (dotted).

mitigation effect into account and reduces the price of the contract in comparison to the comonotone case, leading to a substantial gain in price. This is shown in Figure 18 without discounting effect and in Figure 19 with the structure of interest rates published by the French *Institut des Actuaires*. The corresponding numerical results are given in Table 1.

Numerical results show that correlations have a significant impact on the price, and that a naive, cautious model may lead to a significant overestimation of the price of such contracts, while the independent model significantly underestimates it. This confirms that it is worth taking inter-age correlations into

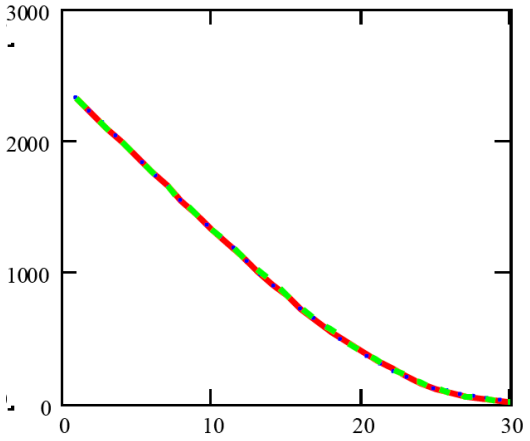


Figure 21. Average cash-flows of each year (real-life portfolio), independent model (solid), stochastic logit's Deltas model (dashed), comonotone model (dotted).

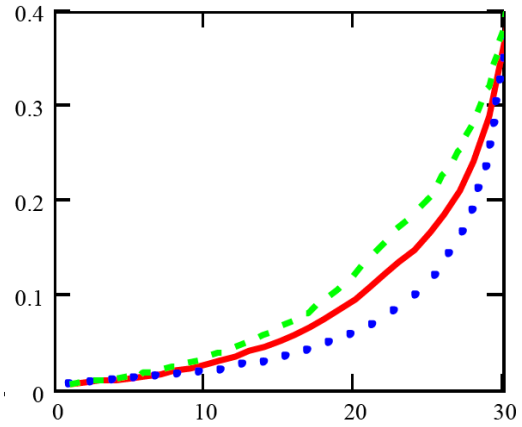


Figure 22. Coefficient of variation of cash-flows of each year (real-life portfolio), independent model (solid), stochastic logit's Deltas model (dashed), comonotone model (dotted).

account.

The part of variance coming from systemic risk is of course more important in our model than in the independent case (see Figure 20). In each model, the part of variance that comes from systemic risk first increases with the maturity of cash-flows as randomness increases, and then decreases as the size of the portfolio becomes smaller and sampling risk becomes more important. Besides, in our example, the distribution of the present value of the sum of the cashflows can be fitted quite well either by lognormal or Gaussian distributions (see Figure 19), which simplifies the computation of the Wang-transform and may be useful for aggregation of mortality risk and some financial risks.

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Risk measure	independent model	logit's Deltas model	comonotone model
Standard error (Fig. 18)	5920	10338	13300
99%-VaR (Fig. 18)	246235	257622	265504
Wang-transform, $\lambda = 0.8$ (Fig. 18)	237199	241843	245204
Wang-transform, $\lambda = 1.5$ (Fig. 18)	241343	249079	254524
Wang-transform, $\lambda = 2.1$ (Fig. 18)	244895	255282	262494
Standard error (Fig. 19)	1 4586	7959	10237
99%-VaR (Fig. 19)	217878	228090	233844
Wang-transform, $\lambda = 0.8$ (Fig. 19)	211610	215081	217652
Wang-transform, $\lambda = 1.5$ (Fig. 19)	214820	220652	224817
Wang-transform, $\lambda = 2.1$ (Fig. 19)	217572	225427	230959

Table 1
Impact of dependence on different risk measures of present values of liabilities

Risk measure	independent model	stochastic logit's Deltas model	comonotone model
Mean value	22.45	22.45	22.45
95%-VaR	22.66	23.33	23.80
99%-VaR	22.78	23.57	24.27

Table 2
Impact of dependence on some characteristics of prospective residual life expectation at age 60 (see also Figure 4).