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On the Korteweg-de Vries Approximation for Uneven Bottoms

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Abstract

In this paper we focus on the water waves problem for uneven bottoms on a two-dimensionnal domain. Starting from the symmetric Boussinesq systems derived in [Chazel, *Influence of Bottom Topography on Long Water Waves*, 2007], we recover the uncoupled Korteweg-de Vries (KdV) approximation justified by Schneider and Wayne for flat bottoms, and by Iguchi in the context of bottoms tending to zero at infinity at a substantial rate. The goal of this paper is to investigate the validity of this approximation for more general bathymetries. We exhibit two kinds of topography for which this approximation diverges from the Boussinesq solutions. A topographically modified KdV approximation is then proposed to deal with such bathymetries. Finally, all the models involved are numerically computed and compared.

Key words: Water waves, free surface flows, uneven bottoms, bottom topography, long waves, Korteweg-de Vries approximation, Boussinesq models.

Introduction

The water waves problem consists in describing the evolution of the free surface and velocity field of a layer of ideal, incompressible and irrotationnal fluid under the only influence of gravity. The governing equations - also called free surface Euler equations - are fully non-linear and non-strictly hyperbolic, and their direct study and computation remains a real obstacle. Many authors such as Nalimov ([38], 1974), Yoshihara ([51], 1982), Craig ([17], 1985), Wu ([49], 1997 and [50], 1999), Ambrose-Masmoudi ([4], 2005) and

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Lannes ([35], 2005 and [2], 2007) have successfully tackled the problem of well-posedness of these equations. Nevertheless, the numerical computation of these solutions remains a tough task, especially in 3-D - see the works of Grilli et al. ([24], 2001) and Fochesato-Dias ([22], 2001).
An alternative way to describe these solutions and their time behaviour is to look for approximations via the use of asymptotic models. Such models are usually derived formally from the water waves problem by introducing dimensionless parameters. Making some hypothesis on these parameters reduces the framework to more limited physical regimes but allows the construction of asymptotic models. In this work, we focus on the so-called long waves regime. In this regime, the ratios $\varepsilon = a/h_0$ and $\mu = h_0^2/\lambda^2$ where a denotes the typical amplitude of the waves, h_0 the mean depth and λ the typical wavelength, are small and of the same order. Many models can be found in the litterature corresponding to this regime. Among them, we can quote the works of Boussinesq ([13], 1871 and [14], 1872) who was the first to propose a model that take into account both nonlinear and dispersive effects, the unidirectional models such as the Korteweg-de Vries (KdV) equation ([33], 1895), the Kadomtsev-Petviashvili (KP) equation ([30], 1970) and the Benjamin-Bona-Mahony one ([5], 1972). These historical models have been considerably studied and generalized, and their justification has been investigated among others by Craig ([17], 1985), Schneider-Wayne ([44], 2000), Bona-Colin-Lannes ([12], 2005), Lannes-Saut ([37], 2006) for flat bottoms, and Iguchi ([25], 2006), Chazel ([15], 2007), Alvarez-Lannes ([2], 2007) for uneven bottoms.
The references [25] and [15] are somehow the starting point of the present work. In [25], Iguchi derived two different models : a coupled KdV system for relatively general bottom topographies, and a uncoupled KdV system for bathymetries decaying at a substantial rate at infinity. In [15], the author derived two classes of symmetric Boussinesq systems for two bottom topography scales, corresponding respectively to slightly and largely varying bottoms. The aim of this work is to recover the uncoupled KdV model - also justified by Schneider-Wayne for flat bottoms - starting from any previous Boussinesq system proposed in [15] for slightly varying bottoms, to discuss its validity regarding the bottom topography, and to propose an alternative.

Formulation of the problem

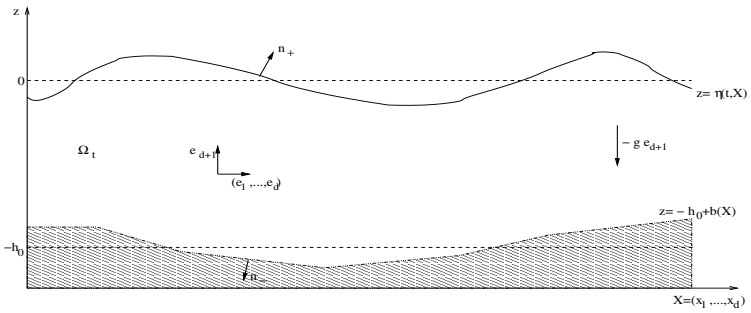
In this paper, we work in two dimensions : x corresponds to the horizontal coordinate and z to the vertical one. We denote by $(t,x) \to \eta(t,x)$ and $x \to b(x)$ the parametrizations of the free surface and bottom, defined respectively over the surface $z = 0$ and the mean depth $z = -h_0$ at the steady state. The time-dependant domain Ω_t of the fluid is thus taken of the form :

$$\Omega_t = \{(x,z),\, x \in \mathbb{R},\, -h_0 + b(x) \leq z \leq \eta(t,x)\} \ .$$

For the sake of simplicity, we assume here that $b \in W^{k,\infty}(\mathbb{R}^d)$, k being as large as needed. In order to avoid some special physical cases such as the presence of islands or beaches, we set a condition of minimal water depth : there exists a strictly positive constant h_{min} such that

$$\eta(t,x) + h_0 - b(x) \geq h_{min} \ , \ \ (t,x) \in \mathbb{R} \times \mathbb{R} \ . \tag{0.1}$$

We introduce the following dimensionless parameters :



$$\varepsilon = \frac{a}{h_0} \ , \ \mu = \frac{h_0^2}{\lambda^2} \ , \beta = \frac{\beta_0}{h_0} \ ,$$

where a denotes the typical amplitude of the waves, h_0 the mean depth, λ the typical wavelength and β_0 the typical amplitude of the bottom topography. In the present long waves regime, one has $\varepsilon \approx \mu \ll 1$, and $\beta = O(\varepsilon)$ since we focus here on slightly varying bottoms. For the sake of simplicity, we set $\mu = \beta = \varepsilon$.

In [15], the author justified a whole class of symmetric Boussinesq systems as being asymptotic models to the water waves problem for slightly varying bottoms in 2-D and 3-D, assuming the existence of the water waves solutions on a large time scale. This assumption has been recently proved by Alvarez-Lannes in [2], where the authors systematically justified the main asymptotics models used in coastal oceanography. The justification of these symmetric Boussinesq models is hence complete. All the details on the construction and justification of these models can be found in [15]; the expression of such symmetric systems in 1-D surface is as follows :

$$(\Sigma) \begin{cases} (1 - \varepsilon a_2 \partial_x^2)\partial_t v + \partial_x \eta + \varepsilon \Big[\frac{1}{2}\eta \partial_x \eta + \frac{3}{2} v \partial_x v - \frac{1}{2} b \partial_x \eta + a_1 \partial_x^3 \eta \Big] = 0 \, , \\ (1 - \varepsilon a_4 \partial_x^2)\partial_t \eta + \partial_x v + \varepsilon \Big[\frac{1}{2} \partial_x \left((\eta - b) v \right) + a_3 \partial_x^3 v \Big] = 0 \, , \end{cases}$$

with

$$\begin{cases} a_1 = -\lambda_1 \dfrac{\theta^2 - 1}{2} \ \ ; \ a_2 = (\lambda_1 - 1) \dfrac{\theta^2 - 1}{2} \, , \\ a_3 = \lambda_2 (\dfrac{\theta^2}{2} - \dfrac{1}{6}) \ ; \ a_4 = (1 - \lambda_2)(\dfrac{\theta^2}{2} - \dfrac{1}{6}) \, , \end{cases}$$

and $(\theta, \lambda_1, \lambda_2)$ being chosen in $[0,1] \times \mathbb{R}^2$ such that $a_1 = a_3$, $a_2 \geq 0$, $a_4 \geq 0$.
Several different set of values for (θ, λ, μ) can be found such that the previous condition on a_1, a_2, a_3, a_4 is verified, f.e. $\theta = \sqrt{2/3}$, $\lambda_1 = \lambda_2 = 1/2$ which gives $a_1 = a_2 = a_3 = a_4 = 1/12$. Throughout this paper, we denote by (Σ) any of this symmetric system. This system is the starting point of our work.

Organization of the paper

The paper is organized as follows. In Section I, we recover the KdV approximation proposed by Schneider-Wayne and Iguchi the following way : we first diagonalize the system (Σ) and then look for an approximation of the solution of the obtained coupled system. This approximation is searched under the form of a couple of waves (U_0, N_0)

moving in opposite directions, plus correcting terms (U_1, N_1) that satisfy a sublinear growth condition. We show that U_0 and N_0 must satisfy a system of two uncoupled KdV equations, as shown by Iguchi in [25]. It is proved that this approximation is correct for sufficiently decaying initial data and topography.
In Section II, we discuss the validity of this approximation - which is equivalent to check the sublinear growth condition on the correctors (U_1, N_1) - for non trivial bathymetries. An analysis of these terms allows us to weaken the decay assumptions on the initial data and the bathymetry. However, it appears clearly that some more general bathymetries can unvalidate the model, and two examples of bottoms are provided for which the approximation diverges : a bottom corresponding to a simple step and a slowly varying sinusoidal bottom. A topographically modified KdV approximation is then proposed by adding bottom-dependent correcting terms to the classical approximation.
In Section III, all these models - the Boussinesq one, the usual uncoupled KdV one and the topographically modified version - are numerically integrated and compared on the two bathymetries introduced in Section II. The numerical schemes are presented, and the results show that both the Boussinesq model and the alternative KdV approximation successfully reproduce the expected physical phenomenons.

1. The classical KdV approximation

In this section, we recover the usual uncoupled KdV approximation justified by Schneider and Wayne in [44] for flat botoms, and by Iguchi in [25] for slightly varying bottoms. To this end, we start with any of the symmetric Boussinesq system (Σ) derived in [15] and look for approximate solutions of the diagonalized version of (Σ) under the form of two waves moving in opposite directions. Each wave is shown to be slightly modulated in time with a dynamic governed by a KdV equation, while the correcting terms must solve an inhomogeneous transport equation. The sum of these two waves is proved to give an approximation of the solutions of (Σ) for sufficiently decaying initial data and topography.

1.1. *Derivation of the approximation*

Let us recall the expression of (Σ) :

$$(\Sigma)\begin{cases} (1-\varepsilon a_2\partial_x^2)\partial_t v + \partial_x \eta + \varepsilon\Big[\frac{1}{2}\eta\partial_x\eta + \frac{3}{2}v\partial_x v - \frac{1}{2}b\partial_x\eta + a_1\partial_x^3\eta\Big] = 0\,, \\ (1-\varepsilon a_4\partial_x^2)\partial_t \eta + \partial_x v + \varepsilon\Big[\frac{1}{2}\partial_x\left((\eta-b)v\right) + a_3\partial_x^3 v\Big] = 0\,, \end{cases}$$

with $a_2 \geq 0$ and $a_4 \geq 0$. We point out that such a system is well posed for sufficiently smooth initial data and provides an approximation of the water waves problem of order $O(\varepsilon)$ for times of order $O(1/\varepsilon)$ (see [15] and [2] for further details).
In order to recover the KdV approximation, we first diagonalize (Σ) by introducing the following unknowns :

$$U = v + \eta \;\;;\;\; N = v - \eta\,.$$

Plugging the relations $v = (U+N)/2$ and $\eta = (U-N)/2$ into (Σ) yields the following coupled system (Γ) in terms of U and N :

$$(\Gamma)\begin{cases} \partial_t U + \partial_x U + \varepsilon\left[\frac{1}{8}\partial_x(3U^2+N^2+2UN) - \frac{1}{2}b\partial_x U - \frac{1}{4}\partial_x b(U+N)\right. \\ \qquad\qquad \left. -\frac{a_2}{2}\partial_x^2\partial_t(U+N) - \frac{a_4}{2}\partial_x^2\partial_t(U-N) + a_1\partial_x^3 U\right] = 0\,, \\ \partial_t N - \partial_x N + \varepsilon\left[\frac{1}{8}\partial_x(U^2+3N^2+2UN) + \frac{1}{2}b\partial_x N + \frac{1}{4}\partial_x b(U+N)\right. \\ \qquad\qquad \left. -\frac{a_2}{2}\partial_x^2\partial_t(U+N) + \frac{a_4}{2}\partial_x^2\partial_t(U-N) - a_1\partial_x^3 N\right] = 0\,. \end{cases}$$

At this step, we choose to look for an approximate solution (U_{app}, N_{app}) of (Γ) of the form :

$$\begin{cases} U_{app}(t,x) = U_0(T, x-t) + \varepsilon U_1(T,t,x)\,, \\ \\ N_{app}(t,x) = N_0(T, x+t) + \varepsilon N_1(T,t,x)\,, \end{cases} \tag{1.1}$$

where T is the slow time variable $T = \varepsilon t$. The use of two different time scales is useful in capturing both the short time evolution of the wave and the nonlinear and dispersive dynamics which occur for larger time scales.
We complement this ansatz with two initial conditions on (U_{app}, N_{app}) and a classical sublinear growth condition on the correctors (U_1, N_1) :

$$U_{app\,|_{t=0}} = U_{|_{t=0}} \quad , \quad N_{app\,|_{t=0}} = N_{|_{t=0}}\,,$$

and for all $T_0 \geq 0$

$$\begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{t}\ |\,U_1(.\,,t,.)|_{L^\infty([0,T_0];L^2(\mathbb{R}))} = 0\,, \\ \\ \lim\limits_{t\to+\infty} \dfrac{1}{t}\ |\,N_1(.\,,t,.)|_{L^\infty([0,T_0];L^2(\mathbb{R}))} = 0\,. \end{cases} \tag{1.2}$$

Such a condition on the correctors is quite usual in multiscales expansions and has been first introduced in the context of nonlinear geometric optics by Joly, Métivier and Rauch in [29]. It forces the correctors εU_1 and εN_1 to be small on the large time scale associated to the KdV dynamics ; namely, it ensures that $\varepsilon U_1 = o(1)$ and $\varepsilon N_1 = o(1)$ in $L^\infty([0, \frac{T_0}{\varepsilon}]; L^2(\mathbb{R}))$.

Plugging this ansatz into (Γ) and neglecting the terms of order $O(\varepsilon)$ yields the following system :

$$\begin{cases} (\partial_t + \partial_x)U_1 = f(T, x-t) - \frac{1}{8}\partial_x N_0^2 - \frac{1}{4}\partial_x(U_0N_0) + \frac{1}{2}b\partial_x U_0 + \frac{1}{4}U_0\partial_x b \\ \qquad\qquad + \frac{1}{4}N_0\partial_x b + \dfrac{a_2-a_4}{2}\,\partial_x^3 N_0\,, \\ (\partial_t - \partial_x)N_1 = g(T, x+t) - \frac{1}{8}\partial_x U_0^2 - \frac{1}{4}\partial_x(U_0N_0) - \frac{1}{2}b\partial_x N_0 - \frac{1}{4}U_0\partial_x b \\ \qquad\qquad - \frac{1}{4}N_0\partial_x b - \dfrac{a_2-a_4}{2}\,\partial_x^3 U_0\,, \end{cases} \tag{1.3}$$

$$\text{where} \quad \begin{cases} f(T,x-t) = -\partial_T U_0 - \dfrac{3}{8}\partial_x U_0^2 - (a_1 + \dfrac{a_2+a_4}{2})\,\partial_x^3 U_0 \ , \\ g(T,x+t) = -\partial_T N_0 - \dfrac{3}{8}\partial_x N_0^2 + (a_1 + \dfrac{a_2+a_4}{2})\,\partial_x^3 N_0 \ . \end{cases} \tag{1.4}$$

N.B. : For the sake of simplicity, we have kept here the notations ∂_x on U_0 and N_0 while we should have written rigorously $\partial_{X_-} U_0$ and $\partial_{X_+} N_0$ with $X_- = x - t$ and $X_+ = x + t$.

At this step, an explicit resolution of (1.3) in terms of U_0 and N_0 shows that U_1 and N_1 have the simplified form :

$$\begin{cases} U_1(T,t,x) = t \times f(T,x-t) + h_b^1(U_0(T,x-t), N_0(T,x+t)) \ , \\ N_1(T,t,x) = t \times g(T,x+t) + h_b^2(U_0(T,x-t), N_0(T,x+t)) \ , \end{cases}$$

the complete expression of (U_1, N_1) being given at the end of this section in (1.9).
Such expressions of U_1 and N_1 include terms that grow linearly in time, which is inconsistent with the sublinear growth conditions (1.2). It follows that both $f(T,x-t)$ and $g(T,x-t)$ must be null quantities for all $(T,t,x) \in [0,T_0] \times \mathbb{R}^2$. Writing explicitly this result, remarking that $a_1 + (a_2+a_4)/2 = (a_1+a_3+a_2+a_4)/2 = 1/6$ and plugging it into (1.3), one obtains the following uncoupled KdV equations on (U_0, N_0) and inhomogeneous transport equations on the correctors (U_1, N_1) :

$$(\Sigma_{KdV}) \quad \begin{cases} \partial_T U_0 + \dfrac{3}{8}\,\partial_x U_0^2 + \dfrac{1}{6}\,\partial_x^3 U_0 = 0 \ , \\ \partial_T N_0 + \dfrac{3}{8}\,\partial_x N_0^2 - \dfrac{1}{6}\,\partial_x^3 N_0 = 0 \ , \end{cases} \tag{1.5}$$

and

$$(\Sigma_{corr}) \quad \begin{cases} (\partial_t + \partial_x) U_1 = -\dfrac{1}{8}\partial_x N_0^2 - \dfrac{1}{4}\partial_x(U_0 N_0) + \dfrac{1}{2} b \partial_x U_0 + \dfrac{1}{4} U_0 \partial_x b \\ \qquad\qquad + \dfrac{1}{4} N_0 \partial_x b + \dfrac{a_2 - a_4}{2}\,\partial_x^3 N_0 \ , \\ (\partial_t - \partial_x) N_1 = -\dfrac{1}{8}\partial_x U_0^2 - \dfrac{1}{4}\partial_x(U_0 N_0) - \dfrac{1}{2} b \partial_x N_0 - \dfrac{1}{4} U_0 \partial_x b \\ \qquad\qquad - \dfrac{1}{4} N_0 \partial_x b - \dfrac{a_2 - a_4}{2}\,\partial_x^3 U_0 \ . \end{cases} \tag{1.6}$$

Finally, we can construct an approximation of the solutions of the initial system (Σ) in a natural way : let $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0 \le \varepsilon \le \varepsilon_0}$ be a family of solutions of (Σ) with initial data (v_0, η_0). One defines $u_0 = v_0 + \eta_0$ and $n_0 = v_0 - \eta_0$, and let (U_0, N_0) be the solutions of the uncoupled KdV equations (Σ_{KdV}) with initial data (u_0, n_0). In the end, the uncoupled KdV approximation of the solutions of (Σ) is given by :

$$(\mathcal{M}) \ : \ v_{KdV}^\varepsilon = \frac{U_0 + N_0}{2} \ ; \ \eta_{KdV}^\varepsilon = \frac{U_0 - N_0}{2} \ . \tag{1.7}$$

This approximation corresponds to the one proposed by Schneider-Wayne for flat bottoms and by Iguchi for uneven bottoms.

1.2. *Validity of the approximation for sufficiently decaying topographies*

Recalling that (U_{app}, N_{app}) is given by (1.1), we introduce the following quantities :

$$v_{app}^{\varepsilon} = \frac{U_{app} + N_{app}}{2} \ , \ \eta_{app}^{\varepsilon} = \frac{U_{app} - N_{app}}{2} \ , \tag{1.8}$$

in order to prove the following proposition.

Proposition 1.1 *Let $s \geq 2$, $\sigma \geq s+5$, $(v_0, \eta_0) \in H^\sigma(\mathbb{R})$ and $b \in W^{1,\infty}(\mathbb{R})$. There exists $T_0 > 0$ and a unique family $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0} \in L^\infty([0, \frac{T_0}{\varepsilon}]; H^\sigma(\mathbb{R}))$ of solutions of (Σ) with initial data (v_0, η_0). We define $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$. Then there exists a unique solution (U_0, N_0) to the system (Σ_{KdV}) with initial data (u_0, n_0) and this solution is bounded in $L^\infty([0, T_0]; H^\sigma(\mathbb{R}))$.*
Moreover, we have the following error estimate for all $t \in [0, \frac{T_0}{\varepsilon}]$:

$$\left|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^{\varepsilon}, \eta_{app}^{\varepsilon})\right|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C(1 + |(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))})\, \varepsilon^2 t \, ,$$

where $(v_{app}^{\varepsilon}, \eta_{app}^{\varepsilon})$ is defined in (1.8).

PROOF. The result on the system (Σ_{KdV}) is a very classical result on the KdV equation and we omit the proof here.
The leading terms (U_0, N_0) and the correcting terms (U_1, N_1) have been chosen such that (U_{app}, N_{app}) is solution of the system (Γ) with a residual of order $O(\varepsilon^2)$. This residual denoted by $(\varepsilon^2 R_1, \varepsilon^2 R_2)$ can be computed explicitly and we get :

$$\begin{aligned} R_1 = &-\frac{3}{4}\partial_x(U_0U_1) - \frac{1}{4}\partial_x(N_0N_1) - \frac{1}{4}\partial_x(U_0N_1) - \frac{1}{4}\partial_x(U_1N_0) + \frac{1}{2}b\partial_x U_1 \\ &+\frac{1}{4}(U_1 + N_1)\partial_x b - a_1\partial_x^3 U_1 + \frac{a_2+a_4}{2}\partial_x^2\partial_t U_1 + \frac{a_2-a_4}{2}\partial_x^2\partial_t N_1 \\ &+\frac{a_2+a_4}{2}\partial_x^2\partial_T U_0 + \frac{a_2-a_4}{2}\partial_x^2\partial_T N_0 + \varepsilon\left[\frac{3}{8}\partial_x U_1^2 + \frac{1}{8}\partial_x N_1^2 + \frac{1}{4}\partial_x(U_1N_1)\right] . \end{aligned}$$

with a similar expression holding for the residual R_2 of the second equation of (Γ). Using the equations (Σ_{KdV}) and (Σ_{corr}) to express $\partial_T U_0$, $\partial_T N_0$, $\partial_t U_1$, $\partial_t N_1$ in terms of spatial derivatives of U_0, N_0, U_1, N_1 and using the fact that $s \geq 2$, this expression yields easily for all $t \in [0, \frac{T_0}{\varepsilon}]$:

$$|(R_1, R_2)|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C(1 + |(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}) \, ,$$

with C depending only on $|(U_0, N_0)|_{L^\infty([0,T_0];H^{s+3}(\mathbb{R}))}$ and $|b|_{W^{1,\infty}(\mathbb{R})}$. Inverting the diagonalization by plugging the relations $U_{app} = v_{app} + \eta_{app}$ and $N_{app} = v_{app} - \eta_{app}$ into (Γ), we easily deduce that $(v_{app}^{\varepsilon}, \eta_{app}^{\varepsilon})$ is solution of the system (Σ) with a residual bounded by $C\varepsilon^2 |(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}$. Standard energy estimates applied on the symmetric Boussinesq system (Σ) yield :

$$|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^{\varepsilon}, \eta_{app}^{\varepsilon})|_{L^\infty([0,t];H^{s+3}(\mathbb{R}))} \leq C(1 + |(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^s(\mathbb{R}))})\, \varepsilon^2 t \, ,$$

which ends the proof.

An easy extension of this proposition is the following corollary which gives an error bound for the KdV approximation.

Corollary 1.1 *Under the same hypothesis as in Proposition 1.1, we have the following error estimate for all* $t \in [0, \frac{T_0}{\varepsilon}]$ *:*

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C(1+|(U_1,N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))})\,\varepsilon(1+\varepsilon t)\,,$$

where $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ *is the uncoupled KdV approximation defined in (1.7).*

PROOF. One has :

$$\begin{aligned}(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) &= (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^\varepsilon, \eta_{app}^\varepsilon) + (v_{app}^\varepsilon, \eta_{app}^\varepsilon) \\ &\quad -(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) \\ &= (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^\varepsilon, \eta_{app}^\varepsilon) + \varepsilon\Big(\frac{U_1+N_1}{2}, \frac{U_1+N_1}{2}\Big)\end{aligned}$$

Using this relation and the error estimate coming from Proposition 1.1 yields the result.

This corollary clearly states that the validity of the uncoupled KdV approximation only depends on the control of the correcting terms (U_1, N_1) in $L^\infty\Big([0,T_0]\times[0,t];\, H^{s+3}(\mathbb{R})\Big)$ norm on the large time scale $1/\varepsilon$. From now on, these correctors become the center of our analysis.
As we saw earlier, the inhomogeneous transport equations that govern the evolution of the correctors (U_1, N_1) can be solved explicitly in terms of U_0 and N_0. We thus get the following expression :

$$\begin{aligned}U_1(t,x) = &-\frac{1}{16}(N_0^2(x+t) - N_0^2(x-t)) - \frac{a_2-a_4}{2}(\partial_x^2 N_0(x+t) \\ &-\partial_x^2 N_0(x-t)) - \frac{1}{8}U_0(x-t)(N_0(x+t) - N_0(x-t)) \\ &+\frac{1}{4}U_0(x-t)(b(x) - b(x-t)) - \frac{1}{4}\partial_x U_0(x-t) \\ &\int_0^t N_0(x-t+2s)ds + \frac{1}{2}\partial_x U_0(x-t)\int_0^t b(x-t+s)ds \\ &+\frac{1}{4}\int_0^t \partial_x b(x-t+s)N_0(x-t+2s)ds\,, \end{aligned} \tag{1.9}$$

and a similar expression holds for N_1. We here only deal with the case of U_1 since all the method can easily be adapted to the case of N_1.
The corrector U_1 is analysed in the following way : let $T_0 \geq 0$, $s \geq 2$, $\sigma \geq s+5$ and (u_0, n_0) in $H^\sigma(\mathbb{R})$. We know that the solutions (U_0, N_0) of the problem (Σ_{KdV}) with initial data (u_0, n_0) are bounded in $L^\infty([0,T_0]; H^\sigma(\mathbb{R}))^2$. We suppose here that the bottom topography b is bounded in $W^{1,\infty}(\mathbb{R})$. Under these circumstances, it clearly appears that the first four terms of the expression of U_1 are bounded in $L^\infty([0,T_0]\times[0,t]; H^{s+3}(\mathbb{R}))$. Only the last four terms can be problematic and deserve a precise treatment.
If (U_0, N_0) and b come with a sufficient decay rate at infinity, we can straightforwardly

control these terms. To this end and following [44], we introduce the following weighted Sobolev space $H^{s,\alpha}$ for all $s \in \mathbb{N}$ and $\alpha \in \mathbb{R}$:

$$H^{s,\alpha} = \left\{ u \in H^s(\mathbb{R}) \,/\, |u|^2_{H^{s,\alpha}} \equiv \sum_{k=0}^{s} \left| (1+x^2)^{\alpha/2} \frac{\partial^k u}{\partial x^k} \right|_{L^2(\mathbb{R})} < \infty \right\} .$$

We can now state our first theorem on the validity of the approximation for sufficiently decaying initial data and bottom topography.

Theorem 1.2 *Let $s \geq 2$, $\sigma \geq s+5$, $(v_0, \eta_0) \in H^{\sigma,1}(\mathbb{R})^2$ and $b \in H^{s+4,1}(\mathbb{R})$. There exists $T_0 > 0$ and a unique family $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0 \leq \varepsilon \leq \varepsilon_0} \in L^\infty([0, \frac{T_0}{\varepsilon}]; H^\sigma(\mathbb{R}))$ of solutions of (Σ) with initial data (v_0, η_0). We define $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$. Then the solution (U_0, N_0) of the system (Σ_{KdV}) with initial data (u_0, n_0) is bounded in $L^\infty([0, T_0]; H^{\sigma,1}(\mathbb{R}))$. Moreover, we have the following error estimate for all $t \in [0, \frac{T_0}{\varepsilon}]$:*

$$\left| (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) \right|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C\varepsilon(1+\varepsilon t) ,$$

where $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ is the uncoupled KdV approximation defined in (1.7).

PROOF. We know from [31] and [44] that the KdV equation propagates the regularity of initial data taken in weighted Sobolev spaces and we omit the proof here. The end of the proof is devoted to the estimate of
$|(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))^2}$. The work of Lannes in [36] is here very useful to control this quantity. Indeed, using the equations (Σ_{corr}), the fact that $U_0(T,.)$, $N_0(T,.)$ and b are bounded in $H^{\sigma,1}(\mathbb{R})$, and Proposition 3.5 of [36], one finally obtains the estimate :

$$|(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))^2} \leq C(|b|_{H^{\sigma,1}(\mathbb{R})}, (U_0, N_0)_{L^\infty([0,t];H^{\sigma,1}(\mathbb{R}))^2}) .$$

Pluging this last estimate into the result of Proposition 2.2 ends the proof.

Remark 1.3 *As specified in [12], this approximation diverges on a large time scale in the periodic framework unless we specify a zero mass assumption on the initial data u_0 and n_0. This drawback is dealt with at the end of the next section. Until then, the results provided can be extended to the periodic framework with this zero mass hypothesis.*
Remark 1.4 *It is worth pointing out that the validity of the uncoupled KdV approximation for the Boussinesq system (Σ) is enough to demonstrate its validity regarding the water waves problem. Indeed, we can deduce from [15] that the error estimate between the solutions $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ of (Σ) and the solutions of the water waves problem is of order $O(\varepsilon(1+\varepsilon t))$. An error estimate between the solutions of the water waves problem and the KdV approximation $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ can thus be immediately deduced from the results of this paper.*

2. A topographically modified KdV approximation

In this section we discuss the validity of the previously derived uncoupled KdV approximation on a large time scale for different bottom topographies. We demonstrate its

validity for less restrictive bottoms, but provide two examples of simple bottoms for which the approximation diverges. A new approximation that takes the bottom into account is finally derived.

2.1. *Discussion on the validity of the approximation*

Starting from the previous theorem, it is worth wondering if this one holds for less restrictive initial data and bottoms, i.e. without any condition of a sufficient decay rate at infinity. In this view, we focus in a more general way on the last three terms of U_1 by supposing that (u_0, n_0) is bounded in $L^\infty([0,t];H^\sigma(\mathbb{R}))^2$, which is propagated by the KdV equation on (U_0, N_0) (see [32]). Using Cauchy-Schwarz inequality on the first two terms and Proposition 3.2 of [36] on the last term, we can write the following controls for all $t \in [0, \frac{T_0}{\varepsilon}]$, $s \geq 2$ and $\sigma \geq s+5$:

$$\Big|\partial_x U_0(T, .-t)\int_0^t N_0(T, .-t+2s)ds\Big|_{H^s(\mathbb{R})} \leq C_1\sqrt{t}\,,$$

$$\Big|\partial_x U_0(T, .-t)\int_0^t b(.-t+s)ds\Big|_{H^s(\mathbb{R})} \leq C_2\,|b|_{L^2(\mathbb{R})}\,\sqrt{t}\,,$$

$$\Big|\int_0^t \partial_x b(.-t+s)N_0(T, .-t+2s)ds\Big|_{H^s(\mathbb{R})} \leq C_3\,|\partial_x b|_{H^s(\mathbb{R})}\,\sqrt{t}\,,$$

where the constants C_1, C_2, C_3 depend exclusively on $|(U_0, N_0)|_{L^\infty([0,t];H^\sigma(\mathbb{R}))^2}$.
These preliminary estimates are at the heart of the proof of the following theorem.

Theorem 2.1 *Let $s \geq 2$, $\sigma \geq s+5$, $(v_0, \eta_0) \in H^\sigma(\mathbb{R})^2$, $b \in H^{s+4}(\mathbb{R})$. There exists $T_0 > 0$ and a unique family $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0} \in L^\infty([0, \frac{T_0}{\varepsilon}]; H^\sigma(\mathbb{R}))$ of solutions of (Σ) with initial data (v_0, η_0). We define $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$. Then the solution (U_0, N_0) of the system (Σ_{KdV}) with initial data (u_0, n_0) is bounded in $L^\infty([0, T_0]; H^\sigma(\mathbb{R}))$. Moreover, we have the following error estimate for all $t \in [0, \frac{T_0}{\varepsilon}]$:*

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C\varepsilon\sqrt{t}(1+\varepsilon t)\,,$$

where $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ are as defined in (1.7).

PROOF. Using the three previous inequalities, one obtains :

$$|(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \leq C\sqrt{t}\,.$$

where $C = C(|b|_{H^{s+4}(\mathbb{R})}, (U_0, N_0)_{L^\infty([0,t];H^\sigma(\mathbb{R}))^2})$. The final result follows from Corollary 1.1.

This theorem proves that the approximation is less precise on a large time scale if we remove the assumption of a sufficient decay rate at infinity. And yet, it is worth pointing

out that the regularity imposed on b in this theorem excludes many physical cases of interest. We focus from now on two simple examples of bottoms which do not fall into the scope of Theorem 2.1 : a regular step, and a slowly varying sinusoidal bottom. Our goal is to emphasize the fact that the approximation $(v^{\varepsilon}_{KdV}, \eta^{\varepsilon}_{KdV})$ diverges from the exact solution $(v^{\varepsilon}_{\Sigma}, \eta^{\varepsilon}_{\Sigma})$ in these two simple cases. To deal with such bathymetries, a topographically modified KdV approximation is derived at the end of the section.

In order to simplify the analysis, we only consider the approximation corresponding to $a_1 = 1/6, a_2 = 0, a_4 = 0$ which is obtained for $\theta = \sqrt{2/3}, \lambda_1 = 1, \lambda_2 = 1$, and the case of a wave propagating to the right. This last condition is realized by taking $n_0 = 0$, which implies that $N_0 = N = 0$.

2.1.1. *The case of a step*
We consider here a bottom whose shape corresponds to a regular step. The interest of such an example is that in this case, $b \notin L^2(\mathbb{R})$.
The bottom is defined as follows :

$$b(x) = \begin{cases} 0 \ , \ \ \forall x \leq 0 \ , \\ \dfrac{A}{2}\left(1 + \sin\left(\dfrac{\pi}{l}(x - \dfrac{l}{2})\right)\right) \ , \ \ \forall x \in [0, l] \ , \\ A \ , \ \ \forall x \geq l \ . \end{cases} \tag{2.1}$$

For a right going wave, the system (Σ_{KdV}) is reduced to the simple KdV equation :

$$\partial_T U_0 + \frac{3}{8}\partial_x U_0^2 + \frac{1}{6}\partial_x^3 U_0 = 0 \ ,$$

and we chose the initial condition u_0 such that the solution of this equation is a positive soliton which propagates to the right.
We write the explicit expression of the corrector U_1 when $N_0 = 0$:

$$U_1(t,x) = \frac{1}{4}U_0(x-t)(b(x) - b(x-t)) + \frac{1}{2}\partial_x U_0(x-t)\int_0^t b(x-t+s)ds \ .$$

In this expression, the only possibly secularly growing term is $\partial_x U_0(T, x-t)\displaystyle\int_0^t b(x - t + s)ds$. The time evolution in amplitude of this term is obviously led by the evolution of $\displaystyle\int_0^t b(x - t + s)$ for all $x \in \mathbb{R}$. When the bottom is a step as defined in (2.1), this integral essentially grows linearly in time. We now prove that because of this, $|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}$ grows linearly in time. Let $s \geq 2$ and $\sigma \geq s + 5$. Starting from the expression of U_1, we get for all $t \in [0, \dfrac{T_0}{\varepsilon}]$ the following estimates :

$$|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} \geq \left|\frac{1}{2}\partial_x U_0(T, \cdot - t)\int_0^t b(\cdot - t + s)ds\right|_{H^{s+3}(\mathbb{R})} - C \ ,$$

with $C = \left|\dfrac{1}{4}U_0(T, \cdot - t)(b(\cdot) - b(\cdot - t))\right|_{H^{s+3}} \leq \dfrac{1}{2}|b|_{L^\infty}|U_0|_{L^\infty([0,t];H^{s+3})} \equiv C_0,$

$$|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} \geq \frac{1}{2}|\partial_x U_0(T,\cdot - t)\int_0^t b(\cdot - t + s)ds|_{L^2(\mathbb{R})} - C_0 \ ,$$
$$= \frac{1}{2}\sqrt{\int_0^\infty |\partial_x U_0(T,x-t)|^2 \ \Big| \int_0^t b(x-t+s)ds\Big|^2 dx} - C_0 \ ,$$

since $\displaystyle\int_0^t b(x-t+s)ds = 0 \ , \ \forall x \leq 0 \ ,$

$$\geq \frac{1}{2}\sqrt{\int_{l+t}^\infty |\partial_x U_0(T,x-t)|^2 \ \Big| \int_{x-t}^x b(s)ds\Big|^2 dx} - C_0 \ ,$$
$$= \frac{1}{2}At\sqrt{\int_{l+t}^\infty |\partial_x U_0(T,x-t)|^2 dx} - C_0 \ ,$$

since $\displaystyle\int_{x-t}^x b(s)ds = At \ , \ \forall x \geq l+t \ ,$

$$= \frac{1}{2}A\,t\sqrt{\int_{l}^\infty |\partial_x U_0(T,x)|^2 dx} - C_0 \ ,$$

which implies that

$$|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \geq C_1 t - C_0 \ , \tag{2.2}$$

where the last constant C_1 only depends on $|\partial_x U_0|_{L^2(\mathbb{R})}$.

This linear growth of $|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}$ is sharp since it follows from the explicit expression of U_1 that this growth is at most linear. Furthermore, we recall that

$$(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) = (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^\varepsilon, \eta_{app}^\varepsilon) + \varepsilon\left(\frac{U_1+N_1}{2}, \frac{U_1-N_1}{2}\right) \ . \tag{2.3}$$

Using this relation, (2.2) and Proposition 1.1, we get that there exists two constants C_2 and C_3 and a time T_1 independent of ε such that $\forall t \in [T_1, \dfrac{T_0}{\varepsilon}]$,

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq \big|C_2(1+t)\varepsilon - C_3(1+t)\varepsilon^2 t\big| \ .$$

We finally deduce that there exists two constants C and C' such that $\forall t \in [T_1, \dfrac{T_0}{\varepsilon}]$,

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq C\varepsilon t|C' - \varepsilon t| \ .$$

This proves that in this case, the error is of order $O(1)$ on times of order $O(1/\varepsilon)$, and the usual KdV approximation is not valid for such a topography.

2.1.2. *The case of a sinusoidal bottom*
We consider here a bottom defined as follows :

$$b(x) = A\sin(\varepsilon x) \ , \ \ \forall x \in \mathbb{R} \ . \tag{2.4}$$

where U_0 is again a soliton propagating to the right.
We mention that such a type of periodic bottom varying on a slow spatial scale has been

studied in [18] by Craig-Guyenne-Nicholls-Sulem, with the difference that the authors authorized the bottom to vary also on a small spatial scale.

Again, the amplitude of the term $\partial_x U_0(T, x-t)\displaystyle\int_0^t b(x-t+s)ds$ evolves in time according to $\displaystyle\int_0^t b(x-t+s)ds$. Let us have a look at this quantity for all $x \in \mathbb{R}$ and $t \geq 0$:

$$\begin{aligned}\int_0^t b(x-t+s)ds &= \int_{x-t}^{x} \sin(\varepsilon x)ds \\ &= -\frac{A}{\varepsilon}\left[\cos(\varepsilon x) - \cos(\varepsilon(x-t))\right] \\ &= \frac{2A}{\varepsilon}\sin\left(\varepsilon(x-\frac{t}{2})\right)\sin\left(\frac{\varepsilon t}{2}\right) \ .\end{aligned}$$

We can see that the amplitude of this term is of order $O(1/\varepsilon)$. We now demonstrate that it is also the case for the corrector U_1 :

$$\begin{aligned}|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} &\geq \left|\frac{1}{2}\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\right|_{H^{s+3}(\mathbb{R})} - C_0 \ , \\ &\geq \frac{1}{2}\left|\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\right|_{L^2(\mathbb{R})} - C_0 \ , \\ &= \frac{A}{\varepsilon}\sqrt{\int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right)\sin^2\left(\frac{\varepsilon t}{2}\right)dx} - C_0 \ , \\ &= \frac{A}{\varepsilon}|\sin^2\left(\frac{\varepsilon t}{2}\right)|\sqrt{\int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right)dx} - C_0 \ .\end{aligned} \tag{2.5}$$

At this point, we remark that

$$0 \leq \int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right)dx \leq \int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 dx \ ,$$

We hence deduce that for all $t \geq 0$ there exists $\alpha(t) \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right)dx = \sin^2\left(\alpha(t)\right)\int_{-\infty}^{\infty}\left|\partial_x U_0(T,x-t)\right|^2 dx \ .$$

Pluging this one into (2.5) leads to

$$|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} \geq \frac{A}{\varepsilon}\left|\sin^2\left(\frac{\varepsilon t}{2}\right)\sin^2\left(\alpha(t)\right)\right|\left|\partial_x U_0(T,\cdot)\right|_{L^2(\mathbb{R})} - C_0 \ ,$$

which finally implies that there exists a constant C_1 such that

$$|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \geq \frac{C_1}{\varepsilon} - C_0 \ .$$

Using this result and the same technique as in the previous example leads to the same conclusion : the uncoupled KdV approximation diverges on a large time scale in this case too.

2.2. *A topographically modified approximation*

Both examples clearly show the invalidity of the approximation on a large time scale if we consider general bottoms topographies b which do not have specific decay properties at infinity. Therefore, we need to modify the usual KdV approximation to be able to handle general bathymetries.
The explicit expression of U_1 has shown that the two terms that may exhibit a secular growth are $\frac{1}{2}\partial_x U_0(T,x-t)\int_0^t b(x-t+s)ds$ and $\frac{1}{4}\int_0^t \partial_x b(x-t+s)N_0(T,x-t+2s)ds$. As far as N_1 is concerned, the same possibly problematic terms are $\frac{-1}{2}\partial_x N_0(T,x+t)$ $\int_0^t b(x+t-s)ds$ and $\frac{-1}{4}\int_0^t \partial_x b(x+t-s)U_0(T,x+t-2s)ds$. The idea is as follows: rather than treating these terms as correcting terms, we can include them with the leading order one terms U_0 and N_0 in the approximation.

This idea leads us to propose the following topographically modified KdV approximation which is an alternative version of $(\mathcal{M})$:

$$(\mathcal{M}_b)\quad \begin{cases} v_{KdV}^{\varepsilon,b} = \dfrac{U_0+N_0}{2} + \dfrac{\varepsilon}{4}\Big[\partial_x U_0(T,x-t)\displaystyle\int_0^t b(x-t+s)ds \\ \qquad -\partial_x N_0(T,x+t)\displaystyle\int_0^t b(x+t-s)ds \\ \qquad +\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x-t+s)N_0(T,x-t+2s)ds \\ \qquad -\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x+t-s)U_0(T,x+t-2s)ds \\ \qquad +\dfrac{1}{2}U_0(T,x-t)\left(b(x)-b(x-t)\right) \\ \qquad +\dfrac{1}{2}N_0(T,x+t)\left(b(x+t)-b(x)\right)\Big]\ , \\ \eta_{KdV}^{\varepsilon,b} = \dfrac{U_0-N_0}{2} + \dfrac{\varepsilon}{4}\Big[\partial_x U_0(T,x-t)\displaystyle\int_0^t b(x-t+s)ds \\ \qquad -\partial_x N_0(T,x+t)\displaystyle\int_0^t b(x+t-s)ds \\ \qquad +\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x-t+s)N_0(T,x-t+2s)ds \\ \qquad -\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x+t-s)U_0(T,x+t-2s)ds \\ \qquad +\dfrac{1}{2}U_0(T,x-t)\left(b(x)-b(x-t)\right) \\ \qquad +\dfrac{1}{2}N_0(T,x+t)\left(b(x+t)-b(x)\right)\Big]\ . \end{cases} \tag{2.6}$$

where U_0 and N_0 are still solutions of the system $(\Sigma_{KdV}^{\varepsilon})$.

Remark 2.2 *We have here also included the terms* $U_0(T, x-t)(b(x)-b(x-t))$ *and* $N_0(T, x+t)\,(b(x+t)-b(x))$ *even if these terms remain bounded indepently of* ε *for all time. The reason of this choice is that we are interested in their physical meaning. Indeed, we further see - in the last section - that they are responsible for the reproduction of the phenomenon of shoaling. We hence decided to include these terms in the approximation.*

The main advantage of this modification relies in the following remark : now that the bottom terms have been included with the leading order terms in the approximation, we can easily see that the correcting terms U_1 and N_1 solve a different equation. Indeed, the equation on U_1 becomes :

$$(\Sigma_{corr}^{\ b}) \quad \begin{cases} (\partial_t + \partial_x)U_1 = -\dfrac{1}{8}\partial_x N_0^2 - \dfrac{1}{4}\partial_x(U_0 N_0) + \dfrac{a_2 - a_4}{2}\,\partial_x^3 N_0\ , \\ (\partial_t - \partial_x)N_1 = -\dfrac{1}{8}\partial_x U_0^2 - \dfrac{1}{4}\partial_x(U_0 N_0) - \dfrac{a_2 - a_4}{2}\,\partial_x^3 U_0\ . \end{cases} \tag{2.7}$$

It is clear here that all the possibly secularly growing terms of the correctors (U_1, N_1) have been removed.

Remark 2.3 *Numerically speaking, this modified version is quite interesting since the topographical terms are computed explicitly from the solution of the KdV equations. We thus expect the numerical simulation of this model to be faster than the one of the symmetric Boussinesq model* (Σ)*. This point is checked in the last section.*

In the periodic framework, we saw that the usual approximation is not valid on a large time scale because of the linear growth in time of the term $\partial_x U_0(T, x-t)\displaystyle\int_0^t N_0(T, x-t+2s)ds$ in U_1, unless we specify a zero mass assumption on the initial data u_0 and n_0. Once more, we can propose a valid approximation just by including this term in the order one terms of the ansatz. We conclude this section with the proposition of a new approximation in the periodic framework :

$$(\mathcal{M}_b^{per}) \quad \begin{cases} v_{KdV}^{\varepsilon,b\,per} = v_{KdV}^{\varepsilon,b} - \dfrac{\varepsilon}{8}\left[\partial_x U_0(T, x-t)\displaystyle\int_0^t N_0(T, x-t+2s)ds \right. \\ \qquad\qquad \left. +\partial_x N_0(T, x+t)\displaystyle\int_0^t U_0(T, x+t-2s)ds\right]\ , \\ \eta_{KdV}^{\varepsilon,b\,per} = \eta_{KdV}^{\varepsilon,b} - \dfrac{\varepsilon}{8}\left[\partial_x U_0(T, x-t)\displaystyle\int_0^t N_0(T, x-t+2s)ds \right. \\ \qquad\qquad \left. -\partial_x N_0(T, x+t)\displaystyle\int_0^t U_0(T, x+t-2s)ds\right]\ , \end{cases} \tag{2.8}$$

which ends this section.

3. Numerical comparison of the models

This section is devoted to the numerical comparison of the different models involved in this article. We compare here three models : the symmetric Boussinesq system (Σ)

coming from [15], the usual uncoupled KdV approximation justified by Schneider-Wayne ([44], flat bottoms) and Iguchi ([25], uneven bottoms), and finally the topographically modified KdV approximation. The aim is here to compare these three models for two non trivial examples of topography : a step and a slowly varying sinusoidal bottom.

3.1. *Numerical schemes*

Our goal is to compare three models, the symmetric Boussinesq one, the usual KdV approximation ($\mathcal{M}$) and its topographically modified version ($\mathcal{M}_b$). The comparison is made for a solitary wave propagating to the right above two topographies : a step and a slowly varying sinusoidal bottom. We use for the Boussinesq system (Σ) and the KdV equations (Σ_{KdV}) a Crank-Nicholson scheme combined with a relaxation method coming from Besse-Bruneau in [8] and justified by Besse in [6]. This type of scheme is of order two in space and time, which is appropriate for our purpose.

3.1.1. *Numerical scheme for the KdV equations*

Due to the identical structure of the two KdV equations of (Σ_{KdV}), we only present the numerical scheme for the first equation. Defining $u(t,x) = U_0(T, x-t)$, we can reformulate this equation as follows

$$\partial_t u + \partial_x u + \varepsilon \left[\frac{3}{4} u \partial_x u + \frac{1}{6} \partial_x^3 u \right] = 0 \, . \tag{3.1}$$

We use a Crank-Nicholson scheme and the relaxation method introduced by Besse-Bruneau in [8] and justified by Besse in [6] which replace the costly numerical treatment of the nonlinear term by a predictive step. This provides us with the following semi-discretized in time equation :

$$\frac{u^{n+1}-u^n}{dt} + \partial_x \left(\frac{u^{n+1}+u^n}{2} \right) + \varepsilon \left[\frac{3}{4} \left(\alpha u^{n+\frac{1}{2}} \partial_x \left(\frac{u^{n+1}+u^n}{2} \right) \right.\right.$$
$$\left.\left. + \; (1-\alpha) \frac{u^{n+1}+u^n}{2} \partial_x u^{n+\frac{1}{2}} \right) + \frac{1}{6} \partial_x^3 \frac{u^{n+1}+u^n}{2} \right] = 0 \, ,$$

where the predictive term $u^{n+\frac{1}{2}}$ is defined as follows

$$u^n = \frac{u^{n+\frac{1}{2}} + u^{n-1/2}}{2} \, .$$

The discretization of the nonlinear term $u\partial_x u$ here takes advantage of the two possible discretizations $u^{n+\frac{1}{2}} \partial_x \left(\frac{u^{n+1}+u^n}{2} \right)$ and $\frac{u^{n+1}+u^n}{2} \partial_x u^{n+\frac{1}{2}}$ by introducing a parameter $\alpha \in [0,1]$ and taking a convex combination of these possibilities. Keeping in mind that we want to preserve the semi-discrete L^2 norm, an easy integration by parts gives us the appropriate value $\alpha = 2/3$. We then choose the spatial discretization so that the discrete L^2 norm is preserved by the complete scheme, which gives the final discretization of (3.1) :

$$\frac{u_i^{n+1}-u_i^n}{\delta t}+\left(D_1\frac{u^{n+1}+u^n}{2}\right)_i+\varepsilon\left[\frac{1}{4}\left(u_i^{n+\frac{1}{2}}+\frac{u_{i+1}^{n+\frac{1}{2}}+u_{i-1}^{n+\frac{1}{2}}}{2}\right)\right.$$
$$\left.\left(D_1\frac{u^{n+1}+u^n}{2}\right)_i+\frac{1}{4}\frac{u_i^{n+1}+u_i^n}{2}\left(D_1u^{n+\frac{1}{2}}\right)_i+\frac{1}{6}\left(D_3\frac{u^{n+1}+u^n}{2}\right)_i\right]=0\,, \tag{3.2}$$

where the matrix D_1 and D_3 are to the classical centered discretizations of the derivatives ∂_x and ∂_x^3.

3.1.2. *Numerical scheme for the Boussinesq system*

As far as the discretization of the Boussinesq system (Σ) is concerned, we consider the same ideas. Using a Crank-Nicholson scheme and the same relaxation method, we aim here at preserving the specific norm $|(v,\eta)|^2_{H^1_\varepsilon}=|v|^2_{L^2}+|\eta|^2_{L^2}+\varepsilon a_2|\partial_x v|^2_{L^2}+\varepsilon a_4|\partial\eta|^2_{L^2}$. This quantity is indeed conserved by (Σ) (see [12] for more details). To this end, the nonlinear terms $v\partial_x v$, $\eta\partial_x\eta$, $\eta\partial_x v$ and $v\partial_x\eta$ are discretized in order to preserve both this specific discrete norm and their symmetric structure. Remarking that the equalities

$$(v\partial_x v,v)_{L^2}=0\ \ ;\ \ (\eta\partial_x\eta,v)_{L^2}+(\eta\partial_x v,\eta)_{L^2}+(v\partial_x\eta,\eta)_{L^2}=0\,,$$

hold for (Σ) and using the same kind of method as for the KdV equation leads to the following semi-discretization of the nonlinear terms :

$$\begin{cases} v\partial_x v(n\delta t)\approx\frac{2}{3}v^{n+\frac{1}{2}}\partial_x(\frac{v^{n+1}+v^n}{2})+\frac{1}{3}\frac{v^{n+1}+v^n}{2}\partial_x v^{n+\frac{1}{2}}\,,\\ \eta\partial_x\eta(n\delta t)\approx\frac{2}{3}\eta^{n+\frac{1}{2}}\partial_x(\frac{\eta^{n+1}+\eta^n}{2})+\frac{1}{3}\frac{\eta^{n+1}+\eta^n}{2}\partial_x\eta^{n+\frac{1}{2}}\,,\\ \eta\partial_x v(n\delta t)\approx\frac{2}{3}\eta^{n+\frac{1}{2}}\partial_x(\frac{v^{n+1}+v^n}{2})+\frac{1}{3}\frac{\eta^{n+1}+\eta^n}{2}\partial_x v^{n+\frac{1}{2}}\,,\\ v\partial_x\eta(n\delta t)\approx\frac{2}{3}v^{n+\frac{1}{2}}\partial_x(\frac{\eta^{n+1}+\eta^n}{2})+\frac{1}{3}\frac{v^{n+1}+v^n}{2}\partial_x\eta^{n+\frac{1}{2}}\,. \end{cases}$$

We then choose the spatial discretization so that the discrete H^1_ε norm is conserved, and these ruminations yield this final scheme :

$$\begin{cases} \left((I-\varepsilon a_2D_2)\frac{v^{n+1}-v^n}{\delta t}\right)_i+\left((I-\frac{\varepsilon}{2}B)D_1\frac{\eta^{n+1}+\eta^n}{2}\right)_i\\ \qquad+\varepsilon\left[\left(M_1\frac{v^{n+1}+v^n}{2}\right)_i+\left(M_2\frac{\eta^{n+1}+\eta^n}{2}\right)_i\right.\\ \qquad\qquad\left.+a_1\left(D_3\frac{\eta^{n+1}+\eta^n}{2}\right)_i\right]=0\\ \left((I-\varepsilon a_4D_2)\frac{\eta^{n+1}-\eta^n}{\delta t}\right)_i+\left((I-\frac{\varepsilon}{2}B)D_1\frac{v^{n+1}+v^n}{2}\right)_i\\ \qquad+\varepsilon\left[\left(M_3\frac{\eta^{n+1}+\eta^n}{2}\right)_i+\left(M_4\frac{v^{n+1}+v^n}{2}\right)_i\right.\\ \qquad\qquad\left.+a_1\left(D_3\frac{v^{n+1}+v^n}{2}\right)_i\right]=0\,, \end{cases} \tag{3.3}$$

where the matrix $(M_i)_{1\leq i\leq 4}$ are as follows :

$$\begin{cases} \left(M_1\,\dfrac{v^{n+1}+v^n}{2}\right)_i = \dfrac{1}{2}(v_i^{n+\frac{1}{2}} + \dfrac{v_{i+1}^{n+\frac{1}{2}}+v_{i-1}^{n+\frac{1}{2}}}{2})\left(D_1\dfrac{v^{n+1}+v^n}{2}\right)_i \\ \qquad\qquad +\dfrac{1}{2}\left(D_1 v^{n+\frac{1}{2}}\right)_i \dfrac{v_i^{n+1}+v_i^n}{2}\ , \\ \left(M_2\,\dfrac{\eta^{n+1}+\eta^n}{2}\right)_i = \dfrac{1}{3}\eta_i^{n+\frac{1}{2}}\left(D_1\dfrac{\eta^{n+1}+\eta^n}{2}\right)_i \\ \qquad\qquad +\dfrac{1}{6}\left(D_1\eta^{n+\frac{1}{2}}\right)_i \dfrac{\eta_i^{n+1}+\eta_i^n}{2}\ , \\ \left(M_3\,\dfrac{v^{n+1}+v^n}{2}\right)_i = \dfrac{1}{3}\dfrac{\eta_{i+1}^{n+\frac{1}{2}}+\eta_{i-1}^{n+\frac{1}{2}}}{2}\left(D_1\dfrac{v^{n+1}+v^n}{2}\right)_i \\ \qquad\qquad +\dfrac{1}{3}\left(D_1\eta^{n+\frac{1}{2}}\right)_i \dfrac{\eta_{i+1}^n+\eta_{i-1}^n-\eta_i^n}{2}\ , \\ \left(M_4\,\dfrac{\eta^{n+1}+\eta^n}{2}\right)_i = \dfrac{1}{6}(v_i^{n+\frac{1}{2}} + \dfrac{v_{i+1}^{n+\frac{1}{2}}+v_{i-1}^{n+\frac{1}{2}}}{2})\left(D_1\dfrac{v^{n+1}+v^n}{2}\right)_i \\ \qquad\qquad +\dfrac{1}{6}\left(D_1 v^{n+\frac{1}{2}}\right)_i \dfrac{v_i^{n+1}+v_i^n}{2}\ . \end{cases}$$

The matrix D_1 and D_3 are as defined in the KdV scheme, and the matrix D_2 is the classical centered discretization of the derivative ∂_x^2.

3.1.3. *Initial data*

Let us now talk about the initialization of the two schemes. First, all the prevision terms are initialized with a simple explicit integration of the equations on a half-step in time. Then, the initial conditions are chosen such that the simulated wave is unidirectional and propagating to the right. To this end, we first take the initial data of the second KdV equation to be zero. Then the system (Σ_{KdV}) reduces to the equation (3.1) for which we know the existence of solitary waves expressed as follows :

$$u(t,x) = \frac{\alpha}{\cosh^2\left(k(x-ct+l)\right)}\ , \tag{3.4}$$

with $c = 1+\dfrac{\varepsilon\alpha}{4}$, $k = \sqrt{\dfrac{3\alpha}{8}}$ and α, l being arbitrary.

It is hence natural to specify the initial condition for the KdV equation (3.1) as follows :

$$u(t=0,x) = u_0(x) = \frac{\alpha}{\cosh^2\left(k(x+d)\right)}\ . \tag{3.5}$$

Finally, and because of the way the KdV approximation was constructed from the Boussinesq model, we specify the initial conditions for this latter as follows :

$$v(t=0,x) = \eta(t=0,x) = \frac{1}{2}\,u_0(x)\ .$$

3.1.4. *Validation of the numerical method*

With the initial data (3.5), the KdV scheme is expected to propagate the corresponding solitary wave to the right, without any deformation for all time. In order to validate this scheme, the numerical results obtained with the initial data (3.5) have been compared with the analytical solution (3.4). The following relative errors on the free surface have been computed in the L^∞ norm for several values of epsilon and for computation times $T = 1/\varepsilon$:

ε	T	L	δx	δt	relative error
0.05	20	80	0.03	0.03	$1.5546.10^{-3}$
0.1	10	80	0.04	0.04	$1.3717.10^{-3}$
0.2	5	80	0.05	0.05	$1.0534.10^{-3}$

where L is the length of the computational domain and δx,δt are respectively the spatial and time discretization steps. These results allow to validate the scheme proposed for the KdV equations.

3.2. *Numerical results and comments*

3.2.1. *Numerical results*

As specified in [15], the choice of the parameters a_1, a_2, a_4 is very interesting in a numerical point of view. Indeed, the parameter a_1 controls the presence of the dispersive terms $\partial_x^3 v$ and $\partial_x^3 \eta$ whereas the parameters a_2 and a_4 correspond to the terms $\partial_x^2 \partial_t v$ and $\partial_x^2 \partial_t \eta$. These last terms have the main advantage of being regularizing terms analytically and numerically speaking, they smooth in some way the solution because they provide a control of the quantities $\partial_x v$ and $\partial_x \eta$ in the L^2 norm. We decided to use here the system (Σ) corresponding to $a_1 = a_2 = a_4 = 1/12$ because it is likely to provide the better results.

All the forthcoming results are expressed in non-dimensionalized variables. We recall that both the free surface and the bottom are of size ε : $z = \varepsilon\eta$ for the free surface and $z = -1 + \varepsilon b$ for the bottom. However, in order to get clear and readable results, we have plotted a rescaled free surface $z = \eta$ and a rescale bottom $z = -1 + b$. A quick word on the duration T of the simulations : the previous secion and [15] provide us with a justification of the models on large time scales of order $O(1/\varepsilon)$. We have decided - only in the first example of the step - to overtake this large time scale and simulate the models on the very large time $T = 1/\varepsilon^{3/2}$, in order to see if the model remains stable on such time scales.

The three models have been tested on two different examples of bottom. The first one correspond to a step at the bottom, defined similarly to [21] by

$$b(x) = \begin{cases} 0\ , \ \ \forall x \in \left[0, \dfrac{L}{2} - \dfrac{3}{2}\right]\ , \\ \dfrac{\beta_0}{2}\left(1 + \sin\left(\dfrac{\pi}{3}(x - \dfrac{L}{2})\right)\right)\ , \ \ \forall x \in \left[\dfrac{L}{2} - \dfrac{3}{2}, \dfrac{L}{2} + \dfrac{3}{2}\right]\ , \\ \beta_0\ , \ \ \forall x \in \left[\dfrac{L}{2} + \dfrac{3}{2}, L\right]\ , \end{cases} \tag{3.6}$$

where β_0 is an arbitrary constant of order $O(1)$ and L is the length of the computation domain.
The second example corresponds to a slowly varying sinusoidal bottom, defined as follows :

$$b(x) = b_0 \sin\left(\frac{\pi}{2} + \frac{2\pi}{l}x\right)\ , \ \ \forall x \in \mathbb{R}\ , \tag{3.7}$$

where l is defined by $l = \dfrac{1 + \varepsilon\alpha/4}{\varepsilon}$ and α is the amplitude of the initial data defined in (3.5).

The following results show the snapshots of the simulations at different times - so that the time evolution is relatively visible - and the evolution of the relative L^∞ error between the free surfaces obtained with the Boussinesq model and respectively the KdV approximation and the topographically modified approximation. The three models have been systematically plotted together in the same pictures in order to compare efficiently their respective behaviours. The numerical simulations have been performed for different values of ε in the case (3.6) of a step : $\varepsilon = 0.05$ and $\varepsilon = 0.2$, which are typical values of the upper part of the range of validity of the long waves approximation. As far as the case (3.7) is concerned, we simulated the models for the value $\varepsilon = 0.1$. For all the simulations, the amplitude α of the initial free surface and the constant β_0 linked to the bottom have been taken equal to 0.5. Here is a global tabular precising all the values of interest used in the simulations.

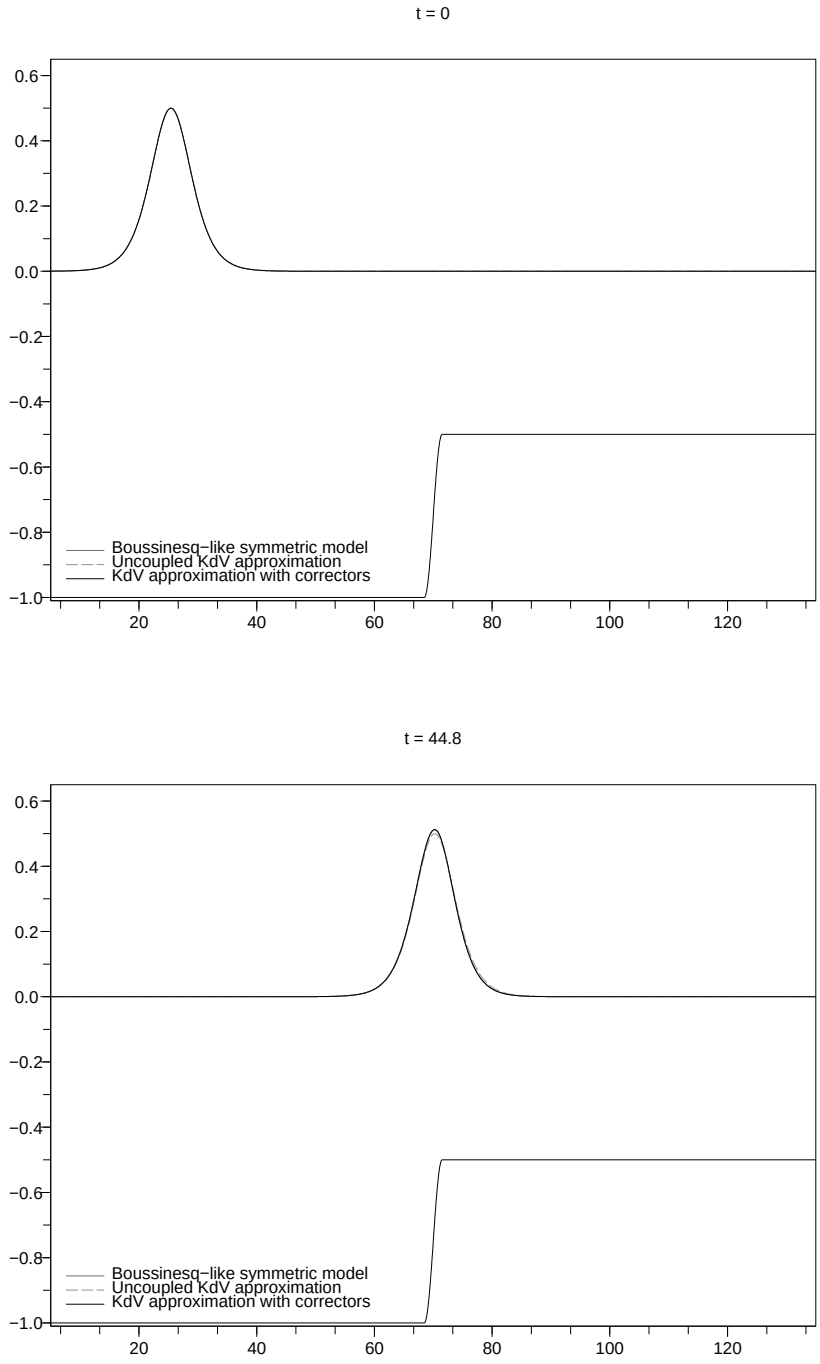
Figure	Bottom	ε	T	L	δx	δt
3.2.1	step	0.05	89	140	0.03	0.03
3.2.3	step	0.2	12	80	0.05	0.05
3.2.5	sinusoidal	0.1	10	20	0.04	0.04

The figures 3.2.2, 3.2.4, 3.2.6 show the relative error between the computed free surfaces of the different models for each value of ε and for the two cases of bottom.

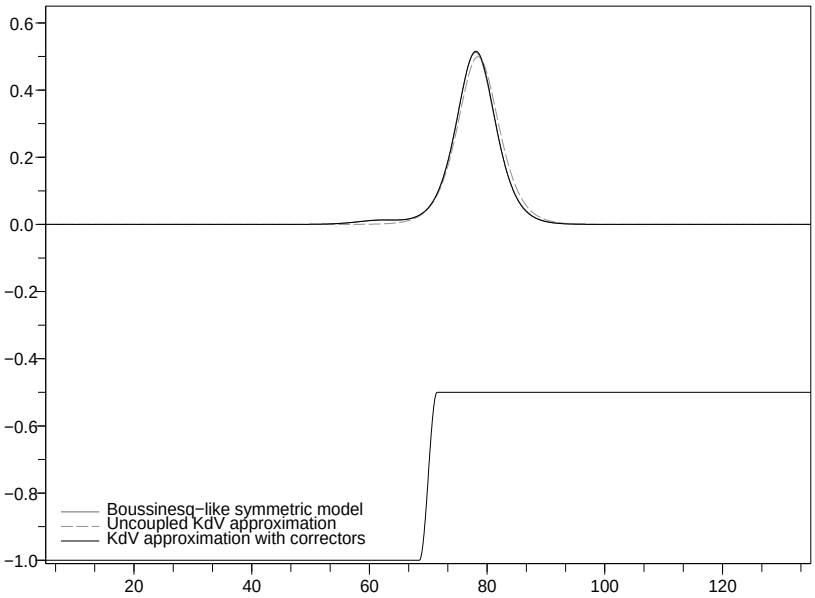
3.2.2. *Comments*

For the sake of readability, we call the computed waves as follows : $\mathcal{B}$ denotes the wave coming from the Boussinesq model (Σ), $\mathcal{K}_{topo}$ denotes the wave produced by the topographically modified KdV approximation ($\mathcal{M}_b$), and finally $\mathcal{K}$ denotes the solitary wave resulting from the usual KdV approximation ($\mathcal{M}$).

Figure 3.2.1. Influence of the step for $\varepsilon = 0.05$



t = 53


0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
20
40
60
80
100
120
Boussinesq−like symmetric model
Uncoupled KdV approximation
KdV approximation with correctors

t = 89

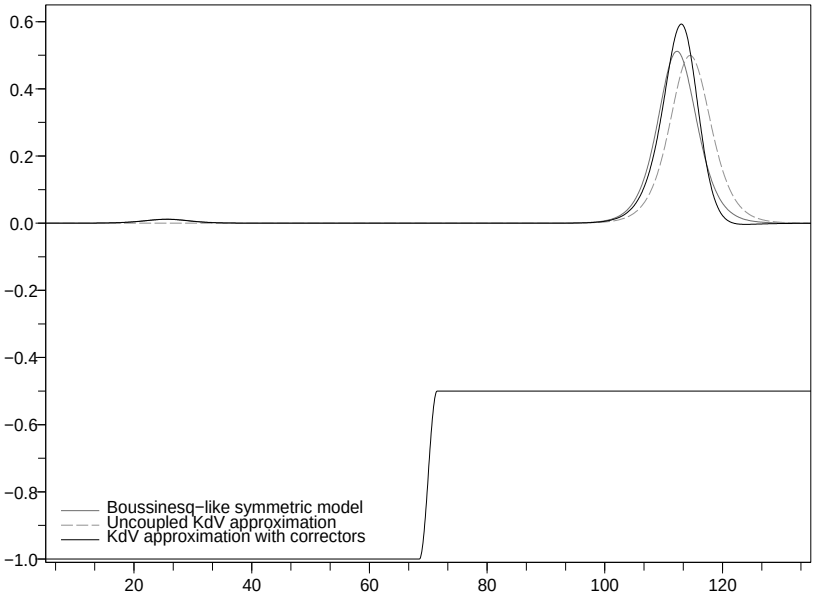
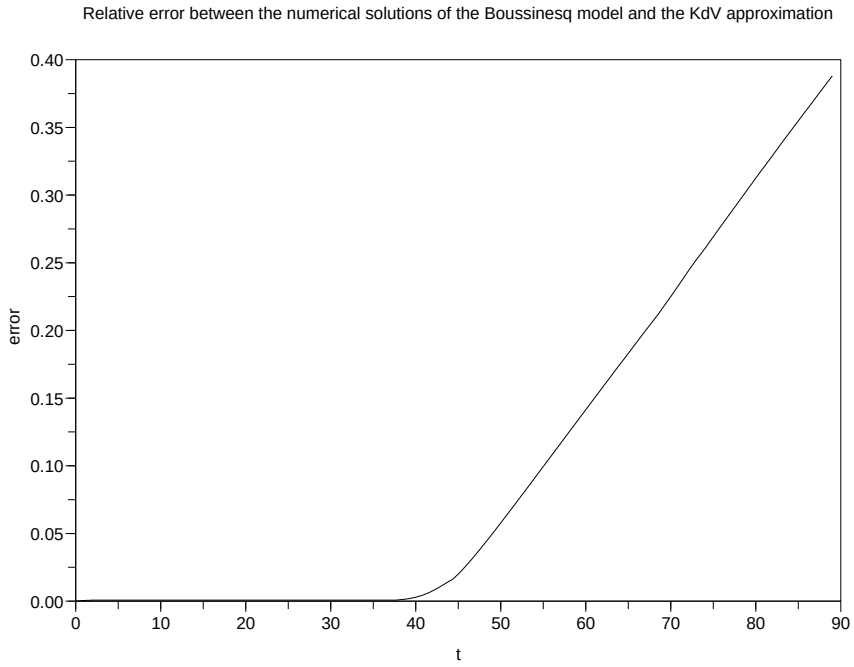

0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
20
40
60
80
100
120
Boussinesq−like symmetric model
Uncoupled KdV approximation
KdV approximation with correctors

Figure 3.2.2. Relative L^∞ error between the free surfaces for $\varepsilon = 0.05$



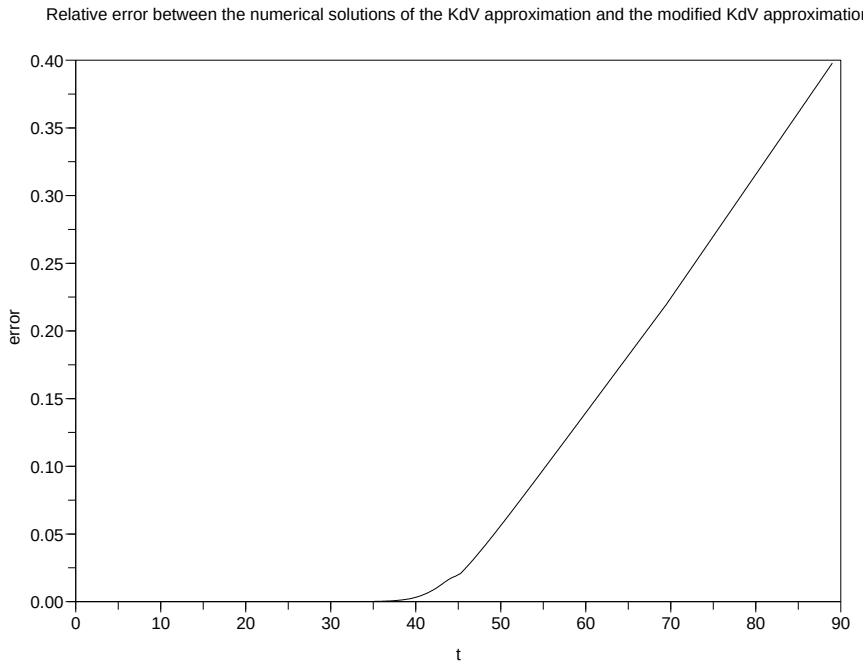
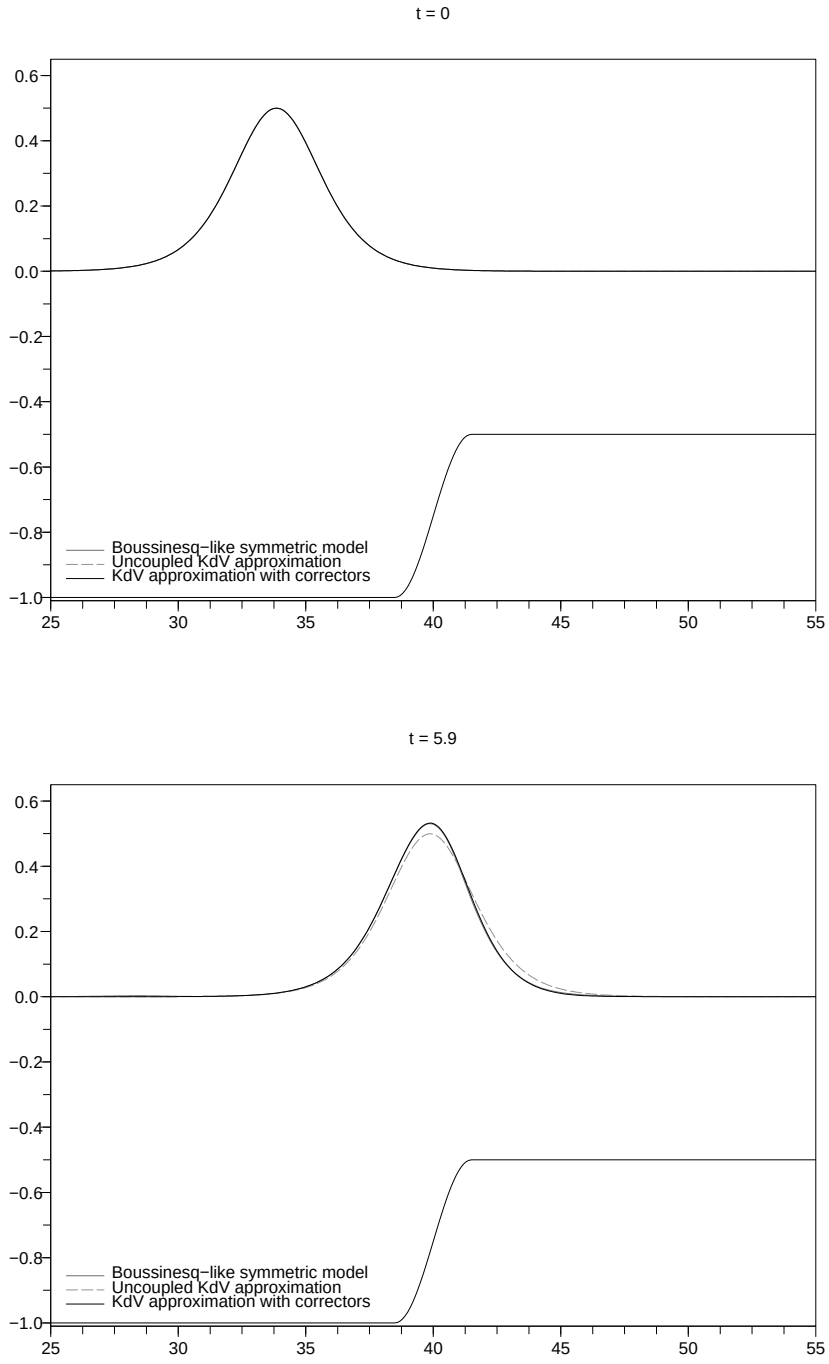
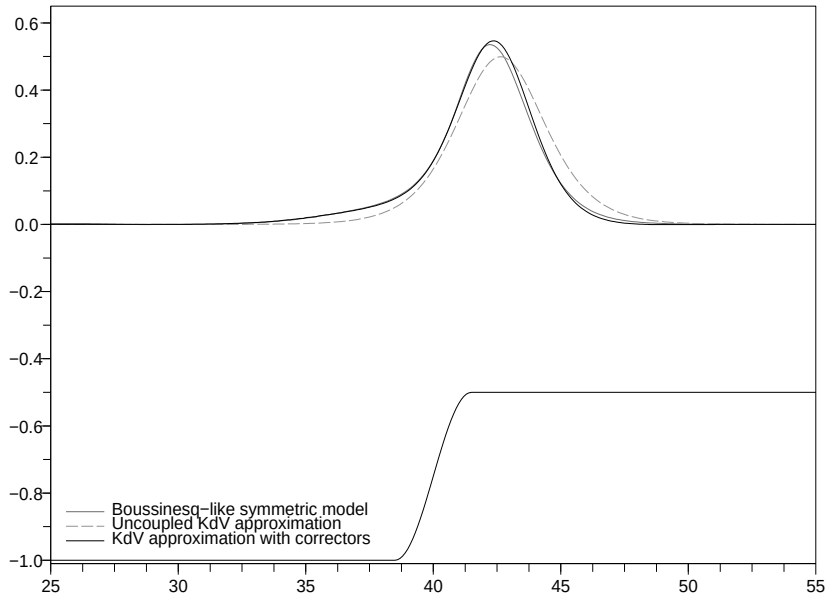


Figure 3.2.3. Influence of the step for $\varepsilon = 0.2$



t = 8.6


0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
25
30
35
40
45
50
55
Boussinesq−like symmetric model
Uncoupled KdV approximation
KdV approximation with correctors

t = 12

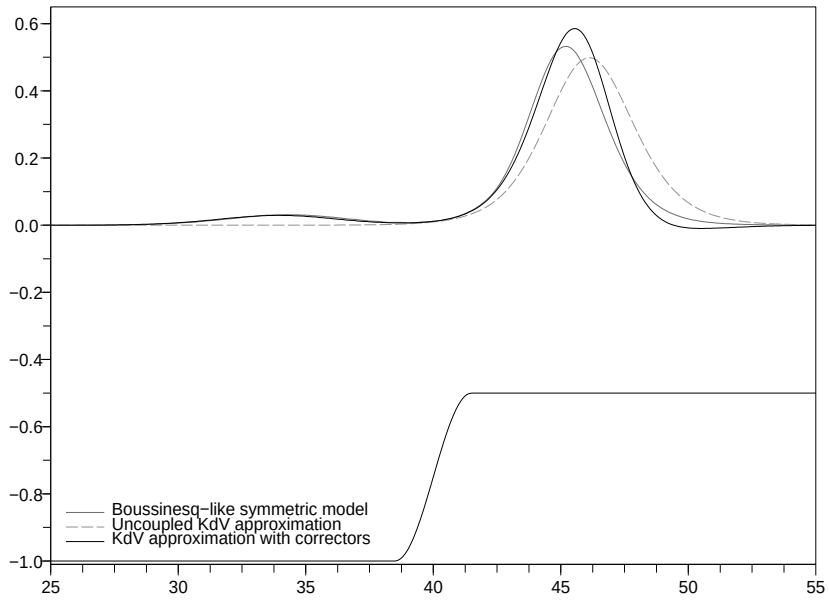
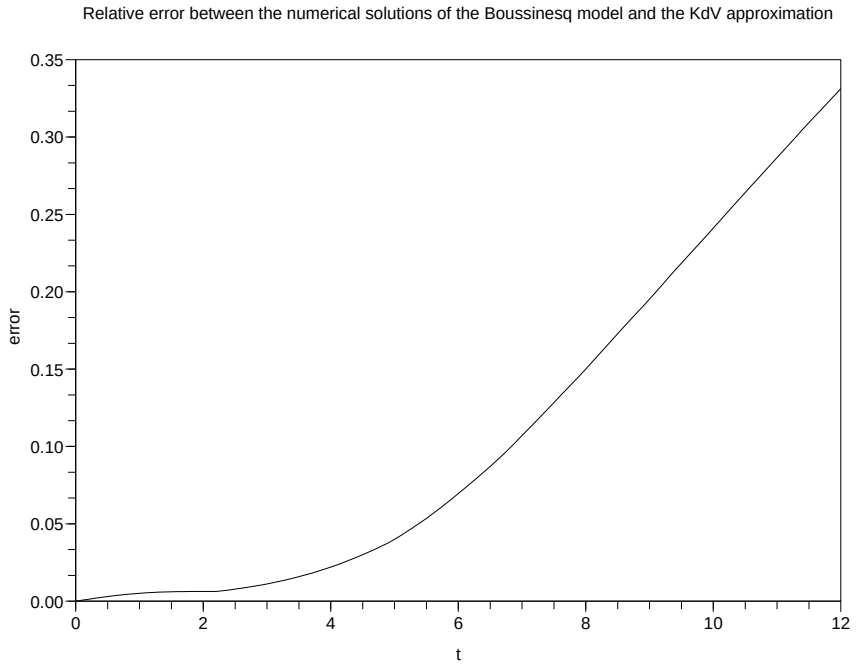

0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
25
30
35
40
45
50
55
Boussinesq−like symmetric model
Uncoupled KdV approximation
KdV approximation with correctors

Figure 3.2.4. Relative L^∞ error between the free surfaces for $\varepsilon = 0.2$



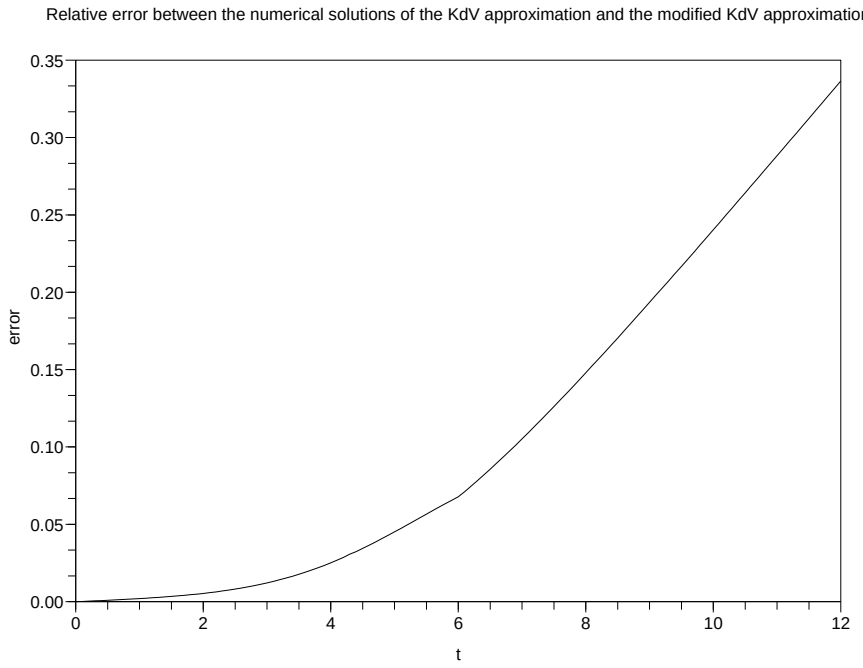
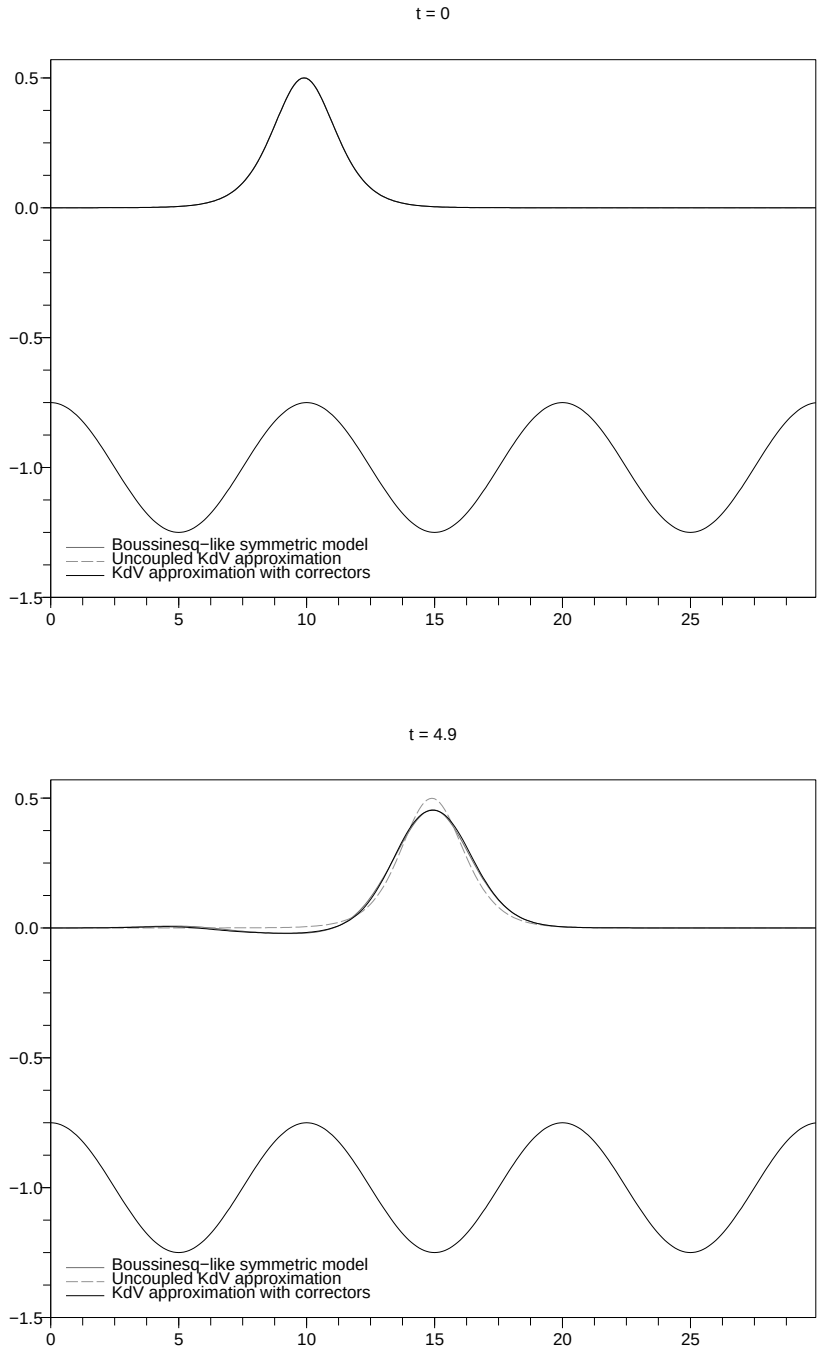


Figure 3.2.5. Influence of a slow sinusoidal bottom for $\varepsilon = 0.1$



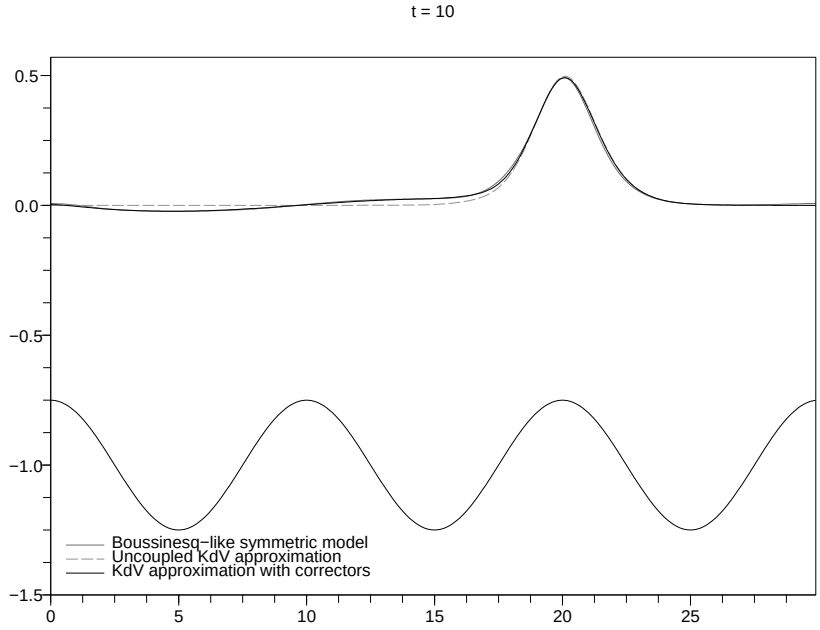
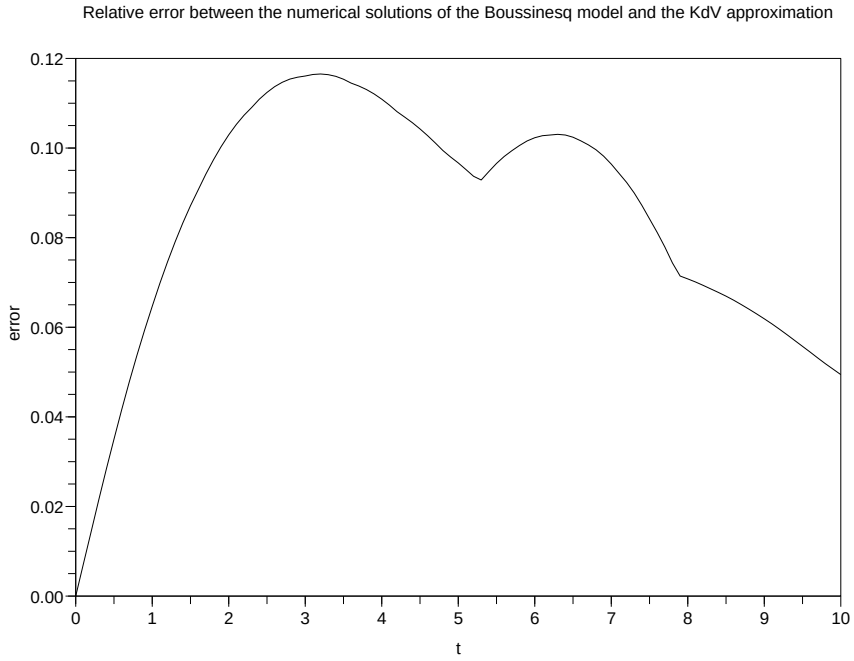
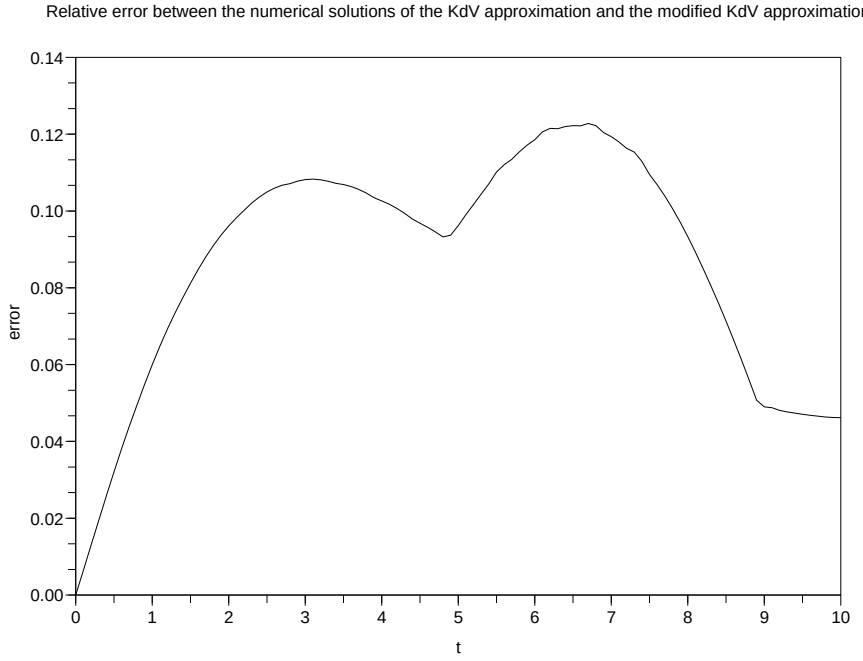


Figure 3.2.6. Relative L^∞ error between the free surfaces for $\varepsilon = 0.1$





In the case of the step, we observe that for all tested values of ε, both the Boussinesq model and alternative version of the KdV approximation succeed in reproducing the phenomenon of reflexion on the bathymetry : a smaller solitary wave appears when the main wave goes over the step. This reflected wave propagates to the left at the same speed as the main wave. The classical uncoupled KdV model cannot - of course - reproduce this phenomenon seing that it does not depend anywhere on the bottom topography. Moreover, the Boussinesq and topographically modified KdV models successfully describe the following expected physical phenomenons : the shoaling which corresponds to the growth in amplitude of the wave after the step ; the deceleration of the wave after the step : the waves $\mathcal{B}$ and $\mathcal{K}_{topo}$ are behind the wave $\mathcal{K}$ which propagates at a constant speed ; and finally the loss of symmetry and the narrowing of the wave, which can be remarked by comparing the distances between several points of the resulting waves at different heights : the solitary wave $\mathcal{K}$ propagates without any deformation and remains symmetric, whereas the symmetry and width of $\mathcal{B}$ and $\mathcal{K}_{topo}$ are modified by the step. All these phenomenons are the premisses of the process of wave breaking, and they are all successfully reproduced by the Boussinesq and new KdV models.

A very interesting remark on our KdV model is that the role of each correcting term in the approximation can be intuitively identified, and these intuitions have been confirmed with several simulations - that are not presented here - in the case of the step. Indeed, it appears clearly that :

– the correcting term $\dfrac{\varepsilon}{8}\displaystyle\int_0^t \partial_x b(x-t+s)U_0(x-t+2s)ds$ is responsible for the birth and propagation of the reflected wave, and for the very beginning of the shoaling effect,

– the correcting term $\frac{\varepsilon}{8}U_0(x-t)(b(x)-b(x-t))$ clearly reproduces the pursuit of the shoaling after the step,
– the correcting term $\frac{\varepsilon}{4}\partial_x U_0(x-t)\int_0^t b(x-t+s)ds$ is reponsible for the deceleration and loss of width and symmetry of the main wave after the step.

In the case of a slowly varying sinusoidal bottom, the effect of the bottom is also clearly visible on $\mathcal{B}$ and $\mathcal{K}_{topo}$. To understand the speed variations of these waves, we have to keep in mind that the comparison is made with the wave $\mathcal{K}$ which evolves as if the bottom was flat and located at the height $z=-1$. Consequently, when $\mathcal{B}$ and $\mathcal{K}_{topo}$ propagate above the downward part of the sinusoidal gap, they cross two different areas : a first area - for a time $t \leq T/4$ - where the depth is lower than for a flat bottom located at $z=-1$, and a second area - for $t \leq T/2$ - where this is the contrary. This explains why the waves $\mathcal{B}$ and $\mathcal{K}_{topo}$ are located for $t=T/2$ at the same position as $\mathcal{K}$: the waves $\mathcal{B}$ and $\mathcal{K}_{topo}$ have speeded up over the first area in comparison with $(\mathcal{K})$, and then have decelerated over the second area. However, we can see that these waves are larger than $(\mathcal{K})$ at $t=T/2$: this is due to the loss of amplitude of the waves $\mathcal{B}$ and $\mathcal{K}_{topo}$ during this downward part which makes the waves be naturally wider. About this loss of amplitude, it is explained by the fact that the amplitude of the bottom decreases over the downward part of the sinusoid, which produces the inverse effect of the shoaling from the previous case of the step. In addition, we can see a reflected wave for $\mathcal{B}$ and $\mathcal{K}_{topo}$ - a depression this time - which goes to the left at the same speed as the main wave, which corresponds to the same phenomenon of bathymetric reflexion as in the case of the step : the amplitude of the bottom decreases and thus produces a depression wave that propagates in the opposite direction. We can see that this depression wave is here larger than for a step because of the slow variations of the bottom. As far as the upward part of the sinusoid is concerned, all the previously described effects happen in an inverted way, and we finally recover three identical main waves for the three models. At this final point, the only remaining visible effects of the crossed topography are the reflected waves.

As specified earlier, we decided to simulate the models on the very large time $T=1/\varepsilon^{3/2}$ in the case of the step. All these models have been proved to be valid on the time scale $O(1/\varepsilon)$ and it is interesting to check numerically their validity - or not - on larger time scales. For a time $T=1/\varepsilon^{3/2}$, we can observe that the wave $\mathcal{K}_{topo}$ goes on growing more and more in amplitude, and that a depression wave deepens in front of the main wave. These effects are obviously not physical and can be explained by the fact that the size of the correcting term $\frac{\varepsilon}{4}\partial_x U_0(x-t)\int_0^t b(x-t+s)ds$ evolves in time like ε/t as we saw in the previous section : on a time $T=1/\varepsilon^{3/2}$, this size become of order $O(1/\sqrt{\varepsilon})$, which explains why this model diverges from the other models on this time scale. This is the main restriction of this model, in comparison with the Boussinesq one which seems to remain stable on very large time scales. An interesting perspective would be to look for higher order terms - like Wright in [48] - in the approximation to deal with this problem.

To sum up, the results on these two examples of bottom show that both Boussinesq and topographically modified KdV models are able to reproduce the expected physical phenomenons : reflection, shoaling, loss of speed and symmetry. This is of course not

the case for the usual KdV approximation which is independent from the bottom topography. Even if we can isolate the role of each correcting terms with our modified KdV aproximation, this one diverges when time goes over the theoretical limit time of validity $T = 1/\varepsilon$. The Boussinesq does not have this drawback and remains stable on time scales of order $O(1/\varepsilon^{3/2})$.

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