Limitation of scaling exponents estimation in turbulence

Bruno Lashermes, Christophe Baudet, Patrice Abry, Pierre Chainais

LIMITATION OF SCALING EXPONENTS ESTIMATION IN TURBULENCE

B. Lashermes$^{(1)}$, *C. Baudet*$^{(2)}$, *P. Abry*$^{(1)}$, *P. Chainais*$^{(3)}$

(1) CNRS, Laboratoire de Physique, École Normale Supérieure, Lyon, France
(2) LEGI, Université Joseph Fourier, Grenoble, France
(3) CNRS, ISIMA-LIMOS, Université Blaise Pascal, Aubière, France

Topic B: intermittency and Scaling

•Scaling in Turbulence. A significant characteristic of fully developed turbulence is scale invariance, i.e., in a wide range of scale ratios a, usually known as the *inertial range*, the moments of order $q > 0$ of the increments of the velocity field $v(x)$ or of the aggregated dissipation field $\epsilon_r(x)$ behave as power laws with respects to scale ratios (see e.g., [1]) :

$$\left\{ \begin{array}{lcl} \mathbb{E}(\frac{1}{ax_0}\int_x^{x+ax_0} \epsilon_r(u)du)^q & = & c_q|a|^{\zeta_\epsilon(q)}, \\ \mathbb{E}|v(x+ax_0)-v(x)|^q & = & c'_q|a|^{\zeta_v(q)}. \end{array} \right. \quad (1)$$

A key issue in the analysis of turbulence data lies in accurately and precisely measuring the scaling exponents. This yields two questions : how can efficient estimators for the $\zeta(q)$ be defined and what are their statistical performance ? While the former question owns classical answers, the latter has been mostly overlooked (cf., a contrario, [2]).

In the present work, we address carefully this question. To do so, we apply the usual multiresolution based estimators for the $\zeta(q)$ to recently proposed multifractal processes used as reference. We show that they undergo a generic *linearisation effect* : there exists a critical q value below which the estimators correctly account for the scaling exponents and above which they significantly depart from the $\zeta(q)$ and necessarily behave as a linear function in q. We also show that this is not a finite observation duration effect. Applied to actual empirical turbulence data, we observe a comparable linearisation effect and estimate the corresponding critical q value. We comment on the implied limitations in the estimation of scaling exponents and consequences in turbulence.

•Estimation. To perfom the scaling exponents $\zeta(q)$ estimation, one defines multiresolution quantities $T_X(a,t) = \langle \psi_{a,t}, X\rangle$ (aggregation, increments or wavelet coefficients), where $\psi_{a,t}(u) = 1/a\psi((u-t)/a)$ are dilated and translated templates from the reference pattern ψ and where X is the process to be analysed. Then, one computes the q-order structure functions, defined as the time averages of the $|T_X(a,t)|^q$, at scale a :

$$S_n(a,q) = \frac{1}{n_a}\sum_{k=1}^{n_a} |T_X(a,t_k)|^q, \quad (2)$$

where n is the process length, and n_a the number of coefficients $T_X(a,t_k)$ available at scale a. When X presents scaling as in Eq. (1), the structure functions follow power laws of the scales : $S_n(a,q) \sim c_q|a|^{\zeta(q)}$. Estimates $\hat{\zeta}(q,n)$ are then obtained from linear regressions in a $\log S_n(a,q)$ vs $\log a$ diagrams. To avoid technical discussions on estimators fully outside the scope of this paper (such as the behaviours of $\hat{\zeta}(q,n)$ when $q \le -1$), we restrict ourselves to real positive orders q.

•Multifractal processes. To characterise the performance of the $\hat{\zeta}_X(q,n)$, they are applied to synthetic multifractal processes, with theoretically perfectly known scaling exponents. Instead of the celebrated Mandelbrot's multiplicative cascades (CMC) [3], we use Compound Poisson Cascades (CPC) recently defined by Barral & Mandelbrot [4] with improved statistical properties, such as continuous scale invariance and stationarity, and with a priori prescribed and known $\zeta(q)$. The corresponding density Q_r, obtained as the product of positive (mean one) multipliers, is a positive process that can be used to model dissipation in turbulence.

Following an idea that goes back to Mandelbrot, together from $Q_r(t)$ and from Fractional Brownian Motion $B_H(t)$ with Hurst parameter H, one can build a *velocity-like* process, called Fractional Brownian Motion in Multifractal Time [5] (FBM(MT)) :

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+, \quad (3)$$

where $A(t) = \lim_{r\to 0}\int_0^t Q_r(s)ds$ is the measure process or multifractal time.

Both processes posses scale invariance properties, with theoretically known $\zeta(q)$ functions :

$$\left\{ \begin{array}{lcl} \mathbb{E}(\frac{1}{a\tau_0}\int_t^{t+a\tau_0} Q_r(u)du)^q & = & c_q|a|^{\zeta_{Q_r}(q)}, \\ \mathbb{E}|V_H(t+a\tau_0)-V_H(t)|^q & = & c'_q|a|^{\zeta_{V_H}(q)}, \end{array} \right. \quad (4)$$

where $\zeta_{V_H}(q) = qH + \zeta_{Q_r}(qH)$.

The Legendre transform of $\zeta(q)$, defined as : $D(h) \equiv 1 + \min_q \ (qh - \zeta(q))$ will be further used.

•Linearisation effect. To study the performance of the $\hat{\zeta}(q,n)$, we apply them to a large number of replications of FBM(MT) processes built on CPC cascades. First, we observe that, for each and every replication, there exists a finite range of q values, denoted $[0,q_0]$, within which $\hat{\zeta}(q,n)$ accounts for the theoretical value $\zeta(q)$ for $V_H(t)$. But outside this range, i.e., when q exceeds the critical order q_0, the $\hat{\zeta}(q,n)$ necessarily present a linear behaviour in q (cf. Fig. 1, left column). Moreover, these individual asymptotic straight lines are distributed around a mean straight line, that depends neither on the resolution of the process under study nor on its observation duration n (this is not a finite size effect) [6] :

$$\left\{ \begin{array}{ll} q \in [-1,q_0], & \hat{\zeta}(q,n) \to \zeta(q) \\ q \ge q_0, & \hat{\zeta}(q,n) = \hat{\alpha}_*^+ + \hat{\beta}_*^+ q \to \overline{\alpha}_*^+ + \overline{\beta}_*^+ q \end{array} \right. \quad (5)$$

This generic and systematic effect (observed with *all processes* and *all estimators*, cf. [6]) will be denoted as *linearisation effect* in the sequel. Fig. 1 (right column) shows the Legendre transforms $\hat{D}(h,n)$ corresponding to $\hat{\zeta}(q,n)$. Each $\hat{D}(h,n)$ is abruptly ended by an accumulation point, (h_0, D_0), and accounts for $D(h)$ (corresponding to the theoretical $\zeta(q)$ function) only when $h \ge h_0$. Furthermore, the accumulation points are spread around the critical point $(h_*^+, D_*^+ = 0)$ (Fig. 1 bottom), defined as the (left) zero

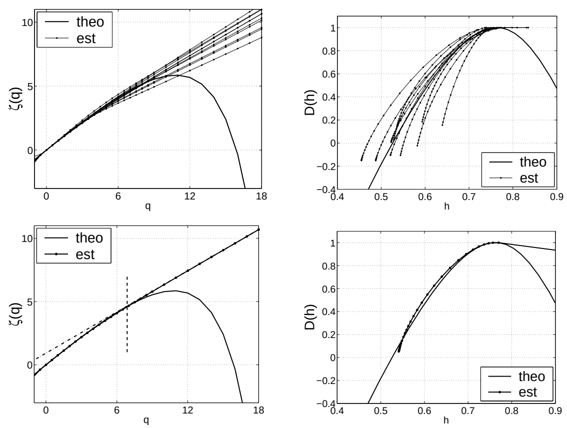

Fig. 1. **Linearisation effect on synthetic data.** Theoretical and estimated $\zeta(q)$ (left column) and corresponding $D(h)$ (right column), for 10 replications (top raw) and averaged over 1000 replications (bottom raw). FBM(MT) built on CMC cascades, with $n = 2^{19}$ and 2^4 integral scales.

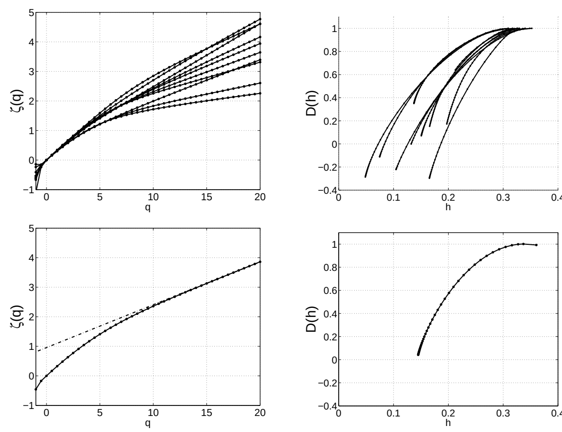

Fig. 2. **Linearisation effect on experimental turbulence (velocity) data.** Estimated $\zeta(q)$ (left column) and corresponding $D(h)$ (right column) for 10 runs (top raw) and averaged over 69 runs (bottom raw), with $n = 2^{20}$ and $\simeq 60$ integral scales per run).

of the theoretical Legendre transform : $h_*^+ \;/\; D(h_*^+) = 0 = D_*^+$. Again these results depend neither on the resolution of the process under study nor on of its duration n.

These results lead us to propose a theoretical definition for the critical order q_*^+, beyond which $\hat{\zeta}(q,n)$ presents a linear behaviour in q, and no longer converges to $\zeta(q)$:

$$q_*^+ \;/\; q\zeta'(q) - \zeta(q) \geq -1, \text{ if } q \in [0, q_*^+] \qquad (6)$$

This criterion has been obtained in the literature [7, 8], but only for the CMC densities. We extend it to other cascade schemes, and to others processes (FBM(MT))[6], showing that the linearisation effect is a very generic and systematic effect in the scaling exponent estimation.

It is worth noting that this critical q_*^+ and hence the linearisation effect is not linked with any statistical moment divergence issue. Indeed, if $\mathbb{E}|T_X(a,t)|^q = \infty, q \geq q_c$, the following unequality can be easily shown : $q_*^+ < q_c$ [6].

•**Estimation of q_*^+.** The next issue that then rises is, given an experimental time serie, to estimate q_*^+. A practical procedure for the estimation of q_*^+ will be given in the extended paper, and numerically probed on synthetic prossesses.

•**Results on turbulence data.** We apply the $\hat{\zeta}(q,n)$ to experimental hot-wire velocity data, obtained in jet turbulence and with a Taylor-based Reynolds number $R_\lambda \sim 580$ (data collected at ENSL[9]). Linear fits are performed in the usual inertial range. Fig. 2 clearly shows that a linearisation effect occurs that is highly comparable to that obtained on synthetic FBM(MT). Then, we applied the estimator for q_*^+ to the data. This yields the following average estimate for the critical value of q : $\overline{q}_*^+ \simeq 9.4 \pm 0.4$. This calls for the following comments. If the turbulent velocity fluctuations were described with two celebrated models [1] that actually satisfactorily fit the $\hat{\zeta}_X(q,n)$, namely the log-normal model — with the commonly accepted value for the intermittency parameter $C_2 \simeq 0.025$ —, and the log-Poisson She-Lévêque model (with no free parameter), then, the theoretical critical q_*^+ derived from the criterion (6) would read :

log-normal ($C_2 = 0.025$)	She-Lévêque
$q_*^+ \simeq 8.94$	$q_*^+ \simeq 12.36$

This is in reasonable agreement with the estimation obtained here directly from the data. Using other experimental data sets, the extended paper will show that the estimated critical order q_*^+ for turbulent velocity does not depend on the Reynolds number R_λ, which is consistent with the fact that the function $\zeta(q)$ is expected to be universal (i.e., independent of R_λ).

As a conclusion, these results tell us that one cannot estimate $\zeta(q)$ when $q \geq q_*^+ \simeq 9.4$ for any fully developed turbulence velocity scalar data, whatever their resolution and observation duration. We again put the emphasis on the fact that larger duration observations will not modify that value.

1. REFERENCES

[1] U. Frisch, *Turbulence, the Legacy of A.N. Kolmogorov*. Cambridge University Press, 1995.

[2] F. Schmitt, D. Schertzer, S. Lovejoy, and Y. Brunet, “Empirical study of multifractal phase transitions in atmospheric turbulence,” *Nonlinear Processes in Geophysics*, vol. 1, pp. 95–104, 1994.

[3] B. B. Mandelbrot, “Intermittent turbulence in self similar cascades : Divergence of high moments and dimension of the carrier,” *J. Fluid. Mech.*, vol. 62, p. 331, 1974.

[4] J. Barral and B. Mandelbrot, “Multiplicative products of cylindrical pulses,” *Probab. Theory Relat. Fields*, vol. 124, pp. 409–430, 2002.

[5] B. B. Mandelbrot, “A multifractal walk down wall street,” *Scientific American*, vol. 280, pp. 70–73, Feb. 1999.

[6] B. Lashermes, P. Abry, and P. Chainais, “New insights on the estimation of scaling exponents,” 2003. preprint.

[7] G. M. Molchan, “Turbulent cascades : Limitations and a statistical test of the lognormal hypothesis,” *Phys. Fluids*, vol. 9, no. 8, pp. 2387–2396, 1997.

[8] M. Ossiander and E. Waymire, “Statistical estimation for multiplicative cascades,” *The Annals of Statistics*, vol. 28, no. 6, pp. 1533–1560, 2000.

[9] G. Ruiz Chavarria, C. Baudet, and S. Ciliberto, “Hierarchy of the energy dissipation moments in fully developed turbulence,” *Phys. Rev. Lett.*, vol. 74, pp. 1986–1989, 1995.