L^p-Inequalities for Scalar Oseen Potential

Chérif Amrouche, Hamid Bouzit

L^p-Inequalities for Scalar Oseen Potential

Chérif Amrouche [a] , Hamid Bouzit [a,b]

a - Laboratoire de Mathématiques Appliquées de Pau
CNRS UMR 5142
Université de Pau et des Pays de l'Adour
IPRA, Avenue de l'Université – 64000 Pau – France
b - Département de Mathématiques
Université de Mostaganem, Algérie.

Abstract

The Oseen equations are obtained by linearizing the Navier-Stokes equations around a nonzero constant vector which is the velocity at infinity. We are interested with the study of the scalar problem corresponding to the anisotropic operator $-\Delta + \frac{\partial}{\partial x_1}$. The Marcinkiewicz interpolation's theorem and the Sobolev embeddings are used to give, in the L^p theory, the continuity's properties of the scalar Oseen potential. The contribution of the term $\frac{\partial}{\partial x_1}$ gives supplementary properties with regard to the Riesz potential.

Key words: Oseen potential, weighted Sobolev spaces, L^p-theory.
AMS Classification: 26D10, 35Q35, 42B20.

1 Introduction

The Oseen system is obtained by linearizing the Navier-Stokes equations around a nonzero constant vector $\boldsymbol{u} = \boldsymbol{u}_\infty$, where $\boldsymbol{u}_\infty = \lambda \boldsymbol{e}_1$ is the velocity at infinity, and can be written as follow (see [14], [15]):

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^n, \\ \operatorname{div} \boldsymbol{u} = g \quad \text{in} \quad \mathbb{R}^n, \end{aligned} \tag{1.1}$$

Email addresses: `cherif.amrouche@univ-pau.fr` (Chérif Amrouche [a]), `hamidbouzit@yahoo.fr` (Hamid Bouzit [a,b]).

where $n \geq 2$. The data are, the external forces $\boldsymbol{f}$, a function g, the positive real number λ and the viscosity of the fluid ν. The unknowns of this system are the velocity $\boldsymbol{u}$ and the pressure π. One of the first works devoted to these equations is due to Finn [6], [7]. Specifically, Finn studied (1.1) Oseen's equations in three and two dimensional exterior domains when $(1 + |\boldsymbol{x}|)\boldsymbol{f} \in L^2(\mathbb{R}^3)$ and $g = 0$. He proved that this system has a unique solution $\boldsymbol{u}$ such that $(1 + |\boldsymbol{x}|)^{-1}\boldsymbol{u} \in L^2(\mathbb{R}^3)$. When $(\boldsymbol{f}, g) \in \boldsymbol{W}^{m,p}(\mathbb{R}^n) \times W^{m+1,p}(\mathbb{R}^n)$ with $m \geq 0$, Galdi [8] proved that Problem (1.1) has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m,p}_{loc}(\mathbb{R}^n) \times W^{m+1,p}_{loc}(\mathbb{R}^n)$ and some results on the derivatives of $\boldsymbol{u}$ and π under conditions on the power p. Recently, Amrouche and Razafison [2] investigated this problem by working in weighted Sobolev spaces.
Taking the divergence of the first equation of (1.1), we observe that the pressure satisfies the Poisson's equation (see for instance [1]):

$$\Delta\pi = \mathrm{div}\boldsymbol{f} + \nu\Delta g - \lambda\frac{\partial g}{\partial x_1} \quad \text{in} \quad \mathbb{R}^n, \tag{1.2}$$

and the vector field $\boldsymbol{u}$ satisfies

$$-\nu\Delta\boldsymbol{u} + \lambda\,\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\,\pi \quad \text{in} \quad \mathbb{R}^n. \tag{1.3}$$

Now observe that each component u_j of the velocity satisfies

$$-\nu\Delta u_j + \lambda\,\frac{\partial u_j}{\partial x_1} = f_j - \frac{\partial \pi}{\partial x_j} \quad \text{in} \quad \mathbb{R}^n. \tag{1.4}$$

Hence, we see that the Oseen problem (1.1) can be reduced to the following scalar equation

$$-\nu\Delta u + \lambda\,\frac{\partial u}{\partial x_1} = f \quad \text{in} \quad \mathbb{R}^n, \tag{1.5}$$

where $f = f_j - \frac{\partial \pi}{\partial x_j}$. Working in the L^2 spaces with anisotropic weights, Farwig [3] treated this equation in three dimensional exterior domain. The purpose of this work is the study of the potential of the scalar Oseen operator:

$$T : u \longmapsto -\Delta u + \frac{\partial u}{\partial x_1}, \tag{1.6}$$

and more precisely the boundedness of the operators $R : f \mapsto \mathcal{O} * f$, $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ and $R_{j,k} : f \mapsto \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)$, for f given in $L^p(\mathbb{R}^n)$ or in $W^{-1,p}_0(\mathbb{R}^n)$. The obtained estimates can be applied to the investigation of qualitative properties of solutions of Navier-Stokes equations with a non-zero constant velocity at infinity.

2 Notations and Functional Spaces

Throughout this paper, p is a real number in the interval $]1, +\infty[$ and p' the conjugate exponent of p. A point in $\mathbb{R}^n$ is denoted by $\boldsymbol{x} = (x_1, .., x_n)$ and we set:

$$r = |\boldsymbol{x}| = (x_1^2 + ... + x_n^2)^{1/2}, \quad s = r - x_1 \quad \text{and} \quad s' = r + x_1.$$

For $R > 0$, B_R denotes the open ball of radius R centered at the origin and $B'_R = \mathbb{R}^2 \backslash \overline{B_R}$. For any $j \in \mathbb{Z}$, $\mathcal{P}_j$ is the space of polynomials of degree lower than or equal to j. If j is a negative integer, we set by convention $\mathcal{P}_j = 0$. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X:

$$B' \perp X = \{f \in B';\ \forall v \in X,\ \langle f, v \rangle = 0\}.$$

We define the weighted Sobolev space

$$W_0^{1,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{D}'(\mathbb{R}^n);\ \ \omega^{-1} u \in L^p(\mathbb{R}^n);\ \ \nabla u \in \boldsymbol{L}^p(\mathbb{R}^n) \right\},$$

where, $\omega = 1 + r$ if $p \neq n$ and $\omega = (1 + r)\ln(2 + r)$ if $p = n$. Equipped with it's natural norm:

$$\| u \|_{W_0^{1,p}(\mathbb{R}^n)} = \left(\| \omega^{-1} u \|^p_{L^p(\mathbb{R}^n)} + \| \nabla u \|^p_{\boldsymbol{L}^p(\mathbb{R}^n)} \right)^{\frac{1}{p}},$$

it is a reflexive Banach space. We denote its semi-norm by: $| u |_{W_0^{1,p}(\mathbb{R}^n)} = \| \nabla u \|_{\boldsymbol{L}^p(\mathbb{R}^n)}$. For more details on these spaces, see [12], [10] and [1]. However, we recall some properties and results that we use in this paper. The space $W_0^{1,p}(\mathbb{R}^n)$ contains constants when $p \geq n$ and no polynomials otherwise. The space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_0^{1,p}(\mathbb{R}^n)$. Then, its dual space denoted by $W_0^{-1,p'}(\mathbb{R}^n)$ is a space of distributions. We recall that there exists a constant C such that (see [1]):

$$\forall u \in W_0^{1,p}(\mathbb{R}^n), \quad \inf_{k \in \mathcal{P}_{[1-\frac{n}{p}]}} \| u + k \|_{W_0^{1,p}(\mathbb{R}^n)} \leq C \| \nabla u \|_{\boldsymbol{L}^p(\mathbb{R}^n)}. \tag{2.1}$$

Which implies that, in particular when $p < n$, the full norm on $W_0^{1,p}(\mathbb{R}^n)$ is equivalent to the semi norm. Inequality (2.1) permits to prove that the following gradient and divergence operators are isomorphisms (see [1]):

$$\nabla : W_0^{1,p}(\mathbb{R}^n)/\mathcal{P}_{[1-\frac{n}{p}]} \longrightarrow \boldsymbol{L}^p(\mathbb{R}^n) \perp \mathbf{H}_{p'}, \tag{2.2}$$

$$\text{div} : \boldsymbol{L}^{p'}(\mathbb{R}^n)/\mathbf{H}_{p'} \longrightarrow W_0^{-1,p'}(\mathbb{R}^n) \perp \mathcal{P}_{[1-\frac{n}{p}]}, \tag{2.3}$$

where, $\boldsymbol{H}_p = \{\boldsymbol{v} \in \boldsymbol{L}^p(\mathbb{R}^n), \ \ \mathrm{div}\boldsymbol{v} = 0\}$. Inequality (2.1) also allows to have the important following result (see [1] and [2] for the expression of $k(u)$ which is given in the case $n = 3$):

Proposition 2.1 *Let u a distribution such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^n)$.*
i) *If $\ 1 < p < n,\ $ there exists a unique constant $k(u) \in \mathbb{R}$ defined by:*

$$k(u) = -\lim_{|\boldsymbol{x}|\to\infty} \frac{1}{\omega_n} \int_{S_{n-1}} u(\sigma|\boldsymbol{x}|)\, d\sigma, \tag{2.4}$$

where ω_n denotes the area of the sphere S_{n-1}, such that $u + k(u) \in W_0^{1,p}(\mathbb{R}^n)$, and

$$\| \, u + k(u) \, \|_{W_0^{1,p}(\mathbb{R}^n)} \leq \ C \|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^n)}. \tag{2.5}$$

ii) *If $p \geq n$, then $u \in W_0^{1,p}(\mathbb{R}^n)$ and*

$$\inf_{k\in\mathbb{R}} \| \, u + k \, \|_{W_0^{1,p}(\mathbb{R}^n)} \leq \ C \|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^n)}. \tag{2.6}$$

We recall the Sobolev embeddings:

$$W_0^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \text{if} \ \ 1 < p < n, \ \ \text{with} \ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \tag{2.7}$$

$$W_0^{1,n}(\mathbb{R}^n) \hookrightarrow VMO(\mathbb{R}^n), \tag{2.8}$$

where,

$$VMO(\mathbb{R}^n) = \overline{\mathcal{D}(\mathbb{R}^n)}^{\|\cdot\|_{BMO}}.$$

The space BMO is defined as follows: A locally integrable function f belongs to BMO if

$$\|f\|_{BMO} =: \sup_Q \frac{1}{|Q|} \int_Q |f(\boldsymbol{x}) - f_Q| \, d\boldsymbol{x} < \infty,$$

where, the supremum is taken on all the cubes and $f_Q = \frac{1}{|Q|}\int_Q f(\boldsymbol{x})\, d\boldsymbol{x}$ is the average of f on Q.
Note that if $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^n)$, with $p > n$ and $u \in L^r(\mathbb{R}^n)$ for some $r \geq 1$ then

$$u \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \tag{2.9}$$

and if $p = n$, then u belongs to $L^q(\mathbb{R}^n)$ for any $q \geq r$.
We introduce also the following space (see [2]):

$$\widetilde{W}_0^{1,p}(\mathbb{R}^n) = \{v \in W_0^{1,p}(\mathbb{R}^n); \ \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^n)\},$$

which is a reflexive Banach space for the norm:

$$\|v\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^n)} = \|v\|_{W_0^{1,p}(\mathbb{R}^n)} + \|\frac{\partial u}{\partial x_1}\|_{W_0^{-1,p}(\mathbb{R}^n)}.$$

3 The fundamental solution.

Following the idea of [8], we get the following fundamental solution $\mathcal{O}$:
i) For $n = 3$,

$$\mathcal{O}(\boldsymbol{x}) = \frac{1}{4\pi r} e^{-\frac{s}{2}}. \tag{3.1}$$

ii) For $n = 2$, the fundamental solution has, in a neighbourhood of the origin, the following behaviour:

$$\mathcal{O}(\boldsymbol{x}) = -\frac{1}{2\pi} e^{\frac{x_1}{2}} \left\{ \ln \frac{1}{r} + 2\ln 2 - \gamma + \sigma(r) \right\}, \tag{3.2}$$

where σ satisfies $\frac{d^k \sigma}{dr^k} = or^{-k})$. When r is sufficiently large, we have:

$$\mathcal{O}(\boldsymbol{x}) = -\frac{1}{2\sqrt{\pi r}} e^{-\frac{s}{2}} \left[1 - \frac{1}{4r} + O(r^{-2}) \right].$$

Using the inequality $e^{-s/2} \leq C_b (1+s)^b$, which holds for any real number b, we obtain the following anisotropic estimates for r sufficiently large:

$$\begin{aligned} &|\mathcal{O}(\boldsymbol{x})| \leq C\, r^{-1} (1+s)^{-2}, \ \ |\tfrac{\partial \mathcal{O}}{\partial x_1}(\boldsymbol{x})| \ \leq C\, r^{-2}\, (1+s)^{-\frac{3}{2}}, \\ &|\tfrac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})| \ \leq C\, r^{-\frac{3}{2}}\, (1+s)^{-\frac{3}{2}} (1 + \tfrac{2}{r}), \quad j = 2, 3, \quad \text{if } n = 3, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &|\mathcal{O}(\boldsymbol{x})| \leq C\, r^{-\frac{1}{2}} (1+s)^{-1}, \ \ |\tfrac{\partial \mathcal{O}}{\partial x_1}(\boldsymbol{x})| \leq C\, r^{-\frac{3}{2}}\, (1+s)^{-1}, \\ &|\tfrac{\partial \mathcal{O}}{\partial x_2}(\boldsymbol{x})| \leq C\, r^{-1}\, (1+s)^{-1}, \quad \text{if } n = 2. \end{aligned} \tag{3.4}$$

In order to study the integrability properties of the fundamental solution and its derivatives, we need to estimate the following integral (for the proof, see [11]):

$$\alpha,\, \beta \in \mathbb{R}, \ \ I_{\alpha,\beta} = \int_{|\boldsymbol{x}|>\mu} r^{-\alpha} (1+s)^{-\beta}\, d\boldsymbol{x}, \quad \text{with} \ \ \mu > 0.$$

Lemma 3.1 *Assume that $n - \alpha - \min(\frac{n-1}{2}, \beta) < 0$. Then, there exists a constant $C > 0$ such that, for all $\mu > 1$, we have*

$$I_{\alpha,\beta} \leq C \mu^{n-\alpha-\min(\frac{n-1}{2},\beta)}, \ \ \text{if } \beta \neq \frac{n-1}{2},$$

$$I_{\alpha,\beta} \leq C \mu^{\frac{n+1}{2}-\alpha} \ln r, \qquad \text{if } \beta = \frac{n-1}{2}.$$

This lemma allows us to derive the following integrability properties of $\mathcal{O}$ and its gradient:

$$\forall p > 3, \ \ \mathcal{O} \in L^p(\mathbb{R}^2) \quad \text{and} \quad \forall p \in]\frac{3}{2}, 2[, \ \ \nabla\, \mathcal{O} \in \boldsymbol{L}^p(\mathbb{R}^2), \tag{3.5}$$

$$\forall p \in]2,3[, \quad \mathcal{O} \in L^p(\mathbb{R}^3) \quad \text{and} \quad \forall p \in]\frac{4}{3}, \frac{3}{2}[, \quad \nabla \mathcal{O} \in \boldsymbol{L}^p(\mathbb{R}^3). \tag{3.6}$$

Note also that

$$\mathcal{O} \in L^1_{loc}(\mathbb{R}^n) \quad \text{and} \quad \nabla \mathcal{O} \in \boldsymbol{L}^1_{loc}(\mathbb{R}^n), \quad \text{for } n = 2, 3. \tag{3.7}$$

3.1 Study of the kernel

Using the Fourier's transform, the kernel of the operator T, when it is defined on the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, is given by:

Lemma 3.2 *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution and let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a solution of* (1.6)*. Then u is uniquely determined up to polynomials of $\mathcal{S}_k$, where:*

$$\mathcal{S}_k = \{q \in \mathcal{P}_k; -\Delta q + \frac{\partial q}{\partial x_1} = 0\}. \tag{3.8}$$

Let us notice that $\mathcal{S}_0 = \mathbb{R}$ and $\mathcal{S}_1$ is the space of polynomials of degree lower than or equal one and not depending on x_1.

4 Scalar Oseen Potential in three dimensional

This section is devoted to the L^p estimates of convolutions with Oseen kernels.

Theorem 4.1 *Let $f \in L^p(\mathbb{R}^3)$. Then $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$. Moreover, the following estimate holds*

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^3)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.1}$$

Moreover,
1) *if $1 < p < 2$, then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.2}$$

2) *If $1 < p < 4$, then $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and verifies the estimate*

$$\|\frac{\partial}{\partial x_j}(\mathcal{O} * f)\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.3}$$

Proof : By Fourier's transform, from Equation (1.5) we obtain:

$$\mathcal{F}(\frac{\partial^2}{\partial x_j \partial x_k}\mathcal{O} * f) = \frac{-\xi_j\xi_k}{\boldsymbol{\xi}^2 + i\xi_1}\mathcal{F}(f).$$

Now, the function $\xi \mapsto m(\xi) = \frac{-\xi_j\xi_k}{\boldsymbol{\xi}^2+i\xi_1}$ is of class $\mathcal{C}^2$ in $\mathbb{R}^3 \setminus \{0\}$ and satisfies for every $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$

$$|\frac{\partial^{|\alpha|}m}{\partial \boldsymbol{\xi}^\alpha}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|^{-\alpha},$$

where, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and C is a constant not depending on $\boldsymbol{\xi}$. Then, the linear operator

$$\mathcal{A} : f \mapsto \frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{i\boldsymbol{x}\boldsymbol{\xi}}\frac{-\xi_j\xi_k}{\boldsymbol{\xi}^2 + i\xi_1}\mathcal{F}f(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

is continuous from $L^p(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ (see E. Stein [18], Thm 3.2, p.96). So, $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and satisfies

$$\|\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}.$$

We have also

$$\mathcal{F}(\frac{\partial}{\partial x_1}(\mathcal{O} * f)) = \frac{i\xi_1}{\boldsymbol{\xi}^2 + i\xi_1}\mathcal{F}(f)$$

and since the function $\boldsymbol{\xi} \mapsto m_1(\boldsymbol{\xi}) = \frac{i\xi_1}{\boldsymbol{\xi}^2+i\xi_1}$ admits the same properties that those of $m(\boldsymbol{\xi})$, then $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and satisfies the estimate

$$\|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)},$$

which proves the first part of the proposition and Estimate (4.1). Next, to prove inequalities (4.2) and (4.3), we adapt the technique used by Stein in [18] which studied the convolution of $f \in L^p(\mathbb{R}^n)$ with the kernel $|\boldsymbol{x}|^{\alpha-n}$. Let us decompose the function K as $K_1 + K_\infty$ where,

$$\begin{aligned} &K_1(\boldsymbol{x}) = K(\boldsymbol{x}) \quad \text{if} \quad |\boldsymbol{x}| \leq \mu \quad \text{and} \quad K_1(\boldsymbol{x}) = 0 \quad \text{if} \quad |\boldsymbol{x}| > \mu, \\ &K_\infty(\boldsymbol{x}) = 0 \qquad\quad\; \text{if} \quad |\boldsymbol{x}| \leq \mu \quad \text{and} \quad K_\infty(\boldsymbol{x}) = K(\boldsymbol{x}) \;\; \text{if} \quad |\boldsymbol{x}| > \mu. \end{aligned} \tag{4.4}$$

The function K will denote successively $\mathcal{O}$ and $\frac{\partial \mathcal{O}}{\partial x_j}$ and μ is a fixed positive constant which need not be specified at this instance. Next, we shall show that the mapping $f \mapsto K * f$ is of *weak-type* (p, q), with $q = \frac{2p}{2-p}$ when $K = \mathcal{O}$ and

$q = \frac{4p}{4-p}$ when $K = \frac{\partial \mathcal{O}}{\partial x_j}$, in the sense that:

$$\text{for all } \lambda > 0, \quad \text{mes}\,\{\boldsymbol{x}\,; |(K * f)(\boldsymbol{x})| > \lambda\} \leq \left(C_{p,q} \frac{\|f\|_{L^p(\mathbb{R}^3)}}{\lambda} \right)^q. \tag{4.5}$$

Since $K * f \;=\; K_1 * f \;+\; K_\infty * f$, we have now:

$$\text{mes}\,\{\boldsymbol{x}\,; |K * f| > 2\lambda\} \leq \text{mes}\,\{\boldsymbol{x}\,; |K_1 * f| > \lambda\} + \text{mes}\,\{\boldsymbol{x}\,; |K_\infty * f| > \lambda\}. \tag{4.6}$$

Note that it is enough to prove inequality (4.5) with $\|f\|_{L^p(\mathbb{R}^3)} = 1$. We have also:

$$\text{mes}\,\{\boldsymbol{x}\,; |(K_1 * f)(\boldsymbol{x})| > \lambda\} \;\leq\; \frac{\|K_1 * f\|^p_{L^p(\mathbb{R}^3)}}{\lambda^p} \;\leq\; \frac{\|K_1\|^p_{L^1(\mathbb{R}^3)}}{\lambda^p}, \tag{4.7}$$

and

$$\|K_\infty * f\|_{L^\infty(\mathbb{R}^3)} \leq \; \|K_\infty\|_{L^{p'}(\mathbb{R}^3)}. \tag{4.8}$$

1) Estimate (4.2). According to (3.3), $\mathcal{O}_1 \in L^1(\mathbb{R}^3)$ and by (3.6), $\mathcal{O}_\infty \in L^{p'}(\mathbb{R}^3)$ for $1 \leq p < 2$. Then, the integral $\mathcal{O}_1 * f$ converges almost everywhere and $\mathcal{O}_\infty * f$ converges everywhere. So, $\mathcal{O} * f$ converges almost everywhere. But

$$\forall \mu > 0, \quad \|\mathcal{O}_1\|_{L^1(\mathbb{R}^3)} \;\leq\; C\mu. \tag{4.9}$$

Next, by using (3.3), we have for any $p' > 2$:

$$\forall \mu > 0, \quad \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^3)} \leq C\mu^{\frac{2-p'}{p'}}. \tag{4.10}$$

Choosing now $\lambda = C\mu^{\frac{2-p'}{p'}}$ or equivalently $\mu = C'\lambda^{\frac{p}{p-2}}$. Then from (4.10) and (4.8) we have $\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^3)} < \lambda$ and so $\text{mes}\,\{\boldsymbol{x}\,;\ |\mathcal{O}_\infty * f| > \lambda\} = 0$. Finally, for $1 \leq p < 2$, we get from inequalities (4.9), (4.6) and (4.7):

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^3;\ |(\mathcal{O} * f)(\boldsymbol{x})| > \lambda\} \;\leq\; \left(C_p \frac{1}{\lambda} \right)^{\frac{2p}{2-p}}. \tag{4.11}$$

So, for $1 \leq p < 2$, the operator $R : f \mapsto \mathcal{O} * f$ is of *weak-type* $(p, \frac{2p}{2-p})$.

2) Estimate (4.3). Here we take $K = \frac{\partial \mathcal{O}}{\partial x_j}$. First, according to (4.1), $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in W^{1,p}(\mathbb{R}^3)$ then, by the Sobolev embedding, we have in particular, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. It remains to prove Estimate (4.3) for $j = 2, 3$. Firstly we have:

$$\|\frac{\partial \mathcal{O}}{\partial x_j}\|_{L^1(\mathbb{R}^3)} \leq c\mu, \ \text{ if } \ \mu \leq 1 \ \text{ and } \|\frac{\partial \mathcal{O}}{\partial x_j}\|_{L^1(\mathbb{R}^3)} \leq c\mu^{\frac{1}{2}}, \ \text{ if } \ \mu > 1.$$

Furthermore, we have for $p' > \frac{4}{3}$:

$$\textstyle\int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \,\le\, C\mu^{4-3p'}, \quad \text{if } \ \mu \le 1,$$

$$\textstyle\int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \,\le C\mu^{\frac{4-3p'}{2}}, \quad \text{if } \ \mu > 1.$$

In summary we have:
a) If $0 < \mu < 1$,

$$\int_{|\boldsymbol{x}|<\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|\, d\boldsymbol{x} \,\le c\mu \quad \text{and} \quad \int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \,\le C\mu^{4-3p'},$$

b) if $\mu \ge 1$,

$$\int_{|\boldsymbol{x}|<\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|\, d\boldsymbol{x} \,\le c\mu^{\frac{1}{2}} \quad \text{and} \quad \int_{|\boldsymbol{x}|>\mu} |\frac{\partial \mathcal{O}}{\partial x_j}(\boldsymbol{x})|^{p'}\, d\boldsymbol{x} \,\le C\mu^{\frac{4-3p'}{2}}.$$

Setting $\lambda = C\mu^{\frac{4-3p'}{p'}}$ in the case a) or $\lambda = C\mu^{\frac{4-3p'}{2p'}}$ in the case b), we get in both cases:

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^3;\ |K * f(\boldsymbol{x})| > \lambda\} \,\le\, \left(C_p\frac{1}{\lambda}\right)^{\frac{4p}{4-p}}. \tag{4.12}$$

Thus, for $1 \le p < 4$, the operator $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ is of *weak-type* $(p, \frac{4p}{4-p})$. Applying now the Marcinkiewicz interpolation's theorem, we deduce that, for $1 < p < 2$, the linear operator R is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and for $1 < p < 4$, R_j is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. $\diamondsuit$

Remark 4.2 Another proof of Theorem 4.1 consists in using the Fourier's approach. Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3)$ a sequence which converges to f in $L^p(\mathbb{R}^3)$. Then the sequence $(u_j)_{j\in\mathbb{N}}$ given by:

$$u_j \,= \mathcal{F}^{-1}(m_0(\boldsymbol{\xi})\mathcal{F}f_j), \quad m_0(\boldsymbol{\xi}) = (|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}, \tag{4.13}$$

satisfies the equation $Tu_j = f_j$, where the operator T is defined by (1.6). Let us recall now the:
Lizorkin Theorem. *Let $D = \{\boldsymbol{\xi} \in \mathbb{R}^3;\ |\boldsymbol{\xi}| > 0\}$ and $m : D \longrightarrow \mathbb{C}$, a continuous function such that its derivatives $\frac{\partial^k m}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}\partial\xi_3^{k_3}}$ are continuous and verify*

$$|\xi_1|^{k_1+\beta}\,|\xi_2|^{k_2+\beta}\,|\xi_3|^{k_3+\beta}\left|\frac{\partial^k m}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}\partial\xi_3^{k_3}}\right| \le M, \tag{4.14}$$

where $k_1, k_2, k_3 \in \{0, 1\},\ k = k_1 + k_2 + k_3$ and $0 \le \beta < 1$. Then, the operator

$$\mathcal{A} : g \longmapsto \mathcal{F}^{-1}(m_0\,\mathcal{F}g),$$

is continuous from $L^p(\mathbb{R}^3)$ into $L^r(\mathbb{R}^3)$ with $\frac{1}{r} = \frac{1}{p} - \beta$.
Applying this continuity property with $f_j \in L^p(\mathbb{R}^3)$ and $\beta = \frac{1}{2}$, we show that

(u_j) is bounded in $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ if $1 < p < 2$. So, this sequence admits a subsequence still denoted (u_j) which converges weakly to u and satisfying $Tu = f$. For the derivative of u_j with respect to x_1, the corresponding multiplier is on the form $m(\boldsymbol{\xi}) = i\xi_1(|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}$. So that (4.14) is satisfied for $\beta = 0$ and then $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^3)$. The same property takes place for the second derivatives with $m(\boldsymbol{\xi}) = \xi_k\xi_l(|\boldsymbol{\xi}|^2 + i\xi_1)^{-1}$. Finally, we verify with $\beta = \frac{1}{4}$, that the first derivative of (u_j) with respect to x_k is bounded in $L^{\frac{4p}{4-p}}(\mathbb{R}^3)$, which implies $\frac{\partial u}{\partial x_k} \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. ♢

Theorem 4.1 states that $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$ and under conditions on p, $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$. Now, using these results and the Sobolev embeddings (2.7)-(2.9), we have the following:

Theorem 4.3 *Let $f \in L^p(\mathbb{R}^3)$.*
1) *Assume that $1 < p < 4$. Then $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{4p}{4-p}}(\mathbb{R}^3)$ with the estimate (4.3). Moreover,*
i) *if $1 < p < 3$, then $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^3)$ with the estimate*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.15}$$

ii) *If $p = 3$, then $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^r(\mathbb{R}^3)$ for any $r \geq 12$ and satisfies*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^r(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.16}$$

iii) *If $3 < p < 4$, then $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^\infty(\mathbb{R}^3)$ and verifies the estimate*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.17}$$

2) *Assume that $1 < p < 2$. Then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ with the estimate (4.2). Moreover,*
i) *if $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^3)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.18}$$

ii) *If $p = \frac{3}{2}$, then $\mathcal{O} * f \in L^r(\mathbb{R}^3)$ for any $r \geq 6$ and*

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.19}$$

iii) *If $\frac{3}{2} < p < 2$, then $\mathcal{O} * f \in L^\infty(\mathbb{R}^3)$ and the following estimate holds*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.20}$$

Proof : **1)** When $1 < p < 4$, the previous theorem asserts that $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ and $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f) \in L^p(\mathbb{R}^3)$. If $1 < p < 3$, according to Proposition 2.1, there exists a unique constant $k(f) \in \mathbb{R}$ such that $v = \frac{\partial}{\partial x_j}(\mathcal{O} * f) + k(f) \in W_0^{1,p}(\mathbb{R}^n)$. Then $k(f) = v - \frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n) + L^{\frac{4p}{4-p}}(\mathbb{R}^3)$. As none of both spaces contains constants then $k(f) = 0$, which implies that $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n)$. Now, the Sobolev embedding (2.7) yields $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ and estimate (4.15). If $p \geq 3$, also by the previous theorem and Proposition 2.1, we have $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in W_0^{1,p}(\mathbb{R}^n)$. The Sobolev embedding (2.8) gives $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in VMO(\mathbb{R}^3)$ if $p = 3$. Applying now the interpolation theorem between BMO and L^p (see [9]), we get $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^r(\mathbb{R}^3)$ for any $r \geq 12$. By Sobolev embedding (2.9), when $3 < p < 4$, we have $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in \boldsymbol{L}^\infty(\mathbb{R}^3)$,) and the case 1) is finished.
2) By the previous theorem, when $1 < p < 2$, we have $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^3)$ and $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^3)$. Now by Sobolev embedding (2.7), $\mathcal{O} * f \in L^{p^*}(\mathbb{R}^3)$, where $\frac{1}{p^*} = \frac{3-p}{3p} - \frac{1}{3} = \frac{1}{p} - \frac{2}{3}$ if $1 < p < \frac{3}{2}$, which gives (4.15). For the remainder of the proof, we use the same arguments that in the previous case with $\mathcal{O} * f$ instead of $\frac{\partial}{\partial x_j}(\mathcal{O} * f)$ and $\frac{\partial}{\partial x_j}(\mathcal{O} * f)$ instead of $\frac{\partial^2}{\partial x_j \partial x_k}(\mathcal{O} * f)$. ♢

Remark 4.4 i) We can also find this result by showing that $\mathcal{O} \in L^{2,\infty}(\mathbb{R}^3)$, *i.e.*

$$\sup_{\mu > 0} \; \mu^2 \operatorname{mes}\{\boldsymbol{x} \in \mathbb{R}^3;\ \mathcal{O}(\boldsymbol{x}) > \mu\} < +\infty. \tag{4.21}$$

So that, for any $1 < q < 2$, according to weak Young inequality (*cf.* [17], chap. IX.4), we obtain:

$$\|\mathcal{O} * f\|_{L^{\frac{2q}{2-q},\infty}(\mathbb{R}^3)} \leq C\|\mathcal{O}\|_{L^{2,\infty}(\mathbb{R}^3)}\|f\|_{L^q(\mathbb{R}^3)}. \tag{4.22}$$

Let now $p \in]1, 2[$. There exist p_0 and p_1 such that $1 < p_0 < p < p_1 < 2$ and such that the operator $R : f \longmapsto \mathcal{O} * f$ is continuous from $L^{p_0}(\mathbb{R}^3)$ into $L^{\frac{2p_0}{2-p_0},\infty}(\mathbb{R}^3)$ and from $L^{p_1}(\mathbb{R}^3)$ into $L^{\frac{2p_1}{2-p_1},\infty}(\mathbb{R}^3)$. The Marcinkiewicz theorem allows again to conclude that the operator R is continuous from $L^p(\mathbb{R}^3)$ into $L^{\frac{2p}{2-p}}(\mathbb{R}^3)$
ii) The same remark is true for $\nabla\mathcal{O}$ which belongs to $\boldsymbol{L}^{\frac{4}{3},\infty}(\mathbb{R}^3)$. ♢

Using the Young inequality with the relations (3.6) and (3.7), we get the following result:

Proposition 4.5 *Let $f \in L^1(\mathbb{R}^3)$. Then*
1) *$\mathcal{O} * f \in L^p(\mathbb{R}^3)$ for any $p \in]2, 3[$ and satisfies the estimate*

$$\|\mathcal{O} * f\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^1(\mathbb{R}^3)}, \tag{4.23}$$

2) $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^p(\mathbb{R}^3)$ *for all* $p \in]\frac{4}{3}, \frac{3}{2}[$ *and the following estimate holds*

$$\|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|f\|_{L^1(\mathbb{R}^3)}. \tag{4.24}$$

Remark 4.6 Taking "formally" $p = 1$ in Theorem 4.3, we find that $\mathcal{O} * f \in L^q(\mathbb{R}^3)$ for any $q \in]2, 3[$ and $\nabla(\mathcal{O} * f) \in L^q(\mathbb{R}^3)$ for any $q \in]\frac{4}{3}, \frac{3}{2}[$. We notice that they are the same results obtained in Theorem 4.5 by using the Young inequality.

Now, we are going to study the Oseen potential $\mathcal{O} * f$ when f is given in $W_0^{-1,p}(\mathbb{R}^3)$. For that purpose, we give the following definition of the convolution of f with the fundamental solution $\mathcal{O}$:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \langle \mathcal{O} * f, \varphi \rangle =: \langle f, \breve{\mathcal{O}} * \varphi \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)}, \tag{4.25}$$

where $\breve{\mathcal{O}}(\boldsymbol{x}) = \mathcal{O}(-\boldsymbol{x})$. With the L^∞ weighted estimates obtained in [11] (Thms 3.1 and 3.2), we get an estimate on the convolution of $\breve{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ which we shall use afterward as follows

Lemma 4.7 *For any* $\varphi \in \mathcal{D}(\mathbb{R}^3)$ *we have the estimates*

$$|\breve{\mathcal{O}} * \varphi(\boldsymbol{x})| \leq C_\varphi \frac{1}{|\boldsymbol{x}|(1 + |\boldsymbol{x}| + x_1)}, \tag{4.26}$$

$$\nabla(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})| \leq C_\varphi \frac{1}{|\boldsymbol{x}|^{\frac{3}{2}}(1 + |\boldsymbol{x}| + x_1)^{\frac{3}{2}}}, \tag{4.27}$$

where C_φ *depends on the support of* φ.

Remark 4.8 **1)** The behaviour on $|\boldsymbol{x}|$ of $\breve{\mathcal{O}} * \varphi$ and its first derivatives is the same that of $\breve{\mathcal{O}}$, but the behaviour on $1 + s'$ is a little bit different (see (3.3).
2) By Lemma 3.1 and estimates (4.28-(4.29) we find that

$$\forall q > \frac{4}{3}, \quad \breve{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^3). \tag{4.28}$$

3) In (4.26) and (4.27), when φ tends to zero in $\mathcal{D}(\mathbb{R}^3)$, then C_φ tends to zero in $\mathbb{R}$.

The next theorem studies the continuity of the operators R and R_j when f belongs to $W_0^{-1,p}(\mathbb{R}^3)$.

Theorem 4.9 *Assume that* $1 < p < 4$ *and let* $f \in W_0^{-1,p}(\mathbb{R}^3)$ *satisfying the compatibility condition*

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0, \quad \text{when } 1 < p \leq \frac{3}{2}. \tag{4.29}$$

Then $\mathcal{O} * f \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ *and* $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^p(\mathbb{R}^3)$ *with the following estimate*

$$\|\mathcal{O} * f\|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \tag{4.30}$$

Moreover,
i) *if* $1 < p < 3$*, then* $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ *and the following estimate holds*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \tag{4.31}$$

ii) *If* $p = 3$*, then* $\mathcal{O} * f \in L^r(\mathbb{R}^3)$ *for any* $r \geq 12$ *and satisfies*

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \tag{4.32}$$

iii) *If* $3 < p < 4$*, then* $\mathcal{O} * f \in L^\infty(\mathbb{R}^3)$ *and we have the estimate*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \tag{4.33}$$

Proof : Let $1 < p < 4$. By Lemma 4.7 and Remark 4.8 point 3), if $\varphi \to 0$ in $\mathcal{D}(\mathbb{R}^3)$, then $C_\varphi \to 0$ where C_φ is defined by (4.26). Thus, $\check{\mathcal{O}} * \varphi \to 0$ in $W_0^{1,p'}(\mathbb{R}^3)$ for all $p \in]1, 4[$, what implies that $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^3)$. Next, by Isomorphism (2.3), there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$ such that

$$f = \mathrm{div}\boldsymbol{F} \quad \text{and} \quad \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}. \tag{4.34}$$

According to (4.1), we have for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned}|\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}| &= |\langle \boldsymbol{F}, \nabla \frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi)\rangle_{\boldsymbol{L}^p(\mathbb{R}^3)\times\boldsymbol{L}^{p'}(\mathbb{R}^3)}| \\ &\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}\|\varphi\|_{L^{p'}(\mathbb{R}^3)}.\end{aligned}$$

Then we deduce the second part of (4.30). We also have for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \mathcal{O} * f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = -\langle \boldsymbol{F}, \nabla(\check{\mathcal{O}} * \varphi)\rangle_{\boldsymbol{L}^p(\mathbb{R}^3)\times\boldsymbol{L}^{p'}(\mathbb{R}^3)},$$

and by (4.3): $|\langle \mathcal{O} * f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}| \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^3)}\|\varphi\|_{L^{\frac{4p}{5p-4}}(\mathbb{R}^3)}$. Note that $1 < p < 4 \Longleftrightarrow 1 < \frac{4p}{5p-4} < 4$. Consequently, we have the first part of (4.30). Moreover, by Sobolev embeddings (2.7)-(2.9), $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ if $1 < p < 3$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geq 12$ if $p = 3$ and belongs to $L^\infty(\mathbb{R}^2)$ if $3 < p < 4$. Thus, we showed that if $1 < p < 4$, the operators R and R_j are continuous. $\diamondsuit$

Corollary 4.10 *Assume that* $1 < p < 4$*. If* u *is a distribution such that* $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^3)$ *and* $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$*, then there exists a unique constant* $k(u)$

such that $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ *and*

$$\| u + k(u) \|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C(\| \nabla u \|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \| \frac{\partial u}{\partial x_1} \|_{W_0^{-1,p}(\mathbb{R}^3)}). \tag{4.35}$$

Moreover, if $1 < p < 3$*, then* $u + k(u) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$*, where* $k(u)$ *is defined by:*

$$k(u) = -\lim_{|\boldsymbol{x}|\to\infty} \frac{1}{\omega_3} \int_{S_2} u(\sigma|\boldsymbol{x}|)\, d\sigma, \tag{4.36}$$

where, ω_3 *denotes the area of the sphere* S_2 *and* u *tends to the constant* $-k(u)$ *as* $\boldsymbol{x}$ *tends to infinity in the following sense:*

$$\lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} |u(\sigma|\boldsymbol{x}|) + k(u)|\, d\sigma = 0. \tag{4.37}$$

If $p = 3$*, then* $u + k(u)$ *belongs to* $L^r(\mathbb{R}^3)$ *for any* $r \geq 12$*. If* $3 < p < 4$*, then* u *belongs to* $L^\infty(\mathbb{R}^3)$*, is continuous in* $\mathbb{R}^3$ *and tends to* $-k(u)$ *pointwise.*

Proof : Setting $g = -\Delta u + \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$. Since $\mathcal{P}_{[1-\frac{3}{p'}]}$ contains at most constants and according to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\widetilde{W}_0^{1,p}(\mathbb{R}^3)$ (see [2]), then g satisfies the compatibility condition (4.29). By the previous theorem, there exists a unique $v = \mathcal{O}*g \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ such that $\nabla v \in L^p(\mathbb{R}^3)$ and $\frac{\partial v}{\partial x_1} \in L^p(\mathbb{R}^3)$, satisfying $T(u - v) = 0$, where T is the Oseen operator, with the estimate:

$$\| v \|_{L^{\frac{4p}{4-p}}(\mathbb{R}^3)} \leq C(\| \nabla u \|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \| \frac{\partial u}{\partial x_1} \|_{W_0^{-1,p}(\mathbb{R}^3)}). \tag{4.38}$$

Setting $w = u - v$, we have for all $i = 1, 2, 3$, $\frac{\partial w}{\partial x_i} \in L^p(\mathbb{R}^3)$ and satisfies $T(\frac{\partial w}{\partial x_i}) = 0$. We deduce then by Lemma 3.2 that $\nabla u = \nabla v$ and consequently there exists a unique constant $k(u)$, defined by (4.36), such that $u + k(u) = v$. The last properties are consequence of (2.8) and (2.9). ♢

Remark 4.11 Let $u \in \mathcal{D}'(\mathbb{R}^3)$ such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^3)$.
i) When $1 < p < 3$, according to Proposition 2.1, we know that there exists a unique constant $k(u)$ such that $u + k(u) \in L^{\frac{3p}{3-p}}(\mathbb{R}^3)$. Here, the fact that in addition $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ we have moreover $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^2)$, with $\frac{4p}{4-p} < \frac{3p}{3-p}$.
ii) When $3 \leq p < 4$, by Proposition 2.1, for any constant k, $u + k$ belongs only to $W_0^{1,p}(\mathbb{R}^3)$ but no to the space $L^r(\mathbb{R}^3)$. But, if moreover $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ then, $u + k(u) \in L^{\frac{4p}{4-p}}(\mathbb{R}^3)$ for some unique constant $k(u)$. Moreover $u + k(u) \in L^r(\mathbb{R}^3)$ for any $r \geq \frac{4p}{4-p}$ and $u \in L^\infty(\mathbb{R}^3)$ if $p > 3$.

5 Scalar Oseen potential in two dimensional.

In this section we study also the continuity of the operators R, R_j and $R_{j,k}$ when f is given in $L^p(\mathbb{R}^2)$ or in $W_0^{-1,p}(\mathbb{R}^2)$. We begin by the case where f belongs to $L^p(\mathbb{R}^2)$.

Theorem 5.1 *Let $f \in L^p(\mathbb{R}^2)$. Then $\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$, $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and satisfy the estimate*

$$\|\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.1}$$

Moreover,
1) *if $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and satisfies*

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-2p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.2}$$

2) *If $1 < p < 3$, then $\frac{\partial}{\partial x_j}(\mathcal{O} * f) \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and verifies the estimate*

$$\|\frac{\partial}{\partial x_j}(\mathcal{O} * f)\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.3}$$

Proof : As in three-dimensional case, since the operator

$$\mathcal{A} : \ f \mapsto \frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f)(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{i\boldsymbol{x}\boldsymbol{\xi}} \frac{-\xi_1 \xi_2}{\boldsymbol{\xi}^2 + i\xi_1} \widehat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

is continuous from $L^p(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$, we get $\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$. We have also $\frac{\partial}{\partial x_1}(\mathcal{O} * f) \in L^p(\mathbb{R}^2)$ and the estimate:

$$\|\frac{\partial^2}{\partial x_1 \partial x_2}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} + \|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}, \tag{5.4}$$

which proves the first part of theorem and Estimate (5.1). Now, as in the three-dimensional case, we will show that the operators $R : f \mapsto \mathcal{O} * f$ and $R_j : f \mapsto \frac{\partial}{\partial x_j}(\mathcal{O} * f)$ are *weak-type* $(p, \frac{3p}{3-2p})$ if $1 \leq p < \frac{3}{2}$ and *weak-type* $(p, \frac{3p}{3-p})$ if $1 \leq p < 3$ respectively. Using the decomposition (4.4), according to (3.7) and Estimate (3.3), the integral $K * f = K_1 * f + K_\infty * f$ converges almost everywhere, where K denotes $\mathcal{O}$ and $\frac{\partial \mathcal{O}}{\partial x_j}$ respectively.
1) Estimate (5.2). We observe that:

$$\forall \mu > 0, \quad \|\mathcal{O}_1\|_{L^1(\mathbb{R}^2)} \leq C\mu, \tag{5.5}$$

and for all $p' > 3$:

$$\forall \mu > 0, \ \|\mathcal{O}_\infty\|_{p'} \leq C\mu^{\frac{3-p'}{2p'}}. \tag{5.6}$$

Setting $\lambda = C\mu^{\frac{3-p'}{2p'}}$ or equivalently $\mu = C'\lambda^{\frac{2p'}{3-p'}} = C'\lambda^{\frac{2p}{2p-3}}$, we get from this last inequality that $\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^2)} < \lambda$. Then by Estimate (4.8), we have $\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2;\ |\mathcal{O}_\infty * f(\boldsymbol{x})| > \lambda\} = 0$. So, for any $1 < p < \frac{3}{2}$, we have from (4.6) and (4.7):

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2;\ |\mathcal{O} * f(\boldsymbol{x})| > 2\lambda\} \ \le\ C\frac{\mu^p}{\lambda^p} \ \le\ C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-2p}}, \tag{5.7}$$

which proves that the operator R is of *weak type* $(p, \frac{3p}{3-2p})$.

2) Estimate (5.3). According to (5.1) and the Sobolev embedding, we get that $\frac{\partial}{\partial x_1}(\mathcal{O} * f)$ belongs in particular to $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. It remains then to show (5.3) for $i = 2$. As previously, we have:

$$\forall \mu > 0, \quad \|\frac{\partial \mathcal{O}}{\partial x_2}\|_{L^1(\mathbb{R}^2)} \ \le\ C\mu^{\frac{1}{2}}, \tag{5.8}$$

and for any $p' > \frac{3}{2}$,

$$\|\frac{\partial \mathcal{O}}{\partial x_2}\|_{L^{p'}(\mathbb{R}^2)} \ \le\ C\mu^{\frac{3-2p'}{p'}}. \tag{5.9}$$

We have also, for $1 < p < 3$ and any $\lambda > 0$:

$$\text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2;\ |\frac{\partial}{\partial x_2}(\mathcal{O} * f)(\boldsymbol{x})| > 2\lambda\} \ \le\ C\left(\frac{1}{\lambda}\right)^{\frac{3p}{3-p}}. \tag{5.10}$$

Thus, the operator R_2 is of *weak-type* $(p, \frac{3p}{3-p})$. Now, from inequalities (5.7), (5.10) and by Marcinkiewicz interpolation's Theorem, the operator $R : f \mapsto \mathcal{O} * f$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ and $R_2 : f \mapsto \frac{\partial}{\partial x_2}(\mathcal{O} * f)$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{\frac{3p}{3-p}}(\mathbb{R}^2)$. ♢

Remark 5.2 i) We can also prove this result as follows. We observe first that $\mathcal{O} \in L^{3,\infty}(\mathbb{R}^2)$, *i.e.*

$$\sup_{\mu > 0} \mu^3 \ \text{mes}\,\{\boldsymbol{x} \in \mathbb{R}^2;\ |\mathcal{O}(\boldsymbol{x})| > \mu\} < +\infty. \tag{5.11}$$

So that, when $1 < q < \frac{3}{2}$ and using weak Young inequality (*cf.* [17], chap. IX.4), we obtain:

$$\|\mathcal{O} * f\|_{L^{\frac{3q}{3-2q},\infty}(\mathbb{R}^2)} \le C\|\mathcal{O}\|_{L^{3,\infty}(\mathbb{R}^2)}\|f\|_{L^{q,\infty}(\mathbb{R}^2)}. \tag{5.12}$$

Now, let $1 < p < \frac{3}{2}$. This last estimate shows that there exist p_0 and p_1 such that $1 < p_0 < p < p_1 < \frac{3}{2}$ and such that the operator $R : f \longmapsto \mathcal{O} * f$ is continuous from $L^{p_0}(\mathbb{R}^2)$ into $L^{\frac{3p_0}{3-2p_0},\infty}(\mathbb{R}^2)$ and from $L^{p_1}(\mathbb{R}^2)$ into $L^{\frac{3p_1}{3-2p_1},\infty}(\mathbb{R}^2)$. The Marcinkiewicz theorem allows again to conclude that

$R : L^p(\mathbb{R}^2) \longrightarrow L^{\frac{3p}{3-2p}}(\mathbb{R}^2)$ is bounded.
ii) The same remark is true for $\nabla\,\mathcal{O}$ which belongs to $\boldsymbol{L}^{\frac{3}{2},\infty}(\mathbb{R}^2)$.

Theorem 5.1 and the Sobolev embedding yield the following result.

Theorem 5.3 *Let $f \in L^p(\mathbb{R}^2)$.*
1) *Assume that $1 < p < 3$. Then, $\nabla\,(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and satisfies Estimate (5.3). Moreover,*
i) *if $1 < p < 2$, then $\nabla\,(\mathcal{O} * f) \in \boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)$ with*

$$\|\nabla\,(\mathcal{O} * f)\|_{\boldsymbol{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.13}$$

ii) *When $p = 2$, then $\nabla\,(\mathcal{O} * f) \in \boldsymbol{L}^r(\mathbb{R}^2)$, $r \geq 6$ and the following estimate holds:*

$$\|\nabla\,(\mathcal{O} * f)\|_{\boldsymbol{L}^r(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.14}$$

iii) *Finally, if $2 < p < 3$, then $\nabla\,(\mathcal{O} * f) \in \boldsymbol{L}^\infty(\mathbb{R}^2)$ and we have the inequality:*

$$\|\nabla\,(\mathcal{O} * f)\|_{\boldsymbol{L}^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.15}$$

2) *Assume that $1 < p < \frac{3}{2}$. Then, besides (5.2), $\mathcal{O} * f \in L^\infty(\mathbb{R}^2)$ and satisfies the estimate:*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.16}$$

The proof of this theorem is the same that of Theorem 4.3. However, in the case 2, we have $\frac{2p}{2-p} > 2$ which gives the result by using the Sobolev embedding (2.9).
Using the Young inequality with the relations (3.5) and (3.7), we get the following:

Proposition 5.4 *Let $f \in L^1(\mathbb{R}^2)$. Then*
1) *$\mathcal{O} * f \in L^p(\mathbb{R}^2)$ for any $p > 3$ and satisfies the estimate*

$$\|\mathcal{O} * f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^1(\mathbb{R}^2)}, \tag{5.17}$$

2) *$\nabla\,(\mathcal{O} * f) \in \boldsymbol{L}^p(\mathbb{R}^2)$ for all $p \in]\frac{3}{2}, 2[$ and the following estimate holds*

$$\|\nabla\,(\mathcal{O} * f)\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leq C\|f\|_{L^1(\mathbb{R}^2)}. \tag{5.18}$$

Proof : Since by (3.7) $\mathcal{O} \in L^1_{loc}(\mathbb{R}^2)$ and $\nabla\,\mathcal{O} \in \boldsymbol{L}^1_{loc}(\mathbb{R}^2)$, then $\nabla\,(\mathcal{O} * f) = (\nabla\,\mathcal{O}) * f$. According to the Young inequality and the relation (3.5), this last term belongs to $L^p(\mathbb{R}^3)$ if $\frac{4}{3} < p < \frac{3}{2}$. With the same argument we get the case 1). ♢

Remark 5.5 Taking "formally" $p = 1$ in Theorem 5.3, we find that $\nabla\,(\mathcal{O} * f) \in L^q(\mathbb{R}^3)$ for any $q \in]\frac{3}{2}, 2[$. We find also $\mathcal{O} * f \in L^3(\mathbb{R}^2)$ and

$\nabla(\mathcal{O} * f) \in L^2(\mathbb{R}^2)$. The Sobolev embedding (2.8) gives then $\mathcal{O} * f \in L^q(\mathbb{R}^2)$ for any $q > 3$ and we notice that they are the same results obtained by Theorem 5.4 by using the Young inequality.

In order to study the case when f is given in $W_0^{-1,p}(\mathbb{R}^2)$, we use the L^∞ weighted estimates obtained in [11] (Thms 3.5, 3.7 and 3.8) and we get an estimates on the convolution of $\breve{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ as follows:

Lemma 5.6 *For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have the estimates*

$$|\breve{\mathcal{O}} * \varphi(\boldsymbol{x})| \leq C_\varphi \frac{1}{|\boldsymbol{x}|^{\frac{1}{2}}(1 + |\boldsymbol{x}| + x_1)^{\frac{1}{2}}}, \tag{5.19}$$

$$|\frac{\partial}{\partial x_1}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})| \leq C_\varphi \frac{1}{|\boldsymbol{x}|^{\frac{3}{2}}(1 + |\boldsymbol{x}| + x_1)^{\frac{1}{2}}}, \tag{5.20}$$

$$|\frac{\partial}{\partial x_2}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})| \leq C_\varphi \frac{1}{|\boldsymbol{x}|(1 + |\boldsymbol{x}| + x_1)}, \tag{5.21}$$

where C_φ depends on the support of φ.

Remark 5.7 **1)** The behaviour on $|\boldsymbol{x}|$ of $\breve{\mathcal{O}} * \varphi$ and its first derivatives is the same that that of $\breve{\mathcal{O}}$ but, the behaviour on $1 + s'$ is a little bit different.
2) By Lemma 3.1 and this last estimates, we find that

$$\forall q > \frac{3}{2}, \quad \breve{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^2). \tag{5.22}$$

3) In (5.19)-(5.21), when φ tends to zero in $\mathcal{D}(\mathbb{R}^3)$, then C_φ tends to zero in $\mathbb{R}$.

With the definition (4.25), when f is given in $W_0^{-1,p}(\mathbb{R}^2)$, we have a similar result to Theorem 4.9.

Theorem 5.8 *Assume that $1 < p < 3$ and let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} = 0, \quad \text{when } 1 < p \leq 2. \tag{5.23}$$

Then, $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and $\nabla(\mathcal{O} * f) \in \boldsymbol{L}^p(\mathbb{R}^2)$ with the following estimate:

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} + \|\nabla(\mathcal{O} * f)\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.24}$$

Moreover,
i) *if $1 < p < 2$, then $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and satisfies the following inequality:*

$$\|\mathcal{O} * f\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.25}$$

ii) *If* $p = 2$*, then* $\mathcal{O} * f \in L^r(\mathbb{R}^2)$ *for any* $r \geq 6$ *and*

$$\|\mathcal{O} * f\|_{L^r(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.26}$$

iii) *If* $2 < p < 3$*, then* $\mathcal{O} * f \in L^\infty(\mathbb{R}^2)$ *and we have the estimate*

$$\|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.27}$$

Proof : Because the proof is the same that that of Theorem 4.9, then we give it briefly. Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfying condition (5.23). As in three-dimensional case, we get $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^2)$ if $1 < p < 3$. By Isomorphism (2.3) and (5.1), we have

$$\begin{aligned}|\langle \mathcal{O} * f, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)}| &\leq \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)}\|\frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi)\|_{L^{p'}(\mathbb{R}^2)} \\ &\leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}\|\varphi\|_{L^{\frac{3p}{4p-3}}(\mathbb{R}^2)}.\end{aligned}$$

Note that $1 < p < 3 \iff 1 < \frac{3p}{4p-3} < 3$. Then, $\mathcal{O} * f \in L^{\frac{3p}{3-p}}(\mathbb{R}^2)$ and

$$\|\mathcal{O} * f\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^2)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Moreover, by the Sobolev embedding, $\mathcal{O} * f \in L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ if $1 < p < 2$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geq 6$ if $p = 2$ and belongs to $L^\infty(\mathbb{R}^2)$ if $2 < p < 3$. We thus showed that if $1 < p < 3$, the following operator is continuous:

$$\begin{aligned}R : W_0^{-1,p}(\mathbb{R}^2) \perp \mathcal{P}_{[1-\frac{2}{p'}]} &\longrightarrow W_0^{1,p}(\mathbb{R}^2) \cap L^{\frac{3p}{3-p}}(\mathbb{R}^2), \\ f &\longmapsto \mathcal{O} * f.\end{aligned}$$

References

[1] Amrouche, C., Girault, V. and Giroire, J., Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^n$, *J. Math. Pures Appl.*, **73-6**, 1994, p. 579-606.

[2] Amrouche, C. and Razafison, U., Weighted Sobolev paces for a scalar model Oseen equation in $\mathbb{R}^3$, to appear in *Journal of Math. Fluids Mech.*

[3] Farwig, R. A variational approach in weighted Sobolev spaces to the operator $-\Delta + \partial/\partial x_1$ in exterior domains of $\mathbb{R}^3$, *Mathematische Zeitschrift*, **210-3**, 1992, p. 449–464.

[4] Farwig, R. The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces, *Journal Math. Z.*, **211**, 1992, p. 409-447.

[5] Farwig, R. and Sohr, H. Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains, *Ser. Adv. Math. Appl. Sci.*, **47**, 1998, p. 11-30.

[6] Finn, R., On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems, *Arch. Rational Mech. Anal.*, **19**, 1965, p. 363-406.

[7] Finn, R., Estimates at infinity for stationary solutions of the Navier-Stokes equations, *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine* **3-51**, 1959, p. 387-418.

[8] Galdi, G. P., An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I, Springer Tracts in Natural Philosophy, **38**, Springer-Verlag, 1994.

[9] Hanks, R. Interpolation by the Real Method between $BMO, L^{\alpha}(0 < \alpha < \infty)$ and $H^{\alpha}(0 < \alpha < \infty)$, *Indiana University Mathematics Journal*, **26-4**, 1977, p. 679–689.

[10] Hanouzet, B. Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace *Rend. Sem. Mat. Univ. Padova*, **46**, 1971, p. 227-272.

[11] Kraćmar, S., Novotný, A. and Pokorný, M., Estimates of Oseen kernels in weighted L^p spaces, *J. Math. Soc. Japan*, **53**, 2001, p. 59-111.

[12] Kufner, A., Weighted Sobolev spaces, A Wiley-Interscience Publication, New York, 1985.

[13] Lizorkin, P. I. $(L^p,\, L^q)$ -multipliers of Fourier integrals, *Dokl. Akad. Nauk SSSR*, **152**, 1963, p. 808-811.

[14] Oseen, C. W.,Uber die Stokessesche Formel und Über eine Verwandte Aufgabe inder Hydrodynamik, *Journal Ark. Mat. Astron. Fys.*, **29-6**, 1910, p. 1-20.

[15] Oseen, C. W., Neuere Methoden und Ergebnisse in der Hydrodynamik. (Akadem. Verlagsgesellschaft, Leipzig, 1927).

[16] Perez, C., Two weighted norm inequalities for Riesz potentials and uniform L^p-weighted Sobolev inequalities. *Indiana Univ. Math. J.*, **39-1**, 1990, p. 31 - 44.

[17] M. Reed and B. Simon, Fourier Analysis Self-Adjointness **t. II**, *Academic Press*, (1975).

[18] Stein, Elias M., Singulars Integrals and Differentiability Properties of Functions. Princeton New Jersey, 1970.