Existence of solutions for a model describing the dynamics of junctions between dislocations

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EXISTENCE OF SOLUTIONS FOR A MODEL DESCRIBING THE DYNAMICS OF JUNCTIONS BETWEEN DISLOCATIONS.

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Abstract. We study a dynamical version of a multi-phase field model of Koslowski and Ortiz for planar dislocation networks. We consider a two-dimensional vector field which describes phase transitions between constant phases. Each phase transition corresponds to a dislocation line, and the vectorial field description allows the formation of junctions between dislocations. This vector field is assumed to satisfy a non-local vectorial Hamilton-Jacobi equation with non-zero viscosity. For this model, we prove the existence for all time of a weak solution.

Key words. Dislocation dynamics, non-local equations, junctions, parabolic system of equations.

AMS subject classifications. 35K15, 74K30.

1. Introduction.

1.1. Physical motivation. Dislocations are line defects in crystal, and their motion is at the origin of plastic properties of metals. In these crystals, we can observe self-organised structures, like Frank networks, *i.e.* networks of dislocations related by junctions. See for instance page 190 in Hull, Bacon [13] for such networks in BCC iron, or page 188 for hexagonal networks in FCC crystals. In the present paper, we consider a special case of a network contained in a single slip plane, where the dislocations can move. We are interested in particular in the motion of the junctions between dislocations, which remains a quite open question, both from the modelling point of view, and from the mathematical analysis point of view (see for instance the work of Rodney, Le Bouar, Finel [19]). Let us mention, for the stationary case, the work of Garroni [11]. The goal of the present paper is to propose and to study a model for the dynamics of junctions of dislocations.

The question of junctions has several other physical applications and there is various literature on this subject. Let us mention for instance the problem of crystal growth or grain growth (see Taylor [22, 23] and Bronsard, Reitich [8]). We also refer to Bonnet [7] for problems concerning the minimisation of the Mumford-Shah functional.

1.2. A phase field model for the dynamics of junctions. In a phase field model, the dislocation can be represented as the phase transition of a phase parameter $\rho(x) = \rho_1(x)e^1 + \rho_2(x)e^2 \in \mathbb{R}^2$ defined for $x = x_1e^1 + x_2e^2$ in the plane $\mathbb{R}^2$ with (e^1, e^2) an orthonormal basis. The energy of the dislocations, in the presence of a constant exterior applied stress $\sigma^0 \in \mathbb{R}^2$, is then given (see the model of Koslowski and Ortiz [14]) by

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} -\frac{1}{2}(C^0 \star \rho)\cdot\rho - \sigma^0\cdot\rho + W(\rho), \tag{1.1}$$

where the precise meaning of this expression will be given later.
For any phase transition between two states A and B, the difference $B - A$ needs physically to be the Burgers vector of the dislocation, *i.e.* a vector of the lattice

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$\Lambda = \mathbb{Z}a^1 + \mathbb{Z}a^2$ of the crystal we are considering, with general basis (a^1, a^2). This information is encoded in the potential $W : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ which is is assumed to be minimal on Λ and to have the periodicity of the lattice Λ:

$$W(\rho + a) = W(\rho) \quad \text{for any} \quad a \in \Lambda. \tag{1.2}$$

In this model, junctions of three dislocations of Burgers vectors $b^1, b^2, b^3 \in \Lambda$ with $b^1 + b^2 + b^3 = 0$ is expected, like for instance as the phase transitions between the states $0, b^1, -b^3$ (see Figure 1.1).

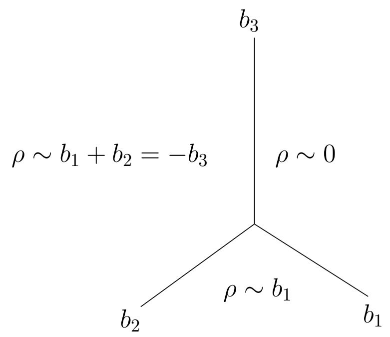

FIG. 1.1. *The junction of three dislocations as phase transitions of ρ.*

When the material is submitted to an exterior shear stress, it makes the dislocations move. The dynamics of a given dislocation line is physically given by its normal velocity, which is called the resolved Peach-Koehler force. This force is the sum of the resolved exterior shear stress and the stress created by all the dislocations lines, including the line itself.
In the expression giving the energy (1.1), the kernel $C^0(x)$ is a 2×2 symmetric matrix which takes into account the long range elastic interactions between dislocations and

$$(C^0 \star \rho)_i = \sum_{j=1,2} C^0_{ij} \star \rho_j, \quad \text{for} \quad i = 1, 2$$

where $\star$ denotes the usual convolution. In (1.1) and throughout the paper, we denote by $A \cdot B$ the scalar product between two vectors $A, B \in \mathbb{R}^2$.
The resolved stress $\sigma[\rho]$ created by all the dislocations is then formally given by the opposite of the gradient of the energy $-\mathcal{E}'(\rho)$, and can be expressed as the following non-local quantity

$$\sigma[\rho] = \sigma^0 + C^0 \star \rho - W'_\rho(\rho). \tag{1.3}$$

The phase parameter $\rho(t, x) \in \mathbb{R}^2$ is then assumed to satisfy the following equation

$$\begin{cases} (\rho_k)_t = |\nabla \rho|^{-1} \displaystyle\sum_{i=1,2} \sum_{j=1,2} (\sigma[\rho])_i \nabla_j \rho_i \nabla_j \rho_k + \varepsilon \Delta \rho_k & \text{for} \quad k = 1, 2 \\ & \text{in} \quad (0, T) \times \mathbb{T}^2, \\ \rho(0, x) = \rho^0(x) & \text{on} \quad \mathbb{T}^2, \end{cases} \tag{1.4}$$

where σ is given in (1.3), $\rho_t = \dfrac{\partial \rho}{\partial t}$ and $\nabla_j \rho_i = \dfrac{\partial \rho_i}{\partial x_j}$ for $i, j = 1, 2$, and

$$|\nabla \rho|^2 = \sum_{i=1,2} \sum_{j=1,2} |\nabla_j \rho_i|^2.$$

Here the parameter $1 > \varepsilon > 0$ is a small viscosity introduced in the model, in order to regularise the equation, but which has no real physical meaning. Given any time $T > 0$, we will study this equation, not on the the whole plane, but on the particular torus $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$, in order to simplify the analysis. This kind of periodic conditions is also meaningful physically, if we want to describe periodic networks of dislocations. This means in particular that $\sigma[\rho]$ is given by (1.3) where the convolution is done on the torus.

Finally, let us mention that our model (1.4) has some similarities with the model of Allen, Cahn [2] on the motion of curved boundaries in which they consider gradient flow associated with a free-energy functional. This led to the study of scalar Ginzburg-Landau type diffusion equation like

$$u_t = \Delta u - W'(u).$$

1.3. Main result. We make the following assumption on the kernel $C^0 : \mathbb{T}^2 \longrightarrow \mathbb{R}^{2\times 2}_{sym}$

(A) We assume that there exists a constant $m > 0$, such that for any $k \in \mathbb{Z}^2$, the Fourier coefficients of the kernel $\widehat{C}^0(k) = \int_{\mathbb{T}^2} dx \; e^{-2i\pi k\cdot x} C^0(x)$ satisfy $\widehat{C}^0(k) = M(k)$, where for any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and any $p = (p_1, p_2) \in \mathbb{R}^2$

$$\left\{ \begin{array}{l} M \in C^\infty(\mathbb{R}^2 \backslash \{0\}\,; \mathbb{R}^{2\times 2}_{sym}), \quad M(-\xi) = M(\xi), \quad M(\xi) = |\xi| M\left(\dfrac{\xi}{|\xi|}\right) \\ \\ \dfrac{|\xi|\ |p|^2}{m} \geq - \displaystyle\sum_{i=1,2} \sum_{j=1,2} p_i \cdot M_{ij}(\xi) \cdot p_j \geq m|\xi|\ |p|^2 \quad \text{with} \quad |p|^2 = \sum_{i=1,2} (p_i)^2 \end{array} \right. \tag{1.5}$$

We also make the following assumption on the potential $W : \mathbb{T}^2 \longrightarrow \mathbb{R}_+$

(B) We assume that $W \in C^2$ and W satisfies (1.2).

Then we have the following result for the model of dynamics of junctions between dislocations:

Theorem 1.1 (Existence of a solution)
Under assumptions (A) and (B), if $\rho^0 \in \left(H^1(\mathbb{T}^2)\right)^2$, then for any constant applied stress $\sigma^0 \in \mathbb{R}^2$ and for any time $T > 0$, there exists a solution ρ of (1.4) with $\rho \in C^0\left([0,T); \left(L^{\frac{4}{3}}(\mathbb{T}^2)\right)^2\right)$.

The uniqueness of the solution is not known, neither the existence of a solution when $\varepsilon = 0$. Let us mention that equation (1.4) is a non-local system of scalar equations, and can be sketched as the following equation

$$v_t = |\nabla v|^2 + \Delta v \tag{1.6}$$

Indeed, this comes from our assumption (A) that the convolution with the kernel behaves like a first order operator. A lot of work has been done on equations (or systems) like equation (1.6). Let us mention for instance the works of Boccardo, Murat, Puel [4, 5, 6] in which they study general equations including equation (1.6) and prove existence result.

Equation (1.6) is also similar to the Navier-Stokes equations written for the potential A such that the velocity of the fluid is given by $u = \text{curl } A$ (see for instance Leray [16]).

1.4. Organisation of the paper. In Section 2, we give some remarks on the modelling. In Section 3, we study an approximate problem of equation (1.4) where the right hand side is approached by some term at most linear in the solution. The main result is proved in Section 4. In a first subsection, we give some *a priori* estimates for the solution of the approximate problem and then in a second subsection we pass to the limit in the approximate problem.

1.5. Notation. In what follows, we will denote by C a generic constant, which will then satisfy $C + C = C$, $C \cdot C = C$, and so on. We also use the following set:

$$W^{2,1;p}(Q_T) = \left\{ u \in L^p(Q_T);\ u_t \in L^p(Q_T) \text{ and } \frac{\partial^2 u}{\partial x_i, \partial x_j} \in L^p(Q_T) \text{ for } i,j = 1,2 \right\}$$

where $Q_T = (0,T) \times \mathbb{T}^2$.

2. Preliminary remarks on the modelling.

2.1. Dynamics of a single dislocation. In this section, we consider a special case where a dislocation of Burgers vector $b^1 \in \Lambda$ is described by the phase transition of a scalar parameter $\bar\rho$ such that

$$\rho = \bar\rho\, b^1$$

Then the resolved shear stress that makes the dislocation move, is given by

$$\bar\sigma[\bar\rho] = b^1 \cdot \sigma[\bar\rho\, b^1]$$

and the dislocation line (described by the phase parameter $\bar\rho$) moves with normal velocity proportional to this shear stress. More precisely, $\bar\rho$ solves the following equation

$$\bar\rho_t = |b^1|^{-1}\ \bar\sigma[\bar\rho]|\nabla\bar\rho| + \varepsilon\Delta\bar\rho$$

where $\bar\rho_t = \dfrac{\partial\bar\rho}{\partial t}$ and $\nabla\bar\rho = \dfrac{\partial\bar\rho}{\partial x_1}\ e^1 + \dfrac{\partial\bar\rho}{\partial x_1}\ e^2$. Physically, we expect to have the eikonal equation with $\varepsilon = 0$, but in order to simplify the analysis and get an existence result, we only consider the case with additional viscosity $\varepsilon > 0$. Then, we easily check that $\rho = \bar\rho\, b^1$ satisfies (1.4).

2.2. Explicit expression of $\widehat{C}^0$ for isotropic materials. Given a particular Burgers vector b^1, let us consider

$$\bar c^0 = (b^1)^T \cdot C^0 \cdot b^1.$$

In the special case of isotropic linear elasticity with constant coefficients, we recall (see for instance a limit case of the Peierls-Nabarro model in Alvarez *et al.* [3]) that we have for $k = (k_1, k_2)$

$$\widehat{\bar c}^0(k) = -\frac{\mu|b^1|^2}{2}\left(\frac{\frac{1}{1-\nu}\,(k\cdot e)^2 + \left(k^\perp \cdot e\right)^2}{|k|}\right) \quad \text{with} \quad e = \frac{b^1}{|b^1|}$$

where $k^\perp = (-k_2, k_1)$ is the vector obtained by a rotation of k of angle $\pi/2$. Here $\mu > 0$ is a Lamé coefficient and $\nu \in (-1, 1/2)$ is the Poisson ratio.
We deduce that

$$\widehat{C}^0(k) = -\frac{\mu}{2|k|}\left(\frac{1}{1-\nu} k \otimes k + k^\perp \otimes k^\perp\right)$$

which satisfies assumption (A).

3. An approximate problem. We first start to approximate the right hand side of equation (1.4) by some term at most linear in the solution. To this end, we introduce a function h^n defined by

$$h^n(r) = h^0(r-n)$$

with

$$h^0(r) = \begin{cases} 1 \quad \text{if} \quad r \le 0 \\ 1-r \quad \text{if} \quad 0 \le r \le 1 \\ 0 \quad \text{if} \quad r \ge 1 \end{cases}$$

We then look at the following approximate problem:

$$\begin{cases} \rho_t - \epsilon\Delta\rho = f^n[\rho] & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ \rho(0,\cdot) = \rho^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{3.1}$$

where

$$f^n[\rho] = h^n(|\nabla\rho|)\ |\nabla\rho|^{-1}\ (\nabla\rho)^T \cdot \nabla\rho \cdot \sigma[\rho]$$

and $\sigma[\rho]$ is given in (1.3), and is at most linear in ρ.

The natural idea to find a solution to equation (3.1), is to define the map Φ which associates to any function u, the solution $\rho = \Phi(u)$ of

$$\begin{cases} \rho_t - \epsilon\Delta\rho = f^n[u] & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ \rho(0,\cdot) = \rho^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{3.2}$$

and to prove that Φ has a fixed point in a suitable space. This way, we will prove the following result

Theorem 3.1 (Existence of a solution for the approximate problem)
If $\rho^0 \in \left(H^1(\mathbb{T}^2)\right)^2$, then for any $n \ge 1$ and any $T > 0$, there exists a solution ρ^n of (3.1) with $\rho^n \in L^2((0,T); \left(H^2(\mathbb{T}^2)\right)^2) \cap C^0([0,T); \left(L^2(\mathbb{T}^2)\right)^2)$.

In this section, we will make the proof of this theorem. In a first subsection, we will collect some preliminary results, and in a second subsection we will prove that Φ has a fixed point.

3.1. Preliminary results. The following estimate on the stress will be important.

Lemma 3.2 (Estimate on $\sigma[\rho]$)
For any $p \in (1, +\infty)$, there exists a constant C (depending on p, on the constant σ^0, on the potential W and on the constant m defined in assumption (A)), such that for any $\rho \in \left(W^{1,p}(\mathbb{T}^2)\right)^2$, we have

$$\|\sigma[\rho]\|_{(L^p(\mathbb{T}^2))^2} \le C\left(1 + \|\nabla\rho\|_{(L^p(\mathbb{T}^2))^{2\times 2}}\right). \tag{3.3}$$

Partial proof of Lemma 3.2
Let us make the proof for $p = 2$. We have with $\sigma = \sigma[\rho]$

$$\begin{aligned}
|\sigma - \sigma^0 + W'(\rho)|^2_{(L^2(\mathbb{T}^2))^2} &= \sum_{k\in\mathbb{Z}^2} |\widehat{(C^0 \star \rho)}(k)|^2 \\
&= \sum_{k\in\mathbb{Z}^2} |\widehat{C}^0(k)\cdot\widehat{\rho}(k)|^2 \\
&\le \frac{1}{m^2}\sum_{k\in\mathbb{Z}^2} |k|^2|\widehat{\rho}(k)|^2 \\
&\le \frac{1}{(2\pi m)^2}\sum_{k\in\mathbb{Z}^2} |\widehat{\nabla\rho}(k)|^2 \\
&= \frac{1}{(2\pi m)^2}|\nabla\rho|^2_{(L^2(\mathbb{T}^2))^{2\times 2}}
\end{aligned}$$

Therefore

$$|\sigma[\rho]|_{(L^2(\mathbb{T}^2))^2} \le |\sigma^0| + |W'|_{(L^\infty(\mathbb{T}^2))^2} + \frac{1}{2\pi m}|\nabla\rho|_{(L^2(\mathbb{T}^2))^{2\times 2}}$$

which provides the result in the case $p = 2$.
The proof for the general case $p \in (1, +\infty)$ is given in Appendix. □

We will also need the following result.

Lemma 3.3 (Estimate on $f^n[u]$)
If $u \in (H^1(\mathbb{T}^2))^2$, then $f^n[u] \in (L^2(\mathbb{T}^2))^2$ with the following estimate:

$$\|f^n[u]\|_{(L^2(\mathbb{T}^2))^2} \le C(n+1)\left(1 + \|\nabla u\|_{L^2(\mathbb{T}^2)^{2\times 2}}\right)$$

where the constant C depends on σ^0, on the potential W and on the constant m defined in assumption (A).

Proof of Lemma 3.3
Since $\mathrm{supp}(h^n) \subset [0, n+1]$, the following holds

$$|f^n[u]| \le (n+1)|\sigma[u]|, \tag{3.4}$$

where we have used the fact that $|B^T \cdot B \cdot p| \le |B|^2|p|$ for $B \in \mathbb{R}^{2\times 2}$ and $p \in \mathbb{R}^2$. Then

$$\begin{aligned}
\|f^n[u]\|_{(L^2(\mathbb{T}^2))^2} \le& (n+1)\|\sigma[u]\|_{(L^2(\mathbb{T}^2))^2} \\
\le& C(n+1)\left(1 + \|\nabla u\|_{L^2(\mathbb{T}^2)^{2\times 2}}\right)
\end{aligned}$$

where we have used Lemma 3.2 □

We now recall some classical results. We start with the following parabolic estimates for the following equation

$$\begin{cases} g_t - \epsilon \Delta g = f & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ g(0,\cdot) = g^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{3.5}$$

Proposition 3.4 (Parabolic estimates for the heat equation)
Let $g^0 \in H^1(\mathbb{T}^2)$ and $f \in L^2(Q_T)$. Then there exists a unique solution g to (3.5) with

$$g \in L^2((0,T); H^2(\mathbb{T}^2)) \cap L^\infty((0,T); H^1(\mathbb{T}^2)), \quad g_t \in L^2(Q_T). \tag{3.6}$$

We have the following estimate

$$\begin{aligned} &\sup_{0\le t\le T} \|g(t)\|_{H^1(\mathbb{T}^2)} + \|g\|_{L^2((0,T);H^2(\mathbb{T}^2))} + \|g_t\|_{L^2((0,T);L^2(\mathbb{T}^2))} \\ \le & C_T \left(\|f\|_{L^2(Q_T)} + \|g^0\|_{H^1(\mathbb{T}^2)} \right), \end{aligned} \tag{3.7}$$

where the constant C_T only depends on T and ε.
Moreover we have

$$\sup_{0\le t\le T} \int_{\mathbb{T}^2} g^2(t) + 4\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla g|^2 \le 2 \int_{\mathbb{T}^2} (g^0)^2 + 4T \int_0^T \int_{\mathbb{T}^2} f^2 \tag{3.8}$$

Proof of Proposition 3.4
For the proof of (3.6)-(3.7), we refer to Evans [10, Theorem 5 page 360].
To prove (3.8), we simply multiply equation (3.5) by g and integrate over $(0,t)$ in time, taking the supremum for $0 \le t \le T$. We get

$$\sup_{0\le t\le T} \int_{\mathbb{T}^2} \frac{g^2(t)}{2} + \varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla g|^2 \le \int_{\mathbb{T}^2} \frac{(g^0)^2}{2} + \int_0^T \int_{\mathbb{T}^2} |g\ f|$$

We now use the fact that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} |g\ f| \quad &\le \quad \left(\int_0^T \int_{\mathbb{T}^2} g^2 \right)^{\frac{1}{2}} \cdot \left(\int_0^T \int_{\mathbb{T}^2} f^2 \right)^{\frac{1}{2}} \\ &\le \quad \left(T \sup_{0\le t\le T} \int_{\mathbb{T}^2} g^2(t) \right)^{\frac{1}{2}} \cdot \left(\int_0^T \int_{\mathbb{T}^2} f^2 \right)^{\frac{1}{2}} \\ &\le \quad \frac{1}{4} \sup_{0\le t\le T} \int_{\mathbb{T}^2} g^2(t) + T \int_0^T \int_{\mathbb{T}^2} f^2 \end{aligned}$$

which implies the result. □

We also recall the

Theorem 3.5 (Schaefer's fixed point theorem)
Let X be a real Banach space. Suppose that

$$\Phi : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u \in X, \quad u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1]\}$$

is bounded. Then Φ has a fixed point.

For the proof of this theorem, we refer to Evans [10, Theorem 4 page 504].

Finally, we will need some compactness argument and a weak continuity property contained in the following two classical results:

Proposition 3.6 (Compactness)
We recall that

$$W^{2,1;2}(Q_T) = \left\{g \in L^2((0,T); H^2(\mathbb{T}^2)), \quad g_t \in L^2(Q_T)\right\}.$$

Then the injection

$$W^{2,1;2}(Q_T) \longrightarrow L^2((0,T); H^1(\mathbb{T}^2)) \quad \text{is compact.}$$

For the proof of this result, we refer to Lions [17, Theorem 5.1 page 58].

Proposition 3.7 (Continuity)
With the notation of Proposition 3.6, let us consider a sequence $(g^m)_m$ such that

$$g^m \rightharpoonup g \quad \text{weakly in} \quad W^{2,1;2}(Q_T)$$

We assume also that $g^m_{|t=0} = \rho^0$. Then

$$g_{|t=0} = \rho^0.$$

This result is classical but for the reader's convenience we give the proof in Appendix.

3.2. Proof of Theorem 3.1. We are now ready to make the proof of Theorem 3.1. To this end, for any $T > 0$, we set

$$X_T = L^2((0,T); H^1(\mathbb{T}^2)).$$

In all what follows the index n is assumed fixed. We first remark that if $u \in X_T^2$, then $f^n[u] \in (L^2(Q_T))^2$, and then we can consider the solution ρ of

$$\begin{cases} \rho_t - \epsilon\Delta\rho = f^n[u] & \text{on} \quad Q_T := (0,T) \times \mathbb{T}^2 \\ \rho(0,\cdot) = \rho^0 & \text{on} \quad \mathbb{T}^2 \end{cases} \tag{3.9}$$

which satisfies $\rho \in X_T^2$ because of the parabolic estimates Proposition 3.4. Then we set $\Phi(u) = \rho$, and see that Φ maps X_T^2 into X_T^2. We will prove that Φ admits a fixed point using Schaefer's fixed point theorem. We do the proof in four steps.

Step 1 : weak continuity of Φ
Let us consider sequences $(u^m)_m, (\rho^m)_m$ such that

$$\begin{cases} u^m \in X_T^2, \quad \rho^m = \Phi(u^m) \\ u^m \longrightarrow u \quad \text{in} \quad X_T^2 \end{cases}$$

From Lemma 3.3, we deduce that

$$||f^n[u^m]||_{(L^2(Q_T))^2} \leq C(n+1)\left(1 + ||u^m||_{X_T^2}\right) \tag{3.10}$$

From the parabolic estimates (Proposition 3.4), we deduce that ρ^m is bounded in $\left(W^{2,1;2}(Q_T)\right)^2$, *i.e.* there exists a constant $C > 0$ such that

$$||\rho^m||_{(W^{2,1;2}(Q_T))^2} \leq C \tag{3.11}$$

Therefore, up to a subsequence, we have

$$\rho^m \rightharpoonup \rho \quad \text{in} \quad \left(W^{2,1;2}(Q_T)\right)^2$$

and from Proposition 3.7, we deduce that

$$\rho_{|t=0} = \rho^0 \quad \text{on} \quad \mathbb{T}^2$$

We now claim that

$$f^n[u^m] \longrightarrow f^n[u] \quad \text{in} \quad L^1(Q_T) \tag{3.12}$$

Indeed, we can write

$$f^n[u] = g^n(\nabla u) \cdot \sigma[u] \quad \text{with} \quad g^n(\nabla u) := h^n(|\nabla u|)\ |\nabla u|^{-1}\ (\nabla u)^T \cdot \nabla u$$

From the proof of Lemma 3.2, for $p = 2$, we already deduce that

$$\sigma[u^m] \longrightarrow \sigma[u] \quad \text{in} \quad L^2(Q_T) \tag{3.13}$$

From the convergence of u^m to u in X_T^2, we deduce that up to a subsequence we have $\nabla u^m \longrightarrow \nabla u$ a.e. in Q_T. Now from the fact that g^n is continuous and bounded, we deduce in particular that

$$g^n(\nabla u^m) \longrightarrow g^n(\nabla u) \quad \text{in} \quad L^2(Q_T) \tag{3.14}$$

Then the convergence (3.12) follows from (3.13) and (3.14).
Therefore we conclude that ρ solves (3.9). Finally, by uniqueness of the solutions of (3.9), we deduce that the limit ρ does not depend on the choice of the subsequence, and then that the full sequence converges:

$$\rho^m \rightharpoonup \rho \quad \text{weakly in} \quad \left(W^{2,1;2}(Q_T)\right)^2, \quad \text{with} \quad \rho = \Phi(u)$$

Step 2 : compactness of Φ
The compactness (and the usual continuity) of Φ follows from the compactness of the injection $\left(W^{2,1;2}(Q_T)\right)^2 \longrightarrow X_T^2$ (see Proposition 3.6).

Step 3 : a priori bounds on the solutions of $u = \lambda\Phi(u)$ for T small
Let us consider a solution u of

$$u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1] \tag{3.15}$$

Then from the parabolic estimates (3.8), we have

$$\begin{aligned}
& \sup_{0\le t\le T} \int_{\mathbb{T}^2} |u(t)|^2 + 4\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \\
\le \quad & 2\int_{\mathbb{T}^2} |\rho^0|^2 + 4T \int_0^T \int_{\mathbb{T}^2} |\lambda f^n[u]|^2 \\
\le \quad & 2\int_{\mathbb{T}^2} |\rho^0|^2 + 8TC^2(n+1)^2 \left(T + \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \right)
\end{aligned}$$

where in the third line we have used Lemma 3.3 and the fact that $|\lambda| \le 1$. Therefore for

$$T \le T^* := \left(4C^2(n+1)^2\right)^{-1} \varepsilon \tag{3.16}$$

we have

$$\sup_{0\le t\le T} \int_{\mathbb{T}^2} |u(t)|^2 + 2\varepsilon \int_0^T \int_{\mathbb{T}^2} |\nabla u|^2 \le 2\int_{\mathbb{T}^2} |\rho^0|^2 + 2\varepsilon T$$

which proves that there exists a constant $C > 0$ such that any solution of (3.15) satisfies

$$||u||_{X_T^2} \le C$$

We can then apply the Schaefer's fixed point Theorem (Theorem 3.5), to deduce that Φ has a fixed point on X_T^2, and therefore there is a solution ρ of (3.1) on the time interval $(0,T)$ if T satisfies (3.16), *i.e.* if T is small enough independently on the initial data ρ^0.

Step 4 : solution for any time
Let us call $\rho\left(\rho^0, t\right)$ the function $\rho(t,\cdot)$ obtained at Step 3 as a solution of (3.1) on the time interval $[0,T^*)$ with initial data ρ^0. From the parabolic estimates (Proposition 3.4), we also know that $\rho(t,\cdot) \in (H^1(\mathbb{T}^2))^2$ for any $t \in [0,T^*)$. Then we can define with $\tau = T^*/2$

$$u(0) = \rho^0 \quad \text{and} \quad u(t) = \rho\left(u(k\tau), t\right) \quad \text{if} \quad k\tau \le t < (k+1)\tau \quad \text{with} \quad k \in \mathbb{N}.$$

Using the fact that $u_t \in L^2_{loc}((0,+\infty); (L^2(\mathbb{T}^2)^2)$, and the fact that the problem is invariant by translation in time, we can easily check that u solves (3.1) for any $T > 0$ and provides the desired solution $\rho^n = u$ of Theorem 3.1.
This ends the proof of Theorem 3.1.

4. A priori estimates and proof of Theorem 1.1.

4.1. A priori estimates. We have the following *a priori* estimates:

Lemma 4.1 (*A priori* estimates)
There exists a constant $C > 0$ such that for all $T > 0, n \geq 1$ and $0 < \varepsilon < 1$, any solution ρ^n of (3.1) given by Theorem 3.1, satisfies

$$\|\rho^n\|^2_{L^\infty((0,T);(H^{\frac{1}{2}}(\mathbb{T}^2))^2)} \quad \leq \quad Ce^{\frac{CT}{\varepsilon}}, \tag{4.1}$$

$$\|\rho^n\|^2_{L^2((0,T);(H^{\frac{3}{2}}(\mathbb{T}))^2)} \quad \leq \quad \frac{C}{\varepsilon}e^{\frac{CT}{\varepsilon}}, \tag{4.2}$$

and

$$\left\| h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}\nabla\rho^n \cdot \sigma[\rho^n] \right\|^2_{(L^2(Q_T))^2} \quad \leq \quad Ce^{\frac{CT}{\varepsilon}}. \tag{4.3}$$

Proof of Lemma 4.1
Step 1: Preliminaries on the energy
We first recall the expression of the energy for a general $\mathbb{Z}^2$-periodic smooth function $\rho(x) = (\rho_1(x), \rho_2(x))$

$$\mathcal{E}(\rho) = \int_{\mathbb{T}^2} -\frac{1}{2}(C^0 \star \rho)\cdot\rho - \sigma^0\cdot\rho + W(\rho)$$

For future use, we start to evaluate from below the first term in the energy, using Fourier series

$$\begin{aligned}\int_{\mathbb{T}^2} -(C^0 \star \rho)\cdot\rho \quad &= \quad \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C^0 \star \rho})(k)\cdot\widehat{\rho}^*(k)\right)\\ &= \quad \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C}^0(k)\cdot\widehat{\rho}(k))\cdot\widehat{\rho}^*(k)\right)\\ &\geq \quad m\sum_{k\in\mathbb{Z}^2}|k|\ |\widehat{\rho}(k)|^2\end{aligned}$$

where we have used in the first line the fact that ρ and C^0 are real, and in the last line we have used assumption (A). Then we define

$$||\rho||^2_{(\dot{H}^{\frac{1}{2}}(\mathbb{T}^2))^2} := \sum_{k\in\mathbb{Z}^2}|k|\ |\widehat{\rho}(k)|^2.$$

Similarly, we compute

$$
\begin{aligned}
&\int_{\mathbb{T}^2} -(C^0 \star (\nabla\rho)^T) : \nabla\rho \\
=&(2\pi)^2 \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -(\widehat{C}^0(k)\cdot\widehat{\rho}(k)\otimes(ik)) : (ik)^* \otimes \widehat{\rho}^*(k)\right) \\
=&(2\pi)^2 \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -|k|^2(\widehat{C}^0(k)\cdot\widehat{\rho}(k)) : \widehat{\rho}^*(k)\right) \\
\geq&(2\pi)^2 m \sum_{k\in\mathbb{Z}^2} |k|^3 \; |\widehat{\rho}(k)|^2
\end{aligned}
$$

where we have used assumption (A) in the last line. Then we define

$$
||\rho||^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2} := \sum_{k\in\mathbb{Z}^2} |k|^3 \; |\widehat{\rho}(k)|^2.
$$

Step 2: Estimate on the time-derivative of the energy
Let us fix $T > 0$. We know that any solution ρ^n given by Theorem 3.1 belongs to the space $W^{2,1;2}(Q_T)$. In particular, using the following general fact (because of assumption (A))

$$
\int_{\mathbb{T}^2} -(C^0 \star \rho)\cdot\rho = \mathrm{Re}\left(\sum_{k\in\mathbb{Z}^2} -|k|(\hat{C}^0\left(\frac{k}{|k|}\right)\cdot\widehat{\rho}(k))\cdot\widehat{\rho}^*(k)\right)
$$

we deduce that the energy $\mathcal{E}(\rho^n(t))$ is well-defined for almost every $t \in [0,T)$, and that for almost every time $t \in [0,T)$, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(\rho^n(t)) =& \int_{\mathbb{T}^2} -\sigma[\rho^n]\cdot\rho^n_t \\
=& \int_{\mathbb{T}^2} -h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1} \; |\nabla\rho^n\cdot\sigma[\rho^n]|^2 - \varepsilon\sigma[\rho^n]\cdot\Delta\rho^n \\
=& \int_{\mathbb{T}^2} -h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1} \; |\nabla\rho^n\cdot\sigma[\rho^n]|^2 \\
&- \int_{\mathbb{T}^2} \varepsilon\left\{W''(\rho^n) : ((\nabla\rho^n)^T\cdot\nabla\rho^n) - (C^0\star(\nabla\rho^n)^T) : \nabla\rho^n\right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\rho^n(t)) + \int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1} \; |\nabla\rho^n\cdot\sigma[\rho^n]|^2 \\
\leq& C\varepsilon\left\{\int_{\mathbb{T}^2} |\nabla\rho^n|^2 + C^0\star(\nabla\rho^n)^T) : \nabla\rho^n\right\} \qquad (4.4)
\end{aligned}
$$

But now (up to change the constant line to line)

$$\begin{aligned}
\|\nabla\rho^n\|^2_{(L^2(\mathbb{T}^2))^{2\times 2}} &\leq C\sum_{k\in\mathbb{Z}^2}|k|^2|\widehat{\rho}^n(k)|^2\\
&\leq C\sum_{k\in\mathbb{Z}^2}|k|^{\frac{3}{2}}|\widehat{\rho}^n(k)|\cdot|k|^{\frac{1}{2}}|\widehat{\rho}^n(k)|\\
&\leq C\left(\sum_{k\in\mathbb{Z}^2}\frac{1}{2\alpha}|k|^3|\widehat{\rho}^n(k)|^2+\frac{\alpha}{2}|k||\widehat{\rho}^n(k)|^2\right)\\
&\leq C\left(\int_{\mathbb{T}^2}-\frac{1}{\alpha}(C^0\star(\nabla\rho^n)^T):\nabla\rho^n+\int_{\mathbb{T}^2}-\alpha(C^0\star\rho^n)\cdot\rho^n\right),
\end{aligned}$$

where $\widehat{\rho}^n(k)$ are the Fourier coefficients of ρ^n and α is a constant which will be precised later. We then deduce finally that:

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\rho^n(t))+\int_{\mathbb{T}^2}h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}\ |\nabla\rho^n\cdot\sigma[\rho^n]|^2\\
&\leq -C\varepsilon\left(1-\frac{1}{\alpha}\right)\int_{\mathbb{T}^2}-(C^0\star(\nabla\rho^n)^T):\nabla\rho^n+C\varepsilon\alpha\int_{\mathbb{T}^2}-(C^0\star\rho^n)\cdot\rho^n \qquad (4.5)\\
&\leq -C\varepsilon\|\rho^n(t)\|^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2}+C\varepsilon\left(1+\mathcal{E}(\rho^n(t))+|\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right).
\end{aligned}$$

for α chosen large enough, with C a suitable positive constant.

Step 3: Estimate on the time-derivative of the mean-value of the solution
Integrating in space equation (3.1), we get

$$\frac{d}{dt}\int_{\mathbb{T}^2}\rho^n(t) = \int_{\mathbb{T}^2}h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}\ (\nabla\rho^n)^T\cdot\nabla\rho^n\cdot\sigma[\rho^n]$$

and then

$$\begin{aligned}
\frac{d}{dt}\left|\int_{\mathbb{T}^2}\rho^n(t)\right| &\leq \int_{\mathbb{T}^2}(h^n(|\nabla\rho^n|)|\nabla\rho^n|)^{\frac{1}{2}}\cdot\left((h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1})^{\frac{1}{2}}|\nabla\rho^n\cdot\sigma[\rho^n]|\right)\\
&\leq \int_{\mathbb{T}^2}(1+|\sigma^0|)h^n(|\nabla\rho^n|)|\nabla\rho^n|\\
&\quad+\tfrac{1}{4(1+|\sigma^0|)}h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2
\end{aligned} \qquad (4.6)$$

Step 4: Estimate on the energy
Setting

$$F^n(t)=1+\mathcal{E}(\rho^n(t))+(1+|\sigma^0|)\left|\int_{\mathbb{T}^2}\rho^n(t)\right|+(1+|\sigma^0|)^4, \qquad (4.7)$$

we deduce from (4.5) and (4.6) that

$$\begin{aligned}
&\frac{d}{dt}F^n(t)+\frac{3}{4}\int h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2\\
&\leq -C\varepsilon\|\rho^n(t)\|^2_{(\dot{H}^{\frac{3}{2}}(\mathbb{T}^2))^2}+C\varepsilon\left(1+\mathcal{E}(\rho^n(t))+|\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right)\\
&\quad+(1+|\sigma^0|)^2\int_{\mathbb{T}^2}h^n(|\nabla\rho^n|)|\nabla\rho^n|.
\end{aligned}$$

Now we remark that

$$(1+|\sigma^0|)^2\int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n| \leq (1+|\sigma^0|)^2\int_{\mathbb{T}^2}|\nabla\rho^n|$$
$$\leq \frac{C\varepsilon}{2}\int_{\mathbb{T}^2}|\nabla\rho^n|^2+\frac{\left(1+|\sigma^0|\right)^4}{2C\varepsilon}$$

Using the fact that (since the domain is bounded)

$$\int_{\mathbb{T}^2}|\nabla\rho^n|^2 \leq \|\rho^n(t)\|^2_{(\dot H^{\frac{3}{2}}(\mathbb{T}^2))^2},$$

we get

$$\begin{aligned}
&\frac{d}{dt}F^n(t)+\frac{3}{4}\int_{\mathbb{T}^2} h^n(|\nabla\rho^n|)|\nabla\rho^n|^{-1}|\nabla\rho^n\cdot\sigma[\rho^n]|^2+\frac{C\varepsilon}{2}\|\rho^n(t)\|^2_{(\dot H^{\frac{3}{2}}(\mathbb{T}^2))^2} \\
&\leq C\varepsilon\left(1+\mathcal{E}(\rho^n(t))+|\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\right)+\frac{(1+|\sigma^0|)^4}{2C\varepsilon} \\
&\leq \frac{C}{\varepsilon}F^n(t).
\end{aligned} \tag{4.8}$$

This implies, using Gronwall Lemma,

$$F^n(t)\leq F^n(0)e^{\frac{C}{\varepsilon}t}. \tag{4.9}$$

Step 5: Estimate on ρ^n
Let us first remark that

$$\mathcal{E}(\rho^n(t))\geq\int -\frac{1}{2}(C^0\star\rho^n)\cdot\rho^n-|\sigma^0|\left|\int_{\mathbb{T}^2}\rho^n(t)\right|. \tag{4.10}$$

Using (4.9), (4.10) and the definition of $F^n(t)$, yields

$$\int_{\mathbb{T}^2}-\frac{1}{2}(C^0\star\rho^n)\cdot\rho^n+\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\leq Ce^{\frac{C}{\varepsilon}t}.$$

Using Step 1, we then get

$$\|\rho^n\|^2_{L^\infty((0,T);(\dot H^{\frac{1}{2}}(\mathbb{T}^2))^2)}\leq Ce^{\frac{C}{\varepsilon}T}\quad\text{and}\quad\left|\int_{\mathbb{T}^2}\rho^n(t)\right|\leq Ce^{\frac{C}{\varepsilon}T}.$$

This implies (4.1). Taking the integral $\int_0^T$ in (4.8) and using the fact that $\forall t\leq T$, $F^n(t)\geq 0$, we get

$$\|h_n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}|\nabla\rho^n\cdot\sigma[\rho^n]|\|^2_{(L^2(Q_T))^2}+\varepsilon C\|\rho^n\|^2_{L^2((0,T);(\dot H^{\frac{3}{2}}(\mathbb{T}^2)))}\leq Ce^{\frac{C}{\varepsilon}T}$$

which implies (4.2) and (4.3).

$\square$

4.2. Proof of Theorem 1.1. We are now able to prove Theorem 1.1. In this section, we denote by C a generic constant which can depend on ρ^0, ε and T but which do not depend on n.

Proof of Theorem 1.1
Let $T > 0$. The idea of the proof is to pass to the limit in Equation (3.1). The only difficulty is to prove that the non-linear term $f^n[\rho^n]$ converges in a certain sense to $|\nabla\rho|^{-1}(\nabla\rho)^T \cdot \nabla\rho \cdot \sigma[\rho]$, where ρ is the limit of ρ^n in an appropriate norm. The proof is decomposed into five steps:

Step 1: a priori bound on $f^n[\rho^n]$
We have the following estimate on $f^n[\rho^n]$:

$$\|f^n[\rho^n]\|_{\left(L^{\frac{4}{3}}(Q_T)\right)^2} \leq C. \tag{4.11}$$

To prove this, let us write

$$f^n[\rho^n] = \left(|\nabla\rho^n|^{-1}(\nabla\rho^n)^T\right) \cdot \left(|\nabla\rho^n|^{\frac{1}{2}}\right) \left(h_n(|\nabla\rho^n|)|\nabla\rho^n|^{-\frac{1}{2}}\nabla\rho^n \cdot \sigma[\rho^n]\right).$$

Using (4.3), we have that the last term is bounded in $(L^2(Q_T))^2$ by C. Moreover, the first term is bounded by 1 in $(L^\infty(Q_T))^{2\times 2}$, then we just have to bound the term $|\nabla\rho^n|^{\frac{1}{2}}$ in $L^4(Q_T)$. Using (4.2), we have

$$\||\nabla\rho^n|^{\frac{1}{2}}\|_{L^4(Q_T)} = \left(\int_{Q_T} |\nabla\rho^n|^2\right)^{\frac{1}{4}} \leq \|\rho^n\|^{\frac{1}{2}}_{L^2((0,T);(H^{\frac{3}{2}}(\mathbb{T}))^2)} \leq C. \tag{4.12}$$

This implies (4.11).

Step 2: Strong convergence of $\nabla\rho^n$ in $L^2((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^{2\times 2})$
Using the parabolic estimates for the heat equation (see [15, ch 4.3 p 80 and ch 4.9 p 341]) and Step 1, we get

$$\|\nabla\rho^n\|_{W^{\frac{1}{2};\frac{4}{3}}((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^{2\times 2})} \leq C \tag{4.13}$$

where for a Banach space B

$$W^{\frac{1}{2};p}((0,T);B) = \left\{ g \in L^p((0,T);B), \ \int_0^T \int_0^T \frac{\|g(t) - g(s)\|_B^p}{|t-s|^{\frac{1}{2}p+1}}\, dt\, ds < \infty \right\}$$

is equiped with the following norm:

$$\|g\|_{W^{\frac{1}{2};p}((0,T);B)} := \left(\int_0^T \int_0^T \frac{\|g(t) - g(s)\|_B^p}{|t-s|^{\frac{1}{2}p+1}}\, dt\, ds\right)^{\frac{1}{p}}.$$

Moreover, using (4.2) we get

$$\|\nabla\rho^n\|_{L^2\left((0,T);\left(H^{\frac{1}{2}}(\mathbb{T}^2)\right)^{2\times 2}\right)} \leq C. \tag{4.14}$$

We then use the following lemma:

Lemma 4.2 (Compactness result)
Let $(g_n)_n$ be a sequence uniformly bounded in

$$L^2\left((0,T);H^{\frac{1}{2}}(\mathbb{T}^2)\right)\cap W^{\frac{1}{2};\frac{4}{3}}\left((0,T);L^{\frac{4}{3}}(\mathbb{T}^2)\right),$$

then, for a subsequence,

$$g_n \to g \text{ strongly in } L^2((0,T);L^{\frac{4}{3}}(\mathbb{T}^2)).$$

Formally, the proof uses the fact that $H^{\frac{1}{2}} \subset L^{\frac{4}{3}}$ with compact injection in space while the compactness in time comes from (4.13). We refer to Simon [20, Corollary 5, p.86] for a more general result and for the proof of this lemma.

Using (4.13), (4.14) and Lemma 4.2, we then deduce that, for a subsequence, $\nabla\rho^n \to \nabla\rho$ strongly in $L^2\left((0,T);\left(L^{\frac{4}{3}}(\mathbb{T}^2)\right)^2\right)$ and almost everywhere.

Step 3: Weak convergence of $\sigma[\rho^n]$ in $L^2\left((0,T);\left(L^4(\mathbb{T}^2)\right)^2\right)$
We have $H^{\frac{1}{2}}(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$ with continuous injection (see Adams [1, Theorem 7.57 p217]). So $L^2((0,T);H^{\frac{1}{2}}(\mathbb{T}^2)) \subset L^2((0,T);L^4(\mathbb{T}^2))$ with continuous injection. We then deduce from (4.2) that

$$\|\nabla\rho^n\|_{L^2\left((0,T);(L^4(\mathbb{T}^2))^{2\times 2}\right)} \leq C. \tag{4.15}$$

Using Lemma 3.2, we then get

$$\|C^0 \star \rho^n\|_{L^2\left((0,T);(L^4(\mathbb{T}^2))^2\right)} \leq C. \tag{4.16}$$

Using the fact that the application $W^{2,1,\frac{4}{3}}_{x,t}(Q_T) \mapsto L^{\frac{4}{3}}(Q_T)$ is compact and the converse of Lebesgue Theorem, we deduce that $W'(\rho^n) \to W'(\rho)$ almost everywhere. This implies that $\sigma[\rho^n] \rightharpoonup \sigma[\rho]$ in $L^2((0,T);(L^4(\mathbb{T}^2))^2$.

Step 4: Passing to the limit
Using Step 2 and Step 3 and the fact that $|\nabla\rho^n|^{-1}\nabla\rho^n$ is bounded by 1, we deduce that

$$f_n[\rho^n] \to |\nabla\rho|^{-1}(\nabla\rho)^T\cdot\nabla\rho\cdot\sigma[\rho] \quad \text{in the distributions sense.}$$

By passing to the limit in (3.1), we obtain

$$\rho_t - \epsilon\Delta\rho = |\nabla\rho|^{-1}(\nabla\rho)^T\cdot\nabla\rho\cdot\sigma[\rho] \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \tag{4.17}$$

Step 5: Initial Condition
Using the fact that ρ^n_t are bounded uniformly in $L^{\frac{4}{3}}(Q_T)$ (by parabolic estimates for the heat equation and Step 1), we deduce that (uniformly in n)

$$\|\rho^n(t+h)-\rho^n(t)\|_{(L^{\frac{4}{3}}(\mathbb{T}^2))^2} \leq Ch^{\frac{1}{4}}\|\rho^n_t\|_{L^{\frac{4}{3}}((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^2)}$$

and then $\rho \in C^0((0,T);(L^{\frac{4}{3}}(\mathbb{T}^2))^2)$ and $\rho_{|t=0} = \rho^0$.

This achieves the proof of Theorem 1.1. □

5. Appendix. Full proof of Lemma 3.2

Here we do the proof for any $p \in (1, +\infty)$. Under assumption (A), there exists a constant $C > 0$ only depending on p, such that the following result holds for any $\tilde{\rho} \in W^{1,p}(\mathbb{R}^2)$

$$|\tilde{C}^0 \star_{\mathbb{R}^2} \tilde{\rho}|_{(L^p(\mathbb{R}^2))^2} \quad \leq \quad \frac{C}{m} |\nabla \tilde{\rho}|_{(L^p(\mathbb{R}^2))^{2\times 2}}$$

where the Fourier transform of $\tilde{C}^0$ satisfies $\widehat{\tilde{C}}^0 = M$ with M as in (1.5).

This result can be found in the scalar case on $\mathbb{R}^n$ in Stein [21], see proposition 5 page 251, or Coifman, Meyer [9], Theorem 9 page 39 and Proposition 2 page 41. See also Calderon-Zygmund inequalities Theorem 2.7.2 in Morrey [18]. Here the convolution by $\tilde{C}^0$ is a multiplier operator in the class S^1 of pseudo-differential operators. We then get the result in the vectorial case, summing the scalar components. See also the book of Garroni, Menaldi [12] for complements on integro-differential operators.

The fact that the result holds on the torus $\mathbb{T}^2$ is then classical. We prove it for the convenience of the reader. To this end, we consider a smooth function φ such that

$$\varphi(x) = 1 \quad \text{on} \quad [-1/3, 1/3]^2, \quad \text{and} \quad \operatorname{supp} \varphi \subset [-2/3, 2/3]^2, \quad \text{and} \quad 0 \leq \varphi \leq 1$$

such that

$$\sum_{k \in \mathbb{Z}^2} \varphi(x - k) = 1.$$

For any smooth function $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is $\mathbb{Z}^2$-periodic, we then set for $K > 0$

$$(S_{2K}\rho)(x) = \sum_{|k| \leq 2K,\ k \in \mathbb{Z}^2} \varphi(x - k)\rho(x).$$

Therefore we get for $K > 0$ large enough

$$\begin{aligned}
&|B_K| \left\{ |\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{L^p((-1/2,1/2)^2)} + 0(1/K) \right\} \\
\leq & |\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{L^p(B_K)} \\
\leq & |\tilde{C}^0 \star_{\mathbb{R}^2} (S_{2K}\rho)|_{L^p(\mathbb{R}^2)} + |\tilde{C}^0 \star (\rho - (S_{2K}\rho))|_{L^p(B_K)} \\
\leq & \frac{C}{m} |\nabla (S_{2K}\rho)|_{L^p(\mathbb{R}^2)} + + |\tilde{C}^0 \star (\rho - (S_{2K}\rho))|_{L^p(B_K)} \\
\leq & \frac{C}{m} |B_{2K}| \left\{ |\nabla \rho|_{(L^p(-1/2,1/2))^{2\times 2}} + 0(1/K) \right\} + |\rho|_{(L^\infty(\mathbb{R}^2))^2} |B_K| \int_{|z| \geq K-1} |\tilde{C}^0(z)|.
\end{aligned}$$

Using the fact that $\int_{|z|\geq K-1} |\tilde{C}^0(z)| = O(1/K)$, dividing by $|B_K|$ and taking the limit as $K \longrightarrow +\infty$, we get

$$|\tilde{C}^0 \star_{\mathbb{R}^2} \rho|_{(L^p(\mathbb{T}^2))^2} \quad \leq \quad \frac{C}{m} \frac{|B_2|}{|B_1|} |\nabla \rho|_{(L^p(\mathbb{T}^2))^{2\times 2}}$$

i.e.

$$|C^0 \star_{\mathbb{T}^2} \rho|_{(L^p(\mathbb{T}^2))^2} \quad \leq \quad \frac{4C}{m} |\nabla \rho|_{(L^p(\mathbb{T}^2))^{2\times 2}}$$

with

$$C^0(x) = \sum_{k\in\mathbb{Z}^2} \tilde{C}^0(x-k).$$

We then get the final result by density of smooth functions in $(W^{1,p}(\mathbb{T}^2))^2$. $\square$

Proof of Proposition 3.7
For simplicity of notation, we denote by $g(t)$ the function $x \mapsto g(t,x)$. We have

$$\begin{aligned}\|g^m(t)-\rho^0\|_{(L^2(\mathbb{T}^2))^2} \leq & \int_0^t ds\ \|g_t^m(s)\|_{(L^2(\mathbb{T}^2))^2} \\ \leq & \sqrt{t}\ \|g_t^m\|_{(L^2(Q_T))^2}.\end{aligned}$$

Using the fact that g^m is bounded uniformly in $W^{2,1;2}(Q_T)$ (this is a consequence of the fact that $g^m \rightharpoonup g$ in $W^{2,1;2}(Q_T)$), we get

$$\|g^m(t)-\rho^0\|_{(L^2(\mathbb{T}^2))^2} \leq C\sqrt{t}. \tag{5.1}$$

Now let $\varphi \in C_c^\infty([0,+\infty),\mathbb{R})$ be such that $\varphi \geq 0$. Using (5.1), we get that

$$\int_0^t ds\ \|g^m(s)-\rho^0\|^2_{(L^2(\mathbb{T}^2))^2}\varphi(s) \leq C\int_0^t ds\ s\ \varphi(s).$$

Using Fatou's Lemma, we deduce that

$$\int_0^t \Big(\|g(s)-\rho^0\|^2_{(L^2(\mathbb{T}^2))^2} - Cs\Big)\varphi(s) \leq 0.$$

Using that $\varphi \geq 0$ is arbitrary, we deduce that for almost every t, we have

$$\|g(t)-\rho^0\|^2_{(L^2(\mathbb{T}^2))^2} \leq \sqrt{C t}.$$

This implies the result. $\square$

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