$\mathcal{T}$-class algorithms for pseudocontractions and κ-strict pseudocontractions in Hilbert spaces

Jean-Philippe Chancelier

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Abstract

In this paper we study iterative algorithms for finding a common element of the set of fixed points of κ-strict pseudocontractions or finding a solution of a variational inequality problem for a monotone, Lipschitz continuous mapping. The last problem being related to finding fixed points of pseudocontractions. These algorithms were already studied in [1] and [9] but our aim here is to provide the links between these know algorithms and the general framework of $\mathcal{T}$-class algorithms studied in [3].

1 Introduction

Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and P_C be the metric projection from $\mathcal{H}$ onto C. A mapping $Q : C \mapsto C$ is said to be a *strict pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that :

$$\|Qx - Qy\|^2 \leq \|x - y\|^2 + \kappa \|(I - Q)x - (I - Q)y\|^2 , \tag{1}$$

for all x, $y \in C$. A mapping Q for which (1) holds is also called a κ-strict pseudocontraction. As pointed out in [1] iterative methods for finding a common element of the set of fixed points of strict pseudocontractions are far less developed than iterative methods for nonexpansive mappings ($\kappa = 0$) [2, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15]. We will, in section 2 of this article, consider the algorithm 1 studied in [1] and we will show that this algorithm can be viewed as a $\mathcal{T}$-class algorithm as defined and studied in [3].

Section 3 is devoted to the case $\kappa = 1$ for which previous algorithm cannot be used. A mapping A for which (1) holds with $\kappa = 1$ is called *pseudocontractive*. We will see that *pseudocontractive* mappings are related to monotone Lipschitz continuous mappings. A mapping $A : C \mapsto \mathcal{H}$ is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \text{for all} \quad (u, v) \in C^2 .$$

A is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Au - Av\| \leq k \|u - v\| \quad \text{for all} \quad (u, v) \in C^2.$$

Let the mapping $A : C \mapsto \mathcal{H}$ be monotone and Lipschitz continuous. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \quad \text{for all} \quad v \in C\,.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Assume that a mapping $Q : C \mapsto C$ is pseudocontractive and k-Lipschitz-continuous then the mapping $A = I - Q$ is monotone and $(k+1)$-Lipschitz-continuous and moreover $Fix(Q) = VI(C, A)$ [9, Theorem 4.5] where $Fix(Q)$ is the set of fixed points of Q, that is

$$Fix(Q) \stackrel{\text{def}}{=} \{x \in C \,:\, Qx = x\} \tag{2}$$

Thus, to cover the case $\kappa = 1$, algorithms which aims at computing $P_{VI(C,A)}x$ for a monotone and k-Lipschitz-continuous mapping A are investigated. We will, in section 3 mainly use results from [9] to prove that the general algorithm that they use can be rephrased in a slightly extended $\mathcal{T}$-class algorithm framework.

2 $\mathcal{T}$-class iterative algorithm for a sequence of κ-strict pseudocontractions

Let $(Q_n)_{n\geq 0}$ be a sequence of κ-strict pseudocontractions, $\kappa \in [0, 1)$ and $(\alpha_n)_{n\geq 0}$ a sequence of real numbers chosen so that $\alpha_n \in (\kappa, 1)$. We consider as in [1] the following algorithm :

Algorithm 1 *Given $x_0 \in C$, we consider the sequence $(x_n)_{n\geq 0}$ generated by the following algorithm :*

$$\begin{aligned}
&y_n = \alpha_n x_n + (1 - \alpha_n) Q_n x_n, \\
&C_n \stackrel{def}{=} \left\{ z \in C \,|\, \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa)\|x_n - Q_n x_n\|^2 \right\}, \\
&D_n \stackrel{def}{=} \{z \in C \,|\, \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
&x_{n+1} = P_{(C_n \cap D_n)} x_0.
\end{aligned}$$

We will show that this algorithm belong to the $\mathcal{T}$-class algorithms as defined in [3] and deduce its strong convergence to $P_F x_0$ when $F \neq \emptyset$ and where $F \stackrel{\text{def}}{=} \cap_{n\geq 0} Fix(Q_n)$.

For $(x, y) \in \mathcal{H}^2$ define the mappings H as follows :

$$H(x, y) \stackrel{\text{def}}{=} \{z \in \mathcal{H} \quad | \quad \langle z - y, x - y \rangle \leq 0\} \tag{3}$$

and denote by $Q(x, y, z)$ the projection of x onto $H(x, y) \cap H(y, z)$. Note that $H(x, x) = \mathcal{H}$ and for $x \neq y$, $H(x, y)$ is a closed affine half space onto which y is the projection of x.

Lemma 1 *The sequence generated by Algorithm 1 coincide with the sequence given by $x_{n+1} = Q(x_0, x_n, T_n x_n)$ with :*

$$T_n(x) \stackrel{def}{=} \frac{x + R_n y}{2} + \frac{1}{2}\left(\frac{\kappa - \alpha_n}{1 - \alpha_n}\right)(x - R_n y), \text{ and } R_n(x) \stackrel{def}{=} \alpha_n x + (1 - \alpha_n) Q_n(x)\,. \tag{4}$$

Moreover, we have :

$$2T_n - I = \kappa I + (1 - \kappa) Q_n x\,. \tag{5}$$

Proof :Let $\kappa \in [0, 1)$, $\alpha \in (\kappa, 1)$, $y \stackrel{\text{def}}{=} \alpha x + (1 - \alpha) Qx$ for a κ-strict pseudo-contractions Q and define $\Gamma(x, y)$ as follows :

$$\Gamma(x, y) \stackrel{\text{def}}{=} \left\{ z \in \mathcal{H} \quad | \quad \|y - z\|^2 \le \|x - z\|^2 - (1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2 \right\}\,. \tag{6}$$

We first prove that $\Gamma(x, y) = H(x, Tx)$ where T is defined by equation (4).

$$\begin{aligned}
& \|y - z\|^2 - \|x - z\|^2 \le -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2 \\
\Leftrightarrow\ & \langle y - z, y - z\rangle - \|x - z\|^2 \le -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2 \\
\Leftrightarrow\ & \langle y - x, y - z\rangle + \langle x - z, y - z\rangle - \|x - z\|^2 \le -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2 \\
\Leftrightarrow\ & \langle y - x, y - z\rangle + \langle x - z, y - x\rangle \le -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2 \\
\Leftrightarrow\ & \langle y - x, y - z\rangle + \langle x - z, y - x\rangle \le (\alpha - \kappa)\langle y - x, x - Qx\rangle \\
\Leftrightarrow\ & \langle y - x, y + x - 2z + (\kappa - \alpha)(x - Qx)\rangle \le 0 \\
\Leftrightarrow\ & \left\langle y - x, y + x - 2z + \left(\frac{\kappa - \alpha}{1 - \alpha}\right)(x - y)\right\rangle \le 0
\end{aligned}$$

which gives :

$$\left\langle z - \frac{x + y}{2} - \frac{1}{2}\left(\frac{\kappa - \alpha}{1 - \alpha}\right)(x - y), x - y\right\rangle \quad \le \quad 0$$

and since we have $x - Tx = (1/2)(1 - \frac{\kappa - \alpha}{1 - \alpha})(x - y)$ with $(1 - \frac{\kappa - \alpha}{1 - \alpha}) > 0$ this is equivalent to $\langle z - Tx, x - Tx\rangle \le 0$. For $y_n = \alpha_n x_n + (1 - \alpha_n) Q_n x_n$, we thus obtain that $C_n = \Gamma(x_n, y_n) = H(x_n, T_n x_n)$ and since by definition of H we have $D_n = H(x_0, x_n)$ the result follows. The last statement of the lemma (5) is obtained by simple rewrite from equation (4) □

We prove now that T_n for all $n \in \mathbb{N}$ belongs to the $\mathcal{T}$ class of mappings.

Definition 2 $\mathcal{T} \stackrel{def}{=} \{T : \mathcal{H} \mapsto \mathcal{H} \,|\, domT = \mathcal{H} \quad and \quad (\forall x \in \mathcal{H})\, Fix(T) \subset H(x, Tx)\}$

Lemma 3 *for all $n \in \mathbb{N}$ and T_n defined by equation (4) we have $T_n \in \mathcal{T}$.*

Proof :Using Lemma 1 we have $2T_n - I = \kappa I + (1 - \kappa) Q_n$. If we can prove that when Q is a κ-strict pseudocontraction the mapping $\kappa I + (1 - \kappa) Q$ is quasi-nonexpansive then the result will follow from [3, Proposition 2.3 (v)]. For $(x, y) \in \mathcal{H}^2$ we have :

$$\begin{aligned}
\|\kappa x \quad + \quad & (1-\kappa)Qx - y - (1-\kappa)y\|^2 = \|\kappa(x-y) + (1-\kappa)(Qx - Qy)\|^2 \\
= \quad & \kappa\|x-y\|^2 + (1-\kappa)\|Qx-Qy\|^2 - \kappa(1-\kappa)\|x-y-(Qx-Qy)\|^2 \\
= \quad & \kappa\|x-y\|^2 + (1-\kappa)\|Qx-Qy\|^2 - \kappa(1-\kappa)\|x-y-(Qx-Qy)\|^2 \\
\leq \quad & \kappa\|x-y\|^2 + (1-\kappa)\left(\|Qx-Qy\|^2 - \kappa\|(I-Q)x-(I-Q)y\|^2\right) \\
\leq \quad & \kappa\|x-y\|^2 + (1-\kappa)\|x-y\|^2 = \|x-y\|^2
\end{aligned}$$

Thus the mapping $\kappa I + (1-\kappa)Q$ is nonexpansive and thus also quasi-nonexpansive. □

Definition 4 *[3] A sequence $(T_n)_{n\geq 0}$ such that $T_n \in \mathcal{T}$ is* coherent *if for every bounded sequence $\{z_n\}_{n\geq 0} \in \mathcal{H}$ there holds :*

$$\left\{ \begin{array}{l} \sum_{n\geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n\geq 0} \|z_n - T_n z_n\|^2 < \infty \end{array} \right. \Rightarrow \mathcal{M}(z_n)_{n\geq 0} \subset \cap_{n\geq 0} Fix(T_n) \tag{7}$$

where $\mathcal{M}(z_n)_{n\geq 0}$ is the set of weak cluster points of the sequence $(z_n)_{n\geq 0}$.

Lemma 5 *Let $(Q_n)_{n\geq 0}$ be a sequence of κ-strict pseudocontraction such that $Fix(Q_n) = F$ which does not depends on n and for each subsequence $\sigma(n)$ we can find a sub-sequence $\mu(n)$ such that $Q_{\mu(n)} \to Q$ with $Fix(Q) = F$ and Q is a κ-strict pseudocontraction. Then, the sequence $(T_n)_{n\geq 0}$ given by (4) is coherent.*

Proof : Suppose that $(z_n)_{n\geq 0}$ is a bounded sequence such that the left hand side of (7) is satisfied. Using (5) we have $\|z_n - T_n z_n\| = (1-\kappa)/2\|z_n - Q_n z_n\|$ and $Fix(T_n) = Fix(Q_n)$. Thus, verifying the coherence of $(T_n)_{n\geq 0}$ or the coherence of $(Q_n)_{n\geq 0}$ is equivalent. Consider now $u \in \mathcal{M}(z_n)_{n\geq 0}$, by hypothesis $\|z_n - Q_n z_n\| \to 0$. Let $\sigma(n)$ a subsequence such that $z_{\sigma(n)} \rightharpoonup u$, we extract a subsequence $\mu(n)$ such that $Q_{\mu(n)} \to Q$ and we thus obtain that $z_{\mu(n)} \rightharpoonup u$ and $\|z_{\mu(n)} - Qz_{\mu(n)}\| \to 0$. Now, if Q is a κ-strict pseudocontraction, using [1, Proposition 2.6] we have that $I - Q$ is demi-closed and thus $u \in Fix(Q) = F$. □

Remark 6 *Given an integer $N \geq 1$, let, for each $1 \leq i \leq N$, $S_i : C \mapsto C$ be a κ_i-strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa \stackrel{def}{=} \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the common fixed point set $F \stackrel{def}{=} \cap_{i=1}^N Fix(S_i)$ of $\{S_i\}$ is nonempty. Assume also for each n, $\{\lambda_{n,i}\}_{i=1,\ldots,N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_{n,i} = 1$ and $\inf_n \lambda_{n,i} > 0$ for all $1 \leq i \leq N$. Let the mapping $Q_n : C \mapsto C$ be defined by :*

$$Q_n x \stackrel{def}{=} \sum_{i=1}^N \lambda_{n,i} S_i x \,. \tag{8}$$

Then using [1], for all $n \in \mathbb{N}$, Q_n is a κ-strict pseudocontraction and $Fix(Q_n) = F$. Moreover for each subsequence $\lambda_{i,(\sigma_n)}$ we can extract a subsequence $\lambda_{i,\mu(n)}$ and $(\overline{\lambda_i})_{1\leq i\leq N} \in (0,1)^N$ such that $\lambda_{i,\mu(n)} \to \overline{\lambda_i}$ for all $1 \leq i \leq N$. We thus have $Q_{\mu(n)} \to \sum_i \overline{\lambda_i} S_i$ and using previous lemma the sequence $(T_n)_{n\geq 0}$ is coherent.

Given $T_n \in \mathcal{T}$ we can also consider [3] the following algorithm :

Algorithm 2 *Given $\epsilon \in (0,1]$ and $x_0 \in C$ we consider the sequence given by the iterations $x_{n+1} = x_n + (2-\epsilon)(T_n x_n - x_n)$.*

Gathering previous result the strong convergence of Algorithm 1 to $P_F x_0$ and the weak convergence of Algorithm 2 is obtained by [3, Theorem 4.2] that we recall now :

Theorem 7 *[3, Theorem 4.2] Suppose that $(T_n)_{n\geq 0}$ is coherent. Then*
(i) if $F \neq \emptyset$, then every orbit of Algorithm 2 converges weakly to a point in F
(ii) For an arbitrary orbit of Algorithm 1, exactly one of the following alternatives holds :

- *(a) $F \neq \emptyset$ and $x_n \to_n P_F x_0$.*
- *(b) $F = \emptyset$ and $x_n \to_n +\infty$.*
- *(c) $F = \emptyset$ and the algorithm terminates.*

Remark 8 *Note that using previous theorem and Remark 6 we obtain an other proof of [1, Theorem 5.1]. In fact the proofs are very similar but we just hilite here the role played by $\mathcal{T}$-class sequences.*

3 $\mathcal{T}$-class iterative algorithm for a sequence of pseudo contractions

Let F be a closed convex of $\mathcal{H}$ we define $\mathcal{U}_F$ as follows :

$$\mathcal{U}_F \stackrel{\text{def}}{=} \{T : \mathcal{H} \mapsto \mathcal{H} \,|\, \text{dom}T = \mathcal{H} \quad \text{and} \quad (\forall x \in \mathcal{H}) F \subset H(x, Tx)\} \,. \tag{9}$$

Of course we have $T \in \mathcal{T} \Leftrightarrow T \in \mathcal{U}_{Fix(T)}$.

A mapping $Q : \mathcal{H} \mapsto \mathcal{H}$ is said F-quasi-nonexpansive if

$$\forall (x,y) \in \mathcal{H} \times F \quad \|Qx - y\| \leq \|x - y\| \tag{10}$$

and we can characterize elements of $\mathcal{U}_F$ using the following easy lemma :

Lemma 9 $2T - I$ *is F-quasi-nonexpansive is equivalent to $T \in \mathcal{U}_F$.*

*Proof :*The proof follows from the equality [3, (2.6)] :

$$(\forall (x,y) \in \mathcal{H}^2) \quad 4\langle y - Tx, x - Tx\rangle = \|(2T - I)x - y\|^2 - \|x - y\|^2 \,. \tag{11}$$

□

Definition 10 *A sequence* $\{T_n\}_{n\geq 0} \subset \mathcal{U}_F$ *is* F-c*oherent if for every bounded sequence* $\{z_n\}_{n\geq 0} \in \mathcal{H}$ *there holds :*

$$\left\{ \begin{array}{l} \sum_{n\geq 0} \|z_{n+1} - z_n\|^2 < \infty \\ \sum_{n\geq 0} \|z_n - T_n z_n\|^2 < \infty \end{array} \right. \Rightarrow \mathcal{M}(z_n)_{n\geq 0} \subset F \tag{12}$$

We propose now the following extension of [3, Theorem 4.2] for the two algorithms 2 and 3.

Algorithm 3 *Given* $x_0 \in C$ *we consider the sequence given by the iterations*

$$x_{n+1} = Q(x_0, x_n, T_n x_n)$$

Theorem 11 *Suppose that* $(T_n)_{n\geq 0}$ *is* F*-coherent for a closed convex* F *Then (i) if* $F \neq \emptyset$*, then every orbit of Algorithm 2 converges weakly to a point in* F *(ii) For an arbitrary orbit of Algorithm 3, exactly one of the following alternatives holds :*

(a) $F \neq \emptyset$ *and* $x_n \to_n P_F x_0$.

(b) $F = \emptyset$ *and* $x_n \to_n +\infty$.

(c) $F = \emptyset$ *and the algorithm terminates.*

*Proof :*The result is very similar to [3, Theorem 2.9] and a careful reading of the proof and remarks in [3, 4] leads to the conclusion that it remains true as stated here. □

We give now a typical application of this theorem.

Definition 12 *For* $A : C \mapsto C$ *a monotone and k-Lipschitz mapping, let* $T_\lambda : \mathcal{H} \times \mathcal{H} \mapsto \mathcal{H}$ *the mapping defined by* $T_\lambda(x, y) \stackrel{def}{=} P_C(x - \lambda A y)$. *We also define* $T_\lambda^{(1)} x \stackrel{def}{=} T_\lambda(x, x)$ *and* $T_\lambda^{(2)} x \stackrel{def}{=} T_\lambda(x, T_\lambda(x, x)) = T_\lambda(x, T_\lambda^{(1)} x)$.

We assume that $\lambda k \in [a, b] \subset (0, 1)$ and consider $(\lambda_n)_{n\geq 0}$ a sequence of real numbers such that $\lambda_n k \in [a, b]$. To simplify the notations we will use $T_n^{(1)}$ (resp. $T_n^{(2)}$) for denoting $T_{\lambda_n}^{(1)}$ (resp. $T_{\lambda_n}^{(2)}$).

Let $F \stackrel{\text{def}}{=} VI(C, A)$, It is known that F is closed convex and that we have $Fix\, T_\lambda^{(1)} = F$. It is easy to see that $F \subset Fix(T_\lambda^{(2)})$ but the inclusion may be strict and thus we do not expect the mapping $T_\lambda^{(2)}$ to be quasi-nonexpansive. Following inequalities contained in the proof of [9, Theorem 3.1] we obtain F-quasi-nonexpansive property as exposed now.

Lemma 13 $T_\lambda^{(2)}$ *is* F*-quasi-nonexpansive where* $F \stackrel{def}{=} VI(C, A)$ *or using Lemma 9* $(T_\lambda^{(2)} + I)/2 \in \mathcal{U}_F$.

Proof : Let $y = T_\lambda^{(1)}(x)$ and $u \in VI(C, A)$. We use the fact that for all $x \in \mathcal{H}$ and $y \in C$ $P_C x$ can be characterized as follows :

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{13}$$

and since A is a monotone mapping following the steps of the proof of [9, Theorem 3.1] that we reproduce here we obtain :

$$\begin{aligned}
\|T_\lambda^{(2)}(x) - u\|^2 &\leq \|x - \lambda Ay - u\|^2 - \|x - \lambda Ay - T_\lambda^{(2)}(x)\|^2 \\
&= \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 + 2\lambda \left\langle Ay, u - T_\lambda^{(2)}(x) \right\rangle \\
&= \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 \\
&\quad +2\lambda(\langle Ay - Au, u - y\rangle + \langle Au, u - y\rangle + \left\langle Ay, y - T_\lambda^{(2)}(x) \right\rangle) \\
&\leq \|x - u\|^2 - \|x - T_\lambda^{(2)}(x)\|^2 + 2\lambda \left\langle Ay, y - T_\lambda^{(2)}(x) \right\rangle \\
&= \|x - u\|^2 - \|x - y\|^2 - 2\left\langle x - y, y - T_\lambda^{(2)}(x) \right\rangle - \|y - T_\lambda^{(2)}(x)\|^2 \\
&\quad +2\lambda \left\langle Ay, y - T_\lambda^{(2)}(x) \right\rangle \\
&= \|x - u\|^2 - \|x - y\|^2 - \|y - T_\lambda^{(2)}(x)\|^2 \\
&\quad +2\left\langle x - \lambda Ay - y, T_\lambda^{(2)}(x) - y \right\rangle .
\end{aligned}$$

Further, since $y = P_C(x - \lambda Ax)$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned}
\langle x - \lambda Ay - y \quad &, \quad T_\lambda^{(2)}(x) - y \rangle = \left\langle x - \lambda Ax - y, T_\lambda^{(2)}(x) - y \right\rangle \\
&+ \left\langle \lambda Ax - \lambda Ay, T_\lambda^{(2)}(x) - y \right\rangle \leq \left\langle \lambda Ax - \lambda Ay, T_\lambda^{(2)}(x) - y \right\rangle \\
&\leq \lambda k \|x - y\| \|T_\lambda^{(2)}(x) - y\| .
\end{aligned}$$

So, we have ;

$$\begin{aligned}
\|T_\lambda^{(2)}(x) - u\|^2 &\leq \|x - u\|^2 - \|x - y\|^2 - \|y - T_\lambda^{(2)}(x)\|^2 + 2\lambda k \|x - y\| \|T_\lambda^{(2)}(x) - y\| \\
&\leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max\left(\|x - y\|^2, \|T_\lambda^{(2)}(x) - y\|^2\right) \qquad (14) \\
&\leq \|x - u\|^2 .
\end{aligned}$$

□

Corollary 14 *If we consider* $R \overset{def}{=} \alpha I + (1 - \alpha)S$ *where* S *is a non-expansive mapping and define* $\tilde{F} = Fix(S) \cap VI(C, A)$ *then we obtain immediately that* $R \circ T_\lambda^{(2)}$ *is a* $\tilde{F}$*-quasi-nonexpansive mapping.*

*Proof :*Let $u \in \tilde{F}$ then $u = Ru$ and we have $\|R \circ T_\lambda^{(2)} - u\| \leq \|T_\lambda^{(2)} - u\|$ and the previous lemma ends the proof. $\square$

Lemma 15 *The sequence $Q_n = 1/2(T_n^{(2)} + I)$ is F-coherent.*

*Proof :*Let $(y_n)_{n\geq 0}$ a bounded sequence satisfying the left hand side of equation (12) and $\varphi \in \mathcal{M}(y_n)_{n\geq 0}$. We can find a subsequence $y_{\sigma(n)}$ which converges weakly to φ. For simplicity, we use the notation y_n for the subsequence and since it satisfies the left hand side of equation (12) we have $\|y_n - Q_n y_n\| \to 0$. By definition of Q_n we also have $\|y_n - T_n^{(2)} y_n\| \to 0$ and thus $T_n^{(2)} y_n \rightharpoonup u$ From equation (14) we obtain :

$$\|T_\lambda^{(2)} x - u\|^2 \leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max\left(\|x - T_\lambda^{(1)} x\|^2, \|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2\right)$$

Thus :

$$\begin{aligned} \max\left(\|x - T_\lambda^{(1)} x\|^2\right. &, \left.\|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2\right) \leq \frac{1}{1 - \lambda^2 k^2}\left(\|x - u\|^2 - \|T_\lambda^{(2)} x - u\|^2\right) \\ &\leq K\left(\|x - u\| + \|T_\lambda^{(2)} x - u\|\right) \|x - T_\lambda^{(2)} x\| \qquad (15) \end{aligned}$$

Using Lemma 13, the sequence $T_n^{(2)} y_n$ is bounded and we thus have from the previous inequality $\|y_n - T_n^{(1)} y_n\| \to 0$ and $\|T_n^{(2)} y_n - T_n^{(1)} y_n\| \to 0$.

Using next lemma (Lemma 17) we therefore obtain that for $(v, w) \in G(T)$:

$$\langle v - \varphi, w\rangle = \lim_{n\to\infty} \left\langle v - T_n^{(2)} y_n, w\right\rangle \geq 0\,.$$

Thus we obtain that $\langle v - \varphi, w\rangle \geq 0$ which gives $\varphi \in T^{-1}(0)$ since T is maximal monotone and then $\varphi \in F = VI(C, A)$. Thus Q_n is F-coherent. $\square$

Corollary 16 *Let $(R_n)_{n\geq 0}$ a sequence of nonexpansive mappings such that for each subsequence $\sigma(n)$ it is possible to extract a subsequence $\mu(n)$ and find R_μ such that $R_{\mu(n)} y_n \to_{n\to\infty} R_\mu y_n$ for every bounded sequence $(y_n)_{n\geq 0}$ with $Fix\, R_\mu = \mathcal{S}$ a fixed set such that $\mathcal{S} \cap \mathcal{S} \neq \emptyset$. Then, we also have that $Q_n = 1/2((R_n \circ T_n^{(2)}) + I)$ is $F \cap \mathcal{S}$-coherent.*

*Proof :*Let $u \in \mathcal{S} \cap \mathcal{S}$, since R_n is nonexpansive we have : $\|R_n \circ T_\lambda^{(2)} - u\| \leq \|T_\lambda^{(2)} - u\|$, Thus equation (15) can be replaced by :

$$\|R_n \circ T_\lambda^{(2)} x - u\|^2 \leq \|x - u\|^2 + (\lambda^2 k^2 - 1) \max\left(\|x - T_\lambda^{(1)} x\|^2, \|T_\lambda^{(2)} x - T_\lambda^{(1)} x\|^2\right)$$

proceeding as in previous lemma we obtain that for $(y_n)_{n\geq 0}$ a bounded sequence satisfying the left hand side of equation (12) for the sequence of mapping $R_n \circ T_n^{(2)}$ we also have up to subsequences that $\|y_n - T_n^{(1)} y_n\| \to 0$ and

$\|T_n^{(2)}y_n - T_n^{(1)}y_n\| \to 0$ and thus also $\|y_n - T_n^{(2)}y_n\| \to 0$. Thus, as before, if φ is a weak limit of $(y_n)_{n\geq 0}$ we have $\varphi \in F$. Moreover, we have :

$$\begin{aligned} \|T_n^{(2)}y_n - R_\mu \nu\| &\leq \|T_n^{(2)}y_n - y_n\| + \|y_n - R_n \circ T_n^{(2)}y_n\| \\ &\quad + \|R_n \circ T_n^{(2)}y_n - R_\mu \circ T_n^{(2)}y_n\| + \|T_n^{(2)}y_n - \nu\| \quad (16) \end{aligned}$$

Thus

$$\liminf_{n\mapsto\infty} \|T_n^{(2)}y_n - R_\mu\nu\| \leq \liminf_{n\mapsto\infty} \|T_n^{(2)}y_n - \nu\|$$

which by Opial's condition is only possible if $R_\mu\nu = \nu$. We conclude that $\nu \in F \cap \mathcal{S}$ which ends the proof. □

Lemma 17 *[9] Let $T : \mathcal{H} \mapsto H$ the mapping defined by $Tv \stackrel{def}{=} Av + N_C v$ when $v \in C$ and $Tv = 0$ when $v \notin C$ where N_C is the normal cone to C at $v \in C$. Let $G(T)$ be the graph of T and $(v, w) \in G(T)$. Then for $x \in C$ we have the following inequality :*

$$\left\langle v - T_\lambda^{(2)}x, w \right\rangle \geq \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} \right\rangle$$

*Proof :*The proof of this inequality is given in [9], we reproduce it for the sake of completeness. The mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$. On the other hand, from $T_\lambda^{(2)}(x) = P_C(x - \lambda AT_\lambda^{(1)}(x))$ and $v \in C$ we have $\left\langle x - \lambda Ay - T_\lambda^{(2)}(x), T_\lambda^{(2)}(x) - v \right\rangle \geq 0$ and hence $\left\langle v - T_\lambda^{(2)}(x), T_\lambda^{(2)}(x) - x\lambda + AT_\lambda^{(1)}x \right\rangle \geq 0$. From $\langle v - t, w - Av\rangle \geq 0$ for all $t \in C$ and $T_\lambda^{(2)}(x) \in C$, we have

$$\begin{aligned} \left\langle v - T_\lambda^{(2)}x, w \right\rangle &\geq \left\langle v - T_\lambda^{(2)}x, Av \right\rangle \\ &\geq \left\langle v - T_\lambda^{(2)}x, Av \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} + AT_\lambda^{(1)}x \right\rangle \\ &= \left\langle v - T_\lambda^{(2)}x, Av - AT_\lambda^{(2)}x \right\rangle + \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle \\ &\quad - \|v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda}\| \\ &\geq \left\langle v - T_\lambda^{(2)}x, AT_\lambda^{(2)}x - AT_\lambda^{(1)}x \right\rangle - \left\langle v - T_\lambda^{(2)}x, \frac{T_\lambda^{(2)}x - x}{\lambda} \right\rangle \end{aligned}$$

□

We end this section by gathering previous results in a main theorem. The proof is immediate by applying Theorem 11. The first statement is a new result. The second statement when applied to the sequence $R_n = \alpha_n Id + (1 - \alpha_n)S$ with $\alpha_n \in [0, c)$ and $c < 1$ gives the same result as [9, Theorem 3.1].

Theorem 18 *Let $(R_n)_{n\geq 0}$ a sequence of nonexpansive mappings satisfying the hypothesis of Corollary 16 and $(T_n^{(2)})_{n\geq 0}$ the sequence of mappings defined on Definition 12. Then, every orbit of Algorithm 2 applied to the sequence of mappings $R_n \circ T_n^{(2)}$ converges weakly to a point in F and the sequence generated by Algorithm 1 converges strongly to $P_F x_0$.*

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