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1 Introduction

Due to emerging technologies such as optical fiber sensors, temperature measurements are destined to play a major role in petroleum production logging interpretation. Using temperature recordings from a wellbore and a flowrate history on the surface, it can be envisaged to develop new ways to predict flow repartition among each producing layer of a reservoir or to estimate virgin reservoir temperatures.

In order to solve the inverse problem, one first needs to develop a forward model describing the flow of a monophasic compressible fluid (oil or gas) in a reservoir and a well, from both a dynamic and a thermal point of view. This implies to couple a reservoir model (porous medium) and a well model (fluid medium).

We have already studied in [1] a reservoir model, consisting of the Darcy-Forchheimer equation coupled with a non-standard energy balance, which includes notably the temperature effects due to the decompression of the fluid (Joule-Thomson effect) and the frictional heating that occurs in the formation. This problem was written in cylindrical coordinates and discretized by means of the conservative Raviart-Thomas elements. The model was also validated from both a numerical and a physical point of view.

In this paper, we introduce the well model which is based on the compressible Navier-Stokes equations coupled with an energy equation, both written in axisymmetric form. In order to take into account the privileged direction of the flow in the well and to reduce the cost of the calculation, we propose here an approach to derive two pseudo 1D models, by constructing an explicit solution in terms of the radial coordinate. The nonlinear time-discretized problem is solved by means of a fixed point method with respect to the density ρ. More precisely, for a given ρ, we compute at any time step the specific flux from the mass equation and then we recover the velocity and the pressure, as well as the heat flux and the temperature by using mixed variational formulations. Finally, we update the density by means of a thermodynamic module and we loop until the convergence is achieved. The difference between our two 1D models consists in the computation of the radial velocity u_r : in the first one, we neglect the momentum equation corresponding to u_r and we compute it directly from the specific flux by a projection method, while in the second one, the whole velocity (u_r, u_z) is computed by using the Navier-Stokes equation and by neglecting, this time, a boundary condition on the perforations. The first model thus obtained is a little bit simpler, but the second one is well adapted to the coupling with the reservoir.

Once the time discretization is achieved, we propose a well-posed finite element approximation, based on lowest-order Raviart-Thomas elements for the fluxes, Q_0 elements for the pressure and the temperature and Q_1-continuous finite elements for the velocity. Numerical tests are also presented, validating the code from both a numerical and a physical point of view.

The paper is organized as follows. The physical model is introduced in section 2, and its semi-discretization leading to our two pseudo 1D conservative models is presented in section 3. Then, the well-posedness of these two models is established in section 4, while their space approximation by means of a finite element method is addressed in section 5. Finally, section 6 is devoted to numerical tests.

2 Physical problem

We are interested in the modelling of the flow of a monophasic compressible fluid in a petroleum wellbore. Our 3D domain is defined by

$$\Omega_{3D} = \{(x,y,z)\ ;\ 0 \leq x^2 + y^2 \leq R^2,\ z \in [z_{\min}, z_{\max}]\}$$

where R is the radius of the well. In practice, $R \simeq 4inch$ while the length of the pipe can attend several thousands meters.

The governing kinematic equations are the mass conservation law :

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0 \tag{1}$$

and the Navier-Stokes equations :

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + div(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + div \underline{\tau} + \rho \mathbf{g} + \mathbf{F}, \tag{2}$$

where :

$$\underline{\tau} = 2\mu \left[\underline{\varepsilon}(\mathbf{u}) - \frac{1}{3}(div \mathbf{u})\underline{\mathbf{I}} \right]$$

and $\mathbf{F}$ is a source term which takes into account the friction at the surface of the pipe. One generally uses an empirical formula for the modelling of the friction term $\mathbf{F}$:

$$\mathbf{F} = -\kappa \rho |\mathbf{u}| \mathbf{u},$$

with κ a positive coefficient depending on the diameter of the pipe.

We also consider an energy equation

$$\frac{\partial}{\partial t}(\rho E) + div((\rho E + p)\mathbf{u}) = div(\underline{\tau} \mathbf{u}) + div(\lambda \nabla T) + \rho \mathbf{g} \cdot \mathbf{u}, \tag{3}$$

where $E = c_v T + \frac{|\mathbf{u}|^2}{2}$ is the total energy, T the temperature, c_v the specific heat, μ the viscosity and λ the thermal conductivity.

Finally, we close the system by considering the Peng-Robinson state equation (see [10] for more details), which we simply write here as follows :

$$\rho = \rho(p, T). \tag{4}$$

The unknowns of the problem are the density ρ, the pressure p, the fluid velocity $\mathbf{u}$ and the temperature T. We impose initial conditions on ρ, $\rho \mathbf{u}$, ρE and boundary conditions which will be described later.

Due to the geometry of the domain (a cylindrical well surrounded by a reservoir), it is natural to write the above problem in 2D axisymmetric form depending only on the cylindrical coordinates (r, z), and to neglect the angular velocity u_θ.

Thus our domain merely consists of :

$$\Omega = \{(r, z)\ ;\ 0 \le r \le R,\ z \in I\}$$

where $I = [z_{\min}, z_{\max}]$, and the unknowns are supposed to be independent of the angular variable.

Let us recall the expression of some differential operators in cylindrical coordinates (r, θ, z) for a vector function $\mathbf{u}$ and a second order tensor $\underline{\tau}$, written in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ where $\mathbf{e}_z = \mathbf{e}_3$:

$$div\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\left\{\begin{array}{l} (div\underline{\tau})_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r} \\ (div\underline{\tau})_\theta = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{\theta r}}{r} \\ (div\underline{\tau})_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \end{array}\right. ,$$

$$\underline{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}) & \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\ \frac{1}{2}(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}) & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{2}(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) & \frac{1}{2}(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}) & \frac{\partial u_z}{\partial z} \end{bmatrix}.$$

Thanks to our previous hypotheses, our problem can be written in 2D axisymmetric form as follows :

$$\frac{\partial}{\partial t}(r\rho) + \nabla \cdot (r\rho\mathbf{u}) = 0 \tag{5}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}(r\rho u_r) + \nabla \cdot (ru_r\rho\mathbf{u}) + r\frac{\partial p}{\partial r} - \frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\partial}{\partial z}(r\tau_{zr}) + \tau_{\theta\theta} + r\kappa\rho|\mathbf{u}|u_r = 0 \\ \frac{\partial}{\partial t}(r\rho u_z) + \nabla \cdot (ru_z\rho\mathbf{u}) + r\frac{\partial p}{\partial z} - \frac{\partial}{\partial r}(r\tau_{rz}) - \frac{\partial}{\partial z}(r\tau_{zz}) + r\rho g + r\kappa\rho|\mathbf{u}|u_z = 0 \end{array}\right. \tag{6}$$

$$\frac{\partial}{\partial t}(r\rho E) + \nabla \cdot (r(\rho E + p)\mathbf{u}) - \nabla \cdot (r\underline{\tau}\mathbf{u}) - \nabla \cdot (r\lambda\nabla T) + r\rho g u_z = 0 \tag{7}$$

$$\rho = \rho(p, T) \tag{8}$$

where $\lambda_1 \ge \lambda(r, z) \ge \lambda_0 > 0$ a.e. in Ω and where from now on, we denote $\nabla = (\frac{\partial}{\partial r}, \frac{\partial}{\partial z})^t$ and $\mathbf{u} = (u_r, u_z)$. The components of the tensor $\underline{\tau}$ are given (cf. for instance [5] or [8]) by :

$$\tau_{rr} = 2\mu\frac{\partial u_r}{\partial r} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right), \quad \tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right),$$

$$\tau_{zz} = 2\mu\frac{\partial u_z}{\partial z} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right), \quad \tau_{\theta\theta} = 2\mu\frac{u_r}{r} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right).$$

One adds initial conditions o the previous system on ρ, $\rho\mathbf{u}$, ρE as well as boundary conditions, which will be described later.

3 Conservative pseudo 1D models

A 2D finite element approximation, based on the MINI element, has confirmed that the flow in the wellbore is essentially vertical (cf. [4]). In order to take into account the privileged direction of the flow, the particular geometry of the domain, as well as the supply at the perforations, a pseudo 1D modelling of problem (5)-(8) was proposed in [4]. Thus, on the one hand, the calculations are lightened and one can treat pipes of several thousands meters high and on the other hand, one avoids any numerical instability due to the large aspect ratio of a 2D grid.

The derivation of a simplified model is based on the following approach. One first introduces two conservative variables (the specific flux $\mathbf{G} = \rho\mathbf{u}$ and the heat flux $\mathbf{q} = \lambda\nabla T$) and a time discretization which leads to a decoupled system. Then, the 1D model is obtained as a conforming approximation of the 2D semi-discretized problem. From a discrete point of view, this method amounts to consider only one mesh in the radial direction.

Let us now discuss the time discretization of the initial problem (5), (6), (7) and (8). We obtain, at each time-step, a nonlinear system for which we apply a fixed point method with respect to ρ. More precisely, the algorithm consists in solving, for a given ρ, the following three decoupled problems :

$$div(r\mathbf{G}) = -r\frac{\rho - \rho^n}{\Delta t}, \tag{9}$$

$$\left\{ \begin{array}{l} div(r\mathbf{u}) = \dfrac{1}{\rho}(div(r\mathbf{G}) - \dfrac{r}{\rho}\mathbf{G}\cdot\nabla\rho) \\ r\rho\frac{\mathbf{u}}{\Delta t} + r\mathbf{G}\cdot\nabla\mathbf{u} + r\nabla p - div(r\underline{\tau}) + \tau_{\theta\theta}\mathbf{e}_r + r\kappa|\mathbf{G}|\mathbf{u} = r\rho\mathbf{g} + r\rho\frac{\mathbf{u}^n}{\Delta t} \end{array} \right. \tag{10}$$

$$\left\{ \begin{array}{c} rc_v\left(\rho\frac{T}{\Delta t} + \mathbf{G}\cdot\nabla T\right) - div(r\mathbf{q}) \\ = r\rho c_v\frac{T^n}{\Delta t} - \frac{1}{2}r\left(\rho\frac{|\mathbf{u}|^2 - |\mathbf{u}^n|^2}{\Delta t} + \mathbf{G}\cdot\nabla(|\mathbf{u}|^2)\right) - div(rp\mathbf{u}) + div(r\underline{\tau}\mathbf{u}) + r\mathbf{g}\cdot\mathbf{G} \\ \mathbf{q} = \lambda\nabla T. \end{array} \right. \tag{11}$$

Finally, the density is updated by means of a thermodynamic module and one loops until the convergence is achieved.

Before proceeding with the boundary conditions associated to the systems (9), (10) and (11), let us comment their derivation. The first equation of (10) translates the fact that :

$$div(r\rho\mathbf{u}) = div(r\mathbf{G}), \tag{12}$$

while in the other equations we have replaced $\rho\mathbf{u}$ by $\mathbf{G}$. In the general case, relation (12) only implies that :

$$r\rho\mathbf{u} = r\mathbf{G} + \mathbf{curl}\varphi.$$

So, at this stage, solving (10) and (11) is not equivalent to solving the initial Navier-Stokes and energy equations.

Next, in order to specify the boundary conditions, $\partial\Omega$ is divided into five parts $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Sigma$ as shown in Figure 1. It is useful to notice that on the wellbore's

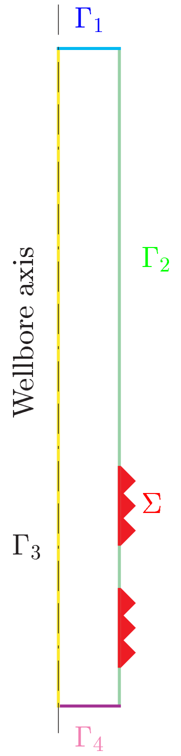


Figure 1: Decomposition of the wellbore boundary

symmetry axis Γ_3 one has $r = 0$. On the perforations Σ, the normal specific flux $\mathbf{G}\cdot\mathbf{n} = G_\Sigma$ is given while an impermeability condition $\mathbf{G}\cdot\mathbf{n} = 0$ is satisfied on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Once $\mathbf{G}$ is computed, we impose

$$\mathbf{u}\cdot\mathbf{n} = \frac{\mathbf{G}\cdot\mathbf{n}}{\rho} \quad \text{on} \quad \partial\Omega.$$

A Neumann condition $\underline{\tau}\mathbf{n}\cdot\mathbf{t} = 0$ is set on the top Γ_1, on the bottom Γ_4 and on the non-perforated lateral boundary Γ_2 while on Σ the tangential velocity is imposed; for the sake of simplicity, we take here $\mathbf{u}\cdot\mathbf{t} = 0$.

Concerning the energy equation, one imposes the temperature $T = T_\Sigma$ on the perforations Σ and the normal heat flux $\mathbf{q}\cdot\mathbf{n} = 0$ elsewhere. At the perforations, one may of course prescribe the values of T_Σ and of G_Σ obtained thanks to the reservoir code.

A relevant issue concerns the boundary condition on the top of the wellbore Γ_1. Let us notice that, even if the flowrate, denoted by $Q_1 = \frac{\mathbf{G}\cdot\mathbf{n}}{\rho}$, is known thanks to recorded data, one cannot impose it on the outflow boundary Γ_1 for the transport equation (9). Indeed, Q_1 and G_Σ are related by a compatibility condition :

$$\int_\Omega r\frac{\rho-\rho^n}{\Delta t}\mathrm{dx} + \int_{\Gamma_1} r\rho Q_1 \mathrm{d}\sigma + \int_\Sigma r G_\Sigma \mathrm{d}\sigma = 0,$$

therefore $\int_{\Gamma_1} r\rho Q_1 d\sigma$ is perfectly determined by G_Σ, at each time step.

Since beyond the perforations, the flow in the pipe is quasi-vertical, and due to the dimensions of the wellbore, we consider an explicit dependence of the unknowns on the radial coordinate. Mainly, the velocity is taken affine with respect to r and hence is determined by two functions of z living on the two lateral boundaries, while the scalar unknowns only depend on z :

$$\begin{aligned}
&\mathbf{u} = \tfrac{r}{R}\overline{\mathbf{u}}(z) + \tfrac{R-r}{R}\widehat{\mathbf{u}}(z),\\
&\mathbf{G} = \begin{pmatrix} G_r \\ G_z \end{pmatrix} = \begin{pmatrix} \frac{r}{R}\bar{G}_r(z) \\ G_z(z) \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q_r \\ q_z \end{pmatrix} = \begin{pmatrix} \frac{r}{R}\bar{q}_r(z) \\ q_z(z) \end{pmatrix},\\
&\rho = \rho(z), \quad p = p(z), \quad T = T(z).
\end{aligned} \tag{13}$$

Thanks to the boundary conditions, one further gets $u_r = \frac{r}{R}\overline{u}_r(z)$ with $\overline{u}_r = 0$ on Γ_2, and $u_z = \frac{r}{R}\overline{u}_z(z) + \frac{R-r}{R}\widehat{u_z}(z)$ with $\overline{u}_z = 0$ on Σ.

Then an integration with respect to r yields a pseudo 1D model. The previous choice (13) allows us to establish the following result, which justifies the proposed algorithm :

Lemma 1. *The relation $div(r\rho\mathbf{u}) = div(r\mathbf{G})$ together with the boundary condition $\rho\mathbf{u}\cdot\mathbf{n} = \mathbf{G}\cdot\mathbf{n}$ on $\partial\Omega$ imply that*

$$\rho\mathbf{u} = \mathbf{G} \quad in\ \Omega_2. \tag{14}$$

Proof. Firstly, one deduces :

$$r\rho\mathbf{u} = r\mathbf{G} + \mathbf{curl}\varphi$$

with φ unique up to a constant. The boundary condition implies :

$$\mathbf{curl}\varphi\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{t} = 0 \text{ on } \partial\Omega,$$

so we can assume that $\varphi = 0$ on $\partial\Omega$.

Secondly, since

$$\partial_r\varphi = r(G_z - \rho u_z),$$

it comes that :

$$\varphi = \frac{r^3}{3R}(-\rho(\overline{u}_z - \widehat{u}_z)) + \frac{r^2}{2}(G_z - \rho\widehat{u}_z) + \psi(z). \tag{15}$$

Moreover,

$$\partial_z \varphi = r(\rho u_r - G_r) = 0$$

due to the boundary conditions on the vertical axes. Therefore, by deriving the relation (15) with respect to z it comes that :

$$\partial_z(\rho\overline{u}_z - \rho\widehat{u}_z) = 0, \quad \partial_z(\rho\widehat{u}_z - G_z) = 0, \quad \psi' = 0.$$

The boundary conditions on the top and on the bottom of the wellbore finally lead to $\varphi = 0$ on Ω. ■

Computation of u_r via the boundary condition

Let us remark that since $u_r = \frac{G_\Sigma}{\rho}$ on Σ, the radial velocity can be completely determined on the whole domain Ω, without solving the corresponding momentum equation. Indeed, it comes that

$$u_r(r,z) = \frac{r}{R}\overline{u}_r(z) = \frac{r}{R}\frac{\widetilde{G}_\Sigma(z)}{\rho} \tag{16}$$

where $\widetilde{G}_\Sigma$ denotes the extension of G_Σ by zero on the interval I.

Therefore, once $\mathbf{G}$ and u_r known, it is sufficient to solve the following problem instead of problem (10) :

$$\begin{cases} \partial_z(ru_z) = -\dfrac{r}{\rho^2}\left(\rho\dfrac{\rho-\rho^n}{\Delta t} + \mathbf{G}\cdot\nabla\rho\right) - \partial_r(ru_r) \\ r(\rho\frac{u_z}{\Delta t} + \mathbf{G}\cdot\nabla u_z) + r\partial_z p - \partial_r(r\tau_{rz}) - \partial_z(r\tau_{zz}) + r\kappa|\mathbf{G}|u_z = -r\rho g + r\rho\frac{u_z^n}{\Delta t}. \end{cases} \tag{17}$$

This variant of the 1D model was developed in [4], under the assumption $\frac{G_\Sigma}{\rho} \in H_0^1(\Sigma)$.

Computation of u_r via the Navier-Stokes equation

In view of the coupling with the reservoir, we also present another variant, where u_r and u_z are both computed from the Navier-Stokes equations. For this purpose, we have to replace the boundary condition on $\mathbf{u}\cdot\mathbf{n}$ on the interface by a Neumann condition, $p - \tau\mathbf{n}\cdot\mathbf{n} = P_\Sigma$ with P_Σ given (for instance) by the reservoir code.

4 Well-posedness of the semi-discretized problem

4.1 First pseudo 1D model

We write here the first time-discretized problem under weak form. More precisely, we propose a Petrov-Galerkin formulation for the equation (9) while for problems (10) and

(11) we write mixed variational formulations. For this purpose, we introduce the following spaces :

$$\begin{aligned}
\mathbb{W} &= \{\mathbf{w} = \begin{pmatrix} \frac{r}{R}\bar{w}_r(z) \\ w_z(z) \end{pmatrix};\ \bar{w}_r \in L^2(I),\ w_z \in H^1(\Omega)\} \subset \mathbb{H}(div, \Omega), \\
V &= \{v;\ v(r,z) = \frac{r}{R}\overline{v}(z) + \frac{R-r}{R}\widehat{v}(z),\ \overline{v}, \widehat{v} \in H^1(I)\} \subset H^1(\Omega), \\
M &= \{q = q(z);\ q \in L^2(I)\} \subset L^2(\Omega),
\end{aligned}$$

as well as :

$$\begin{aligned}
\mathbb{W}^0 &= \{\mathbf{w} \in \mathbb{W};\ \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_1\}, \\
\mathbb{W}^* &= \{\mathbf{w} \in \mathbb{W};\ \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_1 \cup \Sigma),\ \mathbf{w}\cdot\mathbf{n} = G_\Sigma \text{ on } \Sigma\}, \\
\mathbb{H} &= \{\mathbf{w} \in \mathbb{W};\ \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus \Sigma\}, \\
V^0 &= \{v \in V;\ v = 0 \quad \text{on } \Sigma \cup \Gamma_4,\ v = 0 \quad \text{on } \Gamma_1\}, \\
V^* &= \{v \in V;\ v = 0 \quad \text{on } \Sigma \cup \Gamma_4,\ v = Q_1 \text{ on } \Gamma_1\}, \\
M^0 &= M \cap L^2_0(\Omega).
\end{aligned}$$

The spaces $\mathbb{W}$, V and M are respectively endowed with the natural norms of $\mathbb{H}(div, \Omega)$, $H^1(\Omega)$ and $L^2(\Omega)$. One can immediately notice, by means of a simple integration with respect to r, that the norms $\|\mathbf{w}\|_{\mathbb{W}}$, $\|v\|_V$ and $\|q\|_M$ are respectively equivalent to $\|\bar{w}_r\|_{0,I} + \|w_z\|_{0,I}$, $\|v\|_{1,I}$ and $\|q\|_{0,I}$.

We assume in what follows that Q_1 is constant on Γ_1, that $\rho_1 \geq \rho(z) \geq \rho_0 > 0$ a.e. on Σ and also that :

$$\frac{G_\Sigma}{\rho} \in H^1_0(\Sigma).$$

We next consider the problems :

$$\begin{cases} \text{Find } \mathbf{G} \in \mathbb{W}^* \\ \displaystyle\int_\Omega div(r\mathbf{G})\chi \mathrm{dx} = -\int_\Omega r\frac{\rho - \rho^n}{\Delta t}\chi \mathrm{dx} \quad \forall \chi \in M, \end{cases} \tag{18}$$

$$\begin{cases} \text{Find } \overline{u}_r \in H^1_0(\Sigma) \\ \displaystyle\int_\Sigma \rho\overline{u}_r\eta \mathrm{dz} = \int_\Sigma G_\Sigma \eta \mathrm{dz} \quad \forall \eta \in H^1_0(\Sigma), \end{cases} \tag{19}$$

$$\begin{cases} \text{Find } u_z \in V^*,\ p \in M^0 \\ a(u_z, v) \quad + \quad b(p, v) \quad = \quad f_1(v) \quad \forall v \in V^0 \\ b(q, u_z) \quad = \quad f_2(q) \quad \forall q \in M^0, \end{cases} \tag{20}$$

and also the energy balance :

$$\left\{\begin{array}{llll} \text{Find } \mathbf{q}\in\mathbb{H},\ T\in M & & & \\ A(\mathbf{q},\mathbf{w}) + B(T,\mathbf{w}) & = & F_1(\mathbf{w}) & \forall \mathbf{w}\in\mathbb{H} \\ B(S,\mathbf{q}) - C(T,S) - D(T,S) & = & F_2(S) & \forall S\in M. \end{array}\right. \tag{21}$$

The bilinear forms are defined as follows :

$$\begin{aligned} a(u,v) &= \int_\Omega r\left[\left(\frac{\rho}{\Delta t}+\kappa|\mathbf{G}|\right)u+\mathbf{G}\cdot\nabla u\right]v\mathrm{dx}+\int_\Omega \mu r\left(\partial_r u\partial_r v+\frac{4}{3}\partial_z u\partial_z v\right)\mathrm{dx},\\ b(q,v) &= -\int_\Omega q\partial_z(rv)\mathrm{dx},\\ A(\mathbf{q},\mathbf{w}) &= \int_\Omega \frac{r}{\lambda}\mathbf{q}\cdot\mathbf{w}\mathrm{dx}, \qquad B(T,\mathbf{q})=\int_\Omega T div(r\mathbf{q})\mathrm{dx},\\ C(T,S) &= \int_\Omega r\rho c_v\frac{TS}{\Delta t}\mathrm{dx}, \qquad D(T,S)=\int_\Omega rc_v\mathbf{G}\cdot\nabla TS\mathrm{dx} \end{aligned}$$

while the righthand side terms are given by :

$$\begin{aligned} f_1(v) &= -\int_\Omega r\rho g v\mathrm{dx}+\int_\Omega r\rho\frac{u_z^n}{\Delta t}v\mathrm{dx}+\frac{2}{3}\int_\Omega \mu r\left(\frac{u_r}{r}+\partial_r u_r\right)\partial_z v\mathrm{dx}-\int_\Omega \mu r\partial_z u_r\partial_r v\mathrm{dx},\\ f_2(q) &= -\int_\Omega \frac{r}{\rho^2}\left(\rho\frac{\rho-\rho^n}{\Delta t}+\mathbf{G}\cdot\nabla\rho\right)q\mathrm{dx}-\int_\Sigma R\frac{G_\Sigma}{\rho}q\mathrm{d}\sigma,\\ F_1(\mathbf{w}) &= \int_\Sigma RT_\Sigma\mathbf{w}\cdot\mathbf{n}\mathrm{d}\sigma,\\ F_2(s) &= \int_\Omega\left(r\rho c_v\frac{T^n}{\Delta t}-\frac{r}{2}\left(\rho\frac{|\mathbf{u}|^2-|\mathbf{u}^n|^2}{\Delta t}+\mathbf{G}\cdot\nabla(|\mathbf{u}|^2)\right)-div(rp\mathbf{u})\right.\\ &\qquad\left.+div(r\underline{\tau}\mathbf{u})+r\mathbf{g}\cdot\mathbf{G}\right)S\mathrm{dx}. \end{aligned}$$

Remark 1. An integration by parts in $F_2(\cdot)$ is possible but we prefer to keep the above expression, since in the discrete framework $p\mathbf{u}$ and $\underline{\tau}\mathbf{u}$ do not belong to $\mathbb{H}(div,\Omega)$.

In what follows, we are interested in establishing that problems (18), (19), (20) and (21) are well-posed.

Concerning the weak formulation (18), we begin by considering a lifting $\mathbf{G}^*=\begin{pmatrix}\frac{r}{R}\widetilde{G}_\Sigma\\ 0\end{pmatrix}$ of the boundary condition imposed on Σ, such that $\mathbf{G}-\mathbf{G}^*\in\mathbb{W}^0$. Then the following lemma ensures the existence and the uniqueness of the solution of (18), according to a generalization of the Babuška's theorem (Cf. [9]).

Theorem 1. *The next conditions are fulfilled :*

$$\exists c>0,\ \forall\chi\in M,\quad \sup_{\mathbf{w}\in\mathbb{W}^0}\frac{\int_\Omega \chi div(r\mathbf{w})\,dx}{\|\mathbf{w}\|}\geq c\|\chi\|_{0,\Omega},$$

$$\forall \mathbf{w} \in \mathbb{W}^0 \setminus \{\mathbf{0}\}\,, \quad \sup_{\chi \in M} \int_\Omega \chi div(r\mathbf{w})\, dx > 0.$$

Thus, problem (18) has a unique solution.

Proof. Let us first notice that $\mathbf{w} \in \mathbb{W}^0$ translates into $\mathbf{w} = \begin{pmatrix} 0 \\ w_z(z) \end{pmatrix}$, with $w_z \in H^1(I)$ and $w_z(z_{\min}) = 0$. It comes that $div(r\mathbf{w}) = r\partial_z w_z$ and

$$\int_\Omega div(r\mathbf{w})\chi \mathrm{dx} = \frac{R^2}{2} \int_I \chi \partial_z w_z \mathrm{dz}.$$

Associating with any $\chi \in M$ the function

$$w_z(\xi) = \int_{z_{\min}}^{\xi} \chi(z) \mathrm{dz}$$

yields that

$$\int_\Omega div(r\mathbf{w})\chi \mathrm{dx} = \frac{R}{2} \|\chi\|^2_{0,\Omega} \quad \text{and} \quad \|\mathbf{w}\| \le c\|\chi\|_{0,\Omega}.$$

Hence the first inf-sup condition holds.

Concerning the second condition, with any $\mathbf{w} \in \mathbb{W}^0 \setminus \{\mathbf{0}\}$ we associate $\chi = \partial_z w_z$, such that we have :

$$\int_\Omega \chi div(r\mathbf{w})\mathrm{dx} = \frac{R^2}{2} |w|^2_{z1,I} > 0$$

which ends the proof, thanks to the Friedrichs-Poincaré inequality. ∎

Lemma 2. *Problem (19) has a unique solution.*

Proof. The existence of a solution $\overline{u}_r = \frac{G_z}{\rho}$ is obvious, since $\frac{G_z}{\rho}$ belongs to $H^1_0(\Sigma)$ by hypothesis. The uniqueness is also immediate : indeed, by considering the homogeneous problem and by taking the test-function $\eta = \overline{u}_r$, it comes that $\overline{u}_r = 0$ thanks to the positivity of ρ. ∎

So, knowing $\overline{u}_r$ on Σ one can now determine u_r on Ω by means of (16).

Let us next consider the velocity-pressure formulation (20) and establish :

Theorem 2. *Let $\mathbf{G} \in \mathbb{L}^\infty(\Omega)$. Problem (20) has a unique solution, for Δt sufficiently small.*

Proof. As usually when dealing with mixed formulations, we apply the Babuška-Brezzi theorem (Cf. [3]).

We first notice that $a(\cdot,\cdot)$ is well defined for $\mathbf{G} \in \mathbb{L}^\infty(\Omega)$ and then we write that :

$$a(v,v) = \int_\Omega r\left[\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}|\right) v + \mathbf{G}\cdot\nabla v\right] v\mathrm{dx} + \int_\Omega \mu r\left((\partial_r v)^2 + \frac{4}{3}(\partial_z v)^2\right) \mathrm{dx}.$$

We bound the convective term by means of Young's inequality :

$$\int_\Omega r\mathbf{G}\cdot\nabla v v \mathrm{dx} \leq \varepsilon\int_\Omega \mu r|\nabla v|^2\mathrm{dx} + \frac{1}{4\mu\varepsilon}\int_\Omega r|\mathbf{G}|^2 v^2\mathrm{dx}, \quad \forall\varepsilon>0.$$

This leads to :

$$a(v,v) > (1-\varepsilon)\int_\Omega r\mu|\nabla v|^2\mathrm{dx} + \int_\Omega r\left(\frac{\rho}{\Delta t}+\kappa|\mathbf{G}| - \frac{|\mathbf{G}|^2}{4\mu\varepsilon}\right)v^2\mathrm{dx},$$

so $a(\cdot,\cdot)$ is coercive on the whole space V^0 whenever one can choose $\varepsilon < 1$ such that $\frac{\rho}{\Delta t}+\kappa|\mathbf{G}| - \frac{|\mathbf{G}|^2}{4\mu\varepsilon} \geq 0$ a.e. in Ω. Hence, the condition

$$\frac{1}{\Delta t} \geq \frac{|\mathbf{G}|^2 - 4\mu\kappa|\mathbf{G}|}{4\mu\rho} \quad \text{a.e. in } \Omega$$

yields the V^0-coercivity of $a(\cdot,\cdot)$.

One still has to check the inf-sup condition for $b(\cdot,\cdot)$. For this purpose, we apply the Fortin's trick, that is with any $q\in M^0$ we associate a function $v\in V^0$ satisfying :

$$b(q,v)\geq c\|q\|^2_{0,\Omega} \quad \text{and} \quad \|v\|_{1,\Omega}\leq c'\|q\|_{0,\Omega}.$$

A simple computation gives that

$$b(q,v) = -\frac{R^2}{6}\int_I q(2\partial_z\overline{v}+\partial_z\widehat{v})\mathrm{dz},$$

for any $v=\frac{r}{R}\overline{v}(z)+\frac{R-r}{R}\widehat{v}(z)$ and any $q\in M^0$.

By taking $\overline{v}=0$ and $\widehat{v}(\zeta) = -\int_{z_{\min}}^{\zeta} q(z)\mathrm{dz}$, one gets that the previous conditions are fulfilled.

Then the Babuška-Brezzi theorem allows us to conclude. ■

Finally, we study the well-posedness of the energy equation (21).

We begin by neglecting the convective term and we establish :

Theorem 3. *The variational formulation (21) with $D(\cdot,\cdot)=0$ has a unique solution.*

Proof. We apply an extension of the Babuška-Brezzi theorem (Cf. [3]) for a mixed formulation of operator

$$\begin{bmatrix} A & B \\ -B^T & C \end{bmatrix},$$

with both $A(\cdot,\cdot)$ and $C(\cdot,\cdot)$ symmetric.

We first establish, by making use of the Fortin's trick, the inf-sup condition for $B(\cdot,\cdot)$. One has :

$$B(S,\mathbf{w}) = \int_\Omega S div(r\mathbf{w})\mathrm{dx} = \frac{R^2}{2}\int_I S\left(\partial_z w_z + \frac{2}{R}\overline{w}_r\right)\mathrm{dz}$$

where by construction $\mathbf{w} = \begin{pmatrix} \frac{r}{R}\overline{w}_r(z) \\ w_z(z) \end{pmatrix}$. Then, with any $S \in M$ we associate the function $\mathbf{w}$ defined by :

$$\overline{w}_r = 0 \text{ on } \Gamma_2, \ \overline{w}_r = \frac{R}{2|\Sigma|}\int_I S\mathrm{dz} \text{ on } \Sigma,$$

respectively by

$$w_z(\xi) = \int_{z_{\min}}^{\xi} \left(S - \frac{2}{R}\overline{w}_r\right)\mathrm{dz}, \ \forall \xi \in I.$$

This choice ensures that $w_z(z_{\min}) = w_z(z_{\max}) = 0$. So, since $\mathbf{w}\cdot\mathbf{n} = 0$ on $\partial\Omega\setminus\Sigma$, it comes that $\mathbf{w} \in \mathbb{H}$. On the other hand,

$$\partial_z w_z + \frac{2}{R}\overline{w_r} = S \text{ on } I, \quad \|\mathbf{w}\|_{\mathtt{W}} \le c\|S\|_{0,\Omega},$$

which yields the desired inf-sup condition.

Obviously, the bilinear form $A(\cdot,\cdot)$ is positive on the whole space $\mathbb{H}$:

$$A(\mathbf{w},\mathbf{w}) \ge c\left(\|\overline{w}_r\|^2_{0,\Omega} + \|w_z\|^2_{0,\Omega}\right).$$

Moreover, $A(\cdot,\cdot)$ is $\mathbb{H}(div,\Omega)$-elliptic on

$$KerB = \left\{\mathbf{w} \in \mathbb{H}; \ \partial_z w_z + \frac{2}{R}\overline{w}_r = 0\right\},$$

since the elements of $KerB$ are characterized by

$$div\mathbf{w} = \frac{1}{R}\overline{w}_r + \partial_z w_z = -\frac{1}{R}\overline{w}_r.$$

Together with the positivity of $C(\cdot,\cdot)$, the previous properties imply the existence and the uniqueness of the solution to the problem (21). ∎

So, one can now define a linear and continuous operator $\mathcal{L}$, associating with any data $(T_\Sigma, T^n, \mathbf{u}^n, \rho, \mathbf{G}, \mathbf{u}, p)$ the unique solution $(\overline{q}_r, q_z, T) \in L^2(I)\times H^1(I)\times L^2(I)$ to (21).

Let us now return to the study of the global problem (21) and use a similar approach to the one employed for the reservoir model (see [1] for details).

We establish :

Theorem 4. *Let the boundary data T_Σ belong to $H^\delta(\Sigma)$, for a given $\delta \in]0,1]$, and let $\nabla\lambda \in \mathbb{L}^\infty(\Omega)$. Then problem (21) is well posed, for Δt sufficiently small.*

Proof. First of all, let us introduce the space

$$Y = \left\{S \in H^1(I); \ S_{|\Sigma} \in H^{1+\delta}(\Sigma), \ S_{|(I\setminus\Sigma)} \in H^2(I\setminus\Sigma)\right\}$$

and notice that the solution of the variational problem (21) with $D(\cdot,\cdot) = 0$ satisfies $(\overline{q}_r)_{|\Sigma} \in H^\delta(\Sigma)$, $q_z \in H^1(I)$ and $T \in Y$. Indeed, interpreting the first variational equation in the sense of distributions leads to the relations :

$$\partial_z T = \frac{2\overline{q}_r + 3q_z}{3\lambda} \text{ on } I,$$

$$\frac{3\overline{q}_r + 4q_z}{12\lambda} = \frac{1}{R}(T_\Sigma - T) \text{ on } \Sigma.$$

Taking into account that $\overline{q_r} \in L^2(I)$ and $q_z \in H^1(I)$ by construction, it comes on the one hand that the bilinear form $D(\cdot,\cdot)$ is well defined, since $T \in H^1(I)$. On the other hand, the hypothesis $T_\Sigma \in H^\delta(\Sigma)$ gives that $\overline{q}_r \in H^\delta(\Sigma)$. The above mentioned regularity of T comes by virtue of the first equality together with $\overline{q}_r = 0$ on $I \setminus \Sigma$.

In conclusion, for $T_\Sigma \in H^\delta(\Sigma)$ the operator $\mathcal{L}$ associated with the variational formulation (21) takes values in $H^\delta(\Sigma) \times H^1(I) \times Y$ and the linear operator $\mathcal{D}$ associated with the bilinear form $D(\cdot,\cdot)$, defined by $\mathcal{D}(T) = rc_v\mathbf{G}\cdot\nabla T$, is compact from Y into $L^2(I)$. This last assertion comes by applying the Rellich's theorem.

Finally, the operator of the global problem (with convection) can be seen as a compact perturbation of the identity. So, it is now sufficient to prove that the homogeneous problem has only the trivial solution in order to obtain, by means of the Fredholm's alternative, the well-posedness of the non homogeneous problem (21).

For this purpose, we take $(\mathbf{w}, S) = (\mathbf{q}, T)$ as test-function in the homogeneous problem (21) and we get that :

$$\int_\Omega \frac{r}{\lambda}\mathbf{q}\cdot\mathbf{q}\,\mathrm{dx} + \int_\Omega rc_v\rho\frac{T^2}{\Delta t}\mathrm{dx} + \int_\Omega rc_v\mathbf{G}\cdot\nabla T T\,\mathrm{dx} = 0.$$

By replacing ∇T and by integrating with respect to r, it comes that :

$$\int_I \left[\frac{1}{\lambda}\left(\frac{1}{4}\overline{q}_r^2 + \frac{1}{2}q_z^2 + \frac{2}{3}\overline{q}_r q_z\right) + \frac{\rho c_v}{2\Delta t}T^2 + \frac{G_z c_v}{6\lambda}\left(2\overline{q}_r + 3q_z\right)T\right]\mathrm{dz} = 0.$$

Since the expression under the integral sign is a positive definite quadratic form for Δt sufficiently small, one deduces that $(\mathbf{q}, T) = (\mathbf{0}, 0)$.

Therefore, we obtain the announced result. ■

4.2 Second wellbore model

We are now interested in the second pseudo 1D model and therefore we focus on the computation of $(\overline{u}_r, u_z, p)$, which is now achieved by solving :

$$\begin{cases} div(r\mathbf{u}) = \frac{1}{\rho}(div(r\mathbf{G}) - r\mathbf{G}\cdot\nabla\rho) \\ r\rho\frac{\mathbf{u}}{\Delta t} + r\mathbf{G}\cdot\nabla\mathbf{u} + r\nabla p - div(r\underline{\tau}) + \tau_{\theta\theta}\mathbf{e}_r + r\kappa|\mathbf{G}|\mathbf{u} = r\rho\mathbf{g} + r\rho\frac{\mathbf{u}^n}{\Delta t} \end{cases}$$

instead of (16) and (17). To the previous equations, we add a new boundary condition on the perforations :

$$p - \tau \mathbf{n} \cdot \mathbf{n} = P_\Sigma \quad \text{on } \Sigma$$

but we no longer impose $\mathbf{u} \cdot \mathbf{n} = \frac{1}{\rho}\mathbf{G} \cdot \mathbf{n}$ on Σ.

We begin by introducing the spaces :

$$\begin{aligned}
\mathbb{V} &= \{\mathbf{v} = \begin{pmatrix} \frac{r}{R}\bar{v}_r(z) \\ v_z(r,z) \end{pmatrix};\ v_z = \frac{r}{R}\overline{v}_z(z) + \frac{R-r}{R}\widehat{v_z}(z),\ \bar{v}_r, \overline{v}_z, \widehat{v_z} \in H^1(I)\} \subset (H^1(\Omega_2))^2, \\
\mathbb{V}^0 &= \{\mathbf{v} \in \mathbb{V};\ \overline{v}_r = 0 \quad \text{on } \Gamma_2,\ \overline{v}_z = 0 \quad \text{on } \Sigma,\ v_z = 0 \quad \text{on } \Gamma_1 \cup \Gamma_4\}, \\
\mathbb{V}^* &= \{\mathbf{v} \in \mathbb{V};\ \overline{v}_r = 0 \quad \text{on } \Gamma_2,\ \overline{v}_z = 0 \quad \text{on } \Sigma,\ v_z = 0 \quad \text{on } \Gamma_4,\ v_z = Q_1 \text{ on } \Gamma_1\}
\end{aligned}$$

and we next write the associated mixed variational formulation :

$$\begin{cases}
\text{Find } \mathbf{u} \in \mathbb{V}^*,\ p \in M \\
\overline{a}(\mathbf{u},\mathbf{v}) \quad + \quad \overline{b}(p,\mathbf{v}) \quad = \quad \overline{f_1}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{V}^0 \\
\overline{b}(q,\mathbf{u}) \quad = \quad \overline{f_2}(q) \quad \forall q \in M,
\end{cases} \tag{22}$$

Here above, the bilinear, respectively linear forms are defined by :

$$\begin{aligned}
\overline{b}(q,\mathbf{v}) &= -\int_\Omega q\,div(r\mathbf{v})\mathrm{dx} = -\int_\Omega rq(\partial_z v_z + \frac{2}{R}\overline{v}_r)\mathrm{dx}, \\
\overline{a}(\mathbf{u},\mathbf{v}) &= \int_\Omega r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}|\right)(\frac{r^2}{R^2}\overline{u}_r\overline{v}_r + u_z v_z)\mathrm{dx} + \int_\Omega r\mathbf{G} \cdot (v_r \nabla u_r + v_z \nabla u_z)\mathrm{dx} \\
&\quad + \int_\Omega \mu r(\partial_r u_z + \partial_z u_r)(\partial_r v_z + \partial_z v_r)\mathrm{dx} + \frac{4}{3}\int_\Omega \mu r(\partial_z u_z - \frac{1}{R}\overline{u}_r)(\partial_z v_z - \frac{1}{R}\overline{v}_r)\mathrm{dx}, \\
\overline{f_1}(\mathbf{v}) &= \int_\Omega r\rho\mathbf{g} \cdot \mathbf{v}\mathrm{dx} + \int_\Omega r\rho\frac{|\mathbf{u}^n|}{\Delta t} \cdot \mathbf{v}\mathrm{dx} - \int_\Sigma RP_\Sigma \mathbf{v} \cdot \mathbf{n}\mathrm{d}\sigma, \\
\overline{f_2}(q) &= \int_\Omega \frac{r}{\rho^2}\left(\rho\frac{\rho - \rho^n}{\Delta t} + \mathbf{G} \cdot \nabla\rho\right) q\mathrm{dx}
\end{aligned}$$

where $u_r = \frac{r}{R}\overline{u}_r$ and $v_r = \frac{r}{R}\overline{v}_r$. Let us note that due to the imposed normal forces on Σ, the pressure is now looked for in M instead of M^0.

One can then establish :

Theorem 5. *Problem (22) is well-posed, for Δt sufficiently small.*

Proof. We apply once more the Babuška-Brezzi theorem for mixed formulations and we begin by establishing the inf-sup condition for $\overline{b}(\cdot,\cdot)$. After integration with respect to r, one obtains

$$\overline{b}(q,\mathbf{v}) = -\frac{R^2}{6}\int_I q(2\partial_z \overline{v} + \partial_z \hat{v} + \frac{6}{R}\overline{v}_r)\mathrm{dz}.$$

We make use of the Fortin's argument and with any $q \in L^2(I)$, we associate $\mathbf{v} \in \mathbb{V}^0$ as follows. We take $\overline{v} = 0$, $\overline{v}_r \in H_0^1(\Sigma)$ such that

$$\int_\Sigma \overline{v}_r \mathrm{dz} \;=\; -\frac{R}{6}\int_I q\mathrm{dz} \quad \text{and} \quad \|\overline{v}_r\|_{1,\Sigma} \le c\,\|P(q)\|_{0,I}$$

and finally, $\widehat{v}(\zeta) = -\int_{z_{\min}}^{\zeta}(q + \frac{6}{R}\overline{v}_r)\mathrm{dz}$. Here above, the notation $P(q)$ stands for $\frac{1}{|I|}\int_I q\mathrm{dz}$. Obviously, $\widehat{v} \in H_0^1(I)$ since by construction of $\overline{v}_r$, one has $q + \frac{6}{R}\overline{v}_r \in L_0^2(I)$. Then one gets :

$$\overline{b}(q,\mathbf{v}) = \frac{R^2}{6}\,\|q\|_{0,I}^2 \quad \text{and} \quad \|\mathbf{v}\|_{\mathfrak{V}} \le c_1\,\|q\|_{0,I} + c_2\,\|P(q)\|_{0,I} \le c\,\|q\|_{0,I}\,,$$

so the desired inf-sup condition holds.

Let us now look at the coercivity of $\overline{a}(\cdot,\cdot)$ on the kernel of $\overline{b}(\cdot,\cdot)$, given by :

$$Ker\,b = \left\{\mathbf{v} \in \mathbb{V}^0;\; 2\partial_z\overline{v} + \partial_z\hat{v} + \frac{6}{R}\overline{v}_r = 0 \;\; on\; I\right\}.$$

By means of a simple calculation, based on an integration with respect to r, one can notice that the norms $\|\partial_z v_z\|_{0,\Omega}$, $\|\partial_r v_z\|_{0,\Omega}$ and $\|\partial_z v_r\|_{0,\Omega}$ are respectively equivalent to $\left(\|\partial_z\overline{v}\|_{0,I}^2 + \|\partial_z\hat{v}\|_{0,I}^2\right)^{1/2}$, $\|\overline{v} - \hat{v}\|_{0,I}$ and $\|\partial_z\overline{v}_r\|_{0,I}$. Therefore, for any $\mathbf{v} \in Ker\,b$ it comes by replacing $\overline{v}_r$ that :

$$\begin{aligned}\int_\Omega r(\partial_z v_z - \frac{1}{R}\overline{v}_r)^2\mathrm{dx} &= \frac{R^2}{72}\int_I \left(38(\partial_z\overline{v})^2 + 11(\partial_z\hat{v})^2 + 32\partial_z\overline{v}\,\partial_z\hat{v}\right)\mathrm{dz}\\ &\ge c\,\|\partial_z v_z\|_{0,\Omega}^2\end{aligned}$$

while

$$\begin{aligned}\int_\Omega r(\partial_r v_z + \partial_z v_r)^2\mathrm{dx} &= \frac{1}{R}\int_I \left(\frac{R^2}{2}(\overline{v} - \hat{v})^2 + \frac{R^4}{4}(\partial_z\overline{v}_r)^2 + \frac{2R^3}{3}\partial_z\overline{v}_r(\overline{v} - \hat{v})\right)\mathrm{dz}\\ &\ge c(\|\partial_z v_r\|_{0,\Omega}^2 + \|\partial_r v_z\|_{0,\Omega}^2).\end{aligned}$$

We next bound the convective term similarly to Theorem 2. Finally, we deduce for a sufficiently small time-step that :

$$\overline{a}(\mathbf{v},\mathbf{v}) \ge c(\|v_z\|_{1,\Omega}^2 + \|v_r\|_{1,\Omega}^2), \quad \forall\mathbf{v} \in Ker\,b,$$

so the Babuška-Brezzi theorem allows us to conclude that problem (22) has a unique solution. ■

5 Finite element approximation

5.1 First model

For the space discretization of problems (18), (19), (20) and (21), we employ a specific 2D grid.

More precisely, we consider only one cell in the radial direction and we use a regular mesh $\mathcal{T}_h$ in the z direction. We put $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$ with K rectangle of width R. We denote by $\mathcal{E}_0$ the set of internal edges and by $\mathcal{E}_h$ the set of all edges. As usually, h_K represents the diameter of K while h_e is the length of the edge e.

We employ classical conforming finite element spaces which are compatible with the dependence in r prescribed in (13).

The pressure and the temperature will be approximated by piecewise constant finite elements. It goes the same way for the density thanks to its dependency on these two previous variables.

We use a Petrov-Galerkin method for the discretization of problem (18) : $\mathbf{G}$ is approximated by conservative Raviart-Thomas elements of lowest order RT_0, while the test-functions λ are taken piecewise constant. We recall that the degrees of freedom of the RT_0 element are the normal fluxes $\mathbf{G}\cdot\mathbf{n}$ on each edge, and these quantities are constant.

The discrete version of (18) is then written as follows :

$$\left\{ \begin{array}{l} \text{Find } \mathbf{G}_h \in \mathbb{W}_h^* \\ \displaystyle\int_\Omega div(r\mathbf{G}_h)\chi \mathrm{dx} = \int_\Omega r\frac{\rho_h - \rho_h^n}{\Delta t}\chi \mathrm{dx}, \quad \forall \chi \in M_h, \end{array} \right. \tag{23}$$

with :

$$\mathbb{W}_h = \left\{ \mathbf{G} \in \mathbb{H}(div,\Omega)\,;\ \mathbf{G}_{|K} \in RT_0,\ \forall K \in \mathcal{T}_h \right\} \subset \mathbb{W},$$

$$\mathbb{W}_h^* = \left\{ \mathbf{G} \in \mathbb{W}_h\,;\ \mathbf{G}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Sigma),\ \mathbf{G}\cdot\mathbf{n} = \mathcal{I}_h(G_\Sigma) \text{ on } \Sigma \right\},$$

$$M_h = \left\{ q \in L^2(\Omega)\,;\ q_{|K} \in Q_0,\ \forall K \in \mathcal{T}_h \right\} \subset M,$$

and where $\mathcal{I}_h(G_\Sigma)$ is a piecewise constant approximation of G_Σ on $\mathcal{E}_\Sigma$.

Remark 2. The above choice of 2D finite dimensional space for $\mathbf{G}$ is equivalent to taking the 1D functions $\overline{G}_r$ and G_z, respectively piecewise constant and continuous piecewise linear on I.

One has :

Lemma 3. *The discrete problem (23) has a unique solution.*

Proof. The well-posedness of (23) is obvious since one can successively compute $\mathbf{G}_h$ on every rectangle $K \in \mathcal{T}_h$ (from the bottom to the top of the domain Ω) by means of the relation :

$$\int_{\partial K} r\mathbf{G}_h \cdot \mathbf{n} \mathrm{d}\sigma = \int_K r\frac{\rho_h - \rho_h^n}{\Delta t}\mathrm{dx}.$$

■

We next consider the following discrete version of problem (19) :

$$\left\{ \begin{array}{l} \text{Find } \overline{u}_{rh} \in L_h \\ \int_\Sigma \rho_h \overline{u}_{rh} \eta \mathrm{d}\sigma = \int_\Sigma G_\Sigma \eta \mathrm{d}\sigma \quad \forall \eta \in L_h \end{array} \right. \tag{24}$$

where

$$L_h = \left\{ \eta \in H_0^1(\Sigma) \ ; \ \eta \in P_1(e), \forall e \in \mathcal{E}_\Sigma \right\}.$$

The following statement is then obvious :

Lemma 4. *Assume that* $\rho_1 \geq \rho_h(z) \geq \rho_0 > 0$ *a.e. on* Σ*, uniformly with respect to* h*. Then the discrete problem (24) has a unique solution.*

For the approximation of the velocity-pressure formulation (20), we introduce the spaces :

$$V_h = \left\{ v \in H^1(\Omega) \, ; \ v|_K \in Q_1, \ \forall K \in \mathcal{T}_h \right\} \subset V,$$

$$V_h^0 = V_h \cap V^0, \quad V_h^* = V_h \cap V^*, \quad M_h^0 = M_h \cap L_0^2(\Omega).$$

It follows that taking $v \in V_h$ is equivalent to imposing $\widehat{v}, \overline{v}$ continuous and piecewise linear on I.

The convective term is treated by means of the following upwinding scheme, for any ϕ and v belonging to V_h^0 :

$$\int_K r\mathbf{G}_h \cdot \nabla \phi v \mathrm{dx} = \sum_{e \in \partial K^-} (\phi^* - P_K(\phi)) \int_e r\mathbf{G}_h \cdot \mathbf{n} v \mathrm{d}\sigma, \ \forall K \in \mathcal{T}_h$$

where

$$\partial K^- = \{ e \in \partial K \ ; \ \mathbf{G}_h \cdot \mathbf{n} < 0 \}$$

is the set of incoming edges of K, with $\mathbf{n}$ the outward unit normal vector to the edge. We denote the constant projection of ϕ on K by $P_K(\phi) = \frac{1}{|K|} \int_K \phi \mathrm{dx}$ and we put

$$\phi^* = P_{K'}(\phi) \text{ with } K' \in \mathcal{T}_h \text{ such that } K' \cap K = \{e\}.$$

If the edge e is situated on the boundary of Ω, then we agree to take $\phi^* = 0$ if $e \subset (\Gamma_1 \cup \Gamma_4 \cup \Sigma)$, respectively $\phi^* = P_K(\phi)$ if $e \subset (\Gamma_2 \cup \Gamma_3)$.

It is useful to introduce the bilinear form $d_h(\cdot, \cdot)$:

$$d_h(\phi, v) = \sum_{K \in \mathcal{T}_h} \sum_{e \in \partial K^-} (\phi^* - P_K(\phi)) \int_e r\mathbf{G}_h \cdot \mathbf{n} v \mathrm{d}\sigma,$$

which satisfies :

Lemma 5. *Let* $h_{min} = \min_{K \in \mathcal{T}_h} h_K$*. There exists* $c > 0$ *independent of* h *such that:*

$$|d_h(\phi, v)| \ \leq \ \frac{c}{h_{min}} \|\mathbf{G}_h\|_{0,\Omega} \|\phi\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \forall \phi, v \in V_h^0.$$

Proof. Thanks to the Cauchy-Schwarz inequality, one has :

$$\begin{aligned}\left|(\phi^* - P_K(\phi))\int_e r\mathbf{G}_h\cdot\mathbf{n}v\mathrm{d}\sigma\right| &= \frac{1}{\sqrt{h_e}}\|\mathbf{G}_h\cdot\mathbf{n}\|_{0,e}\left|\int_e r\left((\phi^*-\phi)+(\phi-P_K(\phi))\right)v\mathrm{d}\sigma\right| \\ &\le \frac{c}{\sqrt{h_e}}\|\mathbf{G}_h\cdot\mathbf{n}\|_{0,e}\left(\|\phi-\phi^*\|_{0,e}+\|\phi-P_K(\phi)\|_{0,e}\right)\|v\|_{0,e},\end{aligned}$$

where the cells K and K' share the edge e.

A classical passage to the reference finite element yields the estimate :

$$\frac{1}{\sqrt{h_e}}\|v\|_{0,e} \le c\left(\frac{1}{h_K}\|v\|_{0,K} + |v|_{1,K}\right), \quad \forall v \in H^1(K),\ \forall K \in \mathcal{T}_h,\ \forall e \subset \partial K.$$

Moreover, for any v belonging to a finite dimensional space, the following inverse inequality holds :

$$|v|_{1,K} \le \frac{c}{h_K}\|v\|_{0,K}.$$

Therefore, one gets that

$$\frac{1}{\sqrt{h_e}}\|v\|_{0,e} \le \frac{c}{h_K}\|v\|_{0,K}$$

while the Bramble-Hilbert lemma gives :

$$\frac{1}{\sqrt{h_e}}\|\phi - P_K(\phi)\|_{0,e} \le c\,|\phi|_{1,K}\,.$$

The term $\|\phi-\phi^*\|_{0,e}$ is bounded similarly to $\|\phi - P_K(\phi)\|_{0,e}$. Indeed, the only difference appears when $\phi^* = 0$, that is when $e \subset (\Gamma_1 \cup \Gamma_4 \cup \Sigma)$. In this case one employs the Friedrichs-Poincaré inequality on K and a passage to the reference finite element to conclude that :

$$\frac{1}{\sqrt{h_e}}\|\phi\|_{0,e} \le c\,|\phi|_{1,K}\,.$$

Finally, using that for $\mathbf{G}_h \in \mathbb{W}_h$ one has (Cf. [11]) :

$$\left(\sum_{e\in\mathcal{E}_h}\sqrt{h_e}\|\mathbf{G}_h\cdot\mathbf{n}\|_{0,e}\right) \le c\|\mathbf{G}_h\|_{0,\Omega},$$

it comes that :

$$
\begin{aligned}
&|d_h(\phi,v)| \\
&\leq c\left(\sum_{e\in\mathcal{E}_h} h_e\|\mathbf{G}_h\cdot\mathbf{n}\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\|v\|_{0,e}^2\right)^{1/2} \\
&\qquad\left(\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K^-}\frac{1}{h_e}\|\phi-P_K(\phi)\|_{0,e}^2+\frac{1}{h_e}\|\phi-\phi^*\|_{0,e}^2\right)^{1/2} \\
&\leq c\|\mathbf{G}_h\|_{0,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}|v|_{0,K}^2\Big)^{1/2}|\phi|_{1,\Omega} \\
&\leq \frac{c}{h_{min}}\|\mathbf{G}_h\|_{0,\Omega}\|v\|_{0,\Omega}|\phi|_{1,\Omega}.
\end{aligned}
$$

We conclude thanks to the Friedrichs-Poincaré inequality on V_h^0. ■

We are now able to consider the discrete version of problem (20) :

$$
\left\{
\begin{array}{l}
\text{Find } u_{zh}\in V_h^*,\ p_h\in M_h^0 \\
a_h(u_{zh},v)+b(p_h,v)=f_{1h}(v)+g_h(v)\quad \forall v\in V_h^0 \\
b(q,u_{zh})=f_{2h}(q)\quad \forall q\in M_h^0,
\end{array}
\right.
\tag{25}
$$

where we have put for all $\phi, v\in V_h^0$:

$$
a_h(\phi,v)=\int_\Omega r\left(\frac{\rho}{\Delta t}+\kappa|\mathbf{G}_h|\right)\phi v\mathrm{dx}+\int_\Omega \mu r\left(\partial_r\phi\partial_r v+\frac{4}{3}\partial_z\phi\partial_z v\right)\mathrm{dx}+d_h(\phi,v).
$$

Concerning the righthand side terms, $f_{1h}(\cdot)$ and $f_{2h}(\cdot)$ are obtained from $f_1(\cdot)$ and $f_2(\cdot)$, by replacing the quantities $\mathbf{G}$, u_r, ρ by their finite element approximations $\mathbf{G}_h$, u_{rh}, ρ_h respectively. The linear form $g_h(\cdot)$ takes into account the non homogeneous boundary conditions in the treatement of the convective term, more precisely we put :

$$
g_h(v)=-Q_1\int_{\partial K^-\cap\Gamma_1} r\mathbf{G}_h\cdot\mathbf{n}v\mathrm{d}\sigma.
$$

Theorem 6. *The mixed problem (25) is well-posed for Δt sufficiently small.*

Proof. We apply once again the Babuška-Brezzi theorem. The proof of the inf-sup condition for $b(\cdot,\cdot)$ is similar to the one given at the continuous level in Theorem 2.

Indeed, for $q\in M_h^0$ it comes that the function :

$$
\widehat{v}(\zeta)=-\int_{z_{\min}}^{\zeta} q(z)\mathrm{dz},\quad \forall\zeta\in I
$$

is piecewise linear and continuous. So, the function $v = \frac{R-r}{R}\widehat{v}$ associated with q belongs to V_h^0.

Next, we make use of Lemma 5 in order to deduce that $a_h(\cdot,\cdot)$ is coercive on the whole space V_h^0, for

$$\frac{1}{\Delta t} \geq \frac{c\|\mathbf{G}_h\|_{0,\Omega}}{\rho h_{min}} - \frac{\kappa|\mathbf{G}_h|}{\rho} \quad \text{a.e. in } \Omega.$$

So, the statement is established. ■

For the approximation of the energy equation, we employ the spaces M_h and

$$\mathbb{H}_h = \mathbb{W}_h \cap \mathbb{H}.$$

We finally consider the following discrete version of (21), where the righthand side term $F_{2h}(\cdot)$ takes into account the numerical approximation of $\mathbf{G}$, $\mathbf{u}$, ρ and p :

$$\begin{cases} \text{Find } \mathbf{q}_h \in \mathbb{H}_h, \ T_h \in M_h \\ A(\mathbf{q}_h,\mathbf{w}) \ + \ B(T_h,\mathbf{w}) \ = \ F_1(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbb{H}_h \\ B(S,\mathbf{q}_h) \ - \ (C+D_h)(T_h,S) \ = \ F_{2h}(S) \quad \forall S \in M_h. \end{cases} \tag{26}$$

The bilinear form $D_h(\cdot,\cdot)$ takes into account the convective term by means of the following upwinding scheme :

$$D_h(T,S) = \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K^-} (c_v)_{|K} \left(\int_e r\mathbf{G}_h\cdot\mathbf{n}\,\mathrm{d}\sigma\right) \delta(T) S_{|K}$$

where we put, on a given edge e :

$$\delta(T) = \begin{cases} -T_{|K}, & \text{if } e \subset \Sigma \\ 0, & \text{if } e \subset \partial\Omega\setminus\Sigma \\ T^* - T_{|K}, & \text{if } e \in \mathcal{E}_0. \end{cases}$$

We recall some properties of the latter form $D_h(\cdot,\cdot)$, which were established in [1] :

Lemma 6. *There exists a positive constant c independent of h such that :*

$$\forall T,S \in M_h, \quad |D_h(T,S)| \leq \frac{c}{h_{min}^2}\|\mathbf{G}_h\|_{0,\Omega}\|T\|_{0,\Omega}\|S\|_{0,\Omega}.$$

One also has :

$$D_h(T,T) \geq 0, \quad \forall T \in M_h.$$

Then one can show the next result :

Theorem 7. *Problem (26) has a unique solution.*

Proof. Let us first notice that $(C+D_h)(\cdot,\cdot)$ is not symmetric and moreover, its continuity constant depends on both h and Δt, since :

$$(C+D_h)(T,S) \leq \left(\frac{c_1\rho}{\Delta t} + \frac{c_2}{h_{\min}^2}\|\mathbf{G}_h\|_{0,\Omega}\right) \|T\|_{0,\Omega}\|S\|_{0,\Omega}.$$

In order to establish the invertibility of the global operator

$$\mathcal{A}_h = \left[\begin{array}{cc} A & B \\ B^T & -(C+D_h) \end{array}\right]$$

of (26), we employ a result which can be found in Roberts and Thomas [11]. It ensures the well-posedness of problem (26), if the following conditions are satisfied : $(C+D_h)(\cdot,\cdot)$ is positive, $A(\cdot,\cdot)$ is symmetric, positive and coercive on Ker_hB, and $B(\cdot,\cdot)$ satisfies an inf-sup condition. Moreover, if the coercivity of $A(\cdot,\cdot)$ and the inf-sup condition for $B(\cdot,\cdot)$ hold uniformly with respect to the discretization parameter, then one also gets that the norm of the inverse operator $\mathcal{A}_h^{-1}$ is independent of h.

So, let us now check the previous conditions.

The positivity of $(C+D_h)(\cdot,\cdot)$ is obvious according to Lemma 6, as well as the positivity and the symmetry of $A(\cdot,\cdot)$.

By construction, one has :

$$\mathbf{w} = \left(\begin{array}{c} \frac{r}{R}\overline{w}_r(z) \\ w_z(z) \end{array}\right) \in \mathbb{H}_h,$$

that is $\overline{w}_r$ is piecewise constant on I, while w_z is continuous and piecewise linear on I. So, one can characterize the discrete kernel of $B(\cdot,\cdot)$ as follows :

$$Ker_hB = \left\{\mathbf{w} \in \mathbb{H}_h \ ; \ \partial_z w_z + \frac{2}{R}\overline{w}_r = 0 \quad a.e. \text{ in } I\right\} \subset KerB.$$

Then, the uniform coercivity of $A(\cdot,\cdot)$ on Ker_hB results from Theorem 3.

The proof of the inf-sup condition follows the one given in Theorem 3, so the statement is established. ∎

5.2 Second model

We are now interested in the approximation of the second pseudo 1D model. As already mentioned, the only difference concerns the computation of the velocity and the pressure, which are obtained by solving the quasi-Stokes problem (5). So, let us consider its discrete version :

$$\left\{\begin{array}{lllll} \text{Find } \mathbf{u}_h \in \mathbb{V}_h^*, \ p_h \in M_h & & & & \\ \overline{a}_h(\mathbf{u}_h,\mathbf{v}) & + & \overline{b}(p_h,\mathbf{v}) & = & \overline{f_1}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{V}_h^0 \\ \overline{b}(q,\mathbf{u}_h) & & & = & \overline{f_2}(q) \quad \forall q \in M_h \end{array}\right. \tag{27}$$

where now :

$$\mathbb{V}_h = \left\{\mathbf{v} \in \mathbb{V};\ \mathbf{v}|_K \in (Q_1)^2,\ \forall K \in \mathcal{T}_h\right\},$$

$$\mathbb{V}_h^0 = \mathbb{V}_h \cap \mathbb{V}^0, \quad \mathbb{V}_h^* = \mathbb{V}_h \cap \mathbb{V}^*.$$

The bilinear form $\overline{a}_h(\cdot,\cdot)$ takes into account the upwinding scheme for the convective term. For the simplicity of the presentation, let us put :

$$\overline{a}_h(\mathbf{u},\mathbf{v}) = \alpha(\mathbf{u},\mathbf{v}) + d_h(u_r,v_r) + d_h(u_z,v_z),$$

$$\alpha(\mathbf{u},\mathbf{v}) = \int_\Omega r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}_h|\right)\mathbf{u}\cdot\mathbf{v}\mathrm{dx} + \int_\Omega \mu r(\partial_r u_z + \partial_z u_r)(\partial_r v_z + \partial_z v_r)\mathrm{dx} + \frac{4}{3}\int_\Omega \mu r(\partial_z u_z - \frac{1}{R}\overline{u}_r)(\partial_z v_z - \frac{1}{R}\overline{v}_r)\mathrm{dx}.$$

The main result of this section is given in the next Theorem :

Theorem 8. *Problem (27) is well-posed, for Δt sufficiently small.*

Proof. We apply the Babuška-Brezzi theorem for mixed formulations.
First of all, we note that

$$Ker_h\, b = \left\{\mathbf{v} \in \mathbb{V}_h^0;\ 2\partial_z\overline{v} + \partial_z\hat{v} + \frac{6}{R}\overline{v}_r = 0\ \ a.e.\ on\ I\right\} \subset Ker\, b.$$

So, for any $\mathbf{v} \in Ker_h\, b$ one gets, exactly as in the proof of Theorem 5, that :

$$\alpha(\mathbf{v},\mathbf{v}) \geq \int_\Omega r\left(\frac{\rho}{\Delta t} + \kappa|\mathbf{G}_h|\right)|\mathbf{v}|^2\mathrm{dx} + c(\|\partial_z v_r\|_{0,\Omega}^2 + |v_z|_{1,\Omega}^2)$$

with $c > 0$ independent of the triangulation. Then one deduces, thanks to Lemma 5, that $\overline{a}_h(\cdot,\cdot)$ is $\mathbb{V}$-elliptic on $Ker_h\, b$ for Δt sufficiently small, e.g.

$$\frac{1}{\Delta t} \geq \frac{c\|\mathbf{G}_h\|_{0,\Omega}}{\rho h_{min}}.$$

We still have to check the inf-sup condition (of Stokes type) for the $Q_1 - Q_0$ elements, on the particular mesh with only one cell in the radial direction. Following the proof given at the continuous level in Theorem 5, with any $q \in M_h$ we associate the function $\mathbf{v} \in \mathbb{V}_h^0$ defined by $\overline{v} = 0$, $\overline{v}_r \in H_0^1(\Sigma)$, piecewise linear and such that

$$\int_\Sigma \overline{v}_r\mathrm{dz} = -\frac{R}{6}\int_I q\mathrm{dz}, \quad \|\overline{v}_r\|_{1,\Sigma} \leq c\,\|P(q)\|_{0,I}$$

and finally, by

$$\widehat{v}(\zeta) = -\int_{z_{\min}}^{\zeta}(q + \frac{6}{R}P_{M_h}(\overline{v}_r))\mathrm{dz},\ \forall\zeta \in I$$

where $P_{M_h} : L^2(I) \to M_h$ is the $L^2(I)$- orthogonal projection operator on M_h. This choice ensures that $\widehat{v}$ is continuous on I, piecewise linear and it satisfies $\widehat{v}(z_{\min}) = \widehat{v}(z_{\max}) = 0$, therefore the function $\mathbf{v}$ thus constructed belongs to $\mathbb{V}_h^0$. Moreover, one has :

$$\begin{aligned} \|\mathbf{v}\|_{\mathfrak{V}} &\leq c_1 \|q\|_{0,I} + c_2 \|P(q)\|_{0,I} \leq c \|q\|_{0,I} , \\ \overline{b}(q, \mathbf{v}) &= \frac{R^2}{6} \int_I q \left(q - \frac{6}{R} (P_{M_h}(\overline{v}_r) - \overline{v}_r) \right) \mathrm{dz} = \frac{R^2}{6} \|q\|_{0,I}^2 , \end{aligned}$$

which yields the desired inf-sup condition. ■

6 Numerical simulations

In this section, we discuss several numerical tests in order to validate our *conservative pseudo 1D model*. First, we study the stability of our scheme by considering the behavior of the solution with respect to the mesh refinement. Then, we present a more realistic case which interfaces our numerical wellbore with a reservoir simulator already introduced in [1].

6.1 Mesh convergence

As depicted in Figure 2, we consider a three-perforation wellbore.

Figure 2: Vertical section of the 2D axisymetric wellbore

Mesh	Nodes	Edges	Triangles
Mesh $\mathcal{T}_h$	282	421	140
Mesh $\mathcal{T}_{h/2}$	562	841	280
Mesh $\mathcal{T}_{h/4}$	1122	1681	560
Mesh $\mathcal{T}_{h/8}$	2242	3361	1120
Mesh $\mathcal{T}_{h/16}$	4482	6721	2240
Mesh $\mathcal{T}_{h/32}$	8962	13441	4480

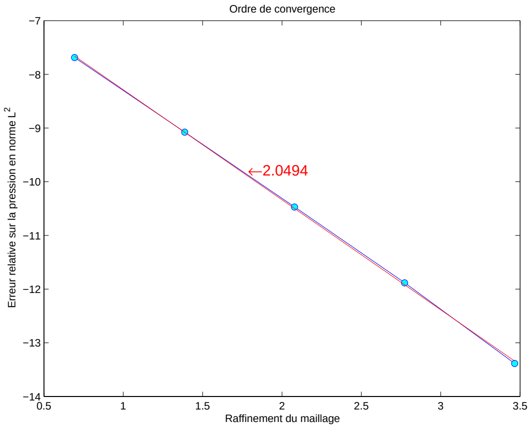
Table 1: Definition of the congruent meshes

Within this framework, we simulate the production of a dry gas by imposing an incoming specific flux at each perforation. In order to study the mesh convergence, the solutions are computed on congruent meshes $(\mathcal{T}_{h/2^i})_{0\leq i\leq 5}$ obtained from an innitial mesh $\mathcal{T}_h$, each element being successively and vertically divided into two congruent ones (Cf. Table 1). Note that, to preserve the pseudo 1D nature of our model, the mesh is not refined in the radial direction.

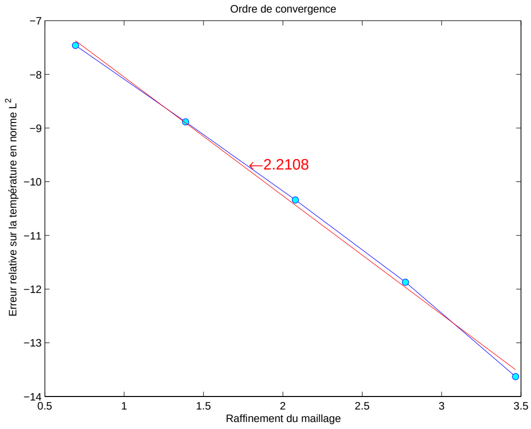
Remark 1. *When treating real cases, the mesh is only refined near the perforations so that a qualitative coupling is ensured with the reservoir. On the other hand, thanks to the conservative property of our scheme, the rest of the wellbore can be discretized with more elongated meshes. Thus, the number of degrees of freedom remains reasonable though considering wellbores of several hundreds meters depth.*

The solution calculated on the most refined mesh $\mathcal{T}_{h/32}$ is assumed to be the reference solution. Then, for each intermediate mesh $(\mathcal{T}_{h/2^i})_{0\leq i\leq 4}$, we evaluate the error between the computed solution and the reference solution. On Figure 3, the latter error is represented in L^2-norm for the pressure and the temperature. By computing the slope of the curves, one can conclude that the convergence of the model with respect to the mesh refinement is in h^2.

For illustration, Figures 4 and 5 represent the reference solutions, *i.e.* the pressure, the temperature, the velocity and the specific flux obtained on $\mathcal{T}_{h/32}$ at the end of the production period. Note that the plots are given with respect to z and that the dotted lines form the perforations. Focussing on Figure 4(a), the pressure in the wellbore clearly appears to be mainly influenced by the gravity. The same observation holds for the temperature when considering the flow beyond the perforations.

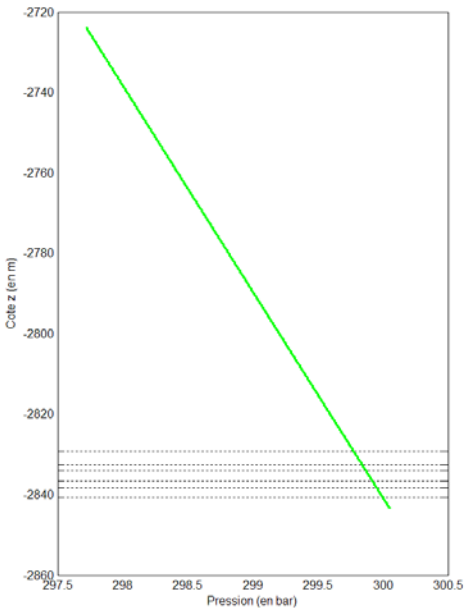


(a) Pressure

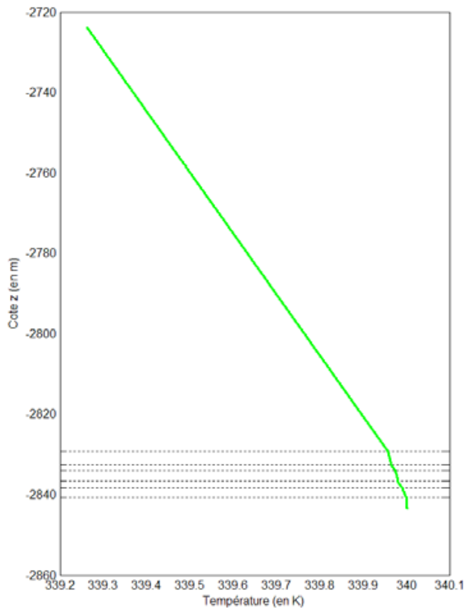


(b) Temperature

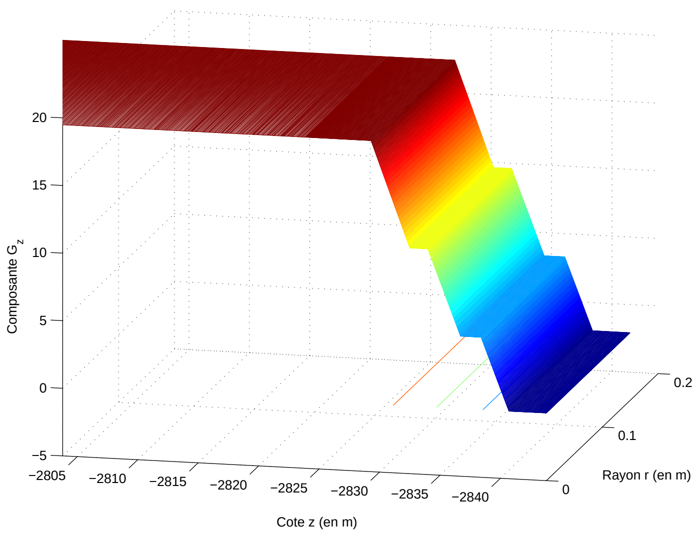
Figure 3: Order of convergence



(a) Pressure

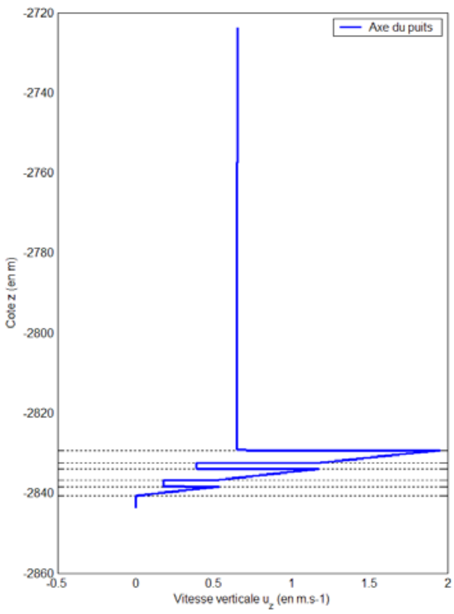


(b) Temperature

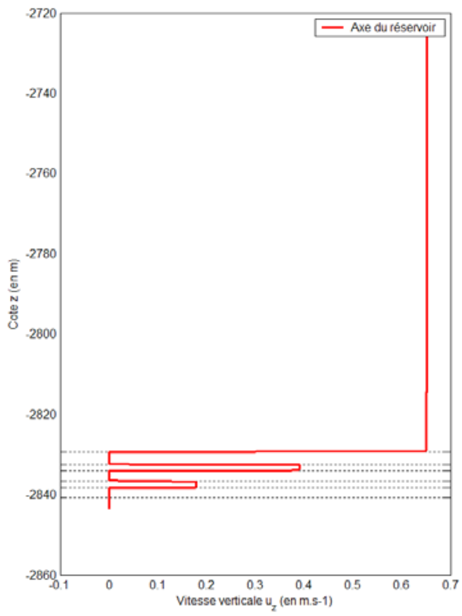


(c) Specific flux G_z, zoom near the perforations

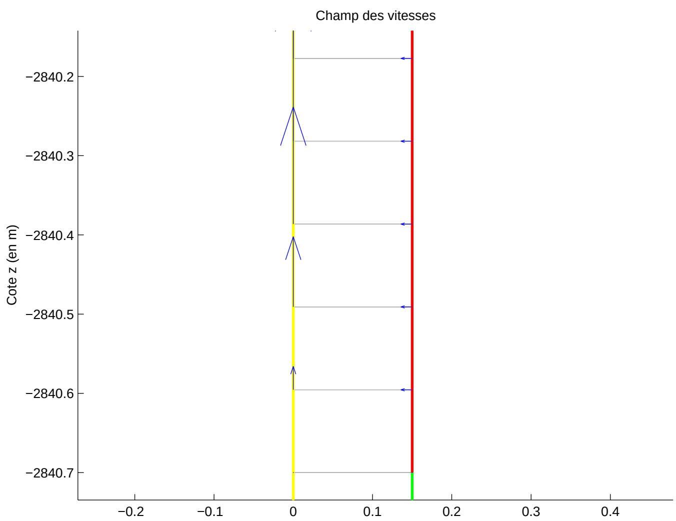
Figure 4: Pressure, temperature and specific flux at the end of the production



(a) u_z on the wellbore axis



(b) u_z on the reservoir axis



(c) Velocity field near a perforation

Figure 5: Velocity

6.2 Synthetic case

As mentioned earlier, our new wellbore model reveals itself when coupled with a reservoir simulator. Thus, in this section we carry out a simulation by imposing at the interface Σ the normal specific flux $\mathbf{G} \cdot \mathbf{n}$ and the temperature T generated by the reservoir model we introduced in [1].

As represented in Figure 6, we now consider a two-perforation wellbore associated with a five-layer reservoir which physical characteristics are detailed in Figure 7. One can notably see that the two producing layers are separated by quasi walls which prevent any cross flow between them in the reservoir.

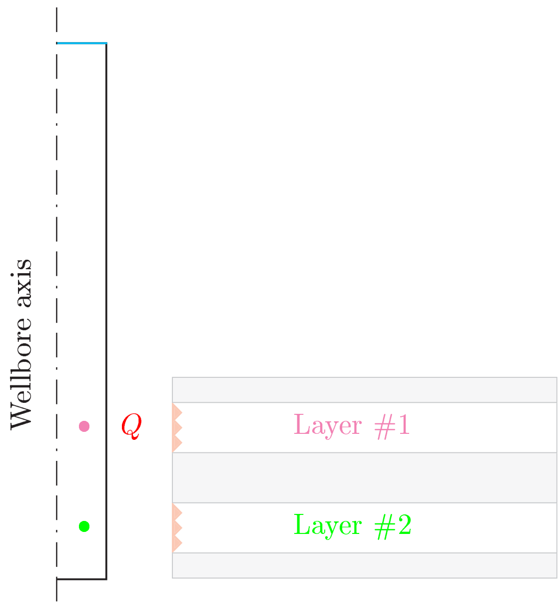


Figure 6: Vertical section of the 2D axisymetric wellbore-reservoir model

Within this framework, a light oil production is simulated by imposing at the exit of each reservoir producing layer, and thus at the entry of each wellbore perforation, the flowrates $(Q_i)_{1 \le i \le 2}$ given by Figure 8. In other words, the wellbore is in production mode during $48h$, and then is shut in the next $48h$.

Concerning the temperature, we impose on Σ the solutions obtained from the reservoir simulation, several of whom are represented on Figure 9 for some selected time-steps. In accordance with the Joule-Thomson effect, the reservoir temperature increases during the production phase and slowly comes back to normal the next hours.

The results of the wellbore simulation are displayed in Figure 10 for two vertical z positions chosen straight in front of each perforation (Purple and green colored dots of Figure 6). For these two z positions, we also draw a parallel (Cf. Figure 11) between

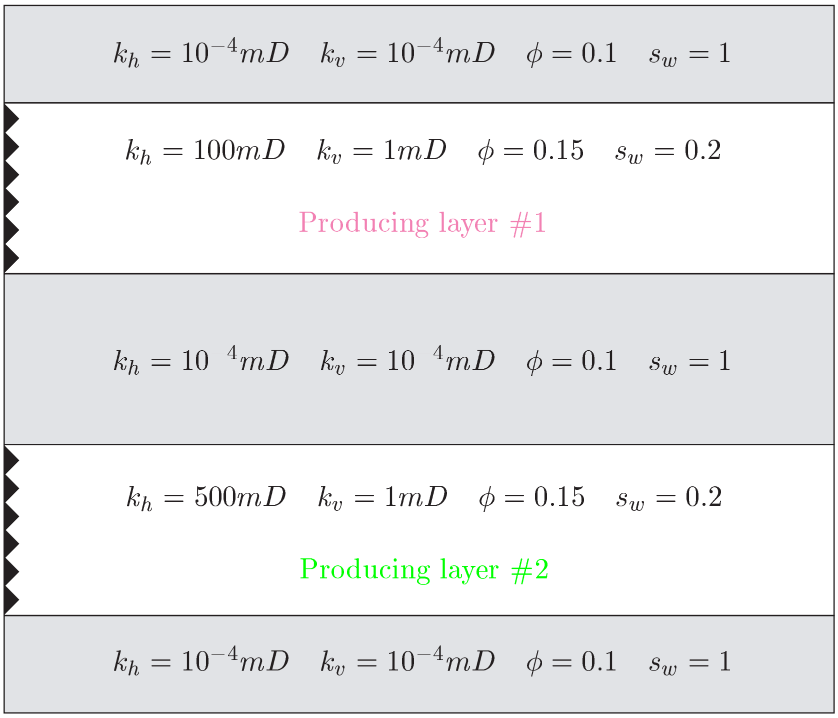


Figure 7: Vertical section of the reservoir

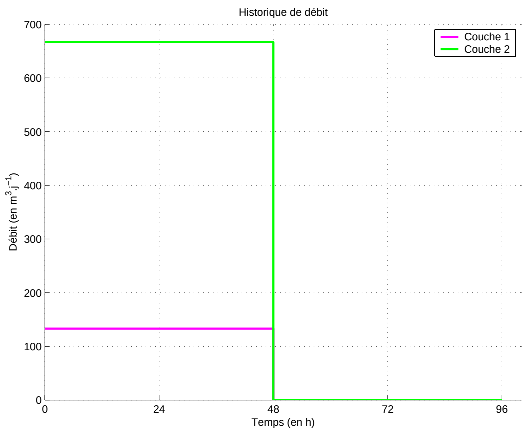


Figure 8: Flow rates imposed at each perforation

the temperature in the wellbore and the one computed inside the reservoir, closed to the perforations.

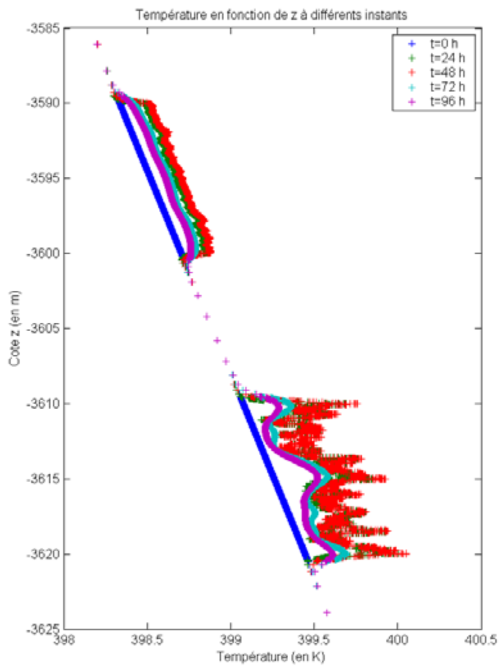


Figure 9: Temperature computed on Σ by the reservoir simulator

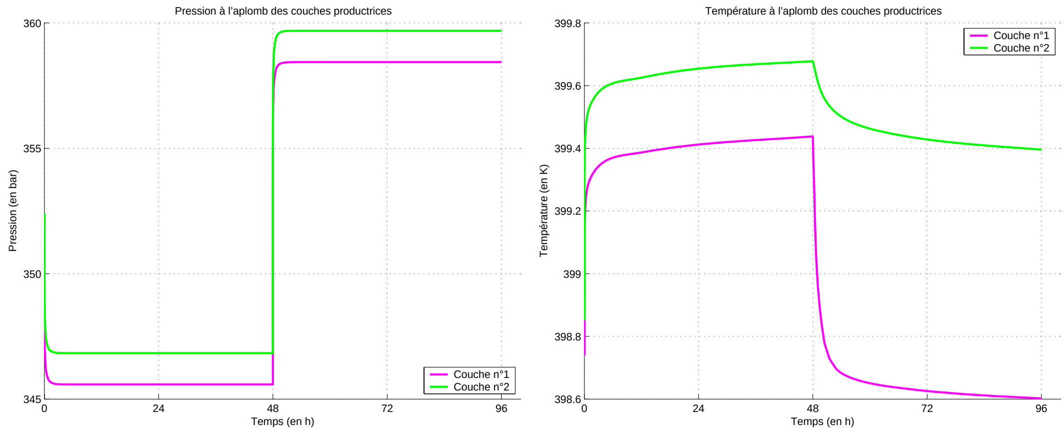
From Figure 10(a), one can notice that the pressure clearly agrees with the theoretical behavior given by softwares dedicated to well-testing analysis such as PIE (Cf. [1]). Namely, we observe on Figure 10(a) that, either during the draw-down or the shut-in period, the flow regime goes through a transitory state to reach a permanent one.

As part of the wellbore temperature, when the production is stopped (during the last $48h$), it decreases and tends to reach the reservoir temperature near the perforation, as enlightened by Figures 11(a) and 11(b).

On the other hand, during the production period, the wellbore temperature turns out to be mainly influenced by the wellbore flow.

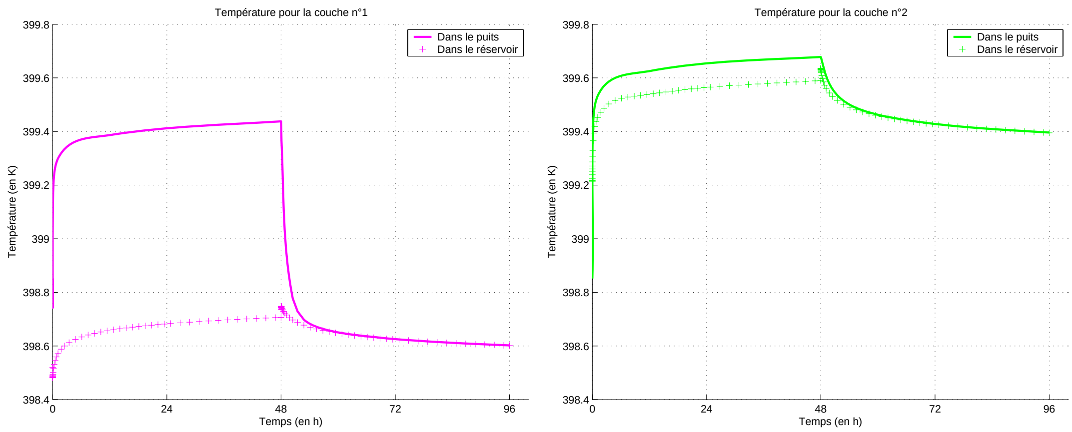
In fact, we first observe that the fluid temperature increases with the depth, which is due to the geothermal gradient imposed in the reservoir. Moreover, the wellbore temperature in front of the shallow perforation is far more higher than the corresponding one in the reservoir side, at the same depth. The fluid extracted from the perforation #1 is warmed in the wellbore when blended with the warmer fluid produced by the perforation #2.

Thus, we come across the empirical idea which states that the temperature of the fluid directly in front a perforation is mainly influenced by the temperature of the fluid



(a) Pressure (b) Temperature

Figure 10: Wellbore results in front of each perforation, with respect to time



(a) Producing layer #1 (b) Producing layer #2

Figure 11: Wellbore and reservoir temperatures at each perforation

coming from the reservoir and by the one of the fluid going up in the pipe, which would be equivalent to say that for a given z :

$$T(z) = \frac{\displaystyle\int_0^z T\mathbf{G} \cdot \mathbf{n} d\sigma}{\displaystyle\int_0^z \mathbf{G} \cdot \mathbf{n} d\sigma}, \ \forall z \in I = I_2 \cup I_{\mathcal{I}}. \tag{28}$$

In fact, the latter relation which is plotted in Figure 12 could be referred to as a first-order approximation of our model in the sense that it provides a good trend in production mode, but fails in accuracy as it does not take into account the whole physical constraints such as the gravity.

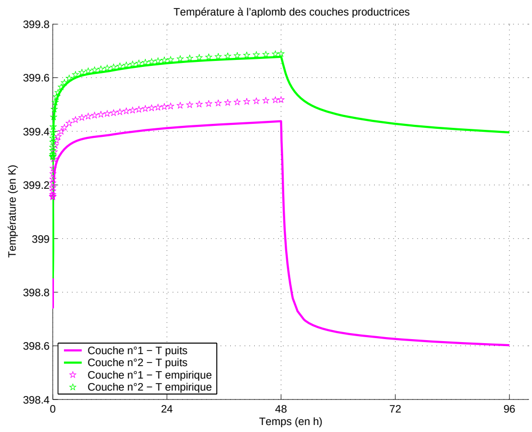


Figure 12: Empirical simplified estimate of the wellbore temperature

References

[1] M. Amara, D. Capatina, B. Denel and P. Terpolilli. *Mixed finite element approximation for a coupled petroleum reservoir model*, M2AN, Vol. 39, No. 2, pp. 349-376, 2005.

[2] G. Bourdarot. *Essais de puits : méthodes d'interprétation*, Editions Technip, Paris, 1996.

[3] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*, Springer Verlag, New York, 1991.

[4] B. Denel. *Simulation numérique et couplage de modèles thermomécaniques puits-milieux poreux*, Ph.D. Thesis, University of Pau, 2004.

[5] G. Duvaut. *Mécanique des milieux continus*, Masson, 1990.

[6] V. Girault and P.A. Raviart. *Finite element method for Navier-Stokes equations, Theory and Algorithms*, Springer Verlag, Berlin, 1986.

[7] F. Maubeuge, M. Didek, E. Arquis, O. Bertrand and J.P. Caltagirone. *A model for interpreting thermometrics*, SPE 28588, 1994.

[8] G. Medic and B. Mohammadi. *NSIKE : An incompressible Navier-Stokes solver for unstructured meshes*, Rapport de Recherche No. 3644, INRIA, 1999.

[9] R.A. Nicolaides. *Existence, uniqueness and approximation for generalized saddle point problems*, SIAM J. Numer. Anal., Vol. 19, No. 2, pp. 349-357, 1982.

[10] D.Y. Peng and D.B. Robinson. *A new two-constant equation of state*, Ind. Eng. Chem. Fundam., Vol. 15, pp. 59-64, 1976.

[11] J.E. Roberts and J.-M. Thomas. *Mixed and Hybrid Methods*, in *Handbook of Numerical Analysis*, Vol. II, North Holland, Amsterdam, pp. 523-639, 1991.