Subdivided graphs as isometric subgraphs of Hamming graphs

Laurent Beaudou, Sylvain Gravier, Kahina Meslem

Institut Fourier

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Unité Mixte de Recherche 5582
CNRS – Université Joseph Fourier

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Institut Fourier
UMR 5582
CNRS – Université Joseph Fourier

Institut Fourier
100, rue des Maths
38402 St-Martin d'Hères – FRANCE

laurent.beaudou@ujf-grenoble.fr
sylvain.gravier@ujf-grenoble.fr
kahina_m2000@yahoo.fr

Abstract/Résumé

It is proven that given G a subdivision of a clique K_n ($n \geq 1$), G is isometrically embeddable in a Hamming graph if and only if G is a partial cube or $G = K_n$. The characterization for subdivided wheels is also obtained.

Keywords: isometric embeddings, subdivided graph, partial cube, Hamming graph.

Dans ce rapport, on prouve qu'étant donné G une clique K_n ($n \geq 1$) subdivisée, G est le sous-graphe isométrique d'un graphe de Hamming si et seulement si G est un cube partiel ou $G = K_n$. La caractérisation est aussi obtenue pour les roues subdivisées.

Mots-clés : plongement isométrique, graphe subdivisé, cube partiel, graphe de Hamming.

Introduction

Isometric subgraphs of Hamming graphs (resp. hypercubes) are called *partial Hamming graphs* (resp. *partial cubes*). Partial cubes have first been investigated by Graham and Pollak [11], and Djokovìc [9]. Later, several algorithmic characterizations were shown using a relation defined on the set of edges (the Θ relation introduced by Djoković [9] and Winkler [18]), or constructive operations.

This relation easily led to a recognition algorithm for partial cubes running in $O(mn)$ (m is the number of edges and n the number of vertices). This complexity was then improved for specific classes of graphs by Brešar et al. [3]. And recently, the general complexity was pulled down to $O(n^2)$ by Eppstein [10]. Partial cubes have found different applications, for instance, in [6, 8, 13], interesting applications in chemical graph theory were established.

The search for structural characterizations was first motivated by a conjecture of Chepoi and Tardif in the 1994 Bielefeld conference on "Discrete Metric Spaces". They asked if bipartite graphs with convex intervals were partial cubes. The negative answer was brought by Brešar and Klavžar in [4], using subdivided wheels. From this point structural characterizations for partial cubes that were subdivisions of wheels and cliques were obtained (see [1, 2, 12]). A natural extension of this work was to study partial Hamming graphs.

One point of interest about partial Hamming graphs, is that they can be labelled in such a way that, given a target vertex, only local information is needed to know which way is a shortest path. Reader can see [13, 14, 15, 17, 18] for further information about partial Hamming graphs.

Moreover, studying subdivided graphs and their isometric embeddings in Hamming graphs, allows us to study partly the λ-scale embeddings in Hamming graphs. For example, if the subdivision of a graph G is isometrically embeddable in a hypercube, then G is 2-scale embeddable in a hypercube (see [7, 16] for more details).

In this paper, we show that, the only subdivision of a clique isometrically embeddable in a Hamming graph which is not a partial cube ; is actually the clique itself. Then, we study the subdivisions of wheels that are isometrically embeddable in Hamming graphs and obtain a structural characterization of them.

1 Preliminary definitions and basic properties

By replacing edges of a graph G by paths, we obtain a *subdivided graph* of G. Given G a graph, $S(G)$ is obtained by subdividing each edge of G exactly once.

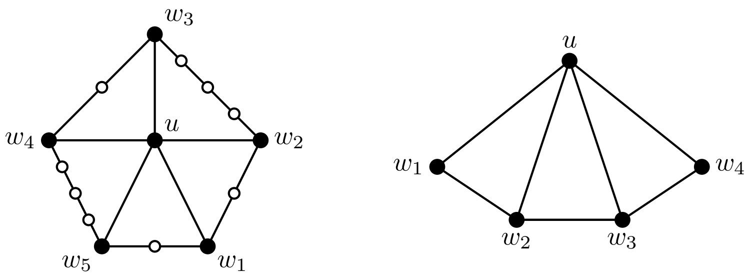
We call *principal vertex*, a vertex of a subdivided graph of G that was not added during the subdivision (it was already in G). We call *universal vertex*, a vertex of a subdivided graph of G such that each edge incident to it, is not subdivided.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. A *Hamming graph* is a cartesian product of cliques.

For a graph G, the distance $d_G(u, v)$ between vertices u and v is defined as the number of edges on a shortest uv-path. A subgraph H of G is called isometric if and only if $d_G(u,v) = d_H(u,v)$ for all $u, v \in V(H)$.

For $k \geq 3$, the *k-fan* F_k is the graph obtained as the join of a vertex u and a path on k vertices $w_1, \ldots, w_k$ (see Fig. 1(a)).

For $k \geq 3$, the *k-wheel* W_k is the graph obtained as the join of a vertex u and a cycle on k vertices $w_1, \ldots, w_k$ (see Fig. 1(b)).



(a) Subdivision of W_5

(b) Fan F_4

Figure 1:

We also remind some former results about partial cubes.

Lemma 1. [12] *Let G be a graph and K an isometric subgraph of G which is isomorphic to a subdivision of F_k ($k \geq 3$) such that at least one of the edges $uw_2, \ldots, uw_{k-1}$ is subdivided. Then G is not a partial cube.*

Theorem 2. [12] *A subdivided wheel (with more than 4 rays) is a partial cube if and only if all its rays are not subdivided and the other edges are oddly subdivided.*

Theorem 3. [2] *Let G be a subdivided graph of a complete graph K_n ($n \geq 4$). G is a partial cube if and only if, G is isomorphic to $S(K_n)$ or G contains a universal vertex u and the number of added vertices to each edge not incident to u in K_n is odd.*

Given G a partial Hamming graph and an isometric embedding ϕ of G into a Hamming graph H, ϕ induces a natural edge labelling ℓ of G : given e an edge of G linking two vertices u and v, as they are adjacent in G and ϕ is isometric, we get that $d_H(\phi(u), \phi(v))$ is 1 and thus $\phi(u)$ and $\phi(v)$ differ in exactly one coordinate i. We define $\ell(e) = i$. x^i will denote the i-th coordinate of the vertex x embedded in a Hamming graph.

Lemma 4. *Given G a partial Hamming graph and a natural labelling of its edges ℓ. If two edges of G, e and f are part of a geodesic path in G, then $\ell(e) \neq \ell(f)$.*

Proof. A shortest path in a Hamming graph cannot use twice the same coordinate, or it would be shorter using it once or not using it at all. □ □

Lemma 5. *Given G a partial Hamming graph and a natural labelling of its edges ℓ. Given C an elementary cycle of G, each label appearing in C, appears at least twice. Moreover, if C is isometric and differs from K_3, then these labels appear exactly twice.*

Proof. For a contradiction let us suppose there is a cycle $(x_0, x_1, \ldots, x_k, x_0)$ such that $\ell(x_0x_1) = i$ and no other edge of the cycle has label i. Then $x_0^i \neq x_1^i$ (since $\ell(x_0x_1) = i$). For all other edges, $x_l^i = x_{l+1}^i$ $(1 \leq l \leq k-1)$ which implies that $x_k^i = x_1^i$. The last edge (x_kx_0) leads to a contradiction since $x_0^i \neq x_1^i$. □ □

Remark 6. *It is straightforward from this Lemma that an isometric cycle in a partial Hamming graph is either even or K_3.*

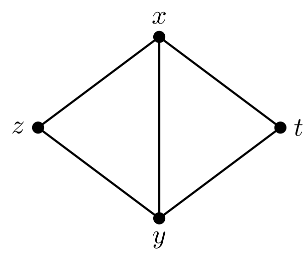


Figure 2: $K_4 - e$

Proposition 7. *K_4-e (Fig. 2) cannot be isometrically embedded in a Hamming graph.*

Proof. x and y differ on exactly one coordinate i, we can assume that $x_i = 0$ and $y_i = 1$. Consequently, all the vertices that are at distance 1 from x and y, differ also on the same coordinate i. Therefore, they are at distance 1 from each other. Thus, t and z should be linked, which is not the case. □ □

2 Structural characterization for cliques

We shall prove the following theorem.

Theorem 8. *Given G a subdivided clique K_n for some n, G is a partial Hamming graph if and only if G is a partial cube or $G = K_n$*

2.1 First case : G is bipartite

Let us show that only two values are used for each coordinate.

For a contradiction, suppose that there exists a coordinate i and three vertices x, y, z such that $x^i = 0$, $y^i = 1$ and $z^i = 2$. Consider shortest paths between these three vertices. Thanks to Lemma 4, a shortest path between x and y is a sequence of vertices u_j with $u_j^i = 0$ followed by a sequence of vertices v_j with $v_j^i = 1$. Same holds for both other shortest paths. The subgraph induced by these shortest paths is represented in Fig. 3.

Therefore, there exists an elementary cycle using exactly three different values for the coordinate i. We can consider the smallest one. Clearly, three of its edges have label i. By Lemma 5, it is either K_3 (impossible since G is bipartite),

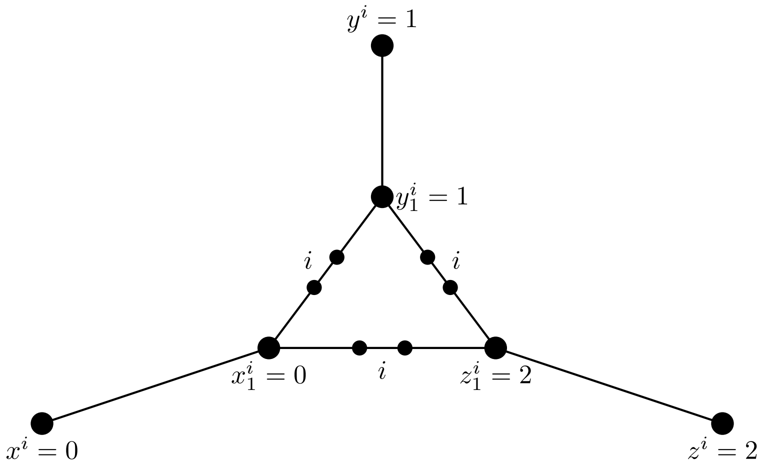


Figure 3: Shortest paths

or it is not isometric. There exists a chord which shorten the cycle, one of the two resulting cycle uses exactly three different values for the coordinate i and is shorter. This is a contradiction.

As a conclusion, every coordinate is used for only two values. This implies that the graph is isometrically embeddable in a hypercube. Therefore, G is a partial cube and the Theorem holds.

Remark 9. *This part of the proof holds for any graph. A bipartite graph that is isometrically embeddable in a Hamming graph, is a partial cube. This result has already been proven in [17].*

2.2 Second case : G is not bipartite

We have to prove that G is then isomorphic to K_n. As G is not bipartite, it contains an odd cycle. Therefore we can consider C a smallest odd cycle of G. C is isometric since if it was not, there would be a chordal chain inducing a strictly smaller odd cycle in G. By Lemma 5 either $C = K_3$ or each label appears exactly twice, but this would give an even cycle. Therefore we can assume that C is K_3.

We now consider a maximal clique in G (with respect to inclusion) containing C, namely K. All its edges have the same label, namely i. If $K = K_n$, then $G = K_n$ and Theorem holds. Let us suppose that $K \neq K_n$. We consider u a principal vertex of G that is nearest to K. Let x be a principal vertex of K nearest to u.

If there exists a vertex y of K distinct from x such that $P(u, y)$ is plain. Then $P(u, x)$ is also plain and $\langle u, x, y\rangle$ is a K_3. Therefore the three edges have the same label which is i. Thus, u is at distance 1 from all the vertices of K which contradicts the maximality of K.

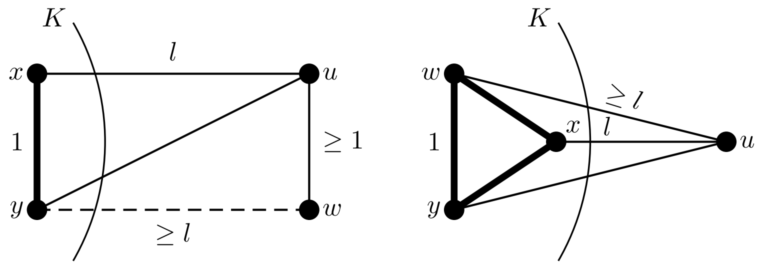
We can thus assume, for every vertex y of K distinct from x, that $P(u, y)$ is subdivided at least once.

Claim 10. *For every vertex y of K distinct from x, $\langle u, x, y\rangle$ is an isometric cycle.*

Proof. For a contradiction, if $\langle u, x, y\rangle$ is not isometric, there must be a strictly shorter path from u to y. Let us denote by l the distance between u and x and consider a shortest path from u to y

- **Case 1** : it goes through principal vertex $w \notin K$ (see Fig. 4(a)). Thus the path from w to y has at least length l and the path from u to w has at least length 1. As it must be strictly less than $l+1$, this is a contradiction.
- **Case 2** : it goes through a principal vertex $w \in K$ (see Fig. 4(b)). Thus the path from u to w has at least length l so that the path from u to y going through w has at least length $l+1$ which is also a contradiction.

Therefore, $\langle u, x, y\rangle$ is isometric. □ □



(a) Path outside K (b) Path in K

Figure 4:

Now consider two vertices of K distinct from x, namely y and w ($|K| \geq 3$). As $P(y, u)$ and $P(w, u)$ are subdivided, we have $\langle u, x, y\rangle$ and $\langle u, x, w\rangle$ two isometric cycle which are not isomorphic to K_3 and thus are even cycle where labels appear exactly twice (by Lemma 5). We then study the subdivision of K_4 $\langle u, x, y, w\rangle$ (see Fig. 5). It is clearly isometric. By considering the outer cycle $\langle u, x, y, w\rangle$ and using Lemma 5, there must be at least one edge labelled with i on the path going from w to y through u. Let us assume that it is on the path $P(u, y)$. Then there cannot be any edge labelled with i on the path $P(u, x)$ since $\langle u, x, y\rangle$ is isometric (or label i would appear three times). We can thus assume that there is no edge labelled with i on the path $P(u, x)$.

Therefore, there must be an edge ab in $P(u, w)$ and an edge de in $P(u, y)$ labelled with i. Moreover, if we consider the label of the first edge of the path $P(x, u)$, namely $j \neq i$, then the following edges bc and ef must be labelled with j (since cycles are even, same labels face each other). To conclude, we just have to consider a shortest path from b to e and observe that it has to go through two edges labelled the same way (either i or j). By use of Lemma 4, this is impossible. We obtain a contradiction.

As a conclusion, $K = K_n$ and Theorem 8 is proven.

By using the characterization of subdivided cliques that are partial cubes obtained in Theorem 3, we get the following characterization.

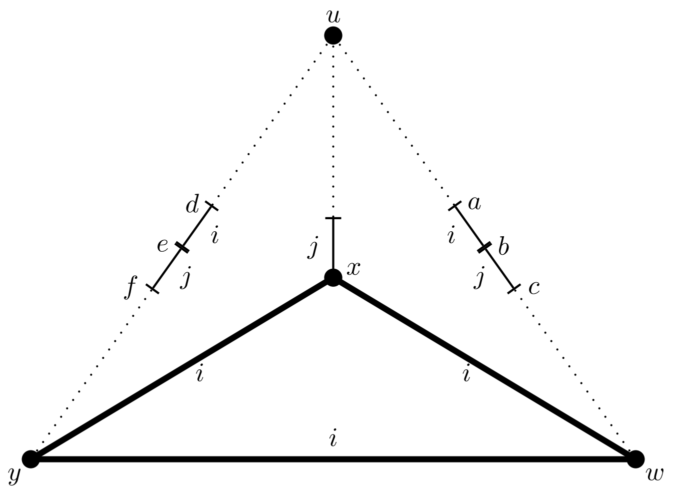


Figure 5: $\langle x, y, u, w \rangle$

Corollary 11. *Given G a subdivided K_n ($n \geq 4$), G is isometrically embeddable in a Hamming graph, if and only if it is either K_n, or $S(K_n)$, or it has a universal vertex and all other edges are oddly subdivided.*

3 Structural characterization for wheels

The wheel W_3 is isomorphic to K_4 studied above, so that we will only study wheels W_k with $k \geq 4$. We state the following theorem.

Theorem 12. *Given W a subdivided wheel, W is isometrically embeddable in a Hamming graph if and only if :*

(i) All its rays are non-subdivided

(ii) Other edges are either non-subdivided or oddly subdivided

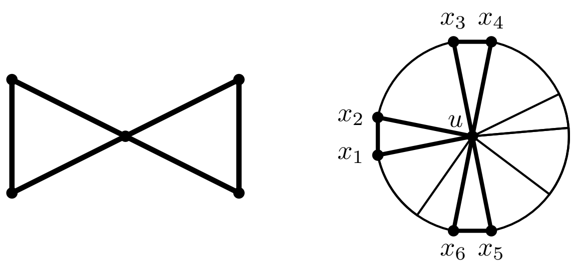
(iii) These last non-subdivided edges are not incident

(iv) For W_4 it does not contain the forbidden pattern (see Fig 6(a))

3.1 Necessary condition

Let us consider W a subdivided wheel with at least four rays which is isometrically embeddable in a Hamming graph. If W is bipartite, it is a partial cube by Remark 9. By Theorem 2, the rays are not subdivided, the other edges are all oddly-subdivided and as there is no triangle (since it is bipartite), the four conditions are verified.

Thus, we can assume that W is not bipartite. It contains an odd cycle. Therefore we can consider C a smallest odd cycle of W. C is isometric since if it was not, there would be a chordal chain inducing a strictly smaller odd



(a) Forbidden pattern (b) Non-bipartite wheel

Figure 6:

cycle in W. By Lemma 5 either $C = K_3$ or each label appears exactly twice, but this would give an even cycle. Therefore we can assume that C is K_3 and C must have u as one of its vertices. Let us denote $x_1, x_2, \ldots, x_{2k-1}, x_{2k}$ the vertices around the wheel which are part of a triangle (see Fig. 6(b)) ; thus we have $\langle u, x_1, x_2\rangle, \ldots, \langle u, x_{2k-1}, x_{2k}\rangle$ triangles. We can assume that all the x_i are different since if $x_{2i} = x_{2i+1}$, it would induce an isometric subgraph isomorphic to $K_4 - e$ forbidden by Proposition 7.

For all $1 \leq l \leq k-1$ the fan induced by u and the path from x_{2l} to x_{2l+1} is isometric in W. The same stands for the fan induced by u and the path from x_{2k} to x_1. Thus, these fans are isometrically embeddable in a Hamming graph. Moreover, as they do not contain any induced K_3, they are bipartite and thus are partial cubes (by Remark 9). We use the Lemma 1 to conclude that every ray of the wheel W is non-subdivided. (i) and (ii) are then proven. Obviously, two triangles cannot have a common edge (or it would be isomorphic to $K_4 - e$) then two incident edges around the wheel cannot be both plain. This proves (iii).

To prove (iv), for a contradiction, let us consider a subdivided W_4 with the forbidden pattern. Each triangle is included in a factor of the Hamming graph, their edges will be respectively labelled with i and j. Both cycles going through u and dashed paths are isometric and distinct from K_3. By Lemma 5, they have even length. Thus each middle vertex (x and y) of subdivided edges is between an edge labelled with i and an edge labelled with j (see Fig. 7). Any shortest path from x to y has to go through two edges labelled the same way. Lemma 4 allows us to conclude it is a contradiction. (iv) is proven.

3.2 Sufficient condition

Let us consider W a subdivision of a wheel (with at least four rays) that fits the conditions from (i) to (iv). We show that it can be isometrically embedded in a power of K_3, which is a Hamming graph.

We first embed all the triangles and single rays such that each is in one different dimension of the product of K_3, which is called the dimension of the ray or triangle. The center vertex u is at the origin.

Then we embed each subdivided edge consecutively with respect to the following induction property : after each step, the result embedding is isometric and given any ray or triangle, the only vertices with a non-zero coordinate in its dimension are part of the halves of the subdivided edges incident to it.

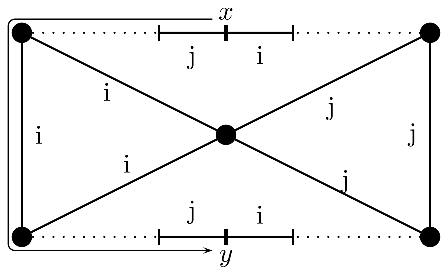


Figure 7: Forbidden pattern

The property is clearly true at the beginning. Let us now add a subdivided edge, ie a path of length $2k$ between two vertices of the wheel, namely a and b in G the current embedded graph. We can suppose that the two involved dimensions correspond to the first two coordinates so that $a = (1, 0, 0 \dots 0)$ and $b = (0, 1, 0 \dots 0)$. Then we add $k - 1$ dimensions. All the precedent vertices have null components in these dimensions. To add the $2k - 1$ new vertices and obtain the new graph G^* we do as follows :

$$\begin{cases} x_i = (1,0,0\dots0|\overbrace{1,\dots,1}^{i \text{ times}},0,\dots,0) & 1 \le i \le k-1 \\ x_i = (1,1,0\dots0|1,1,\dots,1,1) & i = k \\ x_i = (0,1,0\dots0|\underbrace{0,\dots,0}_{i-k-1 \text{ times}},1,\dots,1) & k+1 \le i \le 2k-1 \end{cases}$$

Clearly, the only vertices with a non-zero coordinate in the dimensions of a and b are contained in the halves of the subdivided edges incident to it. Let us now prove that this embedding is still isometric.

Let x, y be to vertices of G^*, we have to prove that $d_{G^*}(x, y) = d_H(x, y)$.

x and y are vertices of G : This means that $d_{G^*}(x, y) = d_G(x, y)$ and as we have only added 0's in the new embedding, the Hamming distance is still the same.

x and y are both part of the new path : It is straightforward that the new vertices are isometrically embedded as the even cycle $\langle b, u, a, x_1 \dots x_{2k-1}\rangle$ is isometric.

x is in the path and y is a vertex of G : Then there exists i such that $x = x_i$ and we have to distinguish two possibilities :

- $i \neq k$: Then we can suppose that x is nearer from a so that $x = (1, 0 \dots 0|1 \dots 1, 0 \dots 0)$. Thus $d_H(x, y) = d_H(a, y) + i = d_G(a, y) + i$. There is a shortest path from x to y going through a so that $d_{G^*}(x, y) = d_{G^*}(x, a) + d_{G^*}(a, y) = i + d_G(a, y)$.
- $i = k$: $x = (1, 1, 0 \dots 0|1 \dots 1)$ and $y = (y|0 \dots 0)$ so that $d_H(x, y) = k - 1 + d_H((1, 1, 0 \dots 0), y)$. If there is a shortest path from y to x going through u, then y must have its first two coordinates equal to 0 otherwise there would be a shortpath to a or b and it would not be isometric. Thus

$d_{G^*}(x, y) = k + 1 + d(u, y) = k + 1 + d_H((1, 1, 0 \ldots 0), y) - 2$. The result is proven. From now on, we suppose that none of the shortest paths goes through u. We consider a shortest path from y to x. It has to go through a or b. We can assume it goes through a. This implies that the first coordinate of y is not 0 else there would be a shorter or equal path going through u (thanks to the induction property). Then $d_{G^*}(x, y) = d_{G^*}(a, y) + k = d_G(a, y) + k = d_H((1, 0 \ldots 0), y) + k$. Therefore we have to prove that $d_H((1, 1, 0 \ldots 0), y) - 1 = d_H((1, 0 \ldots 0), y)$. If the second coordinate of y is 0, it is clearly true. Else, we would have the first two coordinates of y non-equal to 0. Thanks to the induction property, the only possible configuration in which it could happen is a subdivision of W_4 containing the forbidden pattern. In other words this is impossible and the result is proven.

This concludes the proof of the Theorem 12

Conlusion

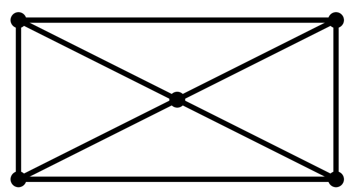


Figure 8: Obstruction

From these results, it appears that partial Hamming graphs are not so far from partial cubes. The main point is that the only non trivially subdivided cliques that are partial Hamming graphs are partial cubes. We remind now a Proposition obtained in [2].

Proposition 13. [2] *Let G be a subdivision of a graph of order n such that each edge contains odd added vertices. K is a graph obtained from G by joining a vertex u adjacent to each principal vertex of G. Then, K is a partial cube.*

This forbids any "minor-free" characterization. Forbidden isometric subgraphs seem to be naturally more appropriate. This gives rise to two new conjectures.

Conjecture 14. *Given G a graph, G is a partial cube if an only if it is bipartite and does not contain any subdivision of $K_4 - e$ as an isometric subgraph.*

Conjecture 15. *Given G a graph, G is a partial Hamming graph if and only if it fits some parity conditions and does not contain any subdivision of the graph H (see Fig. 8) as an isometric subgraph.*

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