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Maximum directed cuts in digraphs with degree restriction

Jenő Lehel ∗ Frédéric Maffray † Myriam Preissmann ‡

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Abstract

For integers \( m, k \geq 1 \), we investigate the maximum size of a directed cut in directed graphs in which there are \( m \) edges and each vertex has either indegree at most \( k \) or outdegree at most \( k \).

1 Introduction

We deal with directed graphs, called here digraphs, without loops and parallel edges. An edge \( xy \) of a digraph is interpreted as an arc or an arrow going from the starting vertex or tail \( x \) to the end vertex or head \( y \). The indegree and the outdegree of a vertex \( v \in V(D) \) is respectively defined as \( d^-_D(v) = |\{zv \in E(D) | z \in V(D)\}| \) and \( d^+_D(v) = |\{vw \in E(D) | w \in V(D)\}| \).

Let \( X, Y \) be a partition of the vertex set \( V(D) \) of a digraph \( D \). The edge set \( \{xy \in E(D) | x \in X, y \in Y\} \) is called a directed cut. Clearly a directed cut of a digraph \( D \) does not contain a directed path on three vertices (a \( P_3 \)). On the other hand every directed \( P_3 \)-free subgraph of \( D \) is the subgraph of some directed cut. Thus when estimating the size of maximum directed cuts we must find directed \( P_3 \)-free subgraphs as large as possible. The size of a cut is its cardinality, the size of a digraph is the cardinality of its edge set.

Discussions in [1] show that a digraph \( D \) of size \( m \) has a cut of size \( \frac{1}{4}m + \Theta(m^{1/2}) \). Furthermore, if the outdegree of each vertex of \( D \) is at most \( k \), then \( D \) has a cut of size at least \( \left( \frac{1}{4} + \frac{1}{8k^2} \right)m \). In [2], lower bounds for the largest directed cuts were asked for a family of digraphs with constrained indegree or outdegree. Let \( D(k, \ell) \) be the family of all digraphs in which every vertex has either indegree at most \( k \) or outdegree at most \( \ell \) (that is \( d^-(v) \leq k \) or \( d^+(v) \leq \ell \),

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for all \( v \in V(D) \). Note that a directed cut (or any directed \( P_3 \)-free graph) forms a graph that belongs to \( D(0,0) \).

In Section 2 we consider the case \( k = \ell = 1 \) and discuss the size of the maximum directed cut of digraphs in \( D(1,1) \). It was proved in [1] that every acyclic digraph of size \( m \) in \( D(1,1) \) has a directed cut of at least \( 2m/5 \) edges.

From a result of Bondy and Locke [2] it is easy to see that the same lower bound holds for maximum directed cuts in triangle-free subcubic digraphs (a graph is subcubic if it has maximum degree at most three). Our main result in Theorem 3 is the extension of this bound for all digraphs in \( D(1,1) \) as follows: if \( D \) contains at most \( t \) pairwise disjoint directed triangles, then \( D \) has a directed cut of size at least \( (2m-t)/5 \). The proof yields a polynomial algorithm which actually finds a directed cut of that size (Corollary 2).

Section 3 concludes with open problems for further consideration. A challenging question whose answer we would like to see the most is whether Theorem 12 remains true for all digraphs in \( D(k,k) \), and for every \( k \geq 3 \).

2 Maximum directed cut of digraphs in \( D(1,1) \)

It was proved in [1] that every acyclic digraph of size \( m \) in \( D(1,1) \) has a directed cut of at least \( 2m/5 \) edges. This is not true for all digraphs in \( D(1,1) \). For example the directed triangle, which is a member of \( D(1,1) \), has no directed cut with two edges. Hence there are digraphs of size \( m \) with maximum directed cut not larger than \( m/3 \). On the other hand, it was shown in [1] that the edge set of every digraph \( D \in D(1,1) \) has a decomposition into three directed cuts (see Theorem 7 below), hence \( D \) always contains a directed cut of size \( m/3 \).
One might conclude that the ratio $m/3$ cannot be improved to $2m/5$ in general, and the graph that consists of disjoint directed triangles is an obvious example showing that. Actually the following example shows that, for infinitely many values of $m$, there are even connected digraphs in $D(1,1)$ of size $m$, that contain no cut of size $3m/8$.

Example 1. For $i = 1, \ldots, k$ let $H_i$ be a directed path with five vertices $(u_i, v_i, w_i, x_i, y_i)$ plus the chord $v_ix_i$, such that the $H_i$’s are pairwise disjoint. Add $k + 1$ directed triangles $(y_i, u_{i+1}, z_i)$, for $i = 0, \ldots, k$, where $y_0, u_{k+1}$ and $z_0, z_1, \ldots, z_k$ are distinct new vertices. The obtained graph $H$ has $m = 8k + 3$ edges and its maximum directed cut has size $3k + 1 = (3m - 1)/8$.

In spite of the evidence that the maximum directed cut size to edge count ratio $2/5$ cannot be achieved, we show in the next theorem that $1/3$ improves to $2/5$, in some sense, for all digraphs in $D(1,1)$.

Theorem 1. Let $D$ be a digraph in $D(1,1)$ with $m$ edges, and let $t$ be the maximum number of pairwise disjoint directed triangles in $D$. Then $D$ has a directed cut of size at least $(2m - t)/5$.

Proof. The claim is clearly true for $m \leq 3$ and $t = 0$. If $m = 3$ and $t = 1$, then $D$ is the directed triangle and any edge of the triangle forms a directed cut of size 1 = $(2m - t)/5$. Now let $D$ be a counterexample with $m \geq 4$ edges, and assume that the theorem is true for all digraphs in $D(1,1)$ with at most $m - 1$ edges. Clearly $D$ is connected.

Let $D^+$ be the subgraph of $D$ induced by $V^+ = \{v \in V(D) | d^+(v) \geq 2\}$; and let $D^-$ be the subgraph of $D$ induced by $V^- = \{v \in V(D) | d^-(v) \geq 2\}$. Notice that $v \in V^+$ implies that $d^-(v) \leq 1$ and $v \in V^-$ implies $d^+(v) \leq 1$.

Because $D \in D(1,1)$, if two directed triangles of $D$ have a common vertex, then they must share a common edge. Moreover, if a triangle intersects with at least two other triangles, then they all share the same common edge. The following property of triangles will be useful.

Claim 1.1. Every directed triangle of $D$ is contained in $D^+$ or in $D^-$. 

Assume that $T = (x, y, z)$ is a directed triangle with $d^-(x) = 1$ and $d^+(y) = 1$. Remove the edges of $T$ from $D$. The graph $D'$ that remains has $m' = m - 3$ edges, and the maximum number $t'$ of disjoint triangles in $D'$ satisfies $t' \leq t - 1$. By induction, $D'$ contains a directed cut $K$ of size at least $(2m' - t')/5 \geq (2m - t)/5 - 1$. Obviously, $K \cup \{xy\}$ is still a directed $P_3$-free subgraph of $D$ containing $(2m - t)/5$ edges, a contradiction. Therefore, either $d^+(w) \geq 2$ for all $w \in \{x, y, z\}$ or $d^-(w) \geq 2$ for all $w \in \{x, y, z\}$. In the first case $T \subset D^+$ and in the second case $T \subset D^-$. Thus Claim 1.1 holds.

Let $A, B \subset E(D)$ be a pair of disjoint edge sets such that every directed $P_3$ in $D$ that has one edge in $A$ has its second edge in $B$. Note that this implies that $A$ contains no directed $P_3$. We call any such pair $A, B \subset E(D)$ a reducing
pair. It is clear that if $K$ is any directed cut in the digraph $D \setminus (A \cup B)$, then $K \cup A$ is a directed $P_3$-free subgraph of $D$. The following claim will be used several times in the induction step.

Claim 1.2. $D$ has no reducing pair $A, B \subseteq E(D)$ with $|B| \leq \frac{2}{3}|A|$.

Suppose that $A, B \subseteq E(D)$ is a reducing pair with $|B| \leq \frac{2}{3}|A|$. Let $K$ be a largest directed cut in the digraph $D' = D \setminus (A \cup B)$. Then $K \cup A$ is a directed $P_3$-free subgraph of $D$. Digraph $D'$ has $m' = m - |A\cup B|$ edges, hence it follows by induction that $|K| \geq (2m' - t)/5$. We obtain

$$|K \cup A| \geq \frac{2m' - t}{5} + |A| \geq \frac{2m - t}{5} - \frac{2}{5}(|A| + |B|) + |A| \geq \frac{2m - t}{5},$$

which contradicts the assumption that $D$ is a counterexample to the theorem. Thus Claim 1.2 holds.

Claim 1.3. Each of $D^+$ and $D^-$ is a disjoint union of directed cycles. Furthermore, every vertex in $D^+$ or $D^-$ is incident with exactly one edge of $D \setminus (D^+ \cup D^-)$.

Let $C$ be any connected component of $D^-$. We show that $C$ is a directed cycle. By the definition of $D^-$, $C$ is either a rooted tree with all edges directed towards the root, or a function graph which is a rooted tree plus an edge from the root to some vertex of the tree.

If $C$ is not a directed cycle, then it is either a singleton vertex $v_0$ or it has a leaf $v_0$. In each case, because $v_0$ is in $V^-$, there exist distinct edges $e_1 = e_1v_0, e_2 = v_2v_0$ of $D$. Furthermore, $d^-(v_0) \leq 1$, thus at most one edge $f_0$ leaves $v_0$. Since $v_1, v_2$ are not in $C$, they are not in $V^-$, hence at most one edge enters each, say $f_1$ and $f_2$, respectively. Note that both edges exist in $A = \{e_1, e_2\}$, but any edge from the set $B = \{f_0, f_1, f_2\}$ might actually not exist. In either case, $A, B$ form a reducing pair with $|B| \leq \frac{2}{3}|A|$, contradicting Claim 1.2. Thus every component of $D^-$ is a directed cycle. Furthermore, if there are two edges $e_1, e_2$ of $D \setminus (D^+ \cup D^-)$ at some vertex $v_0 \in V^-$, then one obtains a contradiction using the same reducing pair.

An analogous argument shows that every connected component $C$ of $D^+$ is a directed cycle with exactly one edge of $D \setminus (D^+ \cup D^-)$ at each vertex of $C$. Thus Claim 1.3 holds.

Note that, due to Claims 1.1 and 1.3 all directed triangles of $D$ are among the cycles of $D^+$ and $D^-$. 

Claim 1.4. All directed cycles in $D^+$ and $D^-$ have odd length.

Suppose the contrary, and let $C = (x_1, x_2, \ldots, x_{2p})$ be a directed cycle, say in $D^+$. For every $i = 1, \ldots, p$ let $e_{2i-1}, e_{2i}$ be the two edges going out from $x_{2i-1}$, such that $e_{2i} = x_{2i}x_{2i+1}$, and call $y_i$ the end vertex of $e_{2i-1}$. Then $y_i \in V \setminus V^+$, therefore there is at most one edge $y_i$ going out from $y_i$. For every $i = 1, \ldots, p$, let $f_{2i-1}, f_{2i}$ be the two edges going out from $x_{2i-1}$, such that $f_{2i-1} = x_{2i-1}x_{2i}$.
Let $A = \{e_1, \ldots, e_{2p}\}$ and $B = \{f_1, \ldots, f_{2p}\} \cup \{g_1, \ldots, g_p\}$. Observe that $A, B$ are disjoint and that $B$ contains one edge of each directed $P_3$ of $D$ that has an edge in $A$. Therefore $A, B$ is a reducing pair, with $|B| \leq \frac{1}{2}|A|$, contradicting Claim 1.2. Thus Claim 1.4 holds.

**Claim 1.5.** $D^+$ and $D^-$ have the same number of vertices, say this number is $k$, and $D \setminus (D^+ \cup D^-)$ is the union of $k$ disjoint edges going from $D^+$ to $D^-$.  

We shall prove that $V^0 = V(D) \setminus (V^- \cup V^+) = \emptyset$. Assume on the contrary that $V^0 \neq \emptyset$. By the connectivity of $D$, there is a vertex $y \in V^0$ adjacent to some vertex of $D^+ \cup D^-$. By symmetry, we may assume that $yz$ is an edge for some $z \in V^-$. Let $C \subseteq D^-$ be the directed cycle containing $z$, let $C$ have length $2\ell + 1$, with $\ell \geq 1$. We call $(\ell + 1)$-set any subset $L \subset V(C)$ such that $V(C) \setminus L$ is a maximum independent set of $C$. Note that for any two vertices $x, y$ of $C$ there exists an $(\ell + 1)$-set that contains both $x, y$. 

If $d^-(y) \neq 0$, then let $e_0 = xy$, and let $g_0$ be an edge going into $x$ if it exists. (Note that $y \notin V^-$ implies $x \notin V^-$.) Let $L \subset V(C)$ be an $(\ell + 1)$-set of $C$ not containing $z$ and define:

$$
B_1 = \{f \in E(C) \mid f = ww' \text{ for some } w \in L\},
$$

$$
A = \{e_0\} \cup (E(C) \setminus B_1) \cup \{e \notin E(C) \mid e = vw \text{ for some } w \in L\},
$$

$$
B_2 = \{g \in (E(D) \setminus B_1) \mid g = uw \text{ such that } uv \in A\}.
$$

Observe that $A$ contains no directed $P_3$ and that every directed $P_3$ with one edge in $A$ has its other edge in $B = B_1 \cup B_2$. So $A, B$ is a reducing pair. Since $|A| = 2\ell + 2$, $|B_1| = \ell + 1$, and $|B_2| \leq 2\ell + 2$, we have $|B| \leq \frac{3}{2}|A|$, contradicting Claim 1.2. 

If $d^-(y) = 0$, then let $L \subset V(C)$ be an $(\ell + 1)$-set of $C$ containing $z$, and define:

$$
B_1 = \{f \in E(C) \mid f = ww' \text{ for some } w \in L\},
$$

$$
A = (E(C) \setminus B_1) \cup \{e \notin E(C) \mid e = vw \text{ for some } w \in L\},
$$

$$
B_2 = \{g \in (E(D) \setminus B_1) \mid g = uw \text{ such that } uv \in A\}.
$$

Again, $A$ and $B = B_1 \cup B_2$ form a reducing pair. We have $|A| = 2\ell + 1$, $|B_1| = \ell + 1$ and $|B_2| = 2\ell$ since no edge enters into $y$. Hence $|B| < \frac{3}{2}|A|$, contradicting Claim 1.2. Then Claim 1.3 follows from the second part of Claim 1.3.

Call $M$ the (loopless) bipartite multigraph obtained by contracting of every directed cycle into one vertex.

**Claim 1.6.** $M$ is a simple graph.

Suppose on the contrary that there are at least two edges from the cycle $C^+ \subseteq D^+$ to the cycle $C^- \subseteq D^-$. By Claim 1.2, $C^+$ is an odd cycle, thus there exist edges $ux, vy \in E(D)$ with $u, v \in V(C^+)$, $x, y \in V(C^-)$ such that $(u, b_1, \ldots, b_{2q}, v)$ is a directed subpath of $C^+$, and no vertex $b_i$ has an edge to $C^-$ ($q = 0$ means that $uv$ is an edge of $C^+$).
Let $C^-$ have length $2\ell + 1$ (it is an odd cycle by Claim 1.5). Obviously there exists an $(\ell + 1)$-set $L \subset V(C^-)$ including $x$ and excluding $y$. Define:

$$B_1 = \{ f \in E(C^-) \mid f = wz \text{ for some } w \in L \},$$

$$A_0 = (E(C^-) \setminus B_1) \cup \{ e \notin E(C^-) \mid e = zw \text{ for some } w \in L \},$$

$$B_2 = \{ g \in (E(D) \setminus B_1) \mid g = wz \text{ such that } zw' \in A_0 \}.$$

Let $e'_0 = ub_1, f'_0 = vw$ where $w \in V(C^+), g'_0 = vy$, and define:

$$A'_0 = \{ e'_0 \} \cup \{ e' \in E(D) \mid e' = b_{2i}z, ~1 \leq i \leq q \},$$

$$B'_1 = \{ f'_0 \} \cup \{ f' \in E(D) \mid f' = b_{2i-1}z, ~1 \leq i \leq q \},$$

$$B'_2 = \{ g' \in (E(D) \setminus B'_1) \mid g' = zw' \text{ such that } bz \in A'_0 \} \setminus \{ g'_0 \}.$$

Observe that the set $A = A_0 \cup A'_0$ contains no directed $P_3$, and every directed $P_3$ with an edge in $A$ has its other edge in $B = B_1 \cup B_2 \cup B'_1 \cup B'_2$ from $D$. Hence $A, B$ form a reducing pair. We have $|A_0| = 2\ell + 1$, $|B_1| = \ell + 1$, $|B_2| = 2\ell + 1, |A'_0| = 2q + 1, |B'_1| = 2q + 1, |B'_2| = q$, so $|A| = 2(\ell + q + 1)$ and $|B| = 3(\ell + q + 1) = 3|A|$, contradicting Claim 1.2. Thus Claim 1.6 holds.

Because every vertex of the contraction graph $M$ has degree at least three, $M$ has a cycle. To conclude the proof of the theorem we show that this leads to a contradiction.

Consider a shortest cycle $\gamma \subset M$, and let $\gamma = (C_{i+}^+, C_i^-, C_{i+}^-, C_{i-}^+, \ldots, C_{i+}^+, C_{i-})$, where, for each $i \in \{1, \ldots, p\}$, $C_{i+}^+ \subset D^+$ and $C_{i-}^- \subset D^-$ are cycles of $D$ of odd length. The edges of $\gamma$ correspond to a matching of $D$ from the set $\bigcup_{i=1}^p \{ u^i, v^i \}$, to the set $\bigcup_{i=1}^p \{ x^i, y^i \}$, where $u^i, v^i \in C_{i+}^+$ and $x^i, y^i \in C_{i-}^-$. Furthermore, by Claim 1.6 and since $\gamma$ has no chords in $M$, no more edges of $D$ are induced between these cycles. We may assume, so we do, that $(u^i, b_1^i, \ldots, b_{2q_i-1}^i, v^i)$, where $q_i \geq 1$, is a directed subpath of $C_{i+}^+$.

Let $2\ell_i + 1$ be the length of $C_{i-}^-$. For every $i = 1, \ldots, p$ select an $(\ell_i + 1)$-set $L_i \subset V(C_{i-}^-)$ of $C^-$ such that $x^i, y^i \in L_i$, and define the following sets:

$$B_{i1}^1 = \{ f \in E(C_{i-}^-) \mid f = wz \text{ for some } w \in L_i \},$$

$$A_{i1}^1 = (E(C_{i-}^-) \setminus B_{i1}^1) \cup \{ e \notin E(C_{i-}^-) \mid e = zw \text{ for some } w \in L_i \},$$

$$B_{i1}^2 = \{ g \in (E(D) \setminus B_{i1}^1) \mid g = wz \text{ such that } zw \in A_{i1}^1 \}.$$

Let $A_i^1 = \bigcup_{i=1}^p A_{i1}^1$ and $B_i^1 = \bigcup_{i=1}^p (B_{i1}^1 \cup B_{i1}^2)$. We have $|A_i^1| = \sum_{i=1}^p |A_{i1}^1| = \sum_{i=1}^p (2\ell_i + 1)$, and because $|B_i^1| = \ell_i + 1, |B_i^2| = 2\ell_i + 1$, we obtain $|B_i^1| = \sum_{i=1}^p (3\ell_i + 2)$.

For every $i = 1, \ldots, p$, let $e_i = u^i b_1^i, f_i = b_{2q_i-1}^i v^i$, and define sets:

$$A_{i2}^1 = \{ e_i \} \cup \{ e \in (E(D) \mid e = b_{2j}w, ~1 \leq j \leq q_i - 1 \},$$

$$B_{i2}^1 = \{ f \in (E(D) \mid f = b_{2j-1}w, ~1 \leq j \leq q_i \} \setminus \{ f_i \},$$

$$B_{i2}^2 = \{ g \in (E(D) \setminus B_{i1}^2) \mid g = wz \text{ such that } bw \in A_{i2}^1 \} \setminus \{ f_i \}.$$
Let $A^2 = \bigcup_{i=1}^{p} A_i^2$ and $B^3 = \bigcup_{i=1}^{p} (B_i^3 \cup B_i^4)$. We have $|A^2| = \sum_{i=1}^{p} |A_i^2| = \sum_{i=1}^{p} (2q_i - 1)$, and because $|B_i^3| = 2q_i - 1$ and $|B_i^4| = q_i$, we obtain $|B^3| = \sum_{i=1}^{p} (3q_i - 2)$. Observe that the sets $A = A^1 \cup A^2$ and $B = B^1 \cup B^3$ form a reducing pair. Furthermore, $|A| = \sum_{i=1}^{p} (2\ell_i + 1 + 2q_i - 1) = 2 \sum_{i=1}^{p} (\ell_i + q_i)$ and $|B| = \sum_{i=1}^{p} (3\ell_i + 2 + 3q_i - 2) = 3 \sum_{i=1}^{p} (\ell_i + q_i) = \frac{3}{2} |A|$, contradicting Claim $[a, 2]$. This concludes the proof of the theorem.

The proof of the theorem can be formulated as an algorithm which, given any digraph $D \in D(1, 1)$ with $m$ edges and at most $t$ disjoint directed triangles, constructs a directed cut $K$ of size at least $(2m - t)/5$. We sketch such an algorithm here. Start from $K := \emptyset$. Then apply the following general step. Find the subgraphs $D^+$ and $D^-$. If there is a directed triangle that is not included in $D^+$ or $D^-$, with the notation of Claim $[a, 3]$, then set $K := K \cup \{xy\}$ and iterate with the subgraph $D \setminus \{xy, yz, zx\}$. (When iterating, the subgraphs $D^+, D^-$ must be updated.) If there is no such directed triangle, then either $D$ violates one of Claims $[a, 3, \ref{cor:3}]$ or $D$ satisfies the conditions described after the proof of Claim $[a, 4]$, and in either case, the proof of the theorem shows how to find a reducing pair $(A, B)$. Then set $K := K \cup A$ and iterate the general step with the subgraph $D \setminus (A \cup B)$. The algorithm terminates when $D$ becomes edgeless. Then at termination $K$ is a directed cut of size at least $(2m - t)/5$. It is easy to see that all the operations (updating $D^+$ and $D^-$, finding a directed triangle, checking whether $D$ violates one of the claims, determining the structure described after the proof of Claim $[a, 4]$) can be done in polynomial time, and there are at most $m$ iterations. Thus we obtain:

**Corollary 2.** There is a polynomial time algorithm which, given any digraph $D \in D(1, 1)$ with $m$ edges and at most $t$ disjoint directed triangles, finds a directed cut in $D$ of size at least $(2m - t)/5$.

**Corollary 3.** If $D \in D(1, 1)$ has $m$ edges, then it contains a directed cut of size at least $m/3$. Moreover $D$ has no directed cut of size larger than $m/3$ if and only if $D$ is the union of disjoint directed triangles.

**Proof.** The number of pairwise disjoint directed triangles satisfies $t \leq m/3$, with equality if and only if $D$ is a union of disjoint directed triangles. Now the claim follows by Theorem $[a, 1]$, because $(2m - t)/5 \geq (2m - m/3)/5 = m/3$.

**Corollary 4.** If $D \in D(1, 1)$ has $m$ edges and no directed triangle, then it contains a directed cut of size at least $2m/5$.

Results by Bondy and Locke $[4]$ on the bipartite density of (undirected) subcubic graphs are reminiscent of our investigations concerning $D(1, 1)$. They proved in $[4]$ that a triangle-free subcubic graph has a bipartite subgraph of size at least $4m/5$. Observe that any triangle-free digraph of maximum degree at most three belongs to $D(1, 1)$, and it is obtained from a subcubic graph by orienting its edges. Hence their result implies that such a $D$ has a directed cut of size at least $2m/5$, the half of $4m/5$. Corollary $[a]$ shows that this bound
is valid for the much larger class of digraphs in $D(1, 1)$ containing no directed triangle.

Now we show that the lower bound $m/3$ in Corollary 3 can be surpassed for connected digraphs of $D(1, 1)$.

**Theorem 5.** If $D \in D(1, 1)$ is a connected digraph with $m$ edges, and $D$ is not a triangle, then it contains a directed cut of size at least $\frac{7m}{20}$.

**Proof.** The proof works by induction on $m$. Let $t$ be the maximum number of pairwise disjoint directed triangles of $D$. By the hypothesis, we have $t = 0$ if $m \leq 3$ and $t \leq 1$ if $m = 4, 5, 6$. Thus by Theorem 1 there is a cut of size at least $1, 1, 2, 2, 2, 3$, respectively, for $m = 1, \ldots, 6$, which matches the corresponding value of $\lceil \frac{7m}{20} \rceil$. Now let $m \geq 7$, and assume that the claim is true for connected graphs with strictly less than $m$ edges. Observe that any $t$ disjoint directed triangles of $D$ have a total of $3t$ edges, furthermore, by the connectivity of $D$, there are at least $t - 1$ more edges between these triangles. Hence we have $m \geq 4t - 1$.

If $m > 4t - 1$, or equivalently, if $t \leq m/4$, then by Theorem 1, $D$ has a cut of size at least $(2m - t)/5 \geq (2m - m/4)/5 = \frac{7m}{20}$ edges as stated.

Assume now that $m = 4t - 1$. So $D$ consists of $t$ disjoint directed triangles connected by $t - 1$ edges in a tree-like manner. Since we cannot have $t = 1$ and $m = 3$, we have $t \geq 2$. So there is a directed triangle $T = (x, y, z)$ that is adjacent to exactly one edge, say $xx'$, which is adjacent to another directed triangle $T' = (x', y', z')$. (The symmetric argument applies if the orientation of the edge between $T$ and $T'$ is $x'x$.) Removing from $D$ the vertices and edges of $T$ together with the two edges $xx', x'y'$, we obtain a connected digraph $D'$ with $m' = m - 5 \geq 2$ edges. By the induction hypothesis, $D'$ has a cut $H'$ of size at least $\frac{7m'}{20} = (7m - 35)/20 > \frac{7m}{20} - 2$ edges. Clearly $H' \cup \{xx', yz\}$ has no directed $P_3$, which yields a cut of size at least $\frac{7m}{20}$ in $D$.

Just like with Theorem 1, the proof of Theorem 5 can be formulated easily as a polynomial time algorithm (we omit the details). So we have:

**Corollary 6.** There is a polynomial time algorithm which, given any digraph $D \in D(1, 1)$ with $m$ edges, such that no component of $D$ is a directed triangle, finds a directed cut in $D$ of size at least $\frac{7m}{20}$.

### 3 Decompositions of $D(k, k)$

The problem of covering the edges of a digraph with cuts was proposed in [1]. Upper bounds were given for digraphs in $D(k, \ell)$, and the only exact value was determined for $k = \ell = 1$.

**Theorem 7 ([1]).** The edge set of any digraph $D \in D(1, 1)$ can be decomposed into at most three cuts.

□
**Theorem 8.** For integers $p_1, p_2 \geq 0$, the edge set of every digraph $D \in D(p_1 + p_2, p_1 + p_2)$ can be decomposed into two subgraphs $D_1 \in D(p_1, p_1)$ and $D_2 \in D(p_2, p_2)$.

**Proof.** Since $D$ is in $D(p_1 + p_2, p_1 + p_2)$, its vertex set $V(D)$ can be partitioned into two sets $X, Y$ such that every vertex $x \in X$ satisfies $d^-(x) \leq p_1 + p_2$ and every vertex $y \in Y$ satisfies $d^+(y) \leq p_1 + p_2$. Consider the set of edges $B = \{yx \in E(D) | y \in Y, x \in X\}$. By the definition of $X$ and $Y$, in the bipartite graph $(X, Y; B)$ every vertex has degree at most $p_1 + p_2$. By a classical corollary of the Kőnig-Hall theorem (see e.g., [7, Prop. 5.3.1]), the edges of $B$ can be colored with $p_1 + p_2$ colors so that any two adjacent edges have different colors. Let $B_1$ be the set of edges of $B$ with the first $p_1$ colors and $B_2$ be the set of edges of $B$ with the remaining colors.

For every vertex $x \in X$, the set $E^- (x)$ of edges with end $x$ has size at most $p_1 + p_2$, so it can be partitioned into two sets $E_1 (x)$ and $E_2 (x)$ such that, for $j = 1, 2$, $|E_j (x)| \leq p_j$ and $E^-(x) \cap B_j \subseteq E_j (x)$. Likewise, for every vertex $y \in Y$, the set $E^+(y)$ of edges with origin $y$ has size at most $p_1 + p_2$, so it can be partitioned into two sets $E_1 (y)$ and $E_2 (y)$ such that, for $j = 1, 2$, $|E_j (y)| \leq p_j$ and $E^+(y) \cap B_j \subseteq E_j (y)$.

Finally let the set $\{xy \in E(D) | x \in X, y \in Y\}$ be partitioned arbitrarily into two sets $F_1, F_2$. Now, for $j = 1, 2$, let $D_j$ be the subgraph of $D$ whose edge set is $B_j \cup F_j \cup \bigcup_{x \in V} E_j (x)$. The definition of these sets implies that each edge of $D$ lies in exactly one of $D_1, D_2$ and that $D_j \in D(p_j, p_j)$ for $j = 1, 2$. More precisely, for $j = 1, 2$, in $D_j$ every vertex $x \in X$ satisfies $d^-(x) \leq p_j$ and every vertex $y \in Y$ satisfies $d^+(y) \leq p_j$.

**Corollary 9.** The edges of every digraph $D \in D(2, 2)$ can be decomposed into two subgraphs $D_1, D_2 \in D(1, 1)$.

From Corollary 9 and Theorem 8 it follows that every digraph $D \in D(2, 2)$ can be covered with six directed cuts. If there was a decomposition of $D$ into a cut and a digraph in $D(1, 1)$, then $D$ would have a cut cover only with four cuts, by Theorem 8 again. Our next example shows that such a decomposition is not always possible.

**Example 2.** Take two disjoint copies of a regular tournament on five vertices, $G_1, G_2$, and include all 25 edges directed from $G_1$ to $G_2$. Thus we obtain a digraph $H \in D(2, 2)$. Assume that $K \subseteq E(H)$ is a cut such that $H' = H \setminus K$ is in $D(1, 1)$. The regular tournament has no cut with more than three edges, hence $G_1$ has a vertex $v_0$ such that every edge going into $v_0$ is in $E(H) \setminus K$ and at least one edge going out of $v_0$ is in $E(H) \setminus K$. Thus $d^-_{H'} (v_0) = 2$, which implies that $v_0 z \in K$ for all $z \in V(G_2)$ in order to obtain $d^+_{H'} (v_0) \leq 1$. Then it follows that no edge of $G_2$ belongs to $K$, thus $d^-_{H'} (z) = 2$ and $d^+_{H'} (z) \geq 2$ for all $z \in V(G_2)$, a contradiction.

How large a subgraph belonging to $D(1, 1)$ can be found in a digraph $D \in D(2, 2)$? Corollary 9 implies that $D$ with $m$ edges contains a subgraph in $D(1, 1)$
with at least \( m/2 \) edges. A larger bound will follow from our more general result.

**Theorem 10.** Every digraph \( D \in D(k, k) \) with \( m \) edges has a subgraph belonging to \( D(k - 1, k - 1) \) with at least \((2k - 1)m/(2k + 1)\) edges.

Proof. Let \( W = \{v \in V(D) | d_D^+(v) \leq k \} \) and \( B = \{v \in V(D) | d_D^-(v) \leq k \} \). Because \( D \in D(k, k) \), we have \( V(D) = W \cup B \). We say that \( v \in W \) is white, and \( v \in B \) is black; note that a vertex may have both colors. An edge \( xy \in E(D) \) is called a black tail arrow if \( x \in B \), and it is called a white head arrow if \( y \in W \). Note that an edge can be both a black tail and a white head arrow. Observe the symmetry of the colors with respect to reversing all arrows in \( D \). Due to this symmetry, if a property is verified for white vertices, then the analogous property is true for black vertices with directions reversed.

Let \( R \subseteq E(D) \) be a set of edges such that (a) the graph \( D' = D \setminus R \) is in \( D(k - 1, k - 1) \), (b) \( R \) is minimum among all sets with property (a), and (c) \( R \) has the maximum number of black tail arrows and white head arrows (each arrow counted once) among all sets that satisfy (a) and (b). Clearly such a set \( R \) exists.

For each edge \( e = xy \in R \), we define a critical vertex of \( e \) as follows:

- \( x \) is a critical vertex for \( e = xy \in R \) if \( d^+_D(x) = k - 1 \) and \( d^+_D(x) \geq k \);
- \( y \) is a critical vertex for \( e = xy \in R \) if \( d^-_D(y) = k - 1 \) and \( d^-_D(y) \geq k \).

The minimality of \( R \) means that at least one of \( x, y \) is critical for each edge \( e = xy \in R \). Note that both \( x, y \) may be critical for \( e \). For each \( e \in R \), let \( \text{Crit}(e) \subseteq \{x, y\} \) be the set of critical vertices of \( e \). For any subset \( X \subseteq R \), define \( \text{Crit}(X) = \cup_{e \in X} \text{Crit}(e) \). From here on, the word critical vertex refers to elements of \( \text{Crit}(R) \). All critical vertices are in the set \( \{v \in V(D) | d_D^+(v), d_D^-(v) \geq k \} \), however not every vertex in that set is critical for some edge of \( R \). The main point of the proof is to establish that:

**Claim 10.1.** \( |\text{Crit}(R)| \geq |R| \).

Assume we already know that \( |\text{Crit}(R)| \geq |R| \). Then the definition of critical vertices implies that, for every \( v \in \text{Crit}(R) \), there are at least \( 2k - 1 \) edges not in \( R \) and incident with \( v \). Thus for the size of \( D \) we have the bound \( m \geq |R| + (2k - 1)|R|/2 \), hence \( |R| \leq 2m/(2k + 1) \). So \( D' \) has at least \( m - |R| \geq (2k - 1)m/(2k + 1) \) edges, and the theorem follows. Therefore the rest of the proof consists in proving Claim 10.1.

The definition of critical vertices implies easily the following two claims, whose proof is omitted.

**Claim 10.2.** If \( x \in B \) and \( d_D^+(x) \geq 2 \), then \( x \notin \text{Crit}(R) \). By symmetry, If \( x \in W \) and \( d_D^-(x) \geq 2 \), then \( x \notin \text{Crit}(R) \).

**Claim 10.3.** If \( e = xy \in R \), \( y \in B \), and \( yz \in R \), then \( \text{Crit}(e) = \{x\} \). By symmetry, if \( e = xy \in R \), \( x \in W \), and \( ux \in R \), then \( \text{Crit}(e) = \{y\} \).
Now we examine the subgraph formed by $R$. Let $A \subseteq R$ be any connected component of $R$ (we use $A$ and $R$ to denote the digraphs defined by the edges in $A$ and $R$, respectively). Note that $\text{Crit}(R) = \text{Crit}(A) \cup \text{Crit}(R \setminus A)$.

Claim 10.4. If $A$ has no cycle, then $|\text{Crit}(A)| \geq |V(A)| - 1 = |A|$.

In this case $A$ is a tree with $|A| + 1$ vertices. We show that at most one non critical vertex may exist in $A$. Suppose on the contrary that $u, v$ are two non-critical vertices in $A$, and let $P = (u, \ldots , v)$ be the (unique) shortest chain between them in $A$. Observe that $P$ has length at least 2 (for otherwise its unique edge $uv$ would satisfy $\text{Crit}(uv) = \emptyset$), and that the inclusion into $D'$ of the two edges of $P$ incident to $u$ and $v$ does not increase their corresponding indegree or outdegree above $k - 1$.

For every white vertex $w$ of $V(P) \setminus \{u, v\}$ select a white head arrow $xw$, and for every black vertex $z$ of $V(P) \setminus \{u, v\}$ select a black tail arrow $zx$ (for two-colored vertices take one such arrow arbitrarily). Let $F$ be the set of selected arrows. So $|F| \leq |P| - 1$. Define $R^* = (R \setminus P) \cup F$. The graph $D^* = D \setminus R^*$ belongs to $D(k - 1, k - 1)$, because the outdegree of every black vertex of $V(P) \setminus \{u, v\}$, and the indegree of every white vertex of $V(P) \setminus \{u, v\}$ is at most $k - 1$, furthermore the corresponding degrees of $u$ and $v$ do not increase above $k - 1$. The set $R^*$ satisfies $|R^*| \leq |R| - 1$, contradicting the minimality of $R$. Thus $A$ has at most one non-critical vertex, and Claim 10.4 holds.

Now we consider an arbitrary cycle $C$ in $R$ (if any).

Claim 10.5. $C$ has no edge $e = xy$ with $x \in W \setminus B$ and $y \in B \setminus W$.

Suppose that there is such an edge $e = xy$. Note that $e$ is neither a white head arrow nor a black tail arrow in $C$. Hence $C$ has at most $|C| - 1$ white head and black tail arrows. For every white vertex $w \in V(C)$ select a white head arrow $xw$, and for every black vertex $v \in V(C)$ select a black tail arrow $vv$ (for two-colored vertices select one arrow arbitrarily). Let $F$ be the set of $|C|$ selected edges, and define $R^* = (R \setminus C) \cup F$. The set $R^*$ satisfies $|R^*| \leq |R|$, and contains more white head and black tail arrows than $R$. Furthermore, the graph $D^* = D \setminus R^*$ belongs to $D(k - 1, k - 1)$, because the outdegree of every black vertex of $C$, and the indegree of every white vertex of $C$ is at most $k - 1$. This contradicts the choice of $R$. Thus Claim 10.5 holds.

Claim 10.6. Let $u, x, y, v$ be four consecutive vertices of $C$.

(1) If $ux, xy, vy \in C$ and $x$ is black, then $y$ is not white.

(2) If $ux, xy, vy \in C$, then either $x$ or $y$ is not white.

(3) If $ux, xy, vy \in C$ and $y$ is black, then $x$ is not white.

Suppose on the contrary that any of (1), (2), (3) fails. Then, in either case, the edge $e = xy$ satisfies $\text{Crit}(e) = \emptyset$, which contradicts the minimality of $R$. Thus Claim 10.6 holds.

Claim 10.7. $C$ is a directed cycle and it is monochromatic, i.e., its vertices are either all in $W \setminus B$ or all in $B \setminus W$. 

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Suppose first that \( C \) is not a directed cycle, and consider a longest directed subpath \((x_1, \ldots, x_q)\) of \( C \), where \( q \geq 2 \).

Suppose that \( q = 2 \), i.e., the directions of the edges alternate on \( C \). Let \( z_1w_1, z_1w_2, z_2w_2 \in C \). If \( z_1 \notin W \setminus B \), then by Claim 10.3 we have \( w_1, w_2 \in W \). If \( z_1 \notin B \), then Claim 10.6 (1) implies that \( w_2 \notin B \setminus W \), and \( z_2 \in B \) follows by Claim 10.5. Thus we obtain that either \( w_1, z_1, w_2 \in W \) or (symmetrically) \( z_1, w_2, z_2 \in B \). We show a contradiction in the first case, then, by symmetry, the second case is impossible as well. So assume that \( w_1, z_1, w_2 \in W \) and set \( e_i = z_1w_i, i = 1, 2 \). Select an arbitrary white head arrow \( f = xz_1 \in E(D) \). The set \( R^* = (R \setminus \{e_1, e_2\}) \cup \{f\} \) satisfies \( |R^*| \leq |R| - 1 \), and the graph \( D^* = D \setminus R^* \) belongs to \( D(k - 1, k - 1) \), because \( d_{D^*}(w_i) = d_{D^*}(w_i) \leq k - 1 \) for \( i = 1, 2 \). This contradicts the minimality of \( R \). Therefore \( q \geq 3 \).

By Claim 10.6 (2), either \( x_q \) or \( x_{q-1} \) is not white on the directed path \((x_1, x_2, \ldots, x_q)\), for \( q \geq 3 \). If \( x_q \notin B \setminus W \), then \( x_{q-1} \in B \) by Claim 10.5. Thus that in each case \( x_{q-1} \) is black.

Suppose that \( q = 3 \). Let \( e_1 = x_1x_2, e_2 = x_1y_2 \in C \), with \( e_1 \neq e_2 \), and let \( y_3 \) be the second neighbor of \( y_2 \) on \( C \) different from \( x_1 \).

Assume first that \( y_2y_3 \in E(D) \). Then, by the argument above, \( x_2 \) and \( y_2 \) are both black. Observe that \( x_1 \notin W \setminus B \), since otherwise the edges \( e_1 \) and \( e_2 \) have no critical vertices. Now select an arbitrary white head arrow \( f = xz_1 \in E(D) \). The set \( R^* = (R \setminus \{e_1, e_2\}) \cup \{f\} \) satisfies \( |R^*| = |R| - 1 \), and the graph \( D^* = D \setminus R^* \) belongs to \( D(k - 1, k - 1) \), contradicting the minimality of \( R \).

Assume now that \( y_2y_3 \notin E(D) \) (where \( y_2y_3 \) might coincide with \( x_2x_3 \), if \( C \) is a triangle). As before, we have \( x_2 \in B \) and \( x_1 \in W \setminus B \). Then, by Claim 10.5, \( y_2 \in W \). Selecting a white head arrow \( f \) at \( x_1 \) and defining the set \( R^* = (R \setminus \{e_1, e_2\}) \cup \{f\} \) we obtain a contradiction in the same way as before. Therefore \( q \geq 4 \).

We already know that \( x_{q-1} \) is black. Hence \( x_{q-2} \notin B \setminus W \) by Claim 10.6 (3). Applying Claim 10.6 (3) repeatedly, we obtain that \( x_{q-2}, \ldots, x_2 \) are in \( B \setminus W \). Then we have \( x_1 \in B \) by Claim 10.5. Hence the edge \( x_1x_2 \) has no critical vertex, because \( d_{D^*}(x_1) \leq k - 2 \) and \( d_{D^*}(x_2) \leq k - 1 \), contradicting the minimality of \( R \). So we have established that \( C \) is a directed cycle.

Now assume without loss of generality that some vertex \( y \) of \( C \) is black. By Claim 10.6 (3), the predecessor \( x \in V(C) \) of \( y \) is not white, i.e., it is in \( B \setminus W \). Applying Claim 10.6 (3) repeatedly we obtain that every vertex of \( C \) is in \( B \setminus W \). So \( C \) is monochromatic. Thus Claim 10.7 holds.

Claim 10.8. If a component \( A \) of \( R \) contains a cycle, then \( A \) is unicyclic and \( |\text{Crit}(A)| = |V(A)| = |A| \).

Let \( A \) contain a cycle \( C \). By Claim 10.7 and by symmetry, \( C \) is a black directed cycle.

For every edge \( e = xy \in C \), by Claim 10.3, we have \( x \in \text{Crit}(e) \), and by Claim 10.2 we have \( d^R_R(x) = 1 \). Let \( A_0 \) be a subgraph of \( A \) that is maximal with the following property: \( A_0 \) contains \( C \), for every \( x \in V(A_0) \) there is a directed path in \( A_0 \) from \( x \) to some vertex of \( C \), and every vertex \( x \in V(A_0) \) is
black and satisfies \( d_R^+(x) = 1 \). Let us prove that \( A_0 = A \). Note that \( A_0 \) exists, because \( C \) itself satisfies all the required properties.

Suppose that \( A_0 \neq A \). Then, since \( A \) is connected, there is a vertex \( x \in V(A) \setminus V(A_0) \) that is adjacent to some \( y \in V(A_0) \). Because \( d_R^+(y) = 1 \) and \( d_R^-(y) = 1 \), we have \( e = xy \in A \). Since \( y \) is black, Claim 10.3 implies \( \text{Crit}(e) = \{x\} \). Observe that \( A_0 \) contains a directed \( P_3 \) from \( y \) containing black vertices. Hence by Claim 10.4 (3), we have \( y \in B \setminus W \). If \( x \in W \setminus B \), then let \( f = ux \in R \) be any white head arrow. Define \( R^* = (R \setminus \{e\}) \cup \{f\} \). The digraph \( D^* = D \setminus R^* \) belongs to \( D(k-1, k-1) \), because \( d_{D^*}^+(y) = d_{D^*}^-(y) \leq k-1 \), and \( d_{D^*}^- (x) \leq k-1 \). This contradicts the choice of \( R \). So \( x \) is black. Because \( x \) is black and \( x \in \text{Crit}(e) \), Claim 10.2 implies \( d_R^+(x) = 1 \). Hence one could include \( x \) to \( A_0 \), contradicting the maximality of \( A_0 \). Therefore \( A_0 = A \).

Since \( A = A_0 \), \( C \) is the only cycle in \( A \) and every vertex \( x \) of \( A \) is black and satisfies \( d_R^+(x) = 1 \). Claim 10.3 implies that every vertex of \( A \) is a critical vertex of its outgoing edge. So \( |\text{Crit}(A)| = |V(A)| = |A| \), and Claim 10.8 holds.

Claims 10.4 and 10.8 show that \( |\text{Crit}(A)| \geq |A| \) is true for every connected component \( A \). If \( A_1, \ldots , A_t \) are the components of \( R \), we have clearly \( \text{Crit}(R) = \text{Crit}(A_1) \cup \cdots \cup \text{Crit}(A_t) \). Thus we obtain \( |\text{Crit}(R)| \geq |R| \), which proves Claim 10.1. This concludes the proof of the theorem. \( \boxdot \)

The regular tournament on \( 2k+1 \) vertices has indegree equal to outdegree for every vertex, hence it is in \( D(k,k) \). To obtain a subgraph belonging to \( D(k-1,k-1) \) one has to remove at least \( k+1 \) from its \( m = \binom{2k+1}{2} \) edges. This shows that the tournament has no subgraph in \( D(k-1,k-1) \) containing more than \( \binom{2k+1}{2} - k+1 = m - (1+1/k)m/(2k+1) = (2k-1-k)m/(2k+1) \) edges.

Corollary 11. Every digraph \( D \in D(2,2) \) with \( m \) edges contains a subgraph belonging to \( D(1,1) \) with at least \( 3m/5 \) edges. \( \boxdot \)

We note that for \( k = 1 \), Theorem 10 yields another proof that every digraph \( D \in D(1,1) \) with \( m \) edges contains a directed cut of size at least \( m/3 \) (cf. Corollary 3).

4 Cuts in \( D(k,k) \)

In [18] it was observed that the \( k \)-regular orientation of the complete graph on \( 2k+1 \) vertices has no directed cut of size more than \( \frac{1}{4} + \frac{1}{2k+1} \binom{2k+1}{2} \). Consequently, in a digraph \( D \in D(k,k) \) with \( m \) edges one cannot guarantee a directed cut of size larger than \( \frac{1}{4} + \frac{1}{2k+1} m \). It was proved in [18] that every digraph with outdegree at most \( k \) does contain a directed cut of that size. Using the same methods we show that it is also true for the acyclic members of \( D(k,k) \).

The basic tool is a lemma in [10] that is proved there by elementary counting.

Lemma 1 ([10]). If a \( \gamma \)-colorable graph \( G \) has \( m \) edges, then it has a bipartite partial graph with at least \( \frac{\gamma^2}{4} |V(G)| \) edges. \( \Box \)
Theorem 12. If $D \in D(k, k)$ is acyclic and has $m$ edges, then $D$ contains a directed cut of size at least $(\frac{1}{4} + \frac{1}{3k+1})m$.

Proof. Let $D^+$ be the subgraph of $D$ induced by the set $X = \{v \in V(D) | d^+(v) \leq k\}$ and let $D^-$ be the subgraph of $D$ induced by $V(D) \setminus X$. Because $D$ is acyclic, every subgraph of $D^+$ has a source, thus its underlying graph $G^+$ is $k$-degenerate. Similarly, every subgraph of $D^-$ has a sink, thus its underlying graph $G^-$ is $k$-degenerate. Consequently, both graphs $G^+$ and $G^-$ are $(k+1)$-colorable, therefore the underlying graph $G$ of $D$ is $(2k+2)$-colorable.

Applying Lemma 1 with $\gamma = 2k+2$, we obtain a bipartite partial graph of $G$ with $\frac{k-1}{2k+2}m$ edges. In $D$ at least half of the edges of that bipartite graph form a directed cut of size at least $\frac{k+1}{4k+2}m = \left(\frac{1}{4} + \frac{1}{3k+1}\right)m$. \hfill \box

We do not know whether Theorem 12 remains true for all digraphs in $D(k, k)$, and for every $k$. The coefficients are $1/3$, $3/10$ and $2/7$ for $k = 1, 2,$ and $3$, respectively. By Theorem 3, a digraph $D \in D(1, 1)$ of size $m$ has a cut with at most $m/3$ edges. The theorem below answers the question affirmatively for $D(2, 2)$.

Theorem 13. Every digraph $D \in D(2, 2)$ with $m$ edges has a directed cut of size at least $3m/10$.

Proof. We prove the theorem by induction on $m$. For $m = 1$ the theorem is true. Now suppose that $m \geq 2$ and that the theorem holds for every digraph with at most $m - 1$ edges. Since $D$ is in $D(2, 2)$, its vertex set can be partitioned into two sets $X, Y$ such that every vertex $x \in X$ satisfies $d^-(x) \leq 2$ and every vertex $y \in Y$ satisfies $d^+(y) \leq 2$. Consider the set of edges $F = \{xy \in E | x \in X, y \in Y\}$.

First suppose that the underlying bipartite graph $(X, Y; F)$ contains a cycle. Let $C$ be any such cycle, say with length $2k$, let $X_C$ and $Y_C$ be the set of vertices of $C$ that lie in $X$ and $Y$ respectively, and let $F_C$ be the set of edges of $C$. So $|F_C| = 2k$. Let $E_C$ be the set of edges such that either their end is in $X_C$ or their origin is in $Y_C$. By the definition of $X, Y$ and the fact that $D$ is in $D(2, 2)$, we have $|E_C| \leq 4k$. Consider the digraph $D' = D \setminus (E_C \cup F_C)$. Clearly, $D' \in D(2, 2)$, and the number $m'$ of edges of $D'$ satisfies $m' \geq m - 6k$. By the induction hypothesis, $D'$ has a directed cut of size at least $3m'/10$. If the edges of $F_C$ are added to such a directed cut, we obtain a directed cut of $D$, because $D'$ does not contain any edge of $E_C$. This directed cut of $D$ has size at least $3m'/10 + |F_C| \geq 3(m - 6k)/10 + 2k \geq 3m/10$. So the theorem holds for $D$.

Now suppose that the bipartite graph $(X, Y; F)$ does not contain any cycle. Thus $|F| \leq n - 1$, where $n$ is the number of vertices of $D$. By the definition of $X$ and $Y$, we have $m \leq 2|X| + 2|Y| + |F| \leq 2n + n - 1 = 3n - 1$, which implies that $D$ has a vertex $v$ of degree at most 5. Actually the same argument can be repeated with $D \setminus \{v\}$, and so on. Thus the underlying graph of $D$ is 5-degenerate and therefore has chromatic number at most 6. Applying Lemma 1 with $\gamma = 6$ we obtain a bipartite subgraph with $3/5$ edges, thus $D$ has a directed cut of size at least $3m/10$. \hfill \box
5 Problems

Let $c_{\text{max}}$ be the ratio of the maximum directed cut size to the edge count $m$ of a digraph. For connected digraphs of $D(1,1)$, Theorem 5 improves the basic estimation $c_{\text{max}} \geq 1/3$ to $c_{\text{max}} \geq 7/20$ provided $m > 3$. On the other hand, Example 1 before Theorem 1 shows infinitely many connected digraphs of $D(1,1)$ with $c_{\text{max}} < 3/8$. We conjecture that the bound $c_{\text{max}} \geq 7/20$ can be improved to $3/8$ in the limit in the following sense.

Problem 1. For every $\varepsilon > 0$, there exists a constant $m_\varepsilon$ such that $c_{\text{max}} > 3/8 - \varepsilon$ holds for every connected digraph of $D(1,1)$ with $m > m_\varepsilon$ edges.

At some point of the investigation in $D(1,1)$ we observed that the presence of source or sink vertices of the digraph increases the size of a maximum directed cut. Corollary 4 might have the following sharpening.

Problem 2. If a connected digraph $D \in D(1,1)$ with $m$ edges contains no directed triangle and has $s$ vertices with indegree or outdegree zero, then $D$ has a directed cut of size at least $(2m + s)/5$.

Bondy and Locke [4] proved that a triangle-free subcubic graph has a cut (a bipartite subgraph) of size at least $4m/5$. A characterization of all extremal graphs for that bound was given by Xu and Yu in [3]. The problem of characterizing the extremal graphs for the bound of Corollary 4 remains open.

Problem 3. Determine the list of all digraphs $D \in D(1,1)$ of size $m$ that contain no directed triangle and have no directed cut with more than $2m/5$ edges.

Bondy and Locke’s result in [4] consists of a polynomial time algorithm that finds a cut with at least $4m/5$ edges in a triangle-free subcubic graph. It is known that finding a maximum cut is NP-hard even in the restricted family of triangle-free cubic graphs (see Yannakakis [2]). Even the approximation of the max cut problem in cubic graphs within the ratio of 0.997 is NP-hard (see Berman and Karpinski [3]). On the other hand, Halperin, Livnat and Zwick [9] give a polynomial time approximation algorithm with ratio 0.9326.

Concerning digraphs in $D(1,1)$, Corollary 6 gives a polynomial time algorithm that produces a cut of size at least $7m/20$ in every digraph in $D(1,1)$ of which no component is a directed triangle; and so this is an approximation algorithm with ratio 0.35. Can a better ratio be obtained? Actually, as far as we know, none of the known results implies that computing the exact value of a maximum directed cut is NP-hard in $D(1,1)$. So we ask:

Problem 4. What is the complexity status of computing the size of a maximum directed cut in a digraph of $D(1,1)$? If it is NP-hard, what is the best value of $\varepsilon$ for which there is a polynomial time approximation algorithm with ratio $1 - \varepsilon$ for this problem?
The same problem can be posed for digraphs of $D(1,1)$ with no directed triangle, or with no triangle at all.

How large a subgraph belonging to $D(1,1)$ can be found in a digraph $D \in D(2,2)$? Corollary 11 says that $D$ with $m$ edges contains a subgraph in $D(1,1)$ with at least $3m/5$ edges. This lower bound is probably not sharp.

**Problem 5.** Determine the largest constant $\lambda$ such that in every digraph $D \in D(2,2)$ with $m$ edges there exists a subgraph $D' \in D(1,1)$ of size at least $\lambda m$.

If $D$ is the regular tournament on five vertices, then $D \in D(2,2)$ and one needs to remove at least three edges to obtain a subgraph $D' \in D(1,1)$. This shows that in the problem above $\lambda \leq 7/10$.

From a result in [1] it follows that the edges of every graph $D \in D(2,2)$ can be decomposed into at most five directed cuts. Furthermore, four cuts are sufficient if $D$ is acyclic. The regular tournament on five vertices shows that four cuts might be necessary. Indeed, it has 10 edges, and the size of a directed cut is at most 3. No example has been found to show that five directed cuts are necessary.

**Problem 6.** The edges of every digraph $D \in D(2,2)$ can be decomposed into at most four directed cuts.

Several problems remain open in $D(2,2)$ pertaining to the ratio $c_{\text{max}}$.

**Problem 7.** If $D \in D(2,2)$ has $m$ edges and contains no copy of the regular tournament on five vertices, then $D$ has a directed cut of size at least $m/3$.

We do not know whether Theorem 12 pertaining to acyclic digraphs remains true for all digraphs in $D(k,k)$, and for every $k$. The coefficients $\frac{1}{2} + \frac{1}{8k+4}$ are equal to $1/3, 3/10, and 2/7$ for $k = 1, 2, and 3$, respectively. By Corollary 8 and by Theorem 13, a digraph $D$ of size $m$ has a cut with $m/3$ and $3m/10$ edges, respectively for $D \in D(1,1)$ and $D \in D(2,2)$. The next case $k = 3$ is proposed here as a question. It is quite possible that the answer is negative. Even if it is not the case we conjecture that Theorem 12 does not extend for every $k$.

**Problem 8.** Is it true that every digraph of $D(3,3)$ with $m$ edges contains a directed cut of size at least $2m/7$?

Digraphs with maximum outdegree $k$ satisfy $c_{\text{max}} \geq \frac{1}{2} + \frac{1}{8k+4}$, and this is the best bound, as shown in [1]. It is worth noting that the same bound was obtained here in Theorem 12 for acyclic members of $D(k,k)$. Furthermore, the regular tournament on $2k+1$ vertices is an example of a digraph with no directed cut larger than $(\frac{1}{2} + \frac{1}{8k+2})m$. We believe that in the larger family $D(k,k)$ there are more examples showing that this bound cannot be achieved, provided $k$ is large enough.

**Problem 9.** There exists a $k_0$ such that for every $k \geq k_0$ there are digraphs in $D(k,k)$ with $c_{\text{max}} < \frac{1}{2} + \frac{1}{8k+4}$. 
In Theorem 10 we are dealing with the largest subgraph of $D \in D(k, k)$ that belongs to the “lower” class $D(k-1, k-1)$. This leads naturally to the investigation of the minimum sets $R \subset E(D)$ to be removed from $D$ in order to lower its class. The proof of Theorem 11 suggests that such minimum sets considered as digraphs have a particular structure reminiscent of forests. Repeating the procedure, one obtains a decomposition of the original digraph $D \in D(k, k)$ into at most $k$ of these structures.

Practical applications motivate the study of decompositions of digraphs into directed stars (see [3]). The directed star arboricity (dst) introduced in [8] is defined as the minimum number of outstar forests (also called galaxy) the edge set of a digraph can be partitioned. For instance it is proved in [3] that a digraph $D$ with indegree at most $k$ has a decomposition into $k$ outforests plus one galaxy. This result implies $dst(D) \leq 2k + 1$, and it is conjectured in [3] that $2k$ is the tight bound, for $k \geq 2$.

As a general problem we propose here a similar decomposition theory of the digraphs of $D(k, k)$ into appropriate forest-like structures.

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References


