



On sets represented by partitions

Jean-Christophe Aval

► **To cite this version:**

Jean-Christophe Aval. On sets represented by partitions. European Journal of Combinatorics, Elsevier, 1999, 20, pp.317-320. <hal-00185506>

HAL Id: hal-00185506

<https://hal.archives-ouvertes.fr/hal-00185506>

Submitted on 6 Nov 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On sets represented by partitions

Jean-Christophe Aval

Abstract: We prove a lemma that is useful to get upper bounds for the number of partitions without a given subsum. From this we can deduce an improved upper bound for the number of sets represented by the (unrestricted or into unequal parts) partitions of an integer n .

1 Introduction

Let n be an integer and let

$$n = n_1 + n_2 + \dots + n_j, \quad n_i \in \mathbb{N}^*, \quad 1 \leq n_1 \leq n_2 \leq \dots \leq n_j$$

be a partition Π of n . We shall say that this partition represents an integer a if there exist $\epsilon_1, \epsilon_2, \dots, \epsilon_j \in \{0, 1\}$ such that $a = \sum_{i=1}^j \epsilon_i n_i$. Let $\mathcal{E}(\Pi)$ denote the set of these integers; we shall call it the set represented by Π . One can easily see that $\mathcal{E}(\Pi)$ is included in $[0, n]$ and symmetric (if it contains a , it also contains $n - a$). For fixed n , let us introduce $p(n)$ the number of partitions of n and $\hat{p}(n)$ the number of different sets amongst the sets $\mathcal{E}(\Pi)$ (where Π runs over the $p(n)$ partitions of n).

Let k be a positive integer. We shall say that a partition is k -reduced if and only if each summand appears at most k times; for instance the 1-reduced partitions are the partitions into unequal parts. We shall use $q(n, k)$ the number of k -reduced partitions of n and $\hat{q}(n, k)$ the number of different $\mathcal{E}(\Pi)$ where Π runs over the $q(n, k)$ k -reduced partitions of n . When k equals 1, we shall note: $q(n) = q(n, 1)$ and $\hat{q}(n) = \hat{q}(n, 1)$.

Following an idea due to P. Erdős, the sets represented by the partitions of an integer n were first studied at the end of the 80's. P. Erdős, J.-L. Nicolas and A. Sárközy (cf. [3]) obtained upper bounds for the number of partitions without a given subsum. P. Erdős then proposed to study the asymptotic behaviour of $\hat{p}(n)$ and $\hat{q}(n)$. In [1] and [5], M. Delglise, P. Erdős, J.-L. Nicolas and A. Sárközy proved the following estimates:

Theorem 1: For n large enough, one has

$$q(n)^{0.51} \leq \hat{q}(n) \leq q(n)^{0.96}$$

and

$$p(n)^{0.361} \leq \hat{p}(n) \leq p(n)^{0.773}.$$

We shall obtain the following improved upper bounds:

Theorem 2: For n large enough, one has

$$\hat{q}(n) \leq q(n)^{0.955} \quad \text{and} \quad \hat{p}(n) \leq p(n)^{0.768}.$$

To get these new exponents, we shall prove in part 2 a lemma improving a result due to J. Dixmier [2], whose application in part 3 gives the announced improvements.

2 The main lemma

Let a be an integer, $a \leq n$. We introduce $\mathcal{R}(n, a)$, the set of partitions of n that do not represent a , and $R(n, a)$ shall denote its cardinality. In the case of partitions into unequal parts, we shall need the same notions, with the similar notations $\mathcal{Q}(n, a)$ and $Q(n, a)$. We shall also define $Q(n, a, 2)$ as the number of 2-reduced partitions Π of n such that a is not represented by Π .

Lemma 1: Let $\epsilon > 0$. Assume there exists $\delta \in]0, 1[$ such that, for any integer n and for any integer a , the following property holds

$$(1) \quad \epsilon\sqrt{n} - 1 \leq a \leq 2\epsilon\sqrt{n} \Rightarrow R(n, a) \leq p(n)^\delta.$$

Then, for n large enough, one has

$$\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n} \Rightarrow R(n, a) \leq (2p(\epsilon\sqrt{n}))^{j-2}p(n)^\delta$$

- for $j = 2, 3, \dots, 2\lceil\sqrt{n}/2\rceil$ if $\epsilon < 1$,
- for $j = 2, 3, \dots, \tau(n)$ with $\tau(n) = o(\sqrt{n})$ for every ϵ .

Remark 1: To obtain a similar conclusion, J. Dixmier [2] assumed that hypothesis (1) is true for $\epsilon\sqrt{n} \leq a \leq 3\epsilon\sqrt{n}$.

Proof: We shall prove Lemma 1 by induction on j . It is true for $j = 2, 3$ by (1). Let us suppose that $j \geq 4$ and that the result is true up to $j - 1$. Let a be such that $\frac{j}{2}\epsilon\sqrt{n} \leq a \leq \frac{(j+1)}{2}\epsilon\sqrt{n}$. Let $\Pi \in \mathcal{R}(n, a)$ and $b = \lfloor \epsilon\sqrt{n} \rfloor$.

If b is not represented by Π , then Π belongs to a set \mathcal{E} such that $|\mathcal{E}| \leq p(n)^\delta$.

If b is represented by Π , then we can write $\Pi = (\Pi', \Pi'')$, where $S(\Pi') = b$, $S(\Pi'') = n - b$ and Π'' does not represent $a - b$. We get

$$a - b \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} = \frac{j-2}{2}\epsilon\sqrt{n} \geq \epsilon\sqrt{n-b}$$

since $j \geq 4$, and

$$a - b \leq \frac{j+1}{2}\epsilon\sqrt{n} - \epsilon\sqrt{n} + 1 = \frac{j-1}{2}\epsilon\sqrt{n} + 1.$$

Moreover we have

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq \frac{j}{2}\epsilon\sqrt{n}\left(1 - \frac{b}{n}\right) \geq \frac{j}{2}\epsilon\sqrt{n} - \epsilon\frac{j}{2}\frac{\epsilon\sqrt{n}}{\sqrt{n}}.$$

We still have to show (at least for n large enough)

$$\frac{j-1}{2}\epsilon\sqrt{n} + 1 \leq \frac{j}{2}\epsilon\sqrt{n} - \frac{j}{2}\epsilon^2.$$

- If $\epsilon < 1$, since $j/2 \leq \sqrt{n}/2$, we have to check the inequality

$$-1/2\epsilon\sqrt{n} + 1 \leq -\epsilon^2\sqrt{n}/2,$$

which is true when n is large enough.

- In the second case, we want to show

$$-1/2\epsilon\sqrt{n} + 1 \leq -1/2\epsilon^2\frac{\tau(n)}{\sqrt{n}}.$$

This is true when n is large enough by using the hypothesis on $\tau(n)$.

We finally get

$$\frac{j}{2}\epsilon\sqrt{n-b} \geq a - b.$$

We deduce from the induction hypothesis that Π'' belongs to a set \mathcal{F} such that

$$|\mathcal{F}| \leq (2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

This implies that Π belongs to a set \mathcal{G} such that

$$|\mathcal{G}| \leq p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta.$$

Hence we have

$$R(n, a) \leq p(n)^\delta + p((\epsilon\sqrt{n}))(2p(\epsilon\sqrt{n}))^{j-3}p(n)^\delta \leq (2p(\epsilon\sqrt{n}))^{j-2}p(n)^\delta$$

which completes the proof of the lemma.

Remark 2: It is easy to see that the result remains true when we replace all the $R(n, a)$'s by $Q(n, a)$'s or by $Q(n, a, 2)$'s, i.e. when we deal with partitions into unequal parts or with 2-reduced partitions (in the proof, if Π is into unequal parts, then Π' and Π'' are also into unequal parts; the same phenomenon occurs when we are dealing with 2-reduced partitions).

3 Applications

This lemma is useful to get upper bounds for $\hat{p}(n)$ and $\hat{q}(n)$ improving those obtained in [1]. Lemma 1 allows us to prove the following lemma:

Lemma 2: *When $n \rightarrow \infty$ we have:*

1. for $1.07\sqrt{n} \leq a \leq n - 1.07\sqrt{n}$,

$$Q(n, a) \leq \exp((1 + o(1))1.732\sqrt{n}),$$

2. for $0.81\sqrt{n} \leq a \leq n - 0.81\sqrt{n}$,

$$Q(n, a, 2) \leq \exp((1 + o(1))1.969\sqrt{n}).$$

To get Lemma 2 (the method is developed in [1]), we find upper bounds for $Q(n, a)$ and $Q(n, a, 2)$ when a ranges over the interval $[\epsilon\sqrt{n}, 2\epsilon\sqrt{n}]$ and we choose the best ϵ ; then we use Lemma 1 and the results in [3].

From Lemma 2, we get Theorem 2 as in [1]. For instance, when studying $\hat{q}(n)$, we distinguish two cases according to whether the partition represents all integers between $1.07\sqrt{n}$ and $n - 1.07\sqrt{n}$ or not. We get this way

$$\hat{q}(n) \leq n \exp((1 + o(1))1.732\sqrt{n}) + 2^{1.07\sqrt{n}} \leq q(n)^{0.955}$$

since $q(n) = \exp((1 + o(1))\pi\sqrt{n/3})$ (cf. [4]).

The method is the same for $\hat{p}(n)$, since $\hat{p}(n) = \hat{q}(n, 2)$ [1, Thorme 1].

Remark 3: The improvement on the exponents in the Theorem 2 is small (5.10^{-3}). This comes from the fact that the functions (cf. [1]) we bound on an interval $[x, 2x]$ (and not $[x, 3x]$, see Remark 1) have slow variations around their minimum value. Indeed, even replacing $[x, 2x]$ by $[x, (1 + \eta)x]$ with η decreasing to 0 would only lead to another small improvement (4.10^{-3} less than our results). To make the exponents in the upper bounds really smaller, we need to find another method.

4 References

1. M. Delglise, P. Erdős and J.-L. Nicolas, Sur les ensembles represents par les partitions d'un entier n , *Discr. Math.*, to appear.
2. J. Dixmier, Partitions avec sous-sommes interdites, *Bull. Soc. Math. Belgique*, **42** (1990), 477-500.
3. P. Erdős, J.-L. Nicolas and A. Sárközy, On the number of partitions of n without a given subsum II, *Analytic Number Theory*, edited by B. Berndt, H. Diamond, H. Halberstam, A. Hildebrand, Birkhaser 1990, 207-236.
4. G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc. (2)*, **17** (1918), 75-115.
5. J.-L. Nicolas and A. Sárközy, On two partition problems, *Acta Math. Hung.*, **77** (1997), 95-121.