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FAILURE OF WIENER'S PROPERTY FOR POSITIVE DEFINITE PERIODIC FUNCTIONS

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Abstract. We say that Wiener's property holds for the exponent $p > 0$ if we have that whenever a positive definite function f belongs to $L^p(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then f necessarily belongs to $L^p(\mathbb{T})$, too. This holds true for $p \in 2\mathbb{N}$ by a classical result of Wiener.

Recently various concentration results were proved for idempotents and positive definite functions on measurable sets on the torus. These new results enable us to prove a sharp version of the failure of Wiener's property for $p \notin 2\mathbb{N}$. Thus we obtain strong extensions of results of Wainger and Shapiro, who proved the negative answer to Wiener's problem for $p \notin 2\mathbb{N}$.

Contre-exemples à la propriété de Wiener pour les fonctions périodi– ques définies-positives.

Résumé. On dit que l'exposant p possède la propriété de Wiener si toute fonction périodique définie-positive qui est de puissance p-ième intégrable au voisinage de 0 l'est sur un intervalle de période. C'est le cas des entiers pairs, d'après un résultat classique de Wiener.

Nous avons récemment obtenu des phénomènes de concentration des polynômes idempotents ou définis-positifs sur un ensemble mesurable du tore qui nous permettent de donner une version forte du fait que les exposants $p \notin 2\mathbb{N}$ n'ont pas la propriété de Wiener, améliorant ainsi les résultats de Wainger et Shapiro.

1. Introduction

Let f be a periodic integrable function which is positive definite, that is, has non negative Fourier coefficients. Assume that it is bounded (in $\|\cdot\|_\infty$) in a neighborhood of 0, then it necessarily belongs to $L_\infty(\mathbb{T})$, too. In fact, its maximum is obtained at 0 and, as $f(0) = \sum_k \widehat{f}(k)$, f has an absolutely convergent Fourier series.

The same question can be formulated in any L^p space. Actually, the following question was posed by Wiener in a lecture, after he proved the L^2 case. We refer to [16] for the story of this conjecture, see also [12], [16] and [18].

Problem 1 (Wiener)**.** *Let $1 \le p < \infty$. Is it true, that if for some $\varepsilon > 0$ a positive definite function $f \in L^p(-\varepsilon, \varepsilon)$, then we necessarily have $f \in L^p(\mathbb{T})$, too?*

The observation that the answer is positive if $p \in 2\mathbb{N}$ has been given by Wainger [17], as well as by Erdős and Fuchs [9]. We refer to Shapiro [16] for the proof, since the constant given by his proof is in some sense optimal, see [12, 13]. Generalizations in higher dimension may be found in [11] for instance. It was shown by Shapiro [16] and Wainger [17] that the answer is to the negative for all other values of p. Negative results were obtained for groups in e.g. [10] and [12].

There is even more evidence that the Wiener property must hold when $p = 2$ and we prescribe large gaps in the Fourier series of f. Indeed, in this case by well-known results of Wiener and Ingham, see e.g. [18, 20], we necessarily have an essentially uniform distribution of the L^2 norm on intervals longer than the reciprocal of the gap, even without the assumption that f be positive definite. As Zygmund pointed out, see the Notes to Chapter V §9, page 380 in [20], Ingham type theorems were not known for $p \neq 2$, nevertheless, one would feel that prescribing large gaps in the Fourier series should lead to better control of the global behavior by means of having control on some subset like e.g. $(-\varepsilon, \varepsilon)$. So the analogous Wiener question can be posed restricting to positive definite functions having gaps tending to ∞. However, we answer negatively as well. In this strong form the question, to the best of our knowledge, has not been dealt with yet. Also we are able to replace the interval $(-\varepsilon, +\varepsilon)$ by any measurable symmetric subset E of the torus of measure $|E| < 1$. Neither extension can be obtained by a straightforward use of the methods of Shapiro and Wainger.

2. L^2 results and concentration of integrals

We use the notation $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the torus. Then $e(t) := e^{2\pi it}$ is the usual exponential function adjusted to interval length 1, and we denote e_h the function $e(hx)$. The set of positive definite trigonometrical polynomials is the set

$$\mathcal{T}^+ := \left\{ \sum_{h\in H} a_k e_k \ : \ H \subset \mathbb{Z} \quad (\text{or } \ \mathbb{N}), \ \ \#H < \infty, \quad a_k \geq 0 \ (k \in H) \right\} \tag{1}$$

For obvious reasons of being convolution idempotents, the set

$$\mathcal{P} := \left\{ \sum_{h\in H} e_h \ : \ H \subset \mathbb{Z} \quad (\text{or } \ \mathbb{N}), \ \ \#H < \infty \right\} \tag{2}$$

is called the set of *(convolution-)idempotent exponential (or trigonometric) polynomials*, or just *idempotents* for short.

Note that multiplying a polynomial by an exponential e_K does not change its absolute value, and the property of belonging to $\mathcal{P}$ or $\mathcal{T}^+$ is not changed either. Therefore, it suffices to consider polynomials with nonnegative spectrum, i.e. $H \subset \mathbb{N}$ in (1) and (2).

Also note that for a positive definite function the function $|f|$ is necessarily even. This is why we consider 0-symmetric (or, just symmetric for short) intervals or sets, (alternatively, we could have chosen to restrict to $[0, 1/2)$ instead of $\mathbb{T}$).

Let us first state the theorem on positive definite functions in L^2. Recall that the direct part is attributed to Wiener, with the constant given by Shapiro in [16]. The converse seems to be well known (see [12, 13]), except, may be, for the fact that counter-examples may be given by idempotents. The fact that the Wiener property fails for arbitrary measurable sets is, to the best of our knowledge, new.

Theorem 2 (Wiener, Shapiro)**.** *For p an even integer, for $0 < a < 1/2$ and for $f \in \mathcal{T}^+$, we have the inequality*

$$\frac{1}{2a} \int_{-a}^{+a} |f|^p \geq \frac{1}{2} \int_{-1/2}^{+1/2} |f|^p. \tag{3}$$

Moreover, the constant $1/2$ cannot be replaced by a smaller one, even when restricting to idempotents. Indeed, for each integer $k > 2$, for $a < 1/k$ and for $b > 1/k$, there exits an idempotent f and such that $\int_{-a}^{+a} |f|^p \leq b \times \int_{-1/2}^{+1/2} |f|^p$.

Proof. We refer to Shapiro for the proof of the inequality (3).

To show sharpness of the constant, let us now give an example, inspired by the examples of [7]. We take $f := D_n * \mu_k$, where D_n is the Dirichlet kernel, defined here as

$$D_n(x) := \sum_{\nu=0}^{n-1} e(\nu x) = e^{\pi i(n-1)x/2} \frac{\sin(\pi n x)}{\sin(\pi x)}, \tag{4}$$

and μ_k is the mean of Dirac masses at each k-th root of unity. Both have Fourier coefficients 0 or 1, so that f is an idempotent. Only one of the point masses of μ_k lies inside the interval $(-a, +a)$ and one can see that the ratio between $\int_{-a}^{+a} |f|^p$ and $\int_{-1/2}^{+1/2} |f|^p$ tends to $1/k$ when n tends to infinity. □

Remark 3. *The interval $(-a, +a)$ cannot be replaced by a measurable set E having 0 as a density point, even if $|E|$ is arbitrarily close to 1. Indeed, assume that the complement of E is the union (modulo 1) of all intervals of radius $1/l^3$ around all irreducible rational numbers k/l, with k different from 0 and $l > L$. Then E has the required properties, while, for the same idempotent $f := D_n * \mu_l$, the ratio between $\int_E |f|^p$ and $\int_{-1/2}^{+1/2} |f|^p$ tends to $1/l$ when n tends to infinity. We get our conclusion noting that l may be arbitrarily large.*

Let us now consider the p-concentration problem, which comes from the following definition.

Definition 4. *Let $p > 0$, and $\mathcal{F}$ be a class of functions on $\mathbb{T}$. We say that for the class $\mathcal{F}$ there is p-concentration if there exists a constant $c > 0$ so that for any symmetric measurable set E of positive measure one can find an idempotent $f \in \mathcal{F}$ with*

$$\int_E |f|^p \geq c \int_{\mathbb{T}} |f|^p. \tag{5}$$

The problem of p-concentration on the torus for idempotent polynomials has been considered in [7], [8], [2]. It was essentially solved recently in [6]. Also, the weaker question of concentration of p^{th} integrals of positive definite functions has been dealt with starting with the works [7, 8]. In this respect we have proved the following result, see [6, Theorem 48]. We will only state that part of the theorems of [6] that we will use.

Theorem 5. *For all $0 < p < \infty$, p not an even integer, whenever a 0-symmetric measurable set E of positive measure $|E| > 0$ is given, then to all $\varepsilon > 0$ there exists some positive definite trigonometric polynomial $f \in \mathcal{T}^+$ so that*

$$\int_{^cE} |f|^p \leq \varepsilon \int_{\mathbb{T}} |f|^p. \tag{6}$$

Moreover, f can be taken with arbitrarily large prescribed gaps between frequencies of its Fourier series.

Remark 6. *The same result is also proved for open symmetric sets and idempotents, and for measurable sets and idempotents when $p > 1$.*

Theorem 5 allows to see immediately that there is no inequality like (3) for p not an even integer. What is new, compared to the results of Shapiro and Wainger, is the fact that this is also the case if f has arbitrarily large gaps, and that we can replace intervals $(-a, +a)$ by arbitrary measurable sets of measure less than 1. We will give a different statement in the next section for E an open set, and also show a strong version of the negative state of Wiener's problem.

3. Negative results in Wiener's problem

Let us start with somewhat strengthening the previous theorem for open sets, which we obtain by an improvement of the methods of Shapiro in [16].

Theorem 7. *For all $0 < q \leq p < 2$, whenever a 0-symmetric open set E of positive measure $|E| > 0$ is given, then for all $\varepsilon > 0$ there exists some positive definite trigonometric polynomial $f \in \mathcal{T}^+$ so that*

$$\int_{{}^cE} |f|^p \leq \varepsilon \left(\int_{\mathbb{T}} |f|^q\right)^{p/q}. \tag{7}$$

The same is valid for $q < p$ with p not an even integer, provided that q is sufficiently close to p, that is $q > q(p)$, where $q(p) < p$.

The construction is closely related to the failure of Hardy Littlewood majorant property. We do not know whether, for $p > 2$ not an even integer, that is $2k < p < 2k + 2$, we can take $q(p) = 2k$. Due to Theorem 2, we cannot take $q(p) < 2k$. We do not know either whether the next statement is valid for functions with arbitrary large gaps.

Proof. Let us first assume that $p < 2$. Then, for D_n the Dirichlet kernel with n sufficiently large depending on ε, there exists a choice of $\eta_k = \pm 1$ such that

$$\|D_n\|_p \leq \varepsilon \|\sum_{k=0}^{n} \eta_k e_k\|_q.$$

Indeed, if it was not the case, taking the q-th power, integrating on all possible signs and using Khintchine's Inequality, we would find that $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq Cn^{1-\frac{1}{p}}$ $(p > 1)$, $c\varepsilon\sqrt{n} \leq \|D_n\|_1 \leq C\log n$ and $c\varepsilon\sqrt{n} \leq \|D_n\|_p \leq C$ $(0 < p < 1)$ which leads to a contradiction.

We assume that E contains $I \cup (-I)$, where $I := (\frac{k}{N}, \frac{k+1}{N})$, and denote

$$g(t) := \sum_{k=0}^{n} \eta_k e_k(t) \qquad\qquad G(t) := D_n(t).$$

Let Δ be a triangular function based on the interval $(-\frac{1}{2N}, +\frac{1}{2N})$, that is, $\Delta(t) := (1 - 2N|t|)_+$. We finally consider the function

$$f(t) := \Delta(t - a)g(2Nt) + \Delta(t + a)g(2Nt) + 2\Delta(t)G(2Nt),$$

where a is the center of the interval I. Then an elementary computation of Fourier coefficients, using the fact that Δ has positive Fourier coefficients while the modulus of those of g and G are equal, allows to see that f is positive definite. Let us prove that one has (7). The left hand side is bounded by $\frac{2}{N}\|G\|_p^p$, while $\int_{\mathbb{T}} |f|^q$ is bounded below by $\frac{1}{2N}\|g\|_q^q - \frac{2}{N}\|G\|_q^q$. We conclude the proof choosing n, N sufficiently large.

Let us now consider $p > 2$ not an even integer. Mockenhaupt and Schlag in [15] have given counter-examples to the Hardy Littlewood majorant conjecture, which are based on the following property: for $j > p/2$ an odd integer, the two trigonometric polynomials

$$g_0 := (1 + e_j)(1 - e_{j+1}) \qquad\qquad G_0 := (1 + e_j)(1 + e_{j+1})$$

satisfy the inequality $\|G_0\|_p < \|g_0\|_p$. By continuity, this inequality remains valid when p is replaced by q in the right hand side, with $q > q(p)$, for some $q(p) < p$. By a standard Riesz product argument, for K large enough, as well as $N_1, N_2, \cdots N_K$, depending on ε, the functions

$$g(t) := g_0(t)g_0(N_1t)\cdots g_0(N_Kt) \quad \text{and} \quad G(t) := G_0(t)G_10(N_1t)\cdots G_0(N_Kt)$$

satisfy the inequality

$$\|G\|_p \leq \varepsilon\|g\|_q.$$

From this point the proof is identical. □

We can now state in two theorems the counter-examples that we obtain for the Wiener conjecture when p is not an even integer.

Theorem 8. *Let $0 < p < \infty$, and $p \notin 2\mathbb{N}$. Then for any symmetric, measurable set $E \subset \mathbb{T}$ with $|E| > 0$ and any $q < p$, there exists a function f in the Hardy space $H^q(\mathbb{T})$ with positive Fourier coefficients, so that its pointwise boundary value f^* is in $L^p({}^cE)$ while $f^* \notin L^p(\mathbb{T})$. Moreover, f can be chosen with gaps tending to ∞ in its Fourier series.*

Here $H^q(\mathbb{T})$ denotes the space of periodic distributions f whose negative coefficients are zero, and such that the function f_r are uniformly in $L^q(\mathbb{T})$ for $0 < r < 1$, where

$$f_r(t) := \sum_n \hat{f}(n)r^{|n|}e^{2i\pi nt}.$$

Moreover, the norm (or quasi-norm) of f is given by

$$\|f\|^q_{H^q(\mathbb{T})} := \sup_{0<r<1}\int_0^1 |f_r|^q.$$

It is well known that, for $f \in H^q(\mathbb{T})$, the functions f_r have an a. e. limit f^* for r tending to 1. The function f^*, which we call the pointwise boundary value, belongs to $L^q(\mathbb{T})$. When $q \geq 1$, then f is the distribution defined by f^*, and $H^q(\mathbb{T})$ coincides with the subspace of functions in $L^q(\mathbb{T})$ whose negative coefficients are zero. In all cases the space $H^q(\mathbb{T})$ identifies with the classical Hardy space when identifying the distribution f with the holomorphic function $\sum_{n\geq 0}\hat{f}(n)z^n$ on the unit disc. This explains the use of the term of boundary value.

The function $f \in H^q$ is said to have gaps (in its Fourier series) tending to ∞ whenever its Fourier series of f can be written as $\sum_{k=0}^{\infty} a_k e^{2i\pi n_k x}$, where n_k is an increasing sequence such that $n_{k+1} - n_k \to \infty$ with k.

In opposite to this theorem, recall that for n_k a *lacunary* series, if the Fourier series is in $L^p(E)$ for some measurable set E of positive measure, then the function f belongs to all spaces $L^q(\mathbb{T})$, see [20]. This has been generalized by Miheev [14] to $\Lambda(p)$ sets for $p > 2$: if f is in $L^p(E)$, then f is in the space $L^p(\mathbb{T})$. See also the expository paper [5].

Proof. The key of the proof is Theorem 5. Remark that we can assume that $p > q > 1$. Indeed, f^ℓ is a positive definite function when f is, and counter-examples for some $p > 1$ will lead to counter-examples for p/ℓ. Now, let us take a sequence E_k of disjoint measurable subsets of E of positive measure, such that $|E_k| < 2^{-\alpha k}$, with α to be given later and let f_k be a sequence of positive definite trigonometric polynomials such that

$$\int_{\mathbb{T}\setminus E_k} |f_k|^p \leq 2^{-kp} \int_{\mathbb{T}} |f_k|^p. \tag{8}$$

Moreover, we assume that f_k's have gaps larger than k. Using Hölder's inequality, we obtain

$$\int_{\mathbb{T}} |f_k|^q \leq 2^{-\alpha(1-q/p)k} \left(\int_{E_k} |f_k|^p\right)^{q/p} + \left(\int_{\mathbb{T}\setminus E_k} |f_k|^p\right)^{q/p} \leq 2 \times 2^{-kq} \left(\int_{\mathbb{T}} |f_k|^p\right)^{q/p},$$

if α is chosen large enough. Finally, we normalize the sequence f_k so that $\int_{\mathbb{T}} |f_k|^p = 2^{\frac{k}{2}}$, and take

$$f(x) := \sum_{k\geq 1} e^{2i\pi m_k x} f_k(x), \tag{9}$$

where the m_k are chosen inductively sufficiently increasing, so that the condition on gaps is satisfied. The series is convergent in $L^q(\mathbb{T})$ and in $L^p({}^cE)$, and the limit f has its Fourier series given by (9). Now, let us prove that f is not in $L^p(\mathbb{T})$. Since the E_j's are disjoint,

$$\|f\|_p \geq \|f\|_{L^p(E_k)} \geq \|f_k\|_p - \sum_j \|f_j\|_{L^p({}^cE_j)} \geq 2^{\frac{k}{2}} - \sum_{j>0} 2^{-\frac{j}{2}},$$

which allows to conclude. □

Using Theorem 7 instead of Theorem 5, we have the following.

Theorem 9. (i) *Let $p > 2$, with $p \notin 2\mathbb{N}$, and let $\ell \in \mathbb{N}$ such that $2\ell < p < 2(\ell+1)$. Then, for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and $q > q(p)$, there exists a positive definite function $f \in L^{2\ell}(\mathbb{T})$, whose negative coefficients are zero, such that $f \notin L^q(\mathbb{T})$ while f is in $L^p({}^cU)$.*

(ii) *Let $0 < p < 2$. Then for any symmetric open set $U \subset \mathbb{T}$ with $|U| > 0$ and any $s < q < p$, there exists a function f in the Hardy space $H^s(\mathbb{T})$ with non negative Fourier coefficients, so that $f \notin H^q(\mathbb{T})$ while f^* is in $L^p({}^cU)$.*

Proof. Let us first prove (i). We can assume that ${}^cU$ contains a neighborhood of 0. So, by Wiener's property, if f is integrable and belongs to $L^p({}^cU)$, then f is in $L^{2\ell}(\mathbb{T})$. Let us prove that there exists such a function, whose Fourier coefficients satisfy the required properties, and which does not belong to $L^q(\mathbb{T})$. The proof follows the same lines as in the previous one. By using Theorem 7, we can find positive definite polynomials f_k such that $\|f_k\|_q = 2^{k/2} \to \infty$, while $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k}$ with $U_k \subset U$ disjoint and of sufficiently small measure, so that $\sum \|f_k\|_{L^p({}^cU)} < \infty$. As before, the function $f := \sum_{k\geq 1} e_{m_k} f_k$ will have the required properties.

Let us now consider $1 \leq p < 2$, from which we conclude for (ii): if $p < 1$, we look for a function of the form f^ℓ, with f satisfying the conclusions for ℓp, with ℓ such that $1 \leq \ell p < 2$. We can assume that $q < 1$. We proceed as before, with f_k's given by Theorem 7, such that $\|f_k\|_q = 2^{k/2}$ and $\|f_k\|_{L^p({}^cU_k)} \leq 2^{-k/2}$. The U_k's are assumed to be disjoint and of small measure, so that $\sum_k \|f_k\|_{H^s}^s < \infty$. It follows that $f \in H^s(\mathbb{T})$. Remark that f is not a function, in general, but a distribution.

Recall that f^* is the boundary value of the corresponding holomorphic function. We write as before

$$\|f\|^q_{H^q(\mathbb{T})} \geq \|f^*\|^q_{L^q(U_k)} \geq \|f_k\|_q^q - \sum_j \|f_j\|^q_{L^q(^cU_j)} \geq 2^{\frac{kq}{2}} - \sum_{j>0} 2^{-\frac{jq}{2}},$$

which allows to conclude for the fact that f is not in $H^q(\mathbb{T})$.

Remark 10. *As Wainger in* [18]*, we can prove a little more: the function f may be chosen such that $\sup_{r<1}|f_r|$ is in $L^p(^cU)$. Let us give the proof in the case (i). We can assume that U may be written as $I\cup(-I)$ for some interval I. Let J be the interval of same center and length half, and take f constructed as wished, but for the open set $J\cup(-J)$. Finally, write $f=\phi+\psi$, with $\phi := f\chi_{^c(J\cup(-J))}$. Then using the maximal theorem we know that $\sup_{r<1}|\phi_r| \in L^p(\mathbb{T})$, while the Poisson kernel $P_t(x-y)$ is uniformly bounded for $x\notin U$ and $y\in J\cup(-J)$, so that $\sup_{r<1}|\psi_r|$ is uniformly bounded outside U.*

In the case (ii), the proof is more technical, f being only a distribution. We use the fact that derivatives of the Poisson kernel $P_t(x-y)$ are also uniformly bounded for $x\notin U$ and $y\in J\cup(-J)$.

□

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