Regularity of solutions of the isoperimetric problem that are close to a smooth manifold

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Contents

1 Introduction

1.1 A regularity theorem

In this article we show some regularity results for solutions (in the sense of geometric measure theory) of the isoperimetric problem in a Riemannian manifold.

Definition 1.1. *Let $\mathcal{M}$ be a Riemannian manifold of dimension n (possibly of infinite volume).*
We denote by $\tau_{\mathcal{M}}$ the class of relatively compact open sets of $\mathcal{M}$ with C^∞ boundary.
The function $I : [0, Vol(\mathcal{M})[\to [0, +\infty[$ such that $I(0) = 0$

$$I : \begin{cases}]0, Vol(\mathcal{M})[& \to \quad [0, +\infty[\\ v & \mapsto \quad Inf_{\Omega\in\tau_{\mathcal{M}}, Vol(\Omega)=v}\{Vol_{n-1}(\partial\Omega)\} \end{cases}$$

is called the isoperimetric profile function (shortly the isoperimetric profile) of the manifold $\mathcal{M}$.

We define *a solution of the isoperimetric problem for volume v* as an integral current T such that $M(T) = v$ and $M(\partial T) = I(v)$. If $\mathcal{M}$ is compact, such currents exist for all $v \in]0, Vol(\mathcal{M})[$. The regularity theory of minimizing currents, inaugurated by Federer and Fleming, that culminated with F. Almgren's works, shows that the solutions of the isoperimetric problem are almost smooth: they are submanifolds with smooth boundary, in the complement of a singular set of codimension at least equal to 7 [Alm76].

For manifolds $\mathcal{M}$ of dimension greater than 8, there are minimizing currents whith non smooth boundary, in $\mathbb{R}^n$ see, for example, [Alm76], [Mor03], [BGG69]. The first result, reached by Bombieri, De Giorgi, Giusti [BGG69] shows that the cone $C := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$ is singular at the origin and has minimal area in $\mathbb{R}^8$. In every ball of $\mathbb{R}^n$, such a current is a solution of the isoperimetric problem. Almgren's theorem is thus optimal. Therefore, additional conditions are required to get more regularity.

The aim of this article is to show that a solution of the isoperimetric problem, sufficiently close in the flat norm to a domain B with smooth boundary ∂B, is also smooth and very close to B in the $C^{k,\alpha}$ topology. For further applications of this theorem, we also require that the Riemannian metric of $\mathcal{M}^n$ is variable. The main result is the following.

Theorem 1. *Let $\mathcal{M}^n$ be a compact Riemannian manifold, g_j a sequence of Riemannian metrics of class C^∞ that converges to a fixed metric g_∞ in the*

C^4 topology. Let B be a domain of $\mathcal{M}$ with smooth boundary ∂B, consider T_j a sequence of solutions of the isoperimetric problem of $(\mathcal{M}^n, g_j)$ such that

$$(*) : \mathbf{M}_{g_\infty}(B - T_j) \rightarrow 0.$$

Then ∂T_j is the graph in normal exponential coordinates of a function u_j on ∂B.

Furthermore, for all $\alpha \in]0,1[$, $u_j \in C^{2,\alpha}(\partial B)$ and $||u_j||_{C^{2,\alpha}(\partial B)} \rightarrow 0$.

1.2 Previous results

We can find a particular case of theorem 1 in an article [MJ00] of David L. Johnson and F. Morgan. Indeed, these authors show that the solutions of the isoperimetric problem in small volumes are close to small balls. The article [Nar06] goes farther. It shows that the solutions of the isoperimetric problem belong to a family of domains, called *pseudo-bubbles*, that we can construct by application of the implicit function theorem. This shows that the isoperimetric problem reduces, naturally (and in particular, in a way compatible with the symmetries of the ambient Riemannian manifold), to a variational problem in finite dimension. We can recover an unpublished result of Bruce Kleiner (dating back to 1985), that can be stated as follows.

Theorem 1.1 (Kleiner). *Let $\mathcal{M}$ be a Riemannian manifold. Let G be a group of isometries that acts transitively on $\mathcal{M}$, $K \leq G$ the isotropy group of a point.*
Then for small $v > 0$, there exists a solution of the isoperimetric profile in volume v that is K invariant.

We learned about this result only in may 2005, as there is no written trace and no other reference than [Tom93]. B. Kleiner persuaded us that he had since 1985 the elements of the proof of theorem 1.

1.3 Future application

From Theorem 1, we can argue that if, for a $\bar{v} > 0$, all the solutions of the isoperimetric problem in volume $\bar{v}$ are smooth, then the solutions of the isoperimetric problem for volumes v close to $\bar{v}$ are smooth too. Under this condition, we could be able to reduce the isoperimetric problem for volumes close to $\bar{v}$ to a variational problem in finite dimension, as developed by [Nar06] for $v_0 = 0$. This will be done in a separate paper.

1.4 Sketch of the proof of Theorem 1

First, assume that the metric is fixed, i.e. $g_j = g$. We make essential use of Allard's theorem [All72] (Theorem 8.1): if a varifold $V = \partial T \ni a$ has, in a ball $B(a, r)$, a weight $||V||(B(a, r))$ sufficiently close to $\omega_{n-1} R^{n-1}$ (where ω_{n-1} is the volume of the unit ball of $\mathbf{R}^{n-1}$), then V is the graph of a function $u \in C^{1,\alpha}$. A regularity theorem for elliptic partial differential equations and a bootstrap argument imply that $u \in C^\infty$ and also give upper bounds for $||u||_{C^{2,\alpha}}$.

In order to show that ∂T_j satisfies the conditions of Allard's regularity theorem, we compare ∂T_j to suitably chosen deformations of ∂T with fixed enclosed volume.

Unfortunately for our purposes, Allard's theorem is stated in Euclidean space. We are forced to give a Riemannian version via isometric embedding of Riemannian manifolds in Euclidean spaces. Furthermore we need to control the second fundamental form of the isometric embeddings relative to different metrics on $\mathcal{M}$. To make this possible we use a fine analysis of the proof of the Nash's isometric embedding theorem that M. Gromov did in [Gro86b], this highlights the fact that free isometric embeddings can be chosen to depend continuously on the metric.

2 Plan of the article

1. Section 1 provides Riemannian versions of 3 classical results of geometric measure theory: Allard's regularity theorem, the link between first variation and mean curvature in the case of currents and varifolds, the monotonicity formula.

2. Section 2 is the core of the paper and gives the proof of theorem 1 in case of a fixed metric. It starts by a detailed sketch of the proof. This part has the aim of elucidating the basic ideas (subsection 2.2).

3. Section 3 deals with the general case of variable metrics.

2.1 Acknowledgements

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3 Regularity Theory

3.1 Introduction

The aim of this section is to adapt several classical results of geometric measure theory stated in Euclidean spaces to arbitrary Riemannian manifolds.

3.2 Notations

In this section we are concerned with a Riemannian manifold $\mathcal{M}$ of class at least C^3 (this condition is needed only to ensure the existence of an isometric embedding via Nash's theorem) and we keep fixed an isometric embedding $i : \mathcal{M} \hookrightarrow \mathbb{R}^N$. We denote

$$\beta_i = ||II_{i\hookrightarrow\mathcal{M}}||_{\infty,\mathcal{M}}$$

where $II_{i\hookrightarrow\mathcal{M}}$ is the second fundamental form of the embedding i and $||.||_{\infty,\mathcal{M}}$ is the supremum norm taken on $\mathcal{M}$. We observe, incidentally, that the second fundamental form depends on first and second derivatives of the embedding i by continuous functions. Hence, if we have 2 embeddings i_1, i_2 that are ε close in the C^2 topology, then β_{i_1}, β_{i_2} will be *const.*ε close and the constant is independent of embeddings i_1, i_2. Indeed the constant depends only on $\mathcal{M}$ and the intrinsic metric.

Remarks: In the rest of this paper we adopt the convention to denote variables with letters without subscripts and constants by letters with subscripts.

3.3 Mean curvature vector based on an hypersurface

For further applications, we give hereby a formula for mean curvature of an hypersurface $\mathcal{N}$ that is a graph in normal exponential coordinates on a neighborhood of ∂B. Let

$$H_\nu^{\mathcal{N}}(y) = \sum_{1\ i}^{n-1} II_y^{\mathcal{N}}(e_i, e_i) = -\sum_{1\ i}^{n-1} < \nabla_{e_i}\nu, e_i >_g (y) \tag{1}$$

where $(e_1, \ldots, e_{n-1})$ is an orthonormal basis of $T_y\mathcal{N}$ such that $(e_1, \ldots, e_{n-2}) = T_y\mathcal{N} \cap T_y\partial B^r$ and B^r is the domain whose boundary is the hypersurface equidistant at distance r to ∂B.
Let ν be a unit normal vector of $\mathcal{N}$, we extend ν to a vector field, that we denote always ν, over an entire neighborhood $\mathcal{U}_{r_0}(\partial B)$ such that $[\theta, \nu] = 0$

where $[\cdot,\cdot]$ indicates the Lie bracket of two vector fields, and θ is the vector field obtained by taking the gradient of the signed distance function to ∂B having positive values outside B.
Introduce a chart ϕ of $\mathcal{M}$. At first we choose a chart Θ in a neighborhood of ∂B then we set

$$\phi : \left\{ \begin{array}{lll}]-r,r[\times\mathcal{U} & \rightarrow & \mathcal{U}_r \subseteq \mathcal{M} \\ (t,x) & \mapsto & exp_{\Theta(x)}\left(t\theta(\Theta(x))\right). \end{array} \right.$$

where $\mathcal{U} \subseteq \mathbb{R}^n$ is the domain of Θ. By choosing r less than the normal injectivity radius of ∂B, we have that ϕ is a diffeomorphism. (t,x) are called Fermi coordinates based at ∂B.
Set $\nu = a + b\theta$ hence $e_{n-1} = \frac{-b}{|a|}a + |a|\theta$. We set ($\tilde{e}_1 = e_1, \ldots, \tilde{e}_{n-1} = e_{n-2}, \tilde{e}_{n-1} = \frac{a}{|a|}$). We observe that the explicit calculation that follows is independent from the extensions choosed for the vector fields e_i.
We have, by a straightforward calculation:

$$-<\nabla_{e_{n-1}}\nu, e_{n-1}>=\frac{-b^2}{|a|^2}<\nabla_a a, a>-\frac{b^2}{|a|^2}+b\nabla_a b \tag{2}$$

$$-\sum_{i=1}^{n-2}<\nabla_{e_i}\nu, e_i>=-\sum_{i=1}^{n-2}<\nabla_{e_i}a, e_i>-\sum_{i=1}^{n-2}b<\nabla_{e_i}\theta, e_i>. \tag{3}$$

Thus

$$-<\nabla_{e_{n-1}}\nu, e_{n-1}>=-div_{\partial B_t}(a)+<\nabla_{\tilde{e}_{n-1}}a, \tilde{e}_{n-1}>+bH_\eta^{\partial B}+b<\nabla_{\tilde{e}_{n-1}}\theta, \tilde{e}_{n-1}> \tag{4}$$

$$H_\nu(t,x) = -div_{\partial B_t}(a)+<\nabla_a a, a>-bII_\theta^{\partial B^t}(a,a)+bH_\theta^{\partial B^t}+b\nabla_a b \tag{5}$$

where $II_\theta^{\partial B^t}$ and $H_\theta^{\partial B^t}$ are respectively the second fundamental form and the mean curvature in the direction of θ of the equidistant hypersurface at distance t from ∂B computed at the point $exp^{\partial B}(t\theta(\Theta(x)))$.

Assume that $\mathcal{N}$ is the graph, in normal exponential coordinates, of a function u. Let

$$W_u := \sqrt{1+\|\overrightarrow{\nabla}_{i_u^*(g)}u\|^2_{i_u^*(g)}}.$$

Then

$$b = \left\{ \begin{array}{lll} \frac{1}{W_u} & <\nu,\theta>\geq 0 & \nu \quad outward \\ -\frac{1}{W_u} & <\nu,\theta>\leq 0 & \nu \quad inward \end{array} \right.$$

$$a = d\phi\left(-bu\mathcal{I}_{(0,x),(u(x),x)}\left(\overrightarrow{\nabla}_{g_u}u\right)\right) \tag{6}$$

where $\mathcal{I}_{(0,x),(u(x),x)}$ is the usual identification in $\mathbb{R}^n \cong]-r,r[\times\mathcal{U}$ of $T_{(0,x)}]-r,r[\times\mathcal{U}$ with $T_{(u(x),x)}]-r,r[\times\mathcal{U}$ that is induced by the chosen system of Fermi coordinates based at ∂B (notation inspired by [Cha95]). In the sequel we always make identifications via ϕ freely. Let $b = -\frac{1}{W_u}$, we can write

$$a = \frac{u}{W_u}\mathcal{I}_{(0,x),(u(x),x)}\left(\overrightarrow{\nabla}_{g_u}u\right). \tag{7}$$

this leads to

$$\begin{aligned} H^{\mathcal{N}}_{\nu_{inw}}(u,x) &= -div_{(\mathbb{S}^{n-1},g_u)}\left(\frac{\overrightarrow{\nabla}_{g_u}u}{W_u}\right) - \frac{1}{W_u^2} < \nabla_{\overrightarrow{\nabla}_{g_u}u}\left(\frac{u\overrightarrow{\nabla}_{g_u}u}{W_u}\right), \overrightarrow{\nabla}_{g_u}u >_{g_u} \\ &+ \frac{u^2}{W_u^3}II^u_\theta(\overrightarrow{\nabla}_{g_u}u, \overrightarrow{\nabla}_{g_u}u) \\ &- \frac{1}{W_u}H^u_\theta(u,x) + \frac{1}{W_u} < \overrightarrow{\nabla}_{g_u}\left(\frac{1}{W_u}\right), u\frac{\overrightarrow{\nabla}_{g_u}u}{W_u} >_{g_u} \end{aligned} \tag{8}$$

with $<\nu_{ext},\theta>\geq 0$.

3.4 Allard's Regularity Theorem

The proof of the theorem 4.1 is mainly based on a regularity theorem for almost minimizing varifolds. In the article [All72], it is stated in an Euclidean ambient context. Using isometric embeddings we can deduce a Riemannian version of it.

We restate, here, for completeness sake, the regularity theorem of chapter 8 page 466 of [All72] that will be of frequent use in the sequel. For this statement we use the notations of the original article [All72]. $\mathcal{X}(\mathcal{M})$ denotes smooth vectorfields on $\mathcal{M}$, $||V||$ the weight of a varifold (a positive measure on $\mathcal{M}$) Θ its density at a point, $\delta V(g)$ the first variation of the varifold V in the direction of vectorfield g.

Theorem 3.1 (Allard's Regularity Theorem, Euclidean version). *Let $p > 1$ be a real number. Let q be the conjugate exponent, $\frac{1}{p}+\frac{1}{q}=1$. Let k be a integer number, $1 \leq k \leq n$. We assume that $k < p < +\infty$ if $k > 1$, and that $p \geq 2$, if $k = 1$.*

For all $\varepsilon \in]0,1[$ there exists $\eta_1 > 0$, (that depends on ε) such that for all reals $R > 0$, for all integer $d \geq 1$, for all varifolds $V \in V_k(\mathbb{R}^n)$ and for all points $a \in spt||V||$, if

1. *$\Theta^k(||V||, x) \geq d$ for $||V||$ almost all $x \in B_{\mathbb{R}^n}(a,R)$;*

2. $||V||(U(a,R)) \leq (1+\eta_1)d\omega_k R^k$;

3. $\delta V(g) \leq \eta_1 d^{\frac{1}{p}} R^{\frac{k}{p}-1} \left(\int_{\mathbb{R}^n} |g|^q ||V||(dx)\right)^{\frac{1}{q}}$ *with* $g \in \mathcal{X}(\mathbb{R}^n)$ *and* $supp(g) \subset U(a,R)$.

Then
there exists a map $F_1 : \mathbb{R}^k \to \mathbb{R}^n$ *such that*

1. $F_1 \in C^1(\mathbb{R}^k, \mathbb{R}^n)$ *and* $F_1 \circ T = Id$*, where* $T : \mathbb{R}^n \to \mathbb{R}^k$ *is an orthogonal projection,*

2. $U(a,(1-\varepsilon)R) \cap spt||V|| = U(a,(1-\varepsilon)R) \cap F_1(\mathbb{R}^k)$,

3. $\forall y,z \in \mathbb{R}^k$, $||dF_1(y) - dF_1(z)|| \leq \varepsilon \left(\frac{|y-z|}{R}\right)^{1-\frac{k}{p}}$.

Theorem 3.2 (Allard's Regularity Theorem, Riemannian version). *Let* $\mathcal{M}^n$ *be a compact Riemannian manifold,* $i : \mathcal{M} \hookrightarrow \mathbb{R}^N$ *an isometric embedding.*
Let $p > 1$ *be a real number. Let* q *be the conjugate exponent,* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* k *be an integer number,* $1 \leq k \leq n$*. We assume that* $k < p < +\infty$ *if* $k > 1$*, and that* $p \geq 2$*, if* $k = 1$*.*

For all $\varepsilon \in]0,1[$ *there exists* $\tilde{\eta}_1$ *(that depends on* ε *), such that there exists a* $\tilde{R}_1 > 0$ *and for all* $0 < \tilde{R} \leq \tilde{R}_1$*, for all integer number* $0 < \tilde{d} < +\infty$*, for all varifolds* $V \in V_k(\mathcal{M}^n)$*, and for all point* $a \in spt||V||$*, if*

1. $\Theta^k(||V||, x) \geq \tilde{d}$ *for* $||V||$ *almost every* $x \in B_{\mathcal{M}}(a, \tilde{R})$;

2. $||V||(B(a,\tilde{R})) \leq (1+\tilde{\eta}_1)\tilde{d}\omega_k \tilde{R}^k$,

3. $\delta V(g) \leq \tilde{\eta}_1 \tilde{d}^{\frac{1}{p}} \tilde{R}^{\frac{k}{p}-1} \left(\int_{\mathcal{M}} |g|^q ||V||(dx)\right)^{\frac{1}{q}}$*, with* $g \in \mathcal{X}(\mathcal{M})$ *et* $supp(g) \subset B(a,\tilde{R})$.

Then
there exists a function $\tilde{F}_1 : \mathbb{R}^k \to \mathcal{M}$*,* $R_0 < \tilde{R}$ *(*$\tilde{F}_1$ *and* R_0 *are mutually independent) such that*

1. $\tilde{F}_1 \in C^1(\mathbb{R}^k, \mathcal{M})$*,* $d\tilde{F}_1(0)$ *is an isometry,*

2. $B(a,(1-\varepsilon)R_0) \cap spt||V|| = B(a,(1-\varepsilon)R_0) \cap \tilde{F}_1(\mathbb{R}^k)$,

3. $||d\tilde{F}_1(y) - d\tilde{F}_1(z)|| \leq \varepsilon \left(\frac{|y-z|}{R_0}\right)^{1-\frac{k}{p}}$ *for all* $y,z \in \mathbb{R}^k$.

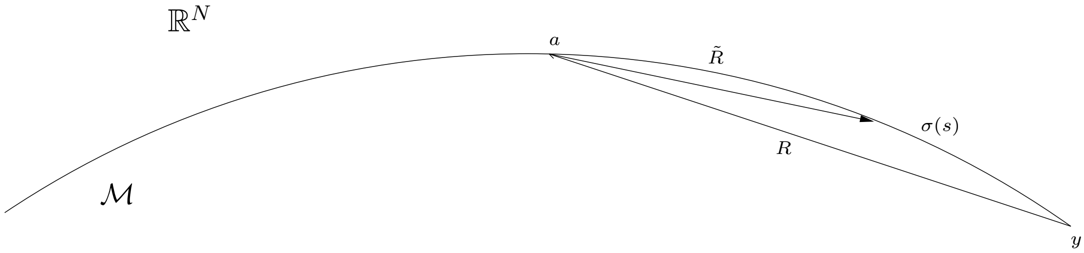
Remark: In the statement of the theorem the constant $\tilde{\eta}_1$ depends on the embedding i and on η_1 produced by theorem 3.1.

Idea of the proof: At this point we try to apply theorem 3.1 to the varifold $i_\#(V)$. Actually, if V satisfies the assumptions 1, 2 and 3 of theorem 3.2, then $i_\#(V)$ satisfies the hypothesis of Allard's Regularity Theorem, Euclidean version (see theorem 3.1) but, with different constants.

To this aim, we need to compare the intrinsic distance of a submanifold and the distance of the ambient manifold restricted to the submanifold.

Lemma 3.1. *Let $\mathcal{M}$ be an embedded manifold into $\mathbb{R}^N$ of arbitrary codimension. $i : \mathcal{M} \hookrightarrow \mathbb{R}^N$ an isometric embedding and β_i his second fundamental form.*
Fix a point $a \in \mathbb{R}^N$, $a \in \mathcal{M}$ and consider a second point $y \neq a$ different from a on $\mathcal{M}$, now take the geodesic σ of $\mathcal{M}$ of length $\tilde{R}$ that joins a and y on $\mathcal{M}$ and the Euclidean segment $[a, y]$ of length R.
Then
there exists a constant $\delta_i > 0$ that depends only on β_i so that $\tilde{R} \leq R(1+\delta_i R^2)$.

Proof: We take as origin of coordinates the point a and parametrize σ



by its arc length s. Consider the function $f(s) = |\sigma(s)|^2$. Then $f(\tilde{R}) = R^2$, $f'(s) = 2 < \sigma', \sigma > (s)$,

$$\begin{aligned} f''(s) &= 2(< \sigma'', \sigma > (s) + < \sigma', \sigma' > (s)) \\ &= 2 + 2 < \sigma'', \sigma > (s) \\ &= 2 + 2 < \sigma'', \sigma - s\sigma' > (s). \end{aligned}$$

Since $(\sigma - s\sigma')' = \sigma' - \sigma' - s\sigma''$, $||(\sigma - s\sigma')'|| \leq s||\sigma''|| \leq s\beta_i$, we get $||\sigma - s\sigma'|| \leq \frac{s^2}{2}\beta_i$. It follows that $f''(s) \geq 2 - s^2\beta_i^2$, $f'(s) - f'(0) = f'(s) \geq 2s - \frac{s^3}{3}\beta_i^2$, $f(s) \geq s^2 - \frac{s^4}{12}\beta_i^2$, which implies

$$f(\tilde{R}) = R^2 \geq \tilde{R}^2 - \frac{\tilde{R}^4}{12}\beta_i^2. \tag{9}$$

Finally $\tilde{R} \leq R(1 + \frac{R^2 const.}{24}\beta_i^2) = R(1 + R^2\delta_i)$ where δ_i is a constant that depends only on β_i. $\square$.

Proof of Allard's Regularity Theorem, Riemannian version. In this context, variables and constants respect the previous convention and furthermore constants and variables relative to intrinsic objects of $\mathcal{M}$ are denoted with a tilda. From the following formula [4.4 (1) in [All72]]:

$$\delta(i_{\#}V)(g) = \delta V(g^{\top}) - \int_{G_k(\mathcal{M})} g^{\perp}(x) \cdot h(\mathcal{M}, (x, S))dV(x, S) \qquad (10)$$

with $g \in \mathcal{X}(U(a, R_0))$, $g(x) = g^{\top}(x) + g^{\perp}(x)$, $g^{\top}(x) \in T_x\mathcal{M}$, $g^{\perp}(x) \in T_x^{\perp}\mathcal{M}$, we can deduce that assumption 3 of theorem 3.1 is satisfied with some suitably chosen constants. To see this, it is sufficient to control the Euclidean mean curvature of $i_{\#}V$.

Now, we assume that R_0, $\tilde{\eta}_1$, $\tilde{R}$ verify the following conditions:

- $0 < R_0 < \min\left\{\inf_{x\in i(B(a,\tilde{R}))}\left\{|x-a|_{\mathbb{R}^N}\right\}, \sqrt{\frac{(1+\eta_1)^{\frac{1}{k}}-1}{\delta_i}}\right\}$,
- $\tilde{d} = d$,
- $0 < \tilde{\eta}_1 \leq \min\left\{\frac{\eta_1}{2}, \frac{1+\eta_1}{(1+\tilde{\delta}_i R_0^2)^k} - 1\right\}$,
- $0 < \tilde{R} \leq \frac{\tilde{\eta}_1}{\beta_i(1+\tilde{\eta}_1)^{\frac{1}{p}}\omega_k^{\frac{1}{p}}} =: \tilde{R}_1$.

Remark: First we choose $R_0 > 0$, then $\tilde{\eta}_1$ and after that $\tilde{R}_1$ with dependences in this order.

The condition $0 < R_0 < \sqrt{\frac{(1+\eta_1)^{\frac{1}{k}}-1}{\delta_i}}$ serves to assert that $\frac{1+\eta_1}{(1+\delta_i R_0^2)^k} - 1 > 0$ and there exists $\tilde{\eta}_1$ such that $(1+\tilde{\eta}_1)\omega_k\tilde{R}^k \leq (1+\eta_1)\omega_k R^k$.

The condition $0 < R_0 < \inf_{x\in i(B(a,\tilde{R}))}\left\{|x-a|_{\mathbb{R}^N}\right\}$ serves to assert that $spt||i_{\#}V|| \cap i(B(a,R_0)) \subseteq i(spt||V|| \cap B(a,\tilde{R}))$.

From what was said, it follows

$$||i_{\#}V||(B_{\mathbb{R}^N}(a,R_0)) \leq ||V||(B_{\mathcal{M}}(a,\tilde{R})) \leq d(1+\tilde{\eta}_1)\omega_k\tilde{R}^k \leq d(1+\eta_1)\omega_k R_0^k. \qquad (11)$$

The first term on the right hand side of equation (10) is estimated thanks to assumption 3,

$$|\delta V(g^{\top})| \leq \tilde{\eta}_1 d^{\frac{1}{p}}\tilde{R}^{\frac{k}{p}-1}\left(\int_{\mathcal{M}} |g^{\top}|^q ||V||(dx)\right)^{\frac{1}{q}} \leq \tilde{\eta}_1 d^{\frac{1}{p}}\tilde{R}^{\frac{k}{p}-1}||g||_{L^q(||V||)}.$$

To the second term, we apply Hölder's inequality,

$$|\int_{G_k(\mathcal{M})} g^{\perp}(x)\cdot h(\mathcal{M},(x,S))dV(x,S)| \leq \beta_i \left\{\int_{Supp(g)} d||V||\right\}^{\frac{1}{p}} ||g||_{L^q(||V||)}.$$

Choosing vectorfields g supported in the R_0-ball makes

$$\left\{\int_{Supp(g)} d||V||\right\}^{\frac{1}{p}} \leq \{||i_{\#}V||(B(a,R_0))\}^{\frac{1}{p}} \leq d^{\frac{1}{p}}(1+\eta_1)\omega_k R_0^k.$$

It follows that

$$\delta(i_{\#}V)(g) \quad \leq \quad \eta d^{\frac{1}{p}} R_0^{\frac{k}{p}-1}\left(\int_{\mathbb{R}^n}|g|^q||V||(dx)\right)^{\frac{1}{q}}. \qquad (12)$$

Now we can apply theorem 3.1 (Allard's Euclidean) to $i_{\#}V$ at point a with $R = R_0$ as described previously to obtain (with a little abuse of notation for i^{-1}), $\tilde{F}_1 = i^{-1}\circ F_1$ where F_1 is given by theorem 3.1 (Allard Euclidean). It can be easily seen that $dF_1(0) = Id$ and that i is an isometric embedding. This implies that $d\tilde{F}_1(0)$ is an isometry.
In the remaining part of this section we assume $\varepsilon = \frac{1}{2}$.

3.5 First Variation of Solutions of the Isoperimetric Problem

In this subsection, we check that varifold solutions of the isoperimetric problem have constant mean curvature. This will used later, in Lemma 4.1, where Levy-Gromov's inequality will be used to verify the third assumption in Allard's theorem.

Lemma 3.2. *Let V be the varifold associated to a current ∂D of dimension $n-1$, that is the boundary of a current D of dimension n solution of the isoperimetric problem.*
Then
there exists a constant H so that for every vector field $X \in \mathcal{X}(\mathcal{M})$ we have

$$\delta\partial D(X) = -H\int_{Spt||\partial D||} < X,\nu > ||\partial D||(x),$$

where ν is the outward normal to the boundary of D.

Proof: As $\mathcal{X}(\mathcal{M})$ is the space of section of the tangent bundle $T\mathcal{M}\to\mathcal{M}$, it has a natural structure of vector space (possibly of infinite dimension). Consider the following linear functionals on this vector space:

$$Flux : \begin{cases} \mathcal{X}(\mathcal{M}) & \to \quad \mathbb{R} \\ X & \mapsto \quad \int_{\partial D} < X,\nu > dVol_{\partial D}(x) \end{cases}$$

$$\delta\partial D : \begin{cases} \mathcal{X}(\mathcal{M}) & \rightarrow & \mathbb{R} \\ X & \mapsto & \delta\partial D(X) \end{cases}$$

Lemma 3.3. *If $Flux(X) = 0$,*
then
there exists a variation $h(t,x)$ such that $\mathbf{M}((h_t)_{\#}D) = \mathbf{M}(D)$ and $\left[\frac{\partial h}{\partial t}\right]_{t=0} =$ X.

Proof: Construction of h. We start with the flow $\tilde{h}(t,x)$ of X (i.e: $X(x) := \frac{\partial}{\partial t}\tilde{h}(t,x)_{|t=0}$) and we make a correction by a flow H_s of a vector field Y that has $Flux(Y) \neq 0$. Now, we consider the function

$$f : \begin{cases} I^2 & \rightarrow & \mathbb{M} \\ (s,t) & \mapsto & \mathbf{M}((H_s \circ h_t)(D)) - Vol(D) \end{cases}$$

where I is an interval of the real line. It is smooth by classical theorems of differentiation of an integral, since we make an integration on rectifiable currents.
We apply the implicit function theorem at point $(0,0)$ to function f in order to find an $s(t)$ that satisfies

$$\mathbf{M}((H_{s(t)} \circ \tilde{h}_t)(D)) - Vol(D) = 0.$$

Such an application of implicit function theorem is possible since

$$\frac{\partial}{\partial s} f(0,0) = Flux(Y) \neq 0.$$

We have also $s'(0) = 0$. Indeed

$$\frac{d}{dt} f(s(t),t) = s'(t) \int_{h_t(D)} div(Y) + \int_D div(H_{s(t)_*}X)$$

and an evaluation at $t = 0$ gives

$$s'(0)Flux(Y) + Flux(X) = 0$$

hence $s'(0) = 0$ since $Flux(Y) \neq 0$ and $Flux(X) = 0$.
Now if we apply the previous argument to $h(t,x) = H_{s(t)} \circ \tilde{h}(t,x)$ we can see that

$$\frac{\partial}{\partial t} h(0,x) = s'(0)Y(h_t(x)) + H_{s(0)_*}X = X,$$

by the fact $s'(0) = 0$. $\square$.

End of the proof of lemma 3.2.

Let X be a vector field with $Flux(X) = 0$. Applying lemma 3.3, there exists a variation $h(t,x)$ satisfing the following two properties

1. $\mathbf{M}((h_t)_{\#}D) = \mathbf{M}(D)$
2. $\frac{\partial h}{\partial t}_{t=0} = X$,

provided $Flux(X) = 0$ and

$$\frac{d}{dt}\left[\mathbf{M}((h_t)_{\#}\partial D)\right]_{t=0} = \delta\partial D(X) = 0.$$

In other words, $Ker(Flux) \subseteq Ker(\delta\partial D)$. Hence there exists $\lambda \in \mathbb{R}$ for which it is true that $\delta\partial D = \lambda Flux$. We set $H = -\lambda$. This notation is justified by the fact that on the smooth part of ∂D, H is equal to the genuine mean curvature. $\square$.

3.6 Riemannian Monotonicity Formula

Theorem 3.3 (Riemannian Monotonicity Formula). *Let $T \in \mathbb{R}V_n(\mathcal{M})$ be a varifold solution of the isoperimetric problem, consider $x \in Spt||\partial T||$, and $R > 0$.*
Then

$$\Theta(||\partial T||, x)\omega_{n-1}R^{n-1}e^{-(|H|+\beta_i)R} \leq ||\partial T||B(x, R), \tag{13}$$

where H is the mean curvature of ∂T .

Proof: When $\mathcal{M}$ is Euclidean space, this result is due to W. K. Allard, [All72]. In order to adapt it to the situation considered here, we make use of an isometric embedding i of $\mathcal{M}$ (whose existence is stated by Nash's theorem) and then we look at the current $i_{\#}T$ in order to apply the Euclidean statement. In this case we see that the term to consider, instead of simply taking into account the mean curvature of T in $\mathcal{M}$, involves the mean curvature of $i_{\#}T$ into $\mathbb{R}^N$. This is not really a problem because of our control on the norm of the second fundamental form of the embedding of $\mathcal{M}$ in $\mathbb{R}^N$ by the upper bound β_i. Therefore

$$\Theta(||\partial T||, x)\omega_{n-1}R^{n-1}e^{-(|H|+\beta_i)R} \leq ||\partial T||B(x, R).$$

$\square$.

4 The Normal Graph Theorem

Theorem 4.1. *Let $\mathcal{M}^n$ be a Riemannian manifold with injectivity radius $inj_{\mathcal{M}} > 0$. Let $i : \mathcal{M} \hookrightarrow \mathbb{R}^N$ be an isometric embedding with second fundamental form bounded by β_i. Let B be a compact domain whose boundary ∂B is smooth with normal injectivity radius $r_0 > 0$ and second fundamental form $||II_{\partial B}||_\infty \leq \beta$.*
Then
there exists $\varepsilon_0(inj_{\mathcal{M}}, \beta_i, r_0, \beta, Vol(B), Vol(\partial B)) > 0$ such that for every current T solution of the isoperimetric problem that satisfies condition $(*)$

$$Vol(B\Delta T) \leq \varepsilon_0, \quad (*)$$

∂T is the normal graph of a function u on ∂B.
Furthermore, for all $\alpha \in]0,1[$, $u \in C^{2,\alpha}(\partial B)$ and $||u||_{C^{2,\alpha}(\partial B)}$ tends to 0 as $Vol(B\Delta T)$ tends to 0. This convergence is uniform. The constants involved only depend on $inj_{\mathcal{M}}$, β_i, r_0 ,β, $Vol(B)$ and $Vol(\partial B)$.

Remark: All the constants that bound the geometry of the ambient space are calculated on a tubular neighborhood of ∂B where the normal exponential map is a diffeomorphism, except for the confinement theorem.
The proof of theorem 4.1 occupies paragraphs 4.1 to 4.7.
We give at first a sketch of this proof and then a series of lemmas that are used in the true proof.

4.1 Sketch of the Proof of theorem 4.1

1. At a first stage we make use of an *a priori* estimate of the mean curvature for solutions of the isoperimetric problem, this is Lévy-Gromov's lemma, stated in 4.1.

2. Secondly, we apply Allard's regularity theorem (Riemannian version) to prove that ∂T is a $C^{1,\alpha}$ submanifold.
   To this aim we proceed as in the following steps:

   (a) We stand on a sufficently small scale R in order to estimate the first variation like required by theorem 3.2.

   (b) We estimate the volume of the intersection of ∂T with a ball $B_{\mathcal{M}}(x, R)$ and we proceed as follows: we cut ∂T with $B_{\mathcal{M}}(x, R)$ and replace T by T' of equal volume thanks to the construction (lemma 4.3) of a one parameter family of diffeomorphisms that perturbes T preserving the volumes of perturbed domains. This leads to the estimates of lemmas 4.3, 4.7.

(c) We apply Allard's theorem and we conclude that ∂T is of class $C^{1,\alpha}$.

Remark: The tangent cone is hence a vector space. As showed by Frank Morgan in [Mor03], it follows that ∂T is as smooth as the metric. We shall give a direct proof of this.

3. We confine ∂T in a tubular neighborhood of ∂B, of sufficiently small thickness, in theorem 4.2. For this, 4.3 is combined with the Riemannian monotonicity formula 3.3.

4. We calculate a bound on r (the tubular neighborhood thickness) so that the projection π, of the tubular neighborhood $\mathcal{U}_{r_0}(\partial B)$ of thickness r on ∂B, restricted to ∂T is a local diffeomorphism and, after, via a topological argument we argue that $\pi_{|\partial T}$ is a global diffeomorphism. This shows that ∂T is the global normal graph on ∂B of a function u. By an application of the implicit function theorem, u is then of class $C^{1,\alpha}$.

5. A geometric argument shows that the C^1 norm of u goes to zero if $r \to 0$. An appeal to Ascoli-Arzelà's theorem is needed to show that $||u||_{C^{1,\alpha}} \to 0$ when $r \to 0$.

6. Finally we use elliptic regularity theory, Schauder's estimates, in order to find upper bounds on $||u||_{C^{2,\alpha}}$ and with the same technique of Ascoli-Arzelà of point 5, we show $||u||_{C^{2,\alpha}} \to 0$ when $r \to 0$.

4.2 A priori estimates on mean curvature

Set

1. $k := Min\left\{-1, \inf_{\overline{\mathcal{U}_{r_0}(\partial B)}} \mathcal{K}^{\mathcal{M}}\right\}$,

2. $\delta := Max\left\{\sup_{\overline{\mathcal{U}_{r_0}(\partial B)}} \mathcal{K}^{\mathcal{M}}, 1\right\}$.

Denote by $H^{\partial T}$ the mean curvature of ∂T. It is constant for isoperimetric domains.

Lemma 4.1. *Let $\mathcal{M}^n$ be a compact Riemannian manifold. Let B a domain whose boundary ∂B is smooth with normal injectivity radius $r_0 > 0$ and second fundamental form $||II_{\partial B}||_\infty \leq \beta < +\infty$.*

Then
there exists $\varepsilon_1 > 0$ *and* $H_1 > 0$ *such that for every current* T *solution of the isoperimetric problem that satisfies the condition*

$$Vol(T\Delta B) \leq \varepsilon_1,$$

$$|H^{\partial T}| \leq H_1(k, n, Vol(B), Vol(\partial B)). \tag{14}$$

Proof: We give first a qualitative argument and after another one that gives an explicit apriori upper bound for $H_1(k, n, Vol_g(B), Vol(\partial B))$.

Qualitative Argument:
Let T_j be a sequence of solutions of the isoperimetric problem for $(\mathcal{M}, g)$ such that $Vol(T_j\Delta B)$ tends to 0. Since $Vol(T_j)$ tends to $Vol(B)$, $Vol(\partial T_j)$ tends to $I_M(Vol(B)) \leq Vol(\partial B)$.

Assume by contradiction that the mean curvature $h_j \geq 0$ of ∂T_j tends to infinity. Using Heintze and Karcher's volume comparison theorem, Gromov proves in [Gro86a] that

$$Vol_g(T_j) \leq Vol_g(\partial T_j) \int_0^{z(h_j,k)} [c_k(t) - h_j s_k(t)]^{n-1}\, dt, \tag{15}$$

where $z(h_j, k)$ is the first positive zero of $c_k(t) - h_j s_k(t)$. Here

$$c_\delta : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ t & \mapsto & \begin{cases} cos(\sqrt{\delta} t) & si\ \delta > 0 \\ 1 & si\ \delta = 0 \\ cosh(\sqrt{\delta} t) & si\ \delta < 0 \end{cases} \end{cases}$$

$$s_\delta : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ t & \mapsto & \begin{cases} \frac{1}{\sqrt{\delta}} sin(\sqrt{\delta} t) & si\ \delta > 0 \\ t & si\ \delta = 0 \\ \frac{1}{\sqrt{\delta}} sinh(\sqrt{\delta} t) & si\ \delta < 0 \end{cases} \end{cases}$$

We easily see that $z(h_j, k) \rightarrow 0$. Furthermore

$$\int_0^{z(h_j,k)} [c_k(t) - h_j s_k(t)]^{n-1}\, dt \leq \int_0^{z(h_j,k)} [c_k(t)]^{n-1}\, dt \rightarrow 0,$$

a contradiction.

Effective Argument: To make the previous argument effective, it is sufficient to explicitly estimate $Vol(\partial T)$ in terms of $Vol(B)$, $Vol(\partial B)$ and curvature bounds. For $s \in [-r_0, r_0]$, let B^s be the domain whose boundary is parallel, at distance s, to ∂B (B^s is inside B if $s < 0$). Let

$$\varepsilon_1 = \max\{Vol(B^{r_0}) - Vol(B), Vol(B) - Vol(B^{-r_0})\}.$$

If $Vol(T\Delta B) < \varepsilon_1$, there exists s such that $Vol(T) = Vol(B^s)$. Then

$$Vol(\partial T) \leq Vol(\partial B^s) \leq (c_k(s) + |H^{\partial B}| s_k(s))^{n-1} Vol(\partial B),$$

by Heintze-Karcher. □.

4.3 Volume of the Intersection of a smooth hypersurface with a ball of the ambient Riemannian manifold

Let $\tau_{\delta,\beta} > 0$ be the first positive zero of the function $c_\delta - \beta s_\delta$.
Set $\lambda(\beta, \delta)(t) = \frac{1}{c_\delta(t) - \beta s_\delta(t)}$ for $t \in [0, \tau_{\delta,\beta}[$.

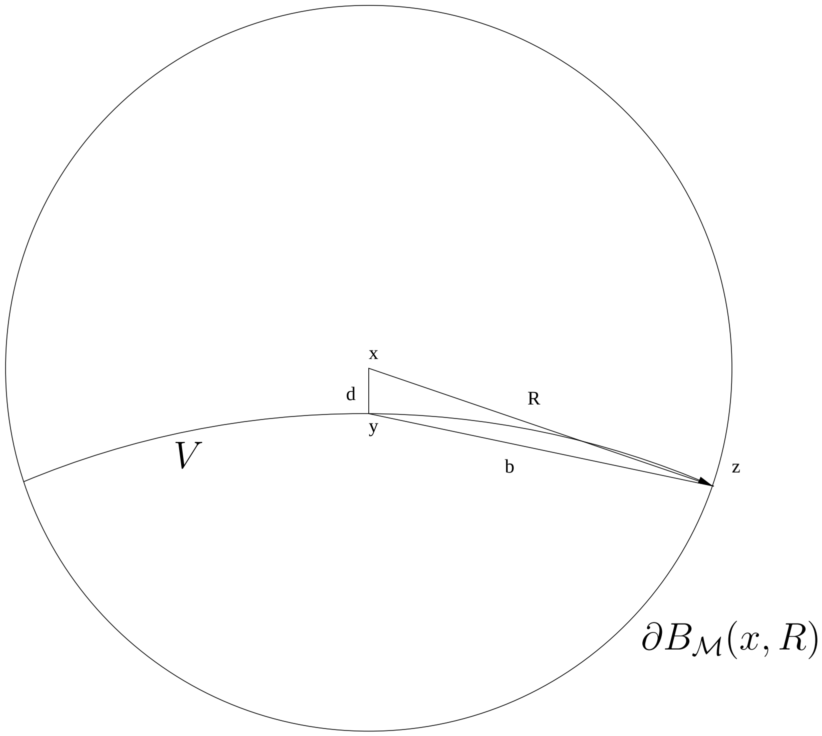
Lemma 4.2. *Let $\mathcal{M}$ be a Riemannian manifold, $V \subset \mathcal{M}$ be a smooth hypersurface. There exists $R_2 > 0$ and $C_2 > 0$ so for every $R < R_2$ and for every $x \in M$ at distance $d < R_2$ from V, if $R' = d + R$, then*

$$Vol(V \cap B(x, R)) \leq (1 + C_2 R')\omega_{n-1} R'^{n-1}.$$

R_2 depends only on β, r_0, $inj_{\mathcal{M}}$ (bound on the second fundamental form of V, normal injectivity radius of V, injectivity radius of $\mathcal{M}$), δ_0 (geometry of the ambient Riemannian manifold) and C_2 depends on the same quantities plus a lower bound on Ricci curvature of V.

Remark: In the proof of theorem 4.1 we apply lemma 4.2 with $V = \partial B$, $d \leq R^3$, but, $d \leq R^2$ is enough too.

Idea of the proof. Using comparison theorems for distortion of the normal exponential map based on a submanifold, we can compare the intrinsic and extrinsic distance functions on $V \hookrightarrow \mathcal{M}$. This allows us to reduce the problem to the estimation of the volume of an intrinsic ball of V, i.e. to Bishop-Gromov's inequality.



Proof:
Whenever $y \in V$ such that $d_{\mathcal{M}}(x,V) = d_{\mathcal{M}}(x,y) = d$ there exists R'' for which

$$V \cap B(x,R) \subseteq B_V(y,R'').$$

We can take for exemple $R'' \leq \sup_{z\in V\cap B(x,R)}\{d_V(y,z)\}$.
Set

$$k_2 := Min\{\frac{\inf\{Ric_V\}}{n-2}, -1\}.$$

then

$$\begin{aligned} Vol(V\cap B(x,R)) &\leq B_V(y,R'') & (16)\\ &\leq Vol_{\mathbb{M}_k^{n-1}}(B(o,R'')) & (17)\\ &= \alpha_{n-2}\int_0^{R''} s_k(t)^{n-2}dt, & (18)\end{aligned}$$

where the first inequality from Bishop-Gromov's.
We have then

$$Vol(V \cap B_{\mathcal{M}}(x,R)) \leq (1 + C'(k_2)(R'')^2)\omega_{n-1}R''^{n-1}$$

after expanding the term

$$\frac{Vol_{\mathbb{M}_k^{n-1}}(B(o,R'')) - Vol_{\mathbb{R}^{n-1}}(o,R'')}{\omega_{n-1}R''^{n-1}}$$

by a Taylor-Lagrange type formula.

We still have to verify that $R'' \leq 1 + CR'$. For this we need to compare the intrinsic and extrinsic distances on V.
Let π be the projection of $\mathcal{U}_{r_0}$ on V.
Following a comparison result of Heintze and Karcher we get

$$(c_\delta(t) - \beta s_\delta(t))^2 g_0 \leq g_t \leq (c_k(t) + \beta s_k(t))^2 g_0, \tag{19}$$

the preceding expression is understood in the sense of quadratic forms. Let $z \in V$ so that $d_{\mathcal{M}}(x,z) = R$, $d_V(y,z) = R''$ and $d_{\mathcal{M}}(x,z) = b$. If we consider the minimizing geodesic γ of $\mathcal{M}$ that joins y to z parameterized by arc length and let us denote $\tilde{\Delta} = Sup_{s\in[o,b]}\{d_{\mathcal{M}}(\gamma(s), \partial B)\}$, there are points $p \in \partial B$, $q \in \gamma$, $p, q \in B_{\mathcal{M}}(y,b)$ for which $\tilde{\Delta} = d_{\mathcal{M}}(p,q)$ and conclude $\tilde{\Delta} \leq b$.
If we take R_2 such that $0 < R_2 := Min\{\frac{\tau_{\delta,\beta}}{2}, \tau_1, r_0, inj_{\mathcal{M}}\}$ where τ_1 is the first positive zero of $(c_\delta - \beta s_\delta)'$, provided that $c_\delta - \beta s_\delta$ be decreasing and positive on $[0, R_2]$ we then infer

$$\begin{aligned} b &\geq \int |c_\delta - \beta s_\delta|(s) ||d\pi(\gamma')||_{g_0} && (20)\\ &\geq (c_\delta(\tilde{\Delta}) - \beta s_\delta(\tilde{\Delta})) l_{g_0}(\pi \circ \gamma) && (21)\\ &\geq (c_\delta(2R) - \beta s_\delta(2R)) R'', && (22) \end{aligned}$$

whence

$$R''(R) \leq \lambda(\beta,\delta)(b) b \leq (1 + C(\beta,\delta)b) b \tag{23}$$

Incidentally we observe that the preceding equation gives us an analogue result to lemma 3.1 in case of an arbitrary Riemannian ambient manifold, but always in codimension 1. We observe also a non sharp estimate

$$R''(R) \leq \lambda(\beta,\delta)(2R) b$$

because $\lambda(\beta,\delta)$ is decreasing in a neighborhood of the origin.
By triangle inequality, we get $b \leq d + R$ and consequently

$$R''(R) \leq \lambda(\beta,\delta)(2R)(d+R).$$

If we look at the Taylor expansion of $\lambda(\beta,\delta)(t) = 1 + \beta t + \mathcal{O}(t^2)$, we notice at a qualitative level that

$$R''(R) \leq (1 + \beta 2R + \mathcal{O}(R^2))(d+R) = (1 + \mathcal{O}(R))(d+R) = (1 + CR)(d+R)$$

where the constant $C = Sup_{R\in[0,R_2]}\{\frac{\lambda(\beta,\delta)(2R)}{R}\}$. So we get

$Vol(V \cap B_{\mathcal{M}}(x,R)) \leq (1+C'(k_2)((1+CR)(d+R))^2)\omega_{n-1}((1+CR)(d+R))^{n-1}$

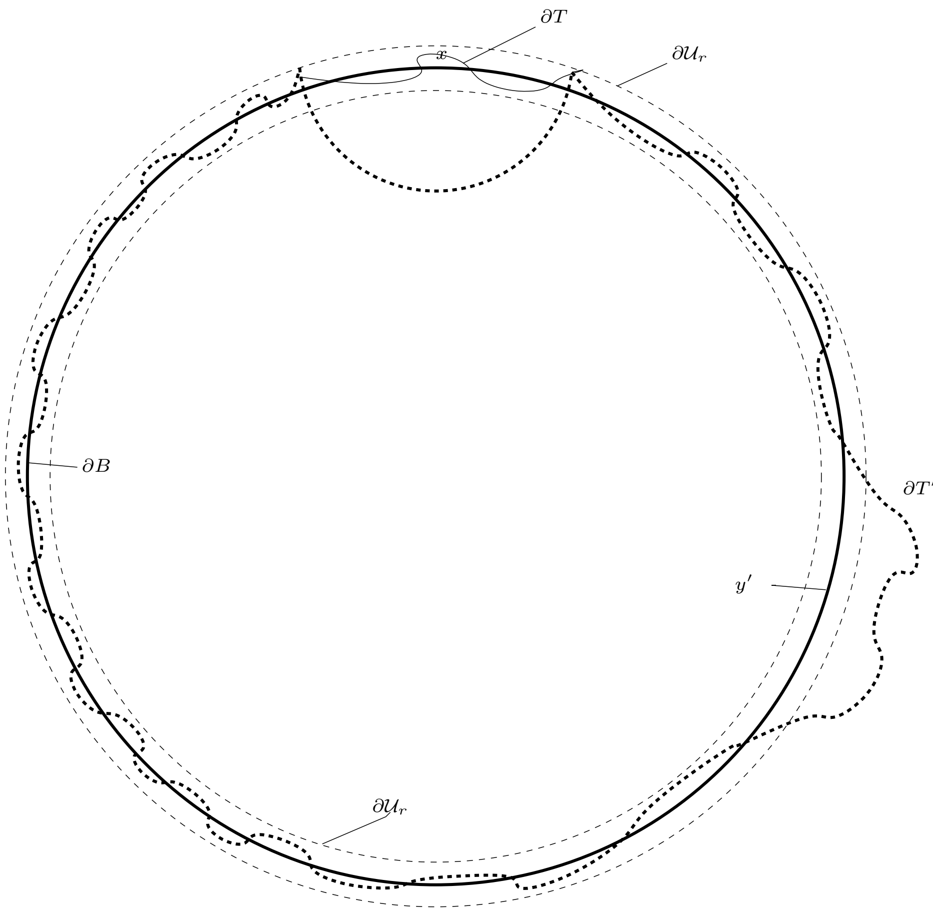
and finally

$$Vol(V \cap B_{\mathcal{M}}(x,R)) \leq (1 + C_2R')\omega_{n-1}R'^{n-1}$$

for C_2 depending on a lower bound on Ricci curvature tensor of V, on an upper bound on the second fundamental formof V and un upper bound on curvature tensor of ambient manifold. □.

4.4 Compensation of Volume Process

Remark: In this subsection we make no assumption on the distance of an arbitrary point x of ∂T. Let $R_3 := Min\{inj_{\mathcal{M}}, r_0, \frac{diam(B)}{4}\}$.



Lemma 4.3. *There exists $C_3 > 0$ such that whenever $R < R_3$, $a < \frac{R}{2}$, there is $\varepsilon_3 > 0$ so that, for every $x \in \partial T$, there exists a vector field ξ_x with the following properties*

1. *the support of ξ_x is disjoint from $B(x, R)$;*

2. *the flow ϕ_t is defined for $t \in [-R, R]$, and for $t \in [-\frac{R}{2}, \frac{R}{2}]$, ξ_x is the gradient of the signed distance function to ∂B;*

3. *the norm of the covariant derivative $|\nabla \xi_x| < C_3$.*

Furthermore, for every solution T of the isoperimetric problem whose boundary contains x, and $Vol(T \Delta B) < \varepsilon_3$, there exists $t \in [-a, a]$ such that $T' = (B \cap B(x, R)) \cup (\phi_t(T) \setminus B(x, R))$ has volume equal to volume of T. In particular,

$$\begin{aligned} Vol(\partial T \cap B(x,R)) &\leq Vol(\partial B \cap B(x,R)) + Vol((T\Delta B) \cap \partial B(x,R)) \\ &+ Vol(\phi_{t\#}(\partial T)) - Vol(\partial T). \end{aligned} \tag{24}$$

Constants C_3 and ε_3 depend only on the geometry of the problem, of the a priori choice of a vector field fixed once and for all on $\mathcal{U}_{\partial B}(r_0)$ and on a bump function ψ defined once at all also.

Remarks:

1. In the proof of theorem 4.1 we use lemma 4.3 with $\varepsilon_0 \leq \varepsilon_3$, among other contraints that will be clear in the sequel.

2. Furthermore if $\delta v := Vol(B \cap B(x, R)) - Vol(T \cap B(x, R)) \leq 0$ then $t \geq 0$ and if $\delta v > 0$ then $t < 0$ (balancing of volume).

3. The parameter a serves to control that t be small, as this t will control the term $|Vol(T' \cap Supp(\varphi)) - Vol(T \cap Supp(\varphi))|$

Idea of proof. The vector field ξ_x is obtained with the classical technique of multiplication by a bump function the metric vector gradient of the signed distance function ∂B. This bump function has support in a neighborhood of a point that belongs to ∂B and that is far away from x. We provide also that the flow of this vector field significantly increases the volume of B. This is sufficient to suitably change the volume of T. We can then operate a balancing of a given volume variation.

Proof: First, we make the following geometric construction of a vector field

ν. Fix a point $y' \in \partial B$ with $B(x,R) \cap B(y,R) = \emptyset$ (it suffices to take y' so that $d(x,y') \geq R + \frac{1}{2}diam(B)$, for example).
Let $\mathcal{U}_{\partial B}(r_0) := \{x \in \mathcal{M} | d(x,\partial B) < r_0\}$. By the choice of r_0, the normal exponential map

$$exp^{\partial B} : \left\{ \begin{array}{lcl} \partial B \times]-r_0, r_0[& \rightarrow & \mathcal{U}_{\partial B}(r_0) \\ (q,t) & \mapsto & exp_q(t\nu(q)) \end{array} \right.$$

is a diffeomorphism.
Let ν be the extension by parallel transport on normal (to ∂B) geodesics of the exterior normal issuing from ∂B (equivalently, ν is the gradient of the signed distance function to ∂B), in a vector field defined on $\mathcal{U}_{r_0}(\partial B)$.
Let

$$\psi : \left\{ \begin{array}{lcl} \mathbb{R} & \rightarrow & [0,1] \\ s & \mapsto & \chi_{[0,1/2]}(|s|) + e^{4/3} e^{\frac{1}{s^2-1}} \chi_{]1/2,1[}(|s|) \end{array} \right.$$

Now, we modulate ν with the smooth function ψ and we set

$$\xi_x := \psi(\frac{d(y',.)}{R})\nu = \psi_1 \nu.$$

It can be seen that $||\nabla_X \xi_x|| \leq ||\psi'||_{\infty,[-1,1]}||X|| + ||\nabla_X \nu|| \leq C_3||X||$, establishing that C_3 depends on geometric quantities and on the choice of ψ.

Let $\{\varphi_t\}$ be the flow (one parameter group of diffeomorphisms of $\mathcal{M}$) of the vector field ξ_x.
It's true that $Supp(\varphi) \subset B_{\mathcal{M}}(y',R)$.

Now, consider, whenever $a \in]0, \frac{R}{2}[$ the functions f, g, h defined as follows:

$$g : \left\{ \begin{array}{lcl} [-a,a] & \rightarrow & \mathbb{R} \\ t & \mapsto & Vol_{g,n}(\varphi_t(B)) \end{array} \right.$$

$$f : \left\{ \begin{array}{lcl} [-a,a] & \rightarrow & \mathbb{R} \\ t & \mapsto & Vol_{g,n}(\varphi_t(\tilde{T})) \end{array} \right.$$

$$h : \left\{ \begin{array}{lcl} [-a,a] & \rightarrow & \mathbb{R} \\ t & \mapsto & Vol_{g,n}(\varphi_t(T)) \end{array} \right.$$

where $\tilde{T} := (T - B(x,R)) \cup (B \cap B(x,R))$.
For the aims of the proof, we need to show that $Vol(T) \in f([-a,a])$ with an argument independent of x as f depends on x.
By construction

$$\begin{array}{rcl} \frac{d}{dt}\left[Vol_{g,n}(\varphi_t(B))\right] & = & \int_{\varphi_t(\partial B)} \psi_1 < \nu, \nu > dVol_{\varphi_t(\partial B)} \\ & \geq & \psi(t) Vol(\partial B_t \cap Supp(\psi_1)) \\ & = & Vol(\partial B_t \cap Supp(\psi_1)), \end{array} \qquad (25)$$

hence letting $R' := \frac{R}{2(c_k+\beta s_k)(\frac{R}{2})}$ and
$(c_\delta - \beta s_\delta)(\frac{R}{2})(Inf_{y' \in \partial B} Vol(\partial B \cap B(y', R')) := C_3'$,

$$\begin{array}{rcl} g'(t) & \geq & Vol(\partial B_t \cap Supp(\psi_1)) \\ & \geq & C_3', \end{array} \tag{26}$$

whenever $t < \frac{R}{2}$.
Hence g is strictly increasing and $g(a) - g(-a) \geq 2aC_3' =: \Delta_3$.
Let

$$J := \left| det \left(\frac{\partial \varphi_t(y)}{\partial y} \right) \right|_{\infty, [-a,a] \times \overline{\mathcal{U}_{r_0}(\partial B)}} \leq e^{nC_3 a},$$

by similar arguments to those of the proof of lemma 4.4.
From

$$\begin{array}{rcl} |f(t) - h(t)| & = & |Vol_n(B \cap B(x,R)) - Vol_n(T \cap B(x,R))| \\ & \leq & Vol((T \Delta B) \cap B(x,R)) \\ & \leq & \varepsilon_3, \\ |h(t) - g(t)| & \leq & |Vol(\varphi_t(T \Delta B))| \\ & \leq & JVol(T \Delta B) \\ & \leq & e^{nC_3 a} \varepsilon_3, \end{array}$$

it follows that

$$|f(t) - g(t)| \leq \varepsilon_3 + J\varepsilon_3 \leq (1 + e^{nC_3 a})\varepsilon_3 =: \sigma,$$

σ is independent on x.
If we take

$$0 < \varepsilon_3 \leq \frac{1}{2(1 + e^{nC_3 a})} aC_3', \tag{27}$$

then

$$\sigma \leq \frac{1}{2} \min\{g(0) - g(-a), g(a) - g(0)\}, \tag{28}$$

therefore

$$[g(-a) + \sigma, g(a) - \sigma] \subseteq f([-a,a]).$$

With this choice for ε_3 we obtain

$$Vol(T) \in [g(-a) + \sigma, g(a) - \sigma],$$

so, there exists $t \in [-a,a]$ depending on x such that $f(t) = Vol(T) = Vol(\varphi_t(\tilde{T}))$ and we conclude the proof by taking $T' := \varphi_t(\tilde{T})$.

Finally

$$\begin{aligned} Vol_{n-1}(\partial T) &= I(Vol(T)) \\ &\leq Vol_{n-1}(\partial T'), \end{aligned}$$

whence

$$\begin{aligned} Vol_{n-1}(\partial T') \leq\ & Vol(\partial B \cap B(x,R)) + Vol((T\Delta B) \cap \partial B(x,R)) \\ & + Vol_{n-1}(\varphi_{t\#}(\partial T)) - Vol(\partial T \cap B(x,R)), \end{aligned}$$

which implies (24) □.

4.5 Comparison of the area of the boundary of an isoperimetric domain with the area of its own variation with constant volume

Lemma 4.4. *Let M be a Riemannian manifold. For every $C > 0$, for every vector field ξ on M such that $|\nabla\xi| < C$, whose flow is denoted by ϕ_t, and whenever V is a hypersurface embedded in M,*

$$Vol(\phi_{t\#}V) \leq e^{(n-1)C|t|} vol(V).$$

Proof: It suffices to majorate the norm of the differential of diffeomorphism ϕ_t.

$$|d_x\phi_t(v)| = (g(x)(v))^{\frac{1}{2}} = (g(\phi_t(x))(d_x\phi_t(v)))^{\frac{1}{2}} = (\phi_t^*(g_{\mathcal{M}})(x)(v))^{\frac{1}{2}}$$

$$(\phi_t^*(g_{\mathcal{M}})(x)(v))^{\frac{1}{2}} \leq e^{C|t|} g(x)(v) = e^{C|t|}|v|.$$

The last inequality comes from the following lemma.

Lemma 4.5. $(\phi_t^*(g_{\mathcal{M}})(x)(v)) \leq e^{2C|t|} g(x)(v)$.

Proof:

$$\frac{\partial}{\partial t}(\phi_t^*(g_{\mathcal{M}})) = \phi_t^* \mathcal{L}_\xi g_{\mathcal{M}}, \tag{29}$$

We assume for the moment that we can show the following inequality:

$$\mathcal{L}_\xi g_{\mathcal{M}} = 2 \times \text{symmetric part of } \nabla\xi. \tag{30}$$

We use this fact to establish

$$\mathcal{L}_\xi g_{\mathcal{M}} \leq 2|\nabla\xi| g_{\mathcal{M}} \leq 2C g_{\mathcal{M}},$$

hence $\phi^*\mathcal{L}_\xi g_\mathcal{M} \leq 2C\phi_t^*(g_\mathcal{M})$.
Set $t \mapsto \phi_t^*(g_\mathcal{M}) = q_t$, on $T_x\mathcal{M}$, then q_t satisfies $\frac{\partial}{\partial t}q_t \leq 2Cq_t$ with $q_0 = g_\mathcal{M}$.
It follows that whenever $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$, $q_t(v) \leq e^{2C|t|}q_0(v)$ we have $(\phi_t^*(g_\mathcal{M})(x)(v)) \leq e^{2C|t|}g(x)(v)$.
It remains to show that $\mathcal{L}_\xi g_\mathcal{M} = 2 \times$ symmetric part of $\nabla\xi$.
Let $A_\xi := \mathcal{L}_\xi - \nabla_\xi$. We look at this operator on 2 covariant tensor fields and evaluate it on the metric $g_\mathcal{M}$. We obtain $\mathcal{L}_\xi g_\mathcal{M} = A_\xi g_\mathcal{M}$ and then

$$0 = A_\xi(g(w_1, w_2)) = (A_\xi g)(w_1, w_2) + g(-\nabla_{w_1}\xi, w_2) + g(-w_1, \nabla_{w_2}\xi)$$

it is obvious that $|\mathcal{L}_\xi g_\mathcal{M}| \leq 2|\nabla\xi|$. $\square$.

End of the proof of lemma 4.4.

We apply the inequality of lemma 4.5 to the members of an orthonormal basis $(v_1, \ldots, v_{n-1})$ of the tangent space T_xV, we find

$$|\phi_{t\#}(v_1 \wedge \cdots \wedge v_{n-1})| \leq e^{(n-1)C|t|}.$$

By an integration on V, one gets

$$Vol(\phi_{t\#}V) \leq e^{(n-1)C|t|}vol(V).$$

$\square$.

Lemma 4.6. *Whenever $R > 0$, $x \in Spt||\partial T||$ there exists R_4, $\frac{R}{2} < R_4 < R$, such that*

$$Vol((T\Delta B) \cap \partial B(x, R_4)) \leq \frac{2}{R}Vol(T\Delta B).$$

Proof: By a straightforward application of the coarea formula and the mean value theorem for integrals. $\square$.

Remark: At this point of the article we cannot put restrictions on the distance of $x \in \partial T$ to ∂B.
This lemma is used in the confinement lemma to majorate the volume of ∂T in a geodesic ball. In lemma 4.7, we need to control the $(n-1)-$*dimensional* volume of the intersection of ∂T with a geodesic ball of radius R centered in x. To make it possible we need to have the quantity $\frac{d(x,\partial B)}{R}$ very small.

Lemma 4.7. *Whenever* $\eta > 0$, *there is* R_5 *such that whenever* $R < R_5$, *there are* R_6, r_6, $\varepsilon_6 > 0$ *(depending only on* R *and on the geometry of the problem) such that* $0 < \frac{R}{2} < R_6 < R$, $0 < r_6 \leq \left(\frac{R}{2}\right)^3$ *and if* T *is a current solution of the isoperimetric problem with the property* $Vol(B\Delta T) \leq \varepsilon_6$,
then,
whenever $x \in Spt||\partial T||$ *with* $d(x, \partial B) \leq r_6$ *we have*

$$Vol(\partial T \cap B_{\mathcal{M}}(x, R_6)) \leq (1+\eta)\omega_{n-1}R_6^{n-1}. \tag{31}$$

Remarks:

1. In this context there are 2 distance scales. The scale of R_6 the radius of the cutting geodesic ball of the ambient Riemannian manifold, that is the same as the scale of R and that of r_6 that is the distance between an arbitrary point of ∂T and a point of ∂B. This is an important point in the estimates required by Allard's theorem, as the proof of lemma 4.2 shows. Without this control on the scales involved we cannot have good control on the volume of the intersection of the hypersurface ∂B with an ambient geodesic ball.

2. The presence of interval $]\frac{R}{2}, R[$ is just a technical complication due to the mean value theorem for integrals in the estimates of the $(n-1)-$ *dimensional* volume of the part of $\partial T \cap B(x, R)$ that is $T\Delta B$.

Proof:
Let $A := C_2 s\left(1+\frac{s^2}{2^3}\right)$, $B := \left(1+\frac{s^2}{2^3}\right)^{n-1} - 1$.
Let R_5 be the greatest positive real number s such that

1. $s \leq Min\{inj_{\mathcal{M}}, r_0, \frac{diam(B)}{4}, R_3\}$,

2. $$AB + B + A \leq \frac{1}{3}\eta. \tag{32}$$

We fix $r_6 > 0$ with $r_6 \leq \left(\frac{R}{2}\right)^3$.
Let $x \in Spt||\partial T||$.
Let a be the greatest positive real number $s < \frac{R}{2}$ with

$$(e^{(n-1)C_3 s} - 1)M \leq \frac{1}{3}\eta\omega_{n-1}\left(\frac{R}{2}\right)^{n-1} \tag{33}$$

where M is the maximum the isoperimetric profile on the interval $[vol(B)/2, 2vol(B)]$.
i.e.

$$a \leq Min\{\frac{1}{(n-1)C_3} log\left[1 + \frac{\eta\omega_{n-1}\left(\frac{R}{2}\right)^{n-1}}{3M}\right], \frac{R}{2}\}.$$

Set $\varepsilon_6 := Min\{\varepsilon_3, \frac{Vol(B)}{2}, \frac{1}{3}\eta\omega_{n-1}\left(\frac{R}{2}\right)^n\}$.
Let T be a solution of the isoperimetric problem such that $Vol(T\Delta B) < \varepsilon_6$.
By (24) we find $t(x) \in [-a, a]$ and ε_3 (given by lemma 4.3) satisfying

$$\begin{aligned} Vol(\partial T \cap B(x,R)) &\leq Vol(\partial B \cap B(x,R)) + Vol((T\Delta B) \cap \partial B(x,R)) \\ &\quad + Vol_{n-1}(\varphi_{t\#}(\partial T)) - Vol_{n-1}(\partial T). \end{aligned} \tag{34}$$

From lemma 4.7 we have

$$\begin{aligned} Vol(\partial T \cap B(x,R)) &\leq Vol(\partial B \cap B(x,R)) + Vol((T\Delta B) \cap \partial B(x,R)) \\ &\quad + (e^{(n-1)C_3 t} - 1)Vol_{n-1}(\partial T). \end{aligned} \tag{35}$$

By lemma 4.6 we get R_4 satisfying

$$\begin{aligned} Vol((T\Delta B) \cap \partial B(x,R_4)) &\leq \tfrac{2}{R} Vol(T\Delta B) \\ &\leq \tfrac{2}{R}\varepsilon_6. \end{aligned}$$

Let $R_6 := R_4$. Lemmas 4.4, 4.6 and 4.2 combined give

$$Vol(\partial T \cap B(x,R_6)) \leq (1+\mathcal{O}(R_6))\omega_{n-1}R_6^{n-1} + \frac{2}{R}Vol(T\Delta B) + (e^{(n-1)C_3 a} - 1)M, \tag{36}$$

as, by lemma 4.2,

$$Vol(\partial B \cap B(x,R)) \leq (1 + \mathcal{O}(R))\omega_{n-1}R^{n-1},$$

and by lemma 4.6 $0 < \frac{R}{2} < R_6 < R$.
By (33), (32) and the choice of ε_6, equation (36) becomes

$$Vol(\partial T \cap B(x,R_6)) \leq (1+\frac{1}{3}\eta)\omega_{n-1}R_6^{n-1} + \frac{1}{3}\eta\omega_{n-1}R_6^{n-1} + \frac{1}{3}\eta\omega_{n-1}R_6^{n-1}. \tag{37}$$

Finally

$$Vol(\partial T \cap B(x,R_6)) \leq (1+\eta)\omega_{n-1}R_6^{n-1} \tag{38}$$

$\square$.

4.6 Confinement of an Isoperimetric Domain by Monotonicity Formula

Theorem 4.2. *Let $\mathcal{M}^n$ be a Riemannian manifold. Let B a compact domain whose boundary ∂B is smooth.*

For every $s > 0$, there exists $\varepsilon_7(s) > 0$ with the property that if T is a current solution of the isoperimetric problem with

$$Vol(B\Delta T) < \varepsilon_7,$$

then ∂T is contained in a tubular neighborhood of thickness s of ∂B.

Idea of the proof: By contradiction, we assume that there is a current T and a point $x \in \partial T$ at distance $> s$ of ∂B. We choose $R \in]s/2, s[$ so that the intersection $T\Delta B$ with the sphere $\partial B(x, R)$ has small area. The mecanism of balancing gives an estimation of the area of $\partial T \cap B(x, R)$, as $\partial B \cap B(x, R) = \emptyset$.This majoration contradicts the minoration given by monotonicity formula (Lemma 3.3), if $vol(T\Delta B)$ is too small.

Proof: Set $s > 0$. Let H_1 be the constant produced by lemma 4.1 (Lévy-Gromov). Let C_3 be the constant given by lemma 4.3. Let M be the maximum of the isoperimetric profile on the interval $[vol(B)/2, 2vol(B)]$. Let β_i be a bound on the second fundamental form of an isometric immersion of $\mathcal{M}$ in $\mathbb{R}^N$ the Euclidean space. We can choose a so that

$$(e^{(n-1)C_3 a} - 1)M < \frac{1}{2}\omega_{n-1}\left(\frac{s}{2}\right)^{n-1} e^{-(H_1+\beta_i)s}. \tag{39}$$

Let ε_3 be the second constant given by lemma 4.3, when, in this lemma, we take $R = s/2$. Let $\varepsilon_7 < \varepsilon_3$, $\varepsilon_7 < vol(B)/2$ and

$$\frac{2\varepsilon_7}{s} < \frac{1}{2}\omega_{n-1}\left(\frac{s}{2}\right)^{n-1} e^{-(H_1+\beta_i)s}.$$

Let T be a current solution of the isoperimetric problem satisfying

$$vol(T\Delta B) < \varepsilon_7.$$

We argue by contradiction. Assume there is a point $x \in \partial T$ placed at distance $> s$ from ∂B.

The balancing of volume (lemme 4.3) gives for all $R \leq Min\{s, R_3\}$

$$Vol(\partial T \cap B(x, R)) \leq Vol((T\Delta B) \cap \partial B(x, R)) + Vol(\phi_{t\#}(\partial T)) - Vol(\partial T),$$

as $B(x, R) \cap B = \emptyset$. We apply lemma 4.4 with $C = C_3$ and we set $R_7 \in]s/2, s[$ defining $R_7 := R_4$ obtained by applying lemma 4.6 with $R = s$ such that

$$vol((T\Delta B) \cap B(x, R_7)) \leq \frac{2}{s} Vol(T\Delta B).$$

It follows

$$vol(\partial T\cap B(x,R_7))\leq\frac{2\varepsilon_7}{s}+(e^{(n-1)C_3a}-1)Vol(\partial T)$$

$$vol(\partial T\cap B(x,R))\leq\frac{2\varepsilon_7}{s}+(e^{(n-1)C_3a}-1)M.$$

Invoking lemma 4.1 (Lévy-Gromov), the mean curvature of ∂T satisfies

$$|H|\leq H_1.$$

Monotonicity inequality (lemma 3.3) gives us

$$vol(\partial T\cap B(x,R_7))\geq\omega_{n-1}R_7^{n-1}e^{-(|H|+\beta_i)R_7},$$

which, by our choice of ε_7, contradicts the preceding inequality. We conclude that ∂T is contained in a tubular neighborhood of thickness s of ∂B. $\square$.

4.7 Proof of Theorem 4.1

Application of Allard's Theorem We give now the proof of theorem 4.1. We must show that solutions T of the isoperimetric problem which are close to B volumewise are graphs of small functions. Therefore, we fix a number r and will find $\varepsilon_0(r)$ such that $Vol(T\Delta B)<\varepsilon_0(r)$ implies that ∂T is the graph of a function u with $||u||_\infty<r$. Later on, stronger norms of u will be estimated in terms of r.

Proof: Set $\alpha\in]0,1[$, $\varepsilon=\frac{1}{2}$, $d=1$ and $p=\frac{n-1}{1-\alpha}$ in the Riemannian Allard's theorem. Theorem 3.2 provides us with a constant $\tilde\eta_1$ and radius $\tilde R_1$. Consider $R_3=Min\{inj_{\mathcal M},r_0,\frac{diam(B)}{4}\}$ as defined in section 4.4 and let $R=Min\{\tilde R_1,R_3,\frac{\tilde\eta_1}{H_1[(1+\tilde\eta_1)\omega_{n-1}]^{\frac{1}{p}}}\}$. Pick a radius $r\leq\left(\frac{R}{2}\right)^3$ and set $\varepsilon_0'=Min\{\varepsilon_6,\varepsilon_7(r)\}$. Let T be a solution of the isoperimetric problem satisfying

$$Vol(T\Delta B)\leq\varepsilon_0'.$$

Then from the comparison lemma 4.7 applied with $\eta=\tilde\eta_1$, we obtain a $R_6\in]\frac{R}{2},R[$ with the property

$$||V||(B(x,R_6))\leq(1+\tilde\eta_1)d\omega_kR_6^k. \tag{40}$$

From lemmas 3.2 and 4.1 we argue that whenever $g \in \mathcal{X}(\mathcal{M})$ with $Supp(g) \subset B_{\mathcal{M}}(x, R_6)$,

$$\delta\partial T(g) \leq H_1(Vol(\partial T \cap B(x, R_6)))^{\frac{1}{p}} ||g||_{L^q(\partial T)}. \tag{41}$$

Hence, an application of comparison lemma 4.7 allows us to get

$$\delta\partial T(g) \leq \left\{ H_1[(1+\tilde{\eta}_1)\omega_{n-1}]^{\frac{1}{p}} R_6 \right\} R_6^{\frac{n-1}{p}-1} ||g||_{L^q(\partial T)} \leq \tilde{\eta}_1 R_6^{\frac{n-1}{p}-1} ||g||_{L^q(\partial T)}, \tag{42}$$

because

$$\left\{ H_1[(1+\tilde{\eta}_1)\omega_{n-1}]^{\frac{1}{p}} R \right\} \leq \tilde{\eta}_1, \tag{43}$$

by the choice of R.
Finally the confinement lemma allows us to state that the support of ∂T is in a tubular neighborhood of thickness r.
The Riemannian version of Allard's theorem applies with $\tilde{R} = R$. It provides us with a radius R_0, and for all $x \in \partial T$, with a C^1 map $F : \mathbb{R}^{n-1} \to \mathcal{M}$ whose image of a neighborhood of the origin is exactly $||i_{\#}(\partial T)|| \cap B_{\mathbb{R}^N}(x, \frac{1}{2}R_0)$ and whose differential satisfies

$$||dF_z - dF_{z'}|| \leq \varepsilon \left(\frac{d(z, z')}{R_0} \right)^{\alpha} \forall z, z' \in \mathbb{R}^{n-1}, \quad |z|, |z'| < R_0.$$

$\pi|_{\partial T}$ **is a local diffeomorphism.** In what follows r indicates again the thickness of a tubular neighborhood of ∂B in which ∂T is confined, π is the projection of $\mathcal{U}_r(\partial B)$ on ∂B, θ is the gradient vector of the signed distance function to ∂B and g_0 the induced metric by that of $\mathcal{M}$ on ∂B.
In addition to $r \leq (R/2)^3$, we shall need that $\sqrt{r} < (1-\varepsilon)R_0 = \frac{1}{2}R_0$ and $c(r) < 1$, for a function c to be defined in the sequel. Therefore we let r_1 be the largest radius satifying these conditions.

Let $\varepsilon_0 = min\{\varepsilon_0', Vol(\{x \in \mathcal{M} | d(x, \partial B) \leq r_1\})\}$. From now on, we assume that $Vol(T \Delta B) < \varepsilon_0$.

Consider the function

$$f : \left\{ \begin{array}{lcl}]-\frac{1}{2}R_0, \frac{1}{2}R_0[& \to & \mathbb{R} \\ t & \mapsto & d_{\mathcal{M}}(F(tv), \partial B) \end{array} \right.$$

where R_0 is given by Allard's theorem, v is a unit vector in $T_x\partial T$.

Allard's theorem gives a $C^{1,\alpha}$ bound on F. Riemannian comparison theorems (Heintze-Karcher) gives C^2 bounds on the distance to ∂B in terms of ambient sectional curvature and the second fundamental form of ∂B. Therefore f is $C^{1,\alpha}$ bounded. If its derivative at 0 were large, then f would take large values. Since f is confined within $]-r,r[$, its derivative $f'(0)$ is small, $|f'(0)| < c(r)$. This shows that $| < dF(v), \theta > | < c(r)$ for all unit vectors v. If $c(r) < 1$, this implies that the differential of the projection to ∂B, restricted to ∂T, is onto, i.e. $\pi|_{\partial T}$ is a local diffeomorphism.

Furthermore, as r gets smaller, the differential of $\pi|_{\partial T}$ gets closer and closer to an isometry.

$\pi|_{\partial T}$ is a global diffeomorphism

Lemma 4.8. *Let $\mathcal{U}$ be a tubular neighborhood of B. There exists $\omega \in \Lambda^{n-1}(\mathcal{U})$ such that $d\omega = dVol_g$.*

Proof: $\mathcal{U}$ being a connected non compact manifold of dimension n implies $H^n(\mathcal{U}, \mathbb{R}) = 0$, see [God71] theorem 6.1 of page 216. □.

By two preceding lemmas we have

$$Vol_g(T) = \int_T d\omega = \int_{\partial T} \omega = \eta Vol_g(B) = \eta \int_{\partial B} \omega$$

with η close to 1, but

$$\int_{\partial T} \omega = m\eta' \int_{\partial B} \omega = m \int_B d\omega = mVol_n(B)$$

with η' close to 1, as $\pi^*(\omega|_{\partial B})$ is close to $\omega|_{\partial T}$ as ∂T is C^1 close to ∂B and

$$\eta' \int_{\partial T} \omega = \int_{\partial T} \pi^*(\omega|_{\partial B}) = m \int_{\partial B} \omega.$$

This establishes that $m = 1$. In other words we have showed that $\pi|_{\partial T}$ is a global diffeomorphism.
Furthermore,

$$u = d(\cdot, \partial B) \circ F \circ (\pi \circ F)^{-1} \tag{44}$$

shows that u belongs to $C^{1,\alpha}(\partial B)$.

$C^{2,\alpha}$ and Higher order Regularity. Let us first give a precise definition of the $C^{\ell,\alpha}$ norms.

Definition 4.1. *Let $\mathcal{M}$ be a compact Riemannian manifold, let u be a function on $\mathcal{M}$. We say that $u \in C^{\ell,\alpha}(\mathcal{M}, \mathbb{R}^m)$ if the representative of u in every coordinates chart is of class $C^{\ell,\alpha}$.*

Definition 4.2. *Let $u \in C^{\ell,\alpha}(\mathcal{M})$. We set*

$$||u||_{C^{\ell,\alpha}(\mathcal{M})} = \max_{l} \left\{ ||u_{|\Omega_l}||_{C^{\ell,\alpha}(\Omega_l)} \right\},$$

where $||u_{|\Omega_l}||_{C^{\ell,\alpha}(\Omega_l)} := ||u \circ \Theta^{-1}||_{C^{\ell,\alpha}(\mathcal{U}_l)}$ *with* $\{\Omega_l \overbrace{\cong}^{\Theta} \mathcal{U}_l \subseteq \mathbb{R}^{n-1}\}$ *be a fixed atlas of* $\mathcal{M}$.

At this point, we continue by attaching the argument of Morgan 3.3 and $C^{2,\alpha}$ regularity and also $C^{\ell,\alpha}$ follows easily. Using these facts, we show $||u||_{C^{1,\alpha}}$ is small and after by Schauder's estimates we can conclude that $||u||_{C^{2,\alpha}}$ is small. In order to show that u is more regular we use the same argument used in [Mor03] proposition 3.3 page 5044 as indicated at the end of the proof of [Mor03] proposition 3.5 page 5047. For reader's convenience, we restate here the theorem.

Proposition 4.1 ([Mor03] prop. 3.3). *Let f be a real function defined on an open set Ω of $\mathbb{R}^{n-1}$ with the property*

$$\frac{d}{dt}\left[\int_{\Omega} F(x, f(x) + tg(x), \nabla(f(x) + tg(x)))dx\right]_{t=0} = 0$$

whenever g is a function with $Supp(g) \subset\subset \Omega$. Assume F and $\frac{\partial F}{\partial f_i}$ are $C^{\ell-1,\alpha}$ and F is elliptic, i.e. the matrix $\frac{\partial F}{\partial f_i \partial f_j}$ is positive definite.
Then
f is $C^{\ell,\alpha}$.

Proof: The proof can be found in [Mor03] □.

In local coordinates, we can see ∂T locally like the graph of a function f of class $C^{1,\alpha}$.
For smooth variations g with compact support the area functional $\mathcal{A}(f) := \int A(x, f, \nabla f(x))dx$ and the volume functional $\mathcal{V}(f) := \int V(x, f(x))dx$ satisfy the relation:

$$\frac{d}{dt}\left[\mathcal{A}(f + tg) - \lambda\mathcal{V}(f + tg)\right]|_{t=0} = 0 \tag{45}$$

for some Lagrange multiplier λ that is the mean curvature of ∂T. The functional $\mathcal{A} - \lambda\mathcal{V}$ then satisfies the regularity and ellipticity assumptions required by 4.1, hence ∂T is as regular as possible and at least of class $C^{2,\alpha}$, which implies by an application of the implicit function theorem that F given by

Allard's theorem belongs to $C^{2,\alpha}$ and therefore that u is also of class $C^{2,\alpha}$. In other words, there exists $\tilde{F}$ of class $C^{2,\alpha}$ such that

$$u = d(\cdot, \partial B) \circ \tilde{F} \circ (\pi \circ \tilde{F})^{-1},$$

and we conclude that u is of class $C^{2,\alpha}$.

$C^{2,\alpha}$ **Estimates.** Now we are in a position to exploit the formula of the first section for the mean curvature of a normal graph. This allows to estimate the $C^{1,\alpha}$ norm and $C^{2,\alpha}$ norm of u. Straightforward computations will show that the $C^{2,\alpha}$ norm of u goes to zero when $r \to 0$. We now give the details of these calculations. We consider a system of Fermi coordinates (r, x) centerd at a point $p \in \partial B$, with x normal coordinates on an open set of ∂B centered in p as in [Gra01]. Let

$$u_i := \frac{\partial u}{\partial x^i}, \quad u_{ij} := \frac{\partial^2 u}{\partial x^i x^j},$$

$$g := dt^2 + g^{ij}(r, x) dx^i dx^j, \tag{46}$$

$$||\nabla_{g_u} u||^2_{g_u} = g^{i,j}(u, x) u_i u_j, \tag{47}$$

$$\begin{aligned} \nabla_{g_u} W_u &= -\frac{1}{2}\frac{1}{\sqrt{(1+||\nabla u||^2)^3}} \left\{ \frac{\partial}{\partial r} g^{lj}(u, x) u_i u_j u_l \right\} \\ &- \frac{1}{2}\frac{1}{\sqrt{(1+||\nabla u||^2)^3}} \left\{ \frac{\partial}{\partial \theta^i} g^{jl}(u, x) u_j u_l + 2 g^{lj}(u, x) u_i u_{ij} u_l \right\} g^{im} \frac{\partial}{\partial \theta^m} \end{aligned} \tag{48}$$

$$\frac{1}{W_u} \left[div_{\partial B^r} \left(\nabla_{g_u} u \right) \right]_{|r=u} = \left[\frac{1}{W_u} g^{ij}(u, x) + f^{ij}(x, u, \nabla u) \right] u_{ij} + f(x, u, \nabla u). \tag{49}$$

Notice that $f(x, u, \nabla u)$, $f^{ij}(x, u, \nabla u) \to 0$, $||u||_{C^1} \to 0$. The functions

$$f, f^{ij} : \left\{ \begin{array}{lcl} \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} & \to & \mathbb{R} \\ (x, y, z) & \mapsto & f(x, y, z) \end{array} \right.$$

have the same regularity than the metric with respect to variables x, y and they are of class C^∞ with respect to z. We carry the same calculations for the remaining 4 terms of formla (8).

$$\begin{aligned} H^{\mathcal{N}}_{\nu_{inw}}(u,x) &= -div_{(\mathbb{S}^{n-1},g_u)}(\frac{\overrightarrow{\nabla}_{g_u}u}{W_u}) - \frac{1}{W_u^2} < \nabla_{\overrightarrow{\nabla}_{g_u}u}(\frac{u\overrightarrow{\nabla}_{g_u}u}{W_u}), \overrightarrow{\nabla}_{g_u}u >_{g_u} \\ &+ \frac{u^2}{W_u^3} II^u_\theta(\overrightarrow{\nabla}_{g_u}u, \overrightarrow{\nabla}_{g_u}u) \\ &- \frac{1}{W_u} H^u_\theta(u,x) + \frac{1}{W_u} < \overrightarrow{\nabla}_{g_u}(\frac{1}{W_u}), u\frac{\overrightarrow{\nabla}_{g_u}u}{W_u}) >_{g_u} \end{aligned}$$

with $< \nu_{inw}, \theta > \leq 0$.

$$\begin{aligned} < \nabla_{-b\nabla u}(-b\nabla u), -b\nabla u) > &= \tfrac{-1}{1+||\nabla u||^2}\left(< g^{im}u_m f_i \nabla u, \nabla u >\right) \\ &+ \tfrac{-1}{1+||\nabla u||^2}\left(< \tfrac{1}{W_u}\nabla_{\nabla u}\nabla u, \nabla u >\right) \end{aligned} \tag{50}$$

where

$$f_i := -\frac{1}{2\sqrt{(1+||\nabla u||^2)^3}}\left\{ g^{lj}_{,1}u_i u_j u_l + g^{lj}_{,i}u_l u_j + 2g^{li}u_l u_{ij} \right\}.$$

$$\begin{aligned} \tfrac{1}{W_u^2} < \nabla_{\overrightarrow{\nabla}_{g_u}u}(\tfrac{u\overrightarrow{\nabla}_{g_u}u}{W_u}), \overrightarrow{\nabla}_{g_u}u >_{g_u} &= \tfrac{1}{W_u^3} < du(\nabla u)\nabla u, \nabla u > \\ &+ < \nabla_{-b\nabla u}(-b\nabla u), -b\nabla u) > \end{aligned} \tag{51}$$

It is easy to see that the expression

$$\frac{1}{W_u^3} < du(\nabla u)\nabla u, \nabla u > = \frac{1}{W_u^3}\left(g^{ij}g^{pl}u_i u_j u_l u_p\right) \tag{52}$$

does not depends on second partial derivatives of u. For this reason we can set

$$\tilde{f}_1(x,u,\nabla u) := \frac{1}{W_u^3} < du(\nabla u)\nabla u, \nabla u > . \tag{53}$$

The following quantities depend only on u, x and on the first derivatives of u.

$$\tilde{f}^{ij}(x,u,\nabla u)u_{ij} + \tilde{f}(x,u,\nabla u) := < \nabla_{-b\nabla u}(-b\nabla u), -b\nabla u) >, \tag{54}$$

$$f^*(x,u,\nabla u) := -\frac{u^2}{W_u^3} II^{\partial B^u}(\nabla u, \nabla u) = -\frac{u^2}{W_u^3} g^{ij}g^{lm}u_i u_l \Gamma^1_{ij}(u,x). \tag{55}$$

The expression

$$-b^2 u \nabla_{\nabla u}\left(\frac{1}{W_u}\right) = -\frac{u}{W_u^2} g^{im}u_m f_i \tag{56}$$

depends linearly on second derivatives of u. We set

$$u(\bar{f}^{ij}u_{ij} + \bar{f}(x,u,\nabla u)) := -b^2 u \nabla_{\nabla u}\left(\frac{1}{W_u}\right). \tag{57}$$

We obtain

$$\left[\frac{1}{W_u} g^{ij}(u,x) + l^{ij}(x,u,\nabla u)\right] u_{ij} = h_1 + h_2 \tag{58}$$

where $h_1 = H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^u}$ and

$$h_2 = f(x,u,\nabla u) - \bar{f}(x,u,\nabla u) - \tilde{f}_1(x,u,\nabla u) - \tilde{f}(x,u,\nabla u) - f^*(x,u,\nabla u).$$

The equation of constant mean curvature for normal graphs takes the form

$$\left[\frac{1}{W_u} g^{ij}(u,x) + l^{ij}(x,u,\nabla u)\right] u_{ij} = h(x,u,\nabla u), \tag{59}$$

with $h(x,u,\nabla u)$, $l^{ij}(x,u,\nabla u)$, f^*, $\bar{f}$, $\tilde{f}_1$, $\tilde{f}$, $\tilde{f}^{ij}$, $\tilde{f} \to 0$, $||u||_{C^1} \to 0$ and

$$h, l^{ij}, f^*, \bar{f}, \bar{f}^{ij}, \tilde{f}^{ij}, \tilde{f} : \begin{cases} \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} & \to & \mathbb{R} \\ (x,y,z) & \mapsto & f(x,y,z) \end{cases}$$

have the same regularity as the Levi-Civita connection with respect to variables x, y and are of class C^∞ with respect to z. If $k \leq K^{\mathcal{M}} \leq \delta$, then by Heintze-Karcher's theorem [HK78] we get

$$(c_\delta(u) - \beta s_\delta(u))^2 g_0 \leq g(u,x) \leq (c_\delta(u) + \beta s_\delta(u))^2 g_0 \tag{60}$$

$$\frac{g_0^{-1}}{(c_\delta(u) - \beta s_\delta(u))^2} \leq g(u,x)^{-1} \leq \frac{g_0^{-1}}{(c_\delta(u) + \beta s_\delta(u))^2}. \tag{61}$$

Consequently, there are $0 < A_1 \leq A_2$ for which

$$\frac{A_1 I_{n-1}}{(c_\delta(u) + \beta s_\delta(u))^2} \leq g(u,x)^{-1} \leq \frac{A_2 I_{n-1}}{(c_\delta(u) - \beta s_\delta(u))^2}, \tag{62}$$

hence the equation

$$Lu := a^{ij} u_{ij} = \tilde{h}(x)$$

with $a^{ij}(x) := \frac{1}{W_u} g^{ij}(u,x) + l^{ij}(x,u,\nabla u)$, $\tilde{h}(x) = h(x,u(x),\nabla u(x))$ is uniformly elliptic as the $l^{ij} \to 0$ when $r \searrow 0$. Using the theory of elliptic partial differential equations, we can obtain L^p estimates on the norm $W^{2,p}$ of the function u.

$$||u||_{2,p} \leq c_1 ||\tilde{h}||_{0,p} + c_2 ||u||_{L^p},$$

where c_1, c_2 depend on ellipticity constants A_1 and A_2, and therefore on the geometry of the situation $(B, \partial B, \beta_i, \beta, \mathcal{M})$ and of the choice of the atlas $\{\Omega_l \overbrace{\cong}^{x} \mathcal{U}_l \subseteq \mathbb{R}^{n-1}\}$ used to define

$$||u||_{C^{\ell,\alpha}(\partial B)} := \max_l \left\{||u_{|\Omega_l}||_{C^{\ell,\alpha}(\Omega_l)}\right\}$$

where $||u_{|\Omega_l}||_{C^{\ell,\alpha}(\Omega_l)} := ||u \circ (x)^{-1}||_{C^{\ell,\alpha}(\mathcal{U}_l)}$.
Finally,

$$||u||_{W^{2,p}(\partial B)} \leq c_1(\partial B, A_1, A_2)||\tilde{h}||_{W^{0,p}(\partial B)} + c_2(\partial B, A_1, A_2)||u||_{L^p(\partial B)}, \quad (63)$$

$$||u||_{C^{1,\alpha}(\partial B)} \leq c_3(\partial B)||u||_{W^{2,p}(\partial B)}. \quad (64)$$

Theorem 4.3 (The Comparison Principle for Curvatures). *Let B_1 and B_2 being two submanifolds with boundary, of dimension n of $\mathcal{M}$, $B_1 \subseteq B_2$, let $x \in \partial B_1 \cap \partial B_2$.*
Then
$< H^{\partial B_1}(x), \nu_{ext} > \leq < H^{\partial B_2}(x), \nu_{ext} >$

Proof: [Ale62] □.

Lemma 4.9.

$$|H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^u}| \leq C(r, ||u||_{C^1}, k, \delta, II_\theta^{\partial B}) \to 2Lip(H_\theta^{\partial B}(\cdot))diam(\partial B)$$

when $r \to 0$.

Proof: Let x_1, $x_2 \in \partial B$ be defined as $u(x_2) := Max_{x \in \partial B}\{u(x)\}$ and $u(x_1) := Min_{x \in \partial B}\{u(x)\}$.
Then

$$B^{u(x_1)} \subseteq T \subseteq B^{u(x_2)}$$

and $B^{u(x_1)}$, $B^{u(x_2)}$ have smooth boundary and are tangent to ∂T at $p_1 = (x_1, u(x_1))$ and $p_2 = (x_2, u(x_2))$. We deduce then, by comparison principle applied to $B^{u(x_1)}$, T, $B^{u(x_2)}$ that

$$H_\theta^{\partial B^{u(x_1)}}(x_1) \leq H_\nu^{\partial T}(x) \leq H_\theta^{\partial B^{u(x_2)}}(x_2). \quad (65)$$

We assume that $r \leq \overline{r} := Min\{\frac{R}{2}, r_1, \dfrac{\pi}{\sqrt{\delta}}\}$.
We substract the same quantity to the sides of (65)

$$\begin{aligned} H_\theta^{\partial B^{u(x_1)}}(x_1) - \tfrac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x) &\leq H_\nu^{\partial T} - \tfrac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x) \\ &\leq H_\theta^{\partial B^{u(x_2)}}(x_2) - \tfrac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x) \end{aligned} \quad (66)$$

$$\begin{array}{rcl} |H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x)| & \leq & |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B^{u(x_2)}}(x_2)| \\ & + & |H_\theta^{\partial B^{u(x_1)}}(x_1) - \frac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x)| \end{array} \tag{67}$$

and after we apply the triangle inequality repeatedly to obtain

$$\begin{array}{rcl} |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B^{u(x_2)}}(x_2)| & \leq & |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B}(x_1)| \\ & + & |H_\theta^{\partial B^{u(x_2)}}(x_2) - H_\theta^{\partial B}(x_2)| \\ & + & |H_\theta^{\partial B}(x_1) - H_\theta^{\partial B}(x_2)| \end{array} \tag{68}$$

$$\begin{array}{rcl} |H_\theta^{\partial B^{u(x_1)}} - \frac{1}{W_u} H_\theta^{\partial B^{u(x)}}| & \leq & |H_\theta^{\partial B^{u(x_1)}}(x_1)| \left(1 - \frac{1}{W_u}\right) \\ & + & \frac{1}{W_u} |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B^{u(x)}}(x)| \end{array} \tag{69}$$

$$\begin{array}{rcl} |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B^{u(x)}}(x)| & \leq & |H_\theta^{\partial B^{u(x_1)}}(x_1) - H_\theta^{\partial B}(x_1)| \\ & + & |H_\theta^{\partial B}(x) - H_\theta^{\partial B^{u(x)}}(x)| \\ & + & |H_\theta^{\partial B}(x) - H_\theta^{\partial B}(x_1)|. \end{array} \tag{70}$$

To majorate the middle term, we need to prove the following lemma.

Lemma 4.10. *There exists $b_3(s)$ such that whenever $y \in \partial B$,*

$$|H_\theta^{\partial B^s}(y) - H_\theta^{\partial B}(y)| \leq b_3(s). \tag{71}$$

Proof: Let

$$b_3'(s, y) := |\sum_{i=1}^{n-1} ctg_\delta(s + c_1(y, \lambda_i(y))) - H_\theta^{\partial B}(y)|$$

$$b_3''(s, y) := |a_k(s + c_2(y, H_\theta^{\partial B}(y))) - H_\theta^{\partial B}(y)|$$

$$b_3(s, y) := Max\,\{b_3'(s, y), b_3''(s, y)\}$$

where $ctg_\delta(c_1(x, s)) = s$, $c_1(x, s) \in]0, \frac{\pi}{\sqrt{\delta}}[$ $ctg_k(c_2(x, s)) = s$,
if $s > \sqrt{-k}$, $tg_k(c_2(x, s)) = s$, if $s < \sqrt{-k}$ and $c_2(x, \sqrt{-k}) = \sqrt{-k}$

$$a_k(s) = \begin{cases} ctg_k(s) & , \quad s > \sqrt{-k} \\ \sqrt{-k} & , \quad s = \sqrt{-k} \\ tg_k(s) & , \quad s < \sqrt{-k} \end{cases}$$

We find $b_3(s) := ||b_3(s, y)||_{\infty, \partial B}$ $\square$.

Continuation of the proof of lemma 4.9. If we set

$$\begin{array}{rcl} C(r, ||u||_{C^1}, k, \delta, II_\theta^{\partial B}) & := & 4b_3(r) + 2Lip(H_\theta^{\partial B}(\cdot))diam(\partial B) \\ & + & \left(1 - \frac{1}{W_u}\right) ||H_\theta^{\partial B}(\cdot)||_{\infty, \partial B}, \end{array}$$

we conclude that

$$|H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^{u(x)}}(x)| \leq C(r, ||u||_{C^1}, k, \delta, II_\theta^{\partial B}) \tag{72}$$

□.

End of $C^{2,\alpha}$ estimates, in case when ∂B has constant mean curvature. The preceding arguments allow us to state that

$$||\tilde{h}||_{W^{0,p}(\partial B)} \to 0 \ , \ r \to 0$$

$$||u||_{L^p(\partial B)} \to 0 \ , \ r \to 0$$

hence

$$||u||_{C^{1,\alpha}(\partial B)} \to 0 \ , \ r \to 0.$$

If we apply the classical Schauder estimates theory we write

$$||u||_{C^{2,\alpha}(\partial B)} \leq c_4(\partial B)||\tilde{h}||_{C^{0,\alpha}(\partial B)} + c_5(\partial B, A_1, A_2)||u||_{C^0(\partial B)}. \tag{73}$$

Therefore

$$||u||_{C^{2,\alpha}(\partial B)} \to 0 \ , \ r \to 0 \ \forall \alpha \in]0,1[\tag{74}$$

as h_1 and h_2 are C^∞ expressions with respect to u and ∇u and hence converge in $C^{0,\alpha}$ topology, necessarily to 0.

Remark: Alternatively, finer calculations would show that

$$||u||_{C^{1,\alpha}(\partial B)} \leq C(geom, \beta_i, \varepsilon, R^\alpha) \ \forall \alpha \in]0,1[.$$

With the aid of the following lemma, this would show that

$$||u||_{C^{1,\alpha}(\partial B)} \to 0 \ , \ r \to 0.$$

Lemma 4.11. *$\forall \varepsilon > 0$, $\forall M > 0$, $\forall \alpha < \beta$ there exists $\delta > 0$ such that if $||u||_{C^k} \leq \varepsilon$ and $||u||_{C^\beta} \leq M$ then $||u||_{C^{\ell,\alpha}} \leq \delta$.*

Proof: Compactness of the injections of $C^{\ell,\beta} \hookrightarrow C^{\ell,\alpha}$ (a direct consequence of Ascoli-Arzelà's theorem). □.

End of $C^{2,\alpha}$ estimates, general case. Qualitative Argument. Whenever r goes to 0, the quantity $|H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^u}|$ goes to 0. To see this it is not necessary to use lemma 4.9, and the end of the proof remains unchanged.

Effective Argument. We need to improve lemma 4.9. As u is bounded in C^2 norm, the second fundamental form of ∂T is bounded. Using the fact that ∂T is everywhere transversal to the vector field $\hat{\theta}$, we show that the normal the injectivity radius of ∂T is bounded from below. We can use a tubular neighborhood of ∂T, and, applying the comparison theorem for curvatures to gives lower and upper bounds for mean curvature of ∂B by mean curvature of equidistant hypersurfaces of ∂T. This gives an upper bound of $|H_\nu^{\partial T} - \frac{1}{W_u} H_\theta^{\partial B^u}|$. The very end of the proof of $C^{2,\alpha}$ estimates remains unchanged.

5 Proof of the Normal Graph Theorem, Variable Metrics Case

Proof: We apply the "Embedding Theorem" of page 223 of [Gro86b] to obtain a free isometric embedding i_∞ for $(\mathcal{M}, g_\infty)$ fixed. Furthermore, an application of the "Main Theorem" of page 118 to this embedding to obtain free isometric embeddings i_j of $(\mathcal{M}, g_j)$ in $\mathbb{R}^N$ close in the C^3 topology (see [Gro86b][page 18]) of i_∞. As constants on which the estimations of theorem 4.1 depend are continuous in the C^3 topology, we apply theorem 4.1 to $(\mathcal{M}_j, g_j)$, with the embeddings i_j, to establish the conclusion. $\square$.

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