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First-order perturbations of the seismic response of fluid-filled stratified poro-elastic media

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SUMMARY

We derive analytical formulas for the first-order effects produced by plane inhomogeneities on the seismic response of a stratified porous medium. The approach used for the derivation is similar to the one employed in the elastic case; it is based on a perturbation analysis of the poro-elastic wave propagation equations. The final forms of the sensitivity operators, which are often referred to as the Fréchet derivatives, are expressed in terms of the Green’s functions of the solid and fluid displacements in the frequency–ray parameter domain. We compute here the Fréchet derivatives with respect to eight parameters, namely, the fluid and mineral density and bulk moduli, porosity, permeability, consolidation parameter and shear modulus.

The accuracy and stability of the derived expressions are checked by comparing differential seismograms computed from the analytical expressions of the Fréchet derivatives with solutions obtained by introducing discrete perturbations into the model properties. We find that the Fréchet derivative approach is generally accurate for perturbations of the medium properties of up to 10%, and for layer thicknesses less than one fifth of the dominant wavelength.

INTRODUCTION

The evaluation of the sensitivity of a seismic wavefield to small perturbations of the properties of a seismological medium is a classical problem in seismology (Aki and Richards, 1980; Tarantola, 1984). The so-called Fréchet derivatives play an important role in least-square inversion schemes, however, their implementation requires a fast and efficient numerical evaluation method. Tarantola (1984) developed the Generalized Reflection and Transmission Matrix Method (Kennett, 1983), already used by Garambois and Dietrich (2002) and Pride et al. (2002) for the computation of theoretical seismograms in porous media.

We first present the governing equations for porous medium before expressing the wave propagation equations for depth-dependent media. Next, we develop the calculation of the Fréchet derivatives in the frequency–ray parameter domain, for the P–SV and SH wave cases. Finally, we check the accuracy of the operators obtained in an infinite medium and in a complex seismic model and conclude with the sensitivity of the seismic waveforms with respect to the different model parameters.

WAVE PROPAGATION IN STRATIFIED POROUS MEDIA

Governing equations

Assuming a $e^{-i\omega t}$ dependence, Pride (1994, 2003) rewrote Biot’s (Biot, 1956) equations of poro-elasticity in the form

\[
\nabla \bar{\tau} = -\omega^2 \left( \bar{\rho} \bar{u} + \rho_i \bar{v} \right)
\]

\[
\tau = [K_U \nabla \bar{u} + C \nabla \bar{v}] f + G \left[ \nabla \bar{u} + (\nabla \bar{u})^T - 2/3(\nabla (\bar{u} \bar{u}^T)) \right]
\]

\[
-P = C \nabla \bar{u} + M \nabla \bar{v}
\]

\[
-\nabla P = -\omega^2 \bar{\rho} \bar{u} - \bar{\rho} \bar{v},
\]

where $\bar{u}$ and $\bar{v}$ respectively denote the average solid displacement and the relative fluid-to-solid displacement. $P$ represents the interstitial pressure and $\tau$ is a $3 \times 3$ stress tensor. $\rho$ is the density of the porous medium. It is related to the fluid density $\rho_f$, solid density $\rho_s$ and porosity $\phi$ via

\[\rho = (1 - \phi) \rho_f + \phi \rho_s.\]

$k_U$ is the undrained bulk modulus and $G$ is the shear modulus. $M$ (fluid storage coefficient) and $C$ (C-modulus) are mechanical parameters. In the quasi-static limit, at low frequencies, these parameters as well as the Lamé parameter $\lambda_c$ defined below are real, frequency-independent and can be expressed in terms of the drained bulk modulus $K_D$, porosity $\phi$, and mineral modulus of the grains $K_s$ and fluid modulus $K_f$.

\[K_U = \frac{\phi K_D + \left[1 - (1 + \phi) K_s \right] K_f}{\phi (1 + \Delta)}, \quad \lambda_c = K_U - \frac{2}{3} G\]

\[C = \frac{1 - (1 + \phi)}{\phi (1 + \Delta)} K_f, \quad M = \frac{K_f}{\phi (1 + \Delta)}\]

with

\[\Delta = 1 - \phi K_c \frac{K_f}{K_s} \left[1 - \frac{K_D}{(1 - \phi) K_f}\right].\]

It is also possible to link the bulk properties $K_D$ and $G$ to the porosity and constitutive mineral properties (Pride, 2003):

\[K_D = K_s \frac{1 - \phi}{1 + c_v \phi} \quad \text{and} \quad G = G_s \frac{1 - \phi}{1 + 3c_v \phi / 2}.\]

The consolidation parameter $c_v$ varies between 2 to 20 in a consolidated medium, but is much greater than 20 in an unconsolidated soil. Finally, the wave attenuation is explained by Darcy’s law which uses a complex, frequency-dependent dynamic permeability (Johnson et al., 1994):

\[\bar{\rho} = i \omega k(\omega) \quad \text{with} \quad k(\omega) = k_0 \left[1 - \frac{4}{3} \frac{\alpha}{\omega \alpha_0} - \frac{\alpha}{\omega_0} \right].\]

The dynamic permeability $k(\omega)$ tends toward the dc permeability $k_0$ at low frequencies (where viscous effects are dominant) and includes
Fréchet derivatives for poro-elastic media

a correction to introduce the inertial effects at higher frequencies. The two domains are separated by the relaxation frequency $\omega_0 = \eta/\mu F_k$. The formation factor $F = \varphi^{-m}$ is expressed in terms of the porosity $\varphi$ and cementation factor $m$. Parameter $n_f$ is considered constant and equal to 8 to simplify the equations. For more information on the parameters used in this study, we refer the reader to the work of Pride (2003).

Coupled second-order equations for plane $P$–$S$–$V$ waves

In a depth-dependent poro-elastic medium and $P$–$S$–$V$ case, equations (1) reduce to the following system of second-order differential equations:

\[
\begin{align*}
F_{1z} &= \frac{\partial}{\partial z} \left[ (\lambda_2 + 2G) \frac{\partial U}{\partial z} + C \frac{\partial W}{\partial z} - \omega p (\lambda_2 + CV + CX) \right] \\
&\quad - \omega p G \frac{\partial V}{\partial z} + \omega^2 \left[ \rho U - \rho^2 GU + \rho p W \right] \\
F_{1r} &= \frac{\partial}{\partial z} \left[ G \frac{\partial V}{\partial z} + \omega p GU + \omega p \left[ \frac{\lambda_2}{\lambda_2 + C} + \frac{2G}{C} \right] \frac{\partial W}{\partial z} \right] \\
&\quad + \omega^2 \left[ \rho V - \rho^2 (\lambda_2 + 2G)V - \rho^2 CX \right] \\
F_{2z} &= \frac{\partial}{\partial z} \left[ C \frac{\partial U}{\partial z} - \omega p \rho V + M \frac{\partial W}{\partial z} - M \omega p X \right] \\
&\quad + \omega^2 \left[ \rho U - \rho^2 U + \rho p W \right] \\
F_{2r} &= \omega p \left[ \frac{\partial U}{\partial z} + \omega \partial \rho X \right] + \omega^2 \left[ \rho V + \rho X \right] \\
&\quad - \rho^2 \left( CV + MX \right)
\end{align*}
\]

These expressions were performed by expressing a series of changes of variables which are described in detail in Kennett (1983) for the elastic case. In the above equations, $U = U(z, \omega; z_2), V = V(z, \omega; z_2), W = W(z, \omega; z_2)$ and $X = X(z, \omega; z_2)$ respectively denote the vertical and horizontal displacements for the solid and for the fluid. Variables $z_2$ for model parameters to carry out a perturbation analysis because of the linear relationship $d = f(m)$. Tarantola (1984) used a Taylor expansion to express the relation between the variations of $m$ and $d$, $f(m + \delta m) = f(m) + D \delta m + o(||\delta m||^2)$ where $D = \partial f / \partial m$ is the matrix of the Fréchet derivatives.

Our aim is to calculate the various Fréchet derivatives corresponding to slight modifications of the model parameters at a given depth $z$. In the $P$–$S$–$V$ case, this problem reduces to finding analytical expressions for the quantities

\[
\begin{align*}
A_1(z, \omega; z) &= \frac{\partial U(z, \omega; z)}{\partial \rho(z)} \\
A_2(z, \omega; z) &= \frac{\partial V(z, \omega; z)}{\partial \rho(z)} \\
A_3(z, \omega; z) &= \frac{\partial W(z, \omega; z)}{\partial \rho(z)} \\
A_4(z, \omega; z) &= \frac{\partial X(z, \omega; z)}{\partial \rho(z)}
\end{align*}
\]

Can we similarly define the Fréchet derivatives $B_i$ for model parameters $\rho_f, \Phi, C, M, \lambda_2$, and $G_i$ for model parameters $\rho_f, \rho_c, \Phi, C, M, \lambda_2$, and $G_i$? Then, the expression for the displacement fields corresponding to a horizontal point force are given by

\[
\begin{align*}
U(z, \omega; z_2) &= U(z, \omega; z_2) + BU(z, \omega; z_2)
\end{align*}
\]
Fréchet derivatives for poro-elastic media

with

\[
\Delta \overline{U} = \int \left[ A_l (z, \omega, z_l) \overline{\delta \rho_l} (z) + B_l (z, \omega, z_l) \overline{\delta \rho_f} (z) + C_l (z, \omega, z_l) \overline{\delta \phi} (z) + \overline{\delta \rho_s} \right] \overline{G_l} (z, \omega, z_l) \overline{\delta \lambda_l} (z) + D_l (z, \omega, z_l) \overline{\delta \rho_s} (z) + E_l (z, \omega, z_l) \overline{\delta \phi} (z) + F_l (z, \omega, z_l) \overline{\delta \lambda_l} (z)
\]

With the model parameterization used, the wave operator \( \overline{\delta \lambda} \) and the Fréchet derivatives for relevant model parameters are obtained from equations (6). As the expressions of \( \overline{\delta \lambda} \) are the wavefields incident on the model perturbations, and \( \overline{\delta \lambda} \) are the Green's functions propagating the scattered wavefields back from the inhomogeneities to the receivers.

The secondary Born sources are deduced from the expressions (19) by changing \( \overline{\delta \lambda} \) to \( \overline{\delta \lambda}_{\text{refletion,}} \) and \( \overline{\delta \lambda} \) to \( \overline{\delta \lambda}_{\text{reflection,}} \). The Fréchet derivatives for the horizontal displacement \( \overline{\delta \lambda} \) are easily deduced from the expressions (19) by changing \( \overline{\delta \lambda} \) to \( \overline{\delta \lambda}_{\text{reflection,}} \). In the same way, the Fréchet derivatives for the vertical and horizontal fluid displacements are obtained by changing \( \overline{\delta \lambda} \) to \( \overline{\delta \lambda}_{\text{reflection,}} \). It can be verified that the expressions corresponding to perturbations of parameters \( \lambda_0, \rho \) and \( G \) are in agreement with the Fréchet derivatives calculated in the elastic case (Dietrich and Kornmei, 1990).

Fréchet derivatives for relevant model parameters

To develop an inversion procedure for porous media, we should ideally concentrate on model parameters that are easily measurable and independent from each other. We consider here the second set of 8 parameters \( \rho_l, \rho_f, k_0, \phi, K_f, K_g, G_s \) and \( c_s \), which more naturally related to the solid and fluid phases. To obtain the corresponding Fréchet derivatives, we use a 8 x 7 Jacobian matrix \( J \) calculated from equations (2) to (5) and defined by

\[
J_{ij} = \frac{\partial p_i}{\partial p_j}
\]

where \( p_i \) is one of the parameters \( \rho_l, \rho_f, k_0, \phi, K_f, K_g, G_s \) and \( c_s \), \( \rho_f \) is one of the parameters \( \rho_l, \rho_f, k_0, \phi, K_f, K_g, G_s \). The Fréchet derivatives can be further simplified by considering that the source and the receivers are located at the same depth \( z_0 = z_8 = z_9 \). It should be noted that the Fréchet derivatives with respect to the permeability and fluid density are complex due to their role in the wave attenuation.

**NUMERICAL SIMULATIONS AND SENSITIVITY OF THE PARAMETERS**

**Computation of Green’s functions**

As mentioned in the introduction, the Green’s function for layered porous media are computed with the Generalized Reflection and Transmission Matrix Method of Kennett (1983) which yields the plane-wave response in the frequency–ray parameter (or horizontal wavenumber) domain. The seismograms in the time–distance domain are finally computed with the discrete wavenumber integration method (Bouchon, 1981). In order to test our analytical formulation and assess its limitations, we compute the differential seismograms computed with the Fréchet derivative approach with those obtained with a discrete perturbation of the medium properties. The similarity between the seismograms is evaluated by computing correlation coefficients.

**Uniform medium**

We first consider the simple case of a uniform infinite medium and small perturbations \( \delta \rho_l, \delta \rho_f, \delta k_0, \delta \phi, \delta K_f, \delta G_s, \delta \phi, \) and \( c_s \) at depth \( z = 50 \) m. Source and receivers are located at the same depth \( z_0 = 0 \) m, and therefore, we observe only reflected waves. Our simulations include \( P_{\text{reflections}}, P_{\text{directs}}, \) and \( S \)-waves whose velocities are respectively equal to 2250, 130, and 750 m/s at a frequency of 43 Hz. Three reflected waves (compressional PP, converted PS and SP, and shear SS) are easily identified. Small contrasts in \( K_f \) and \( K_s \) have no influence on shear waves, whereas changes in the other parameters mainly generate SS reflections, as seen in figure 1.

With this simple model, all correlation coefficients are greater than 99%, except for the permeability in the \( P - SS \) case. However, this derivative is more stable in the \( SH \) case. The analytical expressions of the Fréchet derivatives are thus validated.

We further check the accuracy and stability of the first-order sensitivity operators by modifying the amplitude of the perturbations, with the
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Figure 2: Comparison of the seismic sections obtained with the Fréchet derivatives and with the discrete perturbation approaches for a perturbation in solid density \( \rho_s \) at \( z = 50 \text{ m} \) in a 16-layer model.

Following results: 1) perturbations of consolidation parameter \( c \), and porosity \( \phi \) show very similar wave patterns; 2) the same is true for bulk moduli \( K_0 \) and \( K_f \); 3) Fréchet derivatives of the parameters that only influence \( P \)-waves are more stable than those that influence both \( P \)- and \( S \)-waves; 4) seismic waveforms are not exactly similar for positive and negative perturbations; 5) the first-order approximations are usually stable for parameter perturbations of up to 15%.

In this model, the wavelengths of \( P \)- and \( S \)-waves are respectively equal to 26 m and 9 m at the dominant frequency. The Fréchet derivatives are stable until the thickness of the perturbed layer reaches 20% of the dominant wavelength, which corresponds here to thicknesses of 5 m and 2 m for \( P \)- and \( S \)-waves. Derivatives with respect to \( K_0 \) and \( K_f \) remain accurate for a larger range of layer thicknesses than the derivatives with respect to other parameters which depend more strongly on \( S \)-waves.

We observe in all cases that the accuracy of the Fréchet derivatives decreases when the source-receiver offset increases (i.e., when the angle of incidence increases).

Complex model

Finally, we numerically check the stability and accuracy of the Fréchet derivative formulas in a complex model. Figure 2 is calculated for a 16-layer model and a perturbation in solid density. It is seen that the waveforms obtained with the two computation methods are very similar.

CONCLUSION

We derived analytical expressions of the Fréchet derivatives in a depth-dependent porous medium in the frequency \( \omega \) and ray parameter \( p \) domain. In the \( P - SV \) case, we obtained 64 expressions for 2 different forces, 4 displacements and 8 parameters, while in the \( SH \) case, we obtained 12 sensitivity operators. We checked the accuracy of these expressions with a purely numerical computation method and found good results as long as the Born approximation criteria are satisfied, that is, for weak and localized perturbations of the model parameters. These operators will be especially useful in full waveform inversion algorithms implemented with gradients techniques.

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