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Orthogonal decomposition of derivatives and antiderivatives for easy evaluation of extended Gram matrix

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Abstract

Simple and efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of a function with rational Laplace transform are proposed. Based on a new theorem related to Routh $\alpha - \beta$ expansion, they enable direct evaluation of the extended Gram matrix which has proved to be very useful in model-reduction applications.

1 - Introduction

Since the pioneering work of Jain [1, 2] the Gram matrix has proved to be one of the most reliable tools in the field of model reduction [3-6]; its ability to produce good pole positions has motivated a number of papers. Early work has been devoted to the Gram matrix of successive antiderivatives of a time response. Some of the authors [3] have shown that evaluation of this Gram matrix can be achieved in the frequency domain; in a sequel to [3], Lucas [7] and Sreeram and Goddard [8] presented additional simplifications in computation. Sreeram and Agathoklis [5,9] have shown the connection between the Gram matrix and the other Gramians, whereby the Gram matrix can be obtained by solving Lyapunov equations.

In [4,6] the Gram matrix has been successfully extended to include both derivatives and antiderivatives. It is the purpose of this communication to propose a straightforward orthogonal decomposition of these derivatives and antiderivatives, directly obtained from the so-called Routh $\alpha - \beta$ parameters. Owing to this orthogonal representation, an elementary and simple computation of the extended Gram matrix follows readily

2 - Background

Although the Routh $\alpha - \beta$ tables are familiar to model reduction researchers, to make this paper self contained and to state precisely some notations and numberings, the procedure is first outlined below in polynomial form; for a tabular form see [10].

Let $D(s) = \sum_{i=0}^n d_i s^i$ denote a strictly Hurwitz polynomial. Starting with its even and odd parts, $D_n(s) = d_n s^n + d_{n-2} s^{n-2} + \dots$ and $D_{n-1}(s) = d_{n-1} s^{n-1} + d_{n-3} s^{n-3} + \dots$, let a sequence of polynomials of descending degree be computed recursively by the formula

$$D_{k-1} = D_{k+1} - \alpha_k s D_k, \quad k = n-1, \dots, 1 \quad (1)$$

with $\alpha_k = lc(D_{k+1})/lc(D_k)$, $k = n-1, \dots, 0$, where $lc(P)$ denotes the leading coefficient of the polynomial P . Notice that the numbering as defined in [10] has been modified for convenience. A strictly proper rational Laplace transform $F(s) = N(s)/D(s)$ that is asymptotically stable can always be decomposed into the following form

$$F(s) = \sum_{k=0}^{n-1} \beta_k D_k(s) / D(s) \quad (2)$$

in which the coefficients β_k are uniquely determined by $N(s) = \sum_{k=0}^{n-1} \beta_k D_k(s)$. It is known [11] that the Laplace transforms $\Phi_k(s) \triangleq D_k(s) / D(s)$, $k = 0, \dots, n-1$, define a set of n orthogonal time functions $\varphi_k(t)$:

$$\langle \varphi_k, \varphi_i \rangle \triangleq \int_0^\infty \varphi_k(t) \varphi_i(t) dt = \delta_{ki} / (2\alpha_k) \quad (3)$$

As far as the authors are aware, the following theorem, which is the keystone of the two algorithms to be proposed in this paper, has never been mentioned before.

Theorem 1: Define $\varphi_{-1} \triangleq 0$ and $\varphi_n \triangleq -\varphi_{n-1}$. Then the derivative of $\varphi_k(t)$ satisfies

$$\alpha_k \frac{d\varphi_k(t)}{dt} = \varphi_{k+1}(t) - \varphi_{k-1}(t), \quad k = 0, \dots, n-1 \quad (4)$$

Proof of Theorem 1: On the understanding that $D_{-1} = 0$, eqn. 1 can be rewritten as

$$sD_k = (D_{k+1} - D_{k-1}) / \alpha_k, \quad k = 0, \dots, n-1 \quad (5)$$

Assume $k \leq n-2$. The initial value theorem for the Laplace transform yields $\varphi_k(+0) = 0$, therefore the Laplace transform of $d\varphi_k(t)/dt$ is sD_k/D . Thus, starting from eqn. 5 and dividing throughout by D , eqn. 4 is proved for $k = 0, \dots, n-2$. Now, consider $\varphi_{n-1}(+0) = d_{n-1}/d_n = 1/\alpha_{n-1}$, hence the Laplace transform of $d\varphi_{n-1}(t)/dt$ is $(sD_{n-1}/D_n) - 1/\alpha_{n-1}$ which, on account of eqn. 5 and $D_n + D_{n-1} = D$ may be written $-(D_{n-1} + D_{n-2})/(\alpha_{n-1}D)$. This achieves the proof of eqn. 4 for $k = n-1$.

An extended Gram matrix involves inner products of signals $f_i(t)$, $i \in Z$, recursively defined by $f_0(t) \triangleq f(t)$, $f_{i+1}(t) \triangleq df_i(t)/dt$ and $f_{i-1}(t) \triangleq \int_{-\infty}^t f_i(\tau) d\tau$. Applying the inverse Laplace transform to eqn. 2 yields the orthogonal decomposition $f_0(t) = \sum_{k=0}^{n-1} \beta_k \varphi_k(t)$. More generally, owing to the pole preserving property of operators d/dt and $\int_{-\infty}^t$, any $f_i(t)$ admits an orthogonal decomposition of the following form

$$f_i(t) = \sum_{k=0}^{n-1} \beta_k^i \varphi_k(t) \quad (6)$$

Starting from $\beta_k^0 \triangleq \beta_k$, the coefficients β_k^i can be efficiently computed for $i = 1, 2, \dots$ and $i = -1, -2, \dots$ by algorithms D^+ and D^- proposed below.

Algorithm D^+ : Given α_k , β_k^i and $\theta_k^i \triangleq \beta_k^i / \alpha_k$ relative to $f_i(t)$, let $\theta_{-1}^i \triangleq 0$ and $\theta_n^i \triangleq \theta_{n-1}^i$; then the following algorithm computes the β 's and θ 's relative to the derivative $f_{i+1}(t)$:

For $k = 0, \dots, n-1$

$$\begin{aligned} \beta_k^{i+1} &:= \theta_{k-1}^i - \theta_{k+1}^i \\ \theta_k^{i+1} &:= \beta_k^{i+1} / \alpha_k \end{aligned}$$

Proof of Algorithm D^+ : Differentiating eqn. 6 with respect to t , using eqn. 4 and rearranging yields $f_{i+1}(t) = \sum_{k=0}^{n-1} (\theta_{k-1}^i - \theta_{k+1}^i) \varphi_k(t)$ which achieves the proof. This algorithm takes only $(n-1)$ additive operations $(+,-)$ and n multiplicative operations $(\times, \div)$. For comparison, the standard way to compute β_k^{i+1} is to evaluate the Laplace transform of $f_{i+1}(t)$ from that of $f_i(t)$ and then to construct the related β -table. The first step takes exactly the same number of operations as algorithm D^+ . Hence, the orthogonal decomposition of $f_{i+1}(t)$ via algorithm D^+ can be obtained as cheaply as its Laplace transform.

In short, algorithm D^+ saves the $\left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$ multiplicative operations and $\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right)$ additive operations required by the β -table, where $\lfloor x \rfloor$ denotes the integer part of x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x .

Algorithm D^- : Given α_k , β_k^i and $\theta_k^i \triangleq \beta_k^i / \alpha_k$ relative to $f_i(t)$, define $\theta_{-1}^{i-1} \triangleq 0$; then the following algorithm computes β 's and θ 's relative to the antiderivative $f_{i-1}(t)$:

$$\begin{aligned} & \text{For } k = 0, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \\ & \quad \theta_{2k+1}^{i-1} := \theta_{2k-1}^{i-1} - \beta_{2k}^i \\ & \quad \text{if } n \text{ is odd then } \theta_{n-1}^{i-1} := \theta_{n-2}^{i-1} - \beta_{n-1}^i \\ & \quad \quad \text{else } \theta_{n-2}^{i-1} := \theta_{n-1}^{i-1} + \beta_{n-1}^i \\ & \text{For } k = \left\lfloor \frac{n-1}{2} \right\rfloor, \dots, 1 \\ & \quad \theta_{2k-2}^{i-1} := \theta_{2k}^{i-1} + \beta_{2k-1}^i \\ & \text{For } k = 0, \dots, n-1 \\ & \quad \beta_k^{i-1} := \alpha_k \theta_k^{i-1} \end{aligned}$$

Proof of Algorithm D^- : Replacing i by $i-1$ in algorithm D^+ we have $\beta_k^i = \theta_{k-1}^{i-1} - \theta_{k+1}^{i-1}$, $\theta_k^i = \beta_k^i / \alpha_k$.

Letting $k=0$, with $\theta_{-1}^{i-1} = 0$ in mind, yields $\theta_1^{i-1} = -\beta_0^i$; then it is a simple matter to see that the θ_k^{i-1} can be computed in succession, in the order indicated by algorithm D^- . Once again the orthogonal decomposition is obtained as cheaply as the Laplace transform and all the operations required by the β -table are saved.

Theorem 2: Let $B = [b_{ij}]$ and $\Theta = [\theta_{ij}]$ denote $(m+p+1) \times n$ matrices with (i, j) entries respectively given by $b_{ij} = \beta_{j-1}^{i-p-1}$ and $\theta_{ij} = \theta_{j-1}^{i-p-1}$. Then, denoting transposition by T , the extended Gram matrix involving m antiderivatives and p derivatives is given by

$$G(f_{-m}, \dots, f_0, \dots, f_p) = (1/2) B \Theta^T \quad (7)$$

Proof: Using eqn. 6 and the orthogonality property of eqn. 3, any entry $\langle f_i, f_j \rangle$ of G is readily written as $\langle f_i, f_j \rangle = \sum_{k=0}^{n-1} \beta_k^i \beta_k^j / (2\alpha_k) = \sum_{k=0}^{n-1} \beta_k^i \theta_k^j / 2$ by which eqn. 7 follows

3 – Illustrative examples

We first consider the transfer function given by Krajewski *et al.* in [6]

$$F(s) = \frac{s^2 + 10s + 100}{1.21s^4 + 3s^3 + 110s^2 + 230s + 100}$$

with a view to deriving $G(f_{-1}, f_0, f_1)$ and a second-order reduced model. The entries in rows 2 of B and Θ are readily obtained by the standard α - β Routh algorithm. A run of algorithm D^- yields the entries in rows 1 and a run of algorithm D^+ yields the entries in rows 3:

$$G = \frac{1}{2} \begin{bmatrix} -2.026 & -0.076 & -0.174 & -0.403 \\ 0.942 & 0.047 & 0.058 & 0.000 \\ -0.580 & 0.110 & 0.580 & 0.333 \end{bmatrix} \begin{bmatrix} -0.953 & -0.942 & -1.000 & -1.000 \\ 0.443 & 0.580 & 0.333 & 0.000 \\ -0.273 & 1.354 & 3.333 & 0.826 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1.290 & -0.5 & -0.232 \\ -0.5 & 0.232 & 0 \\ -0.232 & 0 & 1.258 \end{bmatrix}$$

After the matrix product has been computed, the method described in [4] yields the (1,-1,2) approximant whose second-order denominator matches exactly that of eqn. 44 in [6]. The squared L_2 norm of the error is equal to 1.125×10^{-2} to be compared with 1.95×10^{-2} obtained through balancing as pointed out in [6]. Note that the L_2 optimal value obtained via a Gauss-Newton nonlinear optimisation procedure is 1.099×10^{-2} .

In the case of the transfer function given by Hwang and Chen in [13],

$$F(s) = \frac{9s^3 + 42s^2 + 31s + 10}{s^4 + 8s^3 + 21s^2 + 22s + 8}$$

with a view to deriving $G(f_0, f_1, f_2)$ given by

$$G = \begin{bmatrix} 11.40 & -40.50 & 111.7 \\ -40.50 & 158.2 & -450.0 \\ 111.7 & -450.0 & 0.451 \end{bmatrix}$$

and a second order model as in the previous example, the technique described in [4] yields the (0,1,2) approximant with an error norm of 1.5142×10^{-2} while the approximant given by [13] yields 8.2904×10^{-2} .

The optimal L_2 value in this case, calculated by Lucas in [14], is 7.0135×10^{-3} .

We finally refer to Pal's celebrated example [12]:

$$F(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}$$

with the corresponding Gram matrix G :

$$G = \begin{bmatrix} 0.451 & -0.125 & -1.194 \\ -0.125 & 0.694 & -0.5 \\ -1.194 & -0.5 & 9.222 \end{bmatrix}$$

This time we use Jain's model order reduction method [2] as described in [3] which yields an error norm of 3.007×10^{-2} to be compared to Pal's original model yielding 4.098×10^{-2} . We may observe that our result is in fact very close to the optimal value obtained via Gauss-Newton optimisation providing 2.909×10^{-2} .

4 – Conclusion

Efficient algorithms for orthogonal decomposition of derivatives and antiderivatives of functions with rational Laplace transforms have been presented. A simple method for computing the extended Gram matrix follows, whereby model order reduction very close to the optimal can be carried out without any optimising iterative procedure

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